A general theory of Wilf-equivalence for Catalan structures

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joint work with Michael Albert (University of Otago)

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Let $\mathcal{C}$ be any combinatorial class, i.e.
- $\mathcal{C}$ is equipped with a notion of size
- such that for any $n$ there are finitely many objects of size $n$ in $\mathcal{C}$.
- The number of objects of size $n$ in $\mathcal{C}$ is denoted $c_n$.

To $\mathcal{C}$, we associate:
- its enumeration sequence $(c_n)$,
- its generating function $\sum c_n t^n$. 

Wilf-equivalences of Catalan structures
Enumeration sequences and Wilf-equivalence

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To $\mathcal{C}$, we associate:

- its enumeration sequence $(c_n)$,
- its generating function $\sum c_n t^n$.

Sometimes (or very often!), two classes have the same enumeration sequences (or equivalently generating function).

Such enumeration coincidences are called Wilf-equivalences (terminology from the *Permutation Patterns* literature).

Our work: Wilf-equivalences among classes of restricted Catalan objects.
Motivation: from pattern-avoiding permutations

\( \pi \in \mathcal{S}_k \) is a pattern of \( \sigma \in \mathcal{S}_n \) if
\[ \exists \ 1 \leq i_1 < \ldots < i_k \leq n \text{ such that the sequence } \sigma(i_1) \ldots \sigma(i_k) \text{ is in the same relative order as } \pi. \]
Motivation: from pattern-avoiding permutations

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Example: 2 1 3 4 is a pattern of 3 1 2 8 5 4 7 9 6.
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**Example:** 2 1 3 4 is a pattern of
\[ 3 1 2 8 5 4 7 9 6. \]

**Notation:** \( \text{Av}(\pi_1, \pi_2, \ldots) \) is the class of all permutations that do not contain \( \pi_1 \), nor \( \pi_2 \), \ldots as a pattern.
Motivation: from pattern-avoiding permutations

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\( \pi \) and \( \tau \) (or \( Av(\pi) \) and \( Av(\tau) \)) are Wilf-equivalent if \( Av(\pi) \) and \( Av(\tau) \) have the same enumeration.
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For \( R \) and \( S \) sets of permutations, \( R \) and \( S \) (or \( \text{Av}(R) \) and \( \text{Av}(S) \)) are Wilf-equivalent if \( \text{Av}(R) \) and \( \text{Av}(S) \) have the same enumeration.
Some Wilf-equivalences for pattern-avoiding permutations

Small excluded patterns:
- $\text{Av}(123)$ and $\text{Av}(231)$ are Wilf-equivalent, and enumerated by the Catalan numbers $Cat_n$.
- There are three Wilf-equivalence classes for permutation classes $\text{Av}(\pi)$ with $\pi$ of size 4, the enumeration of $\text{Av}(1324)$ being open.
- Check all Wilf-equivalences between $\text{Av}(\pi, \tau)$ when $\pi$ and $\tau$ have size 3 or 4 on Wikipedia.

Some results for arbitrary long patterns:
- $\text{Av}(231 \oplus \pi)$ and $\text{Av}(312 \oplus \pi)$ [West & Stankova 02]

First unbalanced Wilf-equivalences:
- $\text{Av}(1324, 3416725)$ and $\text{Av}(1234)$;
- $\text{Av}(2143, 3142, 246135)$ and $\text{Av}(2413, 3142)$ [Burstein & Pantone 14+]
Old Wilf-equivalences of permutation classes $\text{Av}(231, \pi)$

*Harmless assumption:* In $\text{Av}(231, \pi)$, throughout the talk, $\pi$ avoids 231.
(or we are just studying $\text{Av}(231)$...)*
Old Wilf-equivalences of permutation classes $\mathsf{Av}(231, \pi)$

**Harmless assumption:** In $\mathsf{Av}(231, \pi)$, throughout the talk, $\pi$ avoids $231$. (or we are just studying $\mathsf{Av}(231)$ . . . )

Define $C_0 = 1$ and $C_n = \frac{1}{1-t C_{n-1}}$ for $n \geq 1$.

**Known Wilf-equivalences:** Three families of patterns $\pi$ such that the generating function of $\mathsf{Av}(231, \pi)$ is $C_n$, where $n = |\pi|$,

[Mansour & Vainshtein 01+02; Albert & Bouvel 13]

**Remark:** The generating functions $C_n$ are truncations at level $n$ of the continued fraction defining the generating function of Catalan numbers:

$$C = \frac{1}{1 - \frac{t}{1 - \frac{t}{1 - \frac{t}{\ddots}}}}.$$
Our results: Unification, Generalization, Bijections

- Description of all patterns $\pi$ of size $n$ such that the generating function of $\text{Av}(231, \pi)$ is $C_n$.

