A general theory of Wilf-equivalence for Catalan structures

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joint work with Michael Albert (University of Otago)

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Enumeration sequences and Wilf-equivalence

Let $\mathcal{C}$ be any combinatorial class, *i.e.*

- $\mathcal{C}$ is equipped with a notion of size
- such that for any n there are finitely many objects of size n in $\mathcal{C}$.
- The number of objects of size n in $\mathcal{C}$ is denoted c_n.

To $\mathcal{C}$, we associate:

- its enumeration sequence (c_n),
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Sometimes (or very often!), two classes have the same enumeration sequences (or equivalently generating function).

Such enumeration coincidences are called Wilf-equivalences (terminology from the *Permutation Patterns* literature).

Our work: Wilf-equivalences among classes of restricted Catalan objects.

Motivation: from pattern-avoiding permutations

$\pi \in \mathfrak{S}_k$ is a pattern of $\sigma \in \mathfrak{S}_n$ if
$\exists\ 1 \leq i_1 < \ldots < i_k \leq n$ such that
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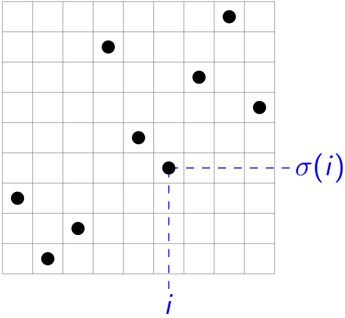

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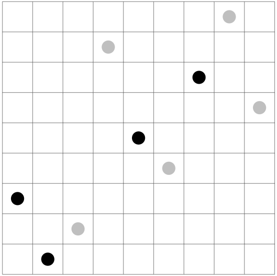

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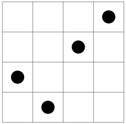

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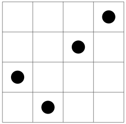

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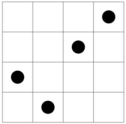

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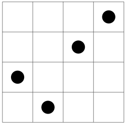

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For R and S sets of permutations, R and S (or $\mathrm{Av}(R)$ and $\mathrm{Av}(S)$) are Wilf-equivalent if $\mathrm{Av}(R)$ and $\mathrm{Av}(S)$ have the same enumeration.

Some Wilf-equivalences for pattern-avoiding permutations

Small excluded patterns:

- $\mathrm{Av}(123)$ and $\mathrm{Av}(231)$ are Wilf-equivalent, and enumerated by the Catalan numbers Cat_n
- There are three Wilf-equivalence classes for permutation classes $\mathrm{Av}(\pi)$ with π of size 4, the enumeration of $\mathrm{Av}(1324)$ being open.
- Check all Wilf-equivalences between $\mathrm{Av}(\pi, \tau)$ when π and τ have size 3 or 4 on Wikipedia.

Some results for arbitrary long patterns:

- $\mathrm{Av}(231 \oplus \pi)$ and $\mathrm{Av}(312 \oplus \pi)$ [West & Stankova 02]

First unbalanced Wilf-equivalences:

- $\mathrm{Av}(1324, 3416725)$ and $\mathrm{Av}(1234)$;
  $\mathrm{Av}(2143, 3142, 246135)$ and $\mathrm{Av}(2413, 3142)$ [Burstein & Pantone 14+]

Old Wilf-equivalences of permutation classes $\mathrm{Av}(231, \pi)$

Harmless assumption: In $\mathrm{Av}(231, \pi)$, throughout the talk, π avoids 231.
(or we are just studying $\mathrm{Av}(231)\ldots$)

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Define $C_0 = 1$ and $C_n = \frac{1}{1 - t\, C_{n-1}}$ for $n \geq 1$.

Known Wilf-equivalences: Three families of patterns π such that the generating function of $\mathrm{Av}(231, \pi)$ is C_n, where $n = |\pi|$,
[Mansour & Vainshtein 01+02; Albert & Bouvel 13]

Remark: The generating functions C_n are truncations at level n of the continued fraction defining the generating function of Catalan numbers:

$$C = \cfrac{1}{1 - \cfrac{t}{1 - \cfrac{t}{1 - \cfrac{t}{1 - \cdots}}}}.$$

New Wilf-equivalences of permutation classes $\mathrm{Av}(231, \pi)$

Our results: Unification, Generalization, Bijections

- Description of all patterns π of size n such that the generating function of $\mathrm{Av}(231, \pi)$ is C_n.
- There are exactly $Motz_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} Cat_k$ such patterns.
- Bijections between $\mathrm{Av}(231, \pi)$ and $\mathrm{Av}(231, \pi')$ for any such patterns.
- For τ of size n, the generating function of $\mathrm{Av}(231, \tau)$ either is C_n or C_n dominates it term by term (and eventually strictly).