- There are exactly $\text{Motz}_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \text{Cat}_k$ such patterns.

- Bijections between $\text{Av}(231, \pi)$ and $\text{Av}(231, \pi')$ for any such patterns.

- For $\tau$ of size $n$, the generating function of $\text{Av}(231, \tau)$ either is $C_n$ or $C_n$ dominates it term by term (and eventually strictly).
New Wilf-equivalences of permutation classes $\text{Av}(231, \pi)$

Our results: Unification, Generalization, Bijections

- Description of all patterns $\pi$ of size $n$ such that the generating function of $\text{Av}(231, \pi)$ is $C_n$.
- There are exactly $\text{Motz}_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \text{Cat}_k$ such patterns.
- Bijections between $\text{Av}(231, \pi)$ and $\text{Av}(231, \pi')$ for any such patterns.
- For $\tau$ of size $n$, the generating function of $\text{Av}(231, \tau)$ either is $C_n$ or $C_n$ dominates it term by term (and eventually strictly).

Most important remark: Classes $\text{Av}(231, \pi)$ are families of Catalan objects ($\text{Av}(231)$) with an additional avoidance restriction.

Main objective: Find all Wilf-equivalences between classes $\text{Av}(231, \pi)$.

Equivalently (but somehow more generally), find all Wilf-equivalences between pattern-avoiding Catalan objects.
Substructures in Catalan objects
Catalan structures, and their substructures

- 231-avoiding permutations
- Dyck paths
- Plane forests
- Arch systems
- Complete binary trees
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Catalan structures, and their substructures

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\[ 31254 = \]

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Wilf-equivalences of Catalan structures
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- 231-avoiding permutations
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Catalan structures, and their substructures

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Catalan structures, and their substructures

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Fact: The usual bijections relating our quartet of Catalan structures preserve the substructure relation.
Catalan structures, and their substructures

- 231-avoiding permutations
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Fact: The usual bijections relating our quartet of Catalan structures preserve the substructure relation.

We will study classes $\text{Av}(A)$ of arch systems avoiding some subsystem $A$, but all results can be translated to other structures via these bijections.
Questions addressed in this talk

- Which arch systems $A$ are Wilf-equivalent? i.e. which classes $Av(A)$ have the same enumeration?

- **Bijections** between $Av(A)$ and $Av(B)$ for Wilf-equivalent arch systems $A$ and $B$?

- How many Wilf-equivalence classes of arch systems are there?
Questions addressed in this talk

- Which arch systems $A$ are Wilf-equivalent?  
  i.e. which classes $A\nu(A)$ have the same enumeration?

- **Bijections** between $A\nu(A)$ and $A\nu(B)$ for Wilf-equivalent arch systems $A$ and $B$?

- How many Wilf-equivalence classes of arch systems are there?

**Observation and terminology:**
An arch system is a concatenation of **atoms**, i.e. (non-empty) arch systems having a single outermost arch.
An equivalence relation strongly related to Wilf-equivalence
The binary relation, \( \sim \), is the finest equivalence relation that satisfies:

\[
\begin{align*}
(0) & \quad A \sim A \\
(1) & \quad A \sim B \implies \overline{A} \sim \overline{B} \\
(2) & \quad a \sim b \implies PaQ \sim PbQ \\
(3) & \quad PabQ \sim PbaQ \\
(4) & \quad a\overline{bc} \sim \overline{ab}c
\end{align*}
\]

where \( A, B, P \) and \( Q \) denote arbitrary arch systems and \( a, b \) and \( c \) denote atoms or empty arch systems.
An equivalence relation refining Wilf-equivalence

The binary relation, $\sim$, is the finest equivalence relation that satisfies:

1. $A \sim A$
2. $A \sim B \implies \overline{A} \sim \overline{B}$
3. $a \sim b \implies PaQ \sim PbQ$
4. $PabQ \sim PbaQ$
5. $a\overline{bc} \sim \overline{ab}c$

where $A$, $B$, $P$ and $Q$ denote arbitrary arch systems and $a$, $b$ and $c$ denote atoms or empty arch systems.