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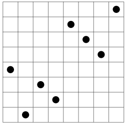
Most important remark: Classes $\mathrm{Av}(231, \pi)$ are families of Catalan objects $(\mathrm{Av}(231))$ with an additional avoidance restriction.

Main objective: Find all Wilf-equivalences between classes $\mathrm{Av}(231, \pi)$.

Equivalently (but somehow more generally), find all Wilf-equivalences between *pattern-avoiding Catalan objects*.

Substructures in Catalan objects

- 231-avoiding permutations

41327658 = 

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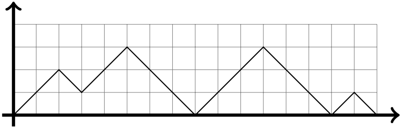

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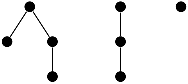

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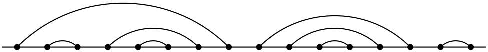

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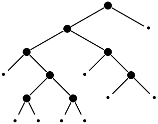

Catalan structures, and their substructures

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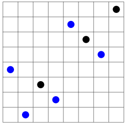
41327658 = 

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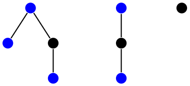

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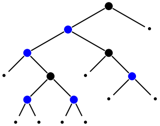

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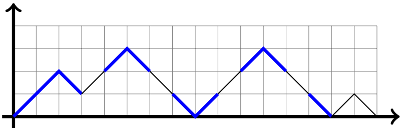

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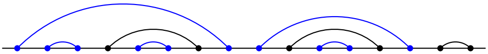

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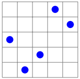
$31254 =$ 

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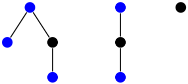

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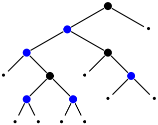

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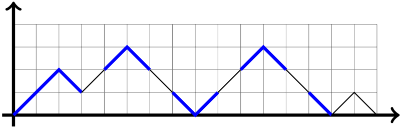

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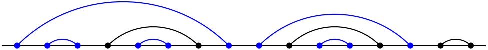

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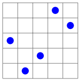
$31254 =$ 

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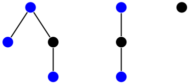

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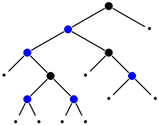

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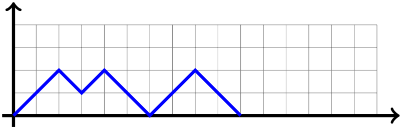

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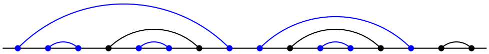

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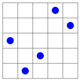
$31254 =$ 

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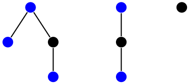

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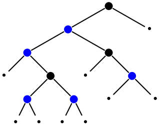

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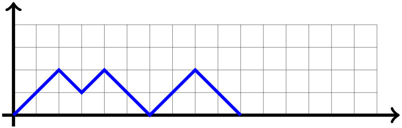

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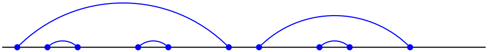

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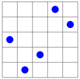
$31254 =$ 

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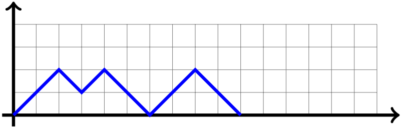

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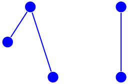

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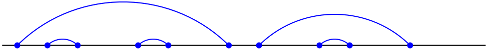

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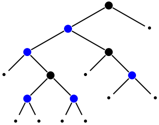

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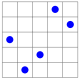
$31254 =$ 

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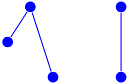

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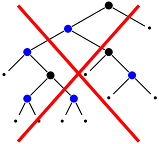

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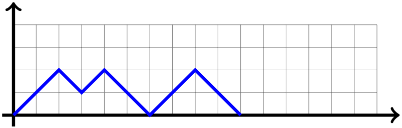

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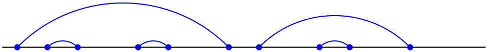

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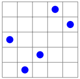
$31254 =$ 

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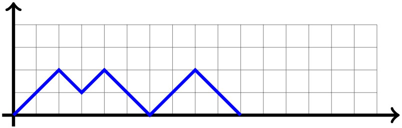

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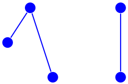

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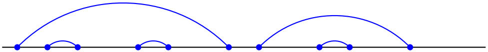

Fact: The usual bijections relating our quartet of Catalan structures preserve the substructure relation.