**Main theorem:** If $A$ and $B$ are arch systems such that $A \sim B$ then $\overline{A\vee(A)}$ and $\overline{A\vee(B)}$ have the same enumeration, i.e. are Wilf-equivalent.
Could $\sim$ be exactly Wilf-equivalence?

In other words, $\sim$ refines Wilf-equivalence.

**Conjecture:** $\sim$ coincides with Wilf-equivalence.

**Data,** obtained with PermLab:
The conjecture holds for arch systems of size up to 15 (where $\sim$ has 16,709 equivalence classes on the $Cat_{15} = 9,694,845$ arch systems).

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Wilf-equivalences of Catalan structures
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**Additional results:**
- Asymptotic enumeration of the number of $\sim$-equivalence classes.
- $\sim$-equivalence class of arch systems of size $n$ contains $\text{Motz}_n$ arch systems, and for $A$ in this $\sim$-class $\text{Av}(A)$ is enumerated by $C_n$.
- Comparison of the enumeration sequences of $\text{Av}(A)$ and $\text{Av}(B)$. 
Idea of the proof
Overview of the proof

Main theorem: If $A$ and $B$ are arch systems such that $A \sim B$ then $A_v(A)$ and $A_v(B)$ have the same enumeration, i.e. are Wilf-equivalent.
Main theorem: If $A$ and $B$ are arch systems such that $A \sim B$ then $\mathcal{A}_v(A)$ and $\mathcal{A}_v(B)$ have the same enumeration, i.e. are Wilf-equivalent.

Base case: If $A = B$ then $\mathcal{A}_v(A)$ and $\mathcal{A}_v(B)$ are Wilf-equivalent.

Inductive case: One case for each rule defining $\sim$.

<table>
<thead>
<tr>
<th>Rule</th>
<th>$A \sim B \implies [A] \sim [B]$</th>
<th>bijective proof</th>
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<td>(2)</td>
<td>$PabQ \sim PbaQ$</td>
<td>yes</td>
<td>_</td>
</tr>
<tr>
<td>(3)</td>
<td>$a(bc) \sim (ab)c$</td>
<td>no</td>
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Overview of the proof... by induction!

Main theorem: If $A$ and $B$ are arch systems such that $A \sim B$ then $A_V(A)$ and $A_V(B)$ have the same enumeration, i.e. are Wilf-equivalent.

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</tr>
<tr>
<td>(4)</td>
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Overview of the proof... by induction!

Main theorem: If $A$ and $B$ are arch systems such that $A \sim B$ then $A \triangledown(A)$ and $A \triangledown(B)$ have the same enumeration, i.e. are Wilf-equivalent.

Base case: If $A = B$ then $A \triangledown(A)$ and $A \triangledown(B)$ are Wilf-equivalent...

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Having only bijective proofs would allow to “unfold” the induction into a bijective proof that $A \triangledown(A)$ and $A \triangledown(B)$ are Wilf-equivalent, for all $A \sim B$. 
(2) \( a \sim b \implies PaQ \sim PbQ \)

Take \( a \sim b \) and suppose that \( Av(a) \) and \( Av(b) \) are Wilf-equivalent.

Take a size-preserving bijection \( \sigma : X \mapsto X^\sigma \) from \( Av(a) \) to \( Av(b) \).

Build a size-preserving bijection \( \tau \) from \( Av(PaQ) \) to \( Av(PbQ) \) as follows:
Bijective proof in case (2)

$$ (2) \quad a \sim b \implies PaQ \sim PbQ $$

Take $a \sim b$ and suppose that $\Av(a)$ and $\Av(b)$ are Wilf-equivalent.

Take a size-preserving bijection $\sigma : X \mapsto X^\sigma$ from $\Av(a)$ to $\Av(b)$.

Build a size-preserving bijection $\tau$ from $\Av(PaQ)$ to $\Av(PbQ)$ as follows:

- If $X$ avoids $PQ$, then take $X^\tau = X$.
- Otherwise, apply $\sigma$ to all intervals determined by the arches having one extremity between the leftmost $P$ and the rightmost $Q$:

  $X = P_L l_1 l_2 \cdots l_k Q_R \mapsto X^\tau = P_L l_1^\sigma l_2^\sigma \cdots l_k^\sigma Q_R$

- $X^\tau$ avoids $PbQ$ if and only if $X$ avoids $PaQ$. 

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Analytic proof in case (4)

\[(4) \quad a\overline{bc} \sim \overline{ab}c\]

Notations: $a = \overline{A}$, $b = \overline{B}$ and $c = \overline{C}$.
$F_X$ = the generating function of $A\nu(X)$.