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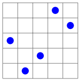
$31254 =$ 

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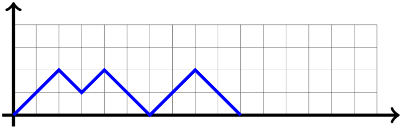

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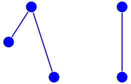

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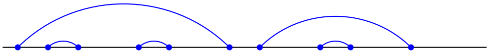

Fact: The usual bijections relating our quartet of Catalan structures preserve the substructure relation.

We will study classes $\mathrm{Av}(A)$ of arch systems avoiding some subsystem A, but all results can be translated to other structures via these bijections.

Questions addressed in this talk

- Which arch systems A are Wilf-equivalent?
  i.e. which classes $\mathrm{Av}(A)$ have the same enumeration?
- Bijections between $\mathrm{Av}(A)$ and $\mathrm{Av}(B)$ for Wilf-equivalent arch systems A and B?
- How many Wilf-equivalence classes of arch systems are there?

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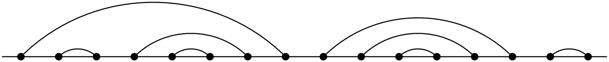
- Which arch systems A are Wilf-equivalent?
  i.e. which classes $\mathrm{Av}(A)$ have the same enumeration?
- Bijections between $\mathrm{Av}(A)$ and $\mathrm{Av}(B)$ for Wilf-equivalent arch systems A and B?
- How many Wilf-equivalence classes of arch systems are there?

Observation and terminology:

An arch system is a concatenation of atoms, i.e. (non-empty) arch systems having a single outermost arch.



An equivalence relation strongly related to Wilf-equivalence

An equivalence relation refining Wilf-equivalence

The binary relation, $\sim$, is the finest equivalence relation that satisfies:

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\begin{array}{ll}
(0) & A \sim A \\
(1) & A \sim B \implies \overparen{A} \sim \overparen{B} \\
(2) & a \sim b \implies PaQ \sim PbQ \\
(3) & PabQ \sim PbaQ \\
(4) & a\overparen{bc} \sim \overparen{ab}c
\end{array}
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where A, B, P and Q denote arbitrary arch systems
and a, b and c denote atoms or empty arch systems.

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Main theorem: If A and B are arch systems such that $A \sim B$ then $\mathrm{Av}(A)$ and $\mathrm{Av}(B)$ have the same enumeration, *i.e.* are Wilf-equivalent.

Could $\sim$ be exactly Wilf-equivalence?

In other words, $\sim$ refines Wilf-equivalence.

Conjecture: $\sim$ coincides with Wilf-equivalence.

Data, obtained with PermLab:
The conjecture holds for arch systems of size up to 15 (where $\sim$ has 16,709 equivalence classes on the $Cat_{15} = 9{,}694{,}845$ arch systems).

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Additional results:

- Asymptotic enumeration of the number of $\sim$-equivalence classes.
- $\sim$-equivalence class of arch systems of size n contains $Motz_n$ arch systems, and for A in this $\sim$-class $\mathrm{Av}(A)$ is enumerated by C_n.
- Comparison of the enumeration sequences of $\mathrm{Av}(A)$ and $\mathrm{Av}(B)$.

Idea of the proof

Overview of the proof

Main theorem: If A and B are arch systems such that $A \sim B$ then $\mathrm{Av}(A)$ and $\mathrm{Av}(B)$ have the same enumeration, *i.e.* are Wilf-equivalent.

Overview of the proof. . . by induction!

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Base case: If $A = B$ then $\mathrm{Av}(A)$ and $\mathrm{Av}(B)$ are Wilf-equivalent. . .

Inductive case: One case for each rule defining $\sim$.

Rule	bijective proof	analytic proof
(1) $A \sim B \implies A \sim B$	yes	_
(2) $a \sim b \implies PaQ \sim PbQ$	yes	_
(3) $PabQ \sim PbaQ$	yes	_
(4) $abc \sim abc$	no	yes

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(4) $a\overset{\frown}{bc} \sim \overset{\frown}{ab}c$	no	yes
(4 weak) $a\overset{\frown}{b} \sim \overset{\frown}{ba}$	yes	_

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Having only bijective proofs would allow to “unfold” the induction into a bijective proof that $\mathrm{Av}(A)$ and $\mathrm{Av}(B)$ are Wilf-equivalent, for all $A \sim B$.

Bijective proof in case (2)

$$(2) \qquad a \sim b \implies PaQ \sim PbQ$$

Take $a \sim b$ and suppose that $\mathrm{Av}(a)$ and $\mathrm{Av}(b)$ are Wilf-equivalent.

Take a size-preserving bijection $\sigma : X \mapsto X^\sigma$ from $\mathrm{Av}(a)$ to $\mathrm{Av}(b)$.

Build a size-preserving bijection τ from $\mathrm{Av}(PaQ)$ to $\mathrm{Av}(PbQ)$ as follows:

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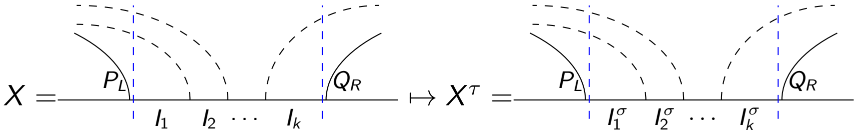
Build a size-preserving bijection τ from $\mathrm{Av}(PaQ)$ to $\mathrm{Av}(PbQ)$ as follows:

- If X avoids PQ, then take $X^\tau = X$.
- Otherwise, apply σ to all intervals determined by the arches having one extremity between the leftmost P and the rightmost Q:



- X^τ avoids PbQ if and only if X avoids PaQ.

Analytic proof in case (4)

$$(4) \quad a\overparen{bc} \sim \overparen{ab}c$$

Notations: $a = \overparen{A}$, $b = \overparen{B}$ and $c = \overparen{C}$.
$F_X =$ the generating function of $\mathrm{Av}(X)$.

We want that $F_{a\overparen{bc}} = F_{\overparen{ab}c}$.

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- Compute a system for $F_{a\overparen{bc}}$:

$$F_{a\overparen{bc}} = 1 + tF_A F_{a\overparen{bc}} + t(F_{a\overparen{bc}} - F_A)F_{\overparen{bc}}$$

$$\mathrm{Av}(a\overparen{bc}) = \varepsilon + \underbrace{\overparen{X}\,Y}_{X \text{ avoids } A} + \underbrace{\overparen{Z}\,T}_{Z \text{ contains } A}$$

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F_{bc} &= 1 + tF_B F_{bc} + t(F_{bc} - F_B)F_c \\
F_c &= 1 + tF_C F_c
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- Consequently, $F_{a\overparen{bc}} = F_{c\overparen{ab}} = F_{\overparen{ab}c}$.
- Using $F_{\overparen{X}} = 1/(1 - tF_X)$, we can write:

$$F_{a\overparen{bc}} = \frac{1 - t(F_aF_b + F_bF_c + F_cF_a - F_aF_bF_c)}{1 - t(F_a + F_b + F_c - F_aF_bF_c)}$$

How many $\sim$-equivalence classes ?

How many Wilf-equivalence classes ?

Enumeration of $\sim$-equivalence classes

Up to size 15, there are as many Wilf-equivalence as $\sim$-equivalence classes:
$1,\ 1,\ 2,\ 4,\ 8,\ 16,\ 32,\ 67,\ 142,\ 307,\ 669,\ 1\,478,\ 3\,290,\ 7\,390,\ 16\,709\ldots$

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For any size n, upper bounds on the number of Wilf-equivalence classes of classes $\mathrm{Av}(A)$, where A is an arch system with n arches are:

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$\hookrightarrow$ because rules (1), (2) and (3) encode non-plane isomorphism.

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$\hookrightarrow$ take rule (4) into account, and use [Harary, Robinson & Schwenk 75] to study the asymptotics of the coefficients of $A(t)$ defined by

$$A = t + tA + \frac{1}{t} MSet_{\geq 2}(t^2 MSet_{\geq 3}(A)) + tMSet_{\geq 3}(A)$$

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Moral of the story:
Many Wilf-equivalences between classes $\mathrm{Av}(A)$ avoiding an arch system A (or equivalently permutation classes $\mathrm{Av}(231, \pi)$)!

Summary of results and open questions

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- Extension to other contexts (*e.g.* Schröder objects and separable permutations [Albert, Homberger, Pantone], ...).