We want that $F_{a\overline{bc}} = F_{\overline{ab}c}$.
Analytic proof in case (4)

(4) $a \, bc \sim ab \, c$

Notations: $a = \overline{A}$, $b = \overline{B}$ and $c = \overline{C}$.
$F_X$ = the generating function of $Av(X)$.

We want that $F_{a\, bc} = F_{ab\, c}$.

- Compute a system for $F_{a\, bc}$:

$$F_{a\, bc} = 1 + tF_AF_{a\, bc} + t(F_{a\, bc} - F_A)F_{bc}$$

$Av(a\, bc) = \varepsilon + \overline{X}\, Y + \overline{Z}\, T$

- $X$ avoids $A$
- $Z$ contains $A$
Analytic proof in case (4)

(4) \[ a \overline{bc} \sim \overline{ab} c \]

Notations: \(a = \overline{A}, \ b = \overline{B} \) and \(c = \overline{C}\).
\(F_X = \) the generating function of \(\Lambda v(X)\).

We want that \(F_{a(bc)} = F_{(ab)c}\).

- Compute a system for \(F_{a(bc)}:\)

\[
\begin{align*}
F_{a(bc)} &= 1 + tF_A F_{a(bc)} + t(F_{a(bc)} - F_A)F_{bc} \\
F_{(bc)} &= 1 + tF_{bc} F_{(bc)} \\
F_{bc} &= 1 + tF_B F_{bc} + t(F_{bc} - F_B)F_c \\
F_c &= 1 + tF_C F_c
\end{align*}
\]
Analytic proof in case (4)

\[ a \overline{bc} \sim \overline{ab}c \]

Notations: \( a = \overline{A} \), \( b = \overline{B} \) and \( c = \overline{C} \).

\( F_X \) = the generating function of \( \text{Av}(X) \).

We want that \( F_{a(\overline{bc})} = F_{\overline{ab}c} \).

- Compute a system for \( F_{a(\overline{bc})} \):
- The solution \( F_{a(\overline{bc})} \) is a terrible mess depending in \( F_A, F_B \) and \( F_C \)
Analytic proof in case (4)

\[(4) \quad a \overline{bc} \sim \overline{ab} c\]

Notations: \(a = \overline{A}\), \(b = \overline{B}\) and \(c = \overline{C}\).

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We want that \(F_{a \overline{bc}} = F_{\overline{ab} c}\).

- Compute a system for \(F_{a \overline{bc}}\):
- The solution \(F_{a \overline{bc}}\) is a terrible mess depending in \(F_A\), \(F_B\) and \(F_C\) . . . but symmetric in \(F_A\), \(F_B\) and \(F_C\)!
- Consequently, \(F_{a \overline{bc}} = F_{c \overline{ab}} = F_{\overline{ab} c}\).
Analytic proof in case (4)

\[ a \text{bc} \sim \text{ab}c \] (4)

Notations: \( a = \overline{A}, \) \( b = \overline{B} \) and \( c = \overline{C}. \)

\( F_X \) = the generating function of \( Av(X) \).

We want that \( F_{a \text{bc}} = F_{\text{ab}c}. \)

- Compute a system for \( F_{a \text{bc}}: \)
- The solution \( F_{a \text{bc}} \) is a terrible mess depending in \( F_A, F_B \) and \( F_C \)
  
  \[ \ldots \text{but symmetric in } F_A, F_B \text{ and } F_C! \]
- Consequently, \( F_{a \text{bc}} = F_{c \text{ab}} = F_{\text{ab}c}. \)
- Using \( F_X = 1/(1 - tF_X) \), we can write:

\[
F_{a \text{bc}} = \frac{1 - t(F_aF_b + F_bF_c + F_cF_a - F_aF_bF_c)}{1 - t(F_a + F_b + F_c - F_aF_bF_c)}
\]
How many $\sim$-equivalence classes?
How many Wilf-equivalence classes?
Enumeration of $\sim$-equivalence classes

Up to size 15, there are as many Wilf-equivalence as $\sim$-equivalence classes: 1, 1, 2, 4, 8, 16, 32, 67, 142, 307, 669, 1 478, 3 290, 7 390, 16 709...
Enumeration of $\sim$-equivalence classes

Up to size 15, there are as many Wilf-equivalence as $\sim$-equivalence classes: $1, 1, 2, 4, 8, 16, 32, 67, 142, 307, 669, 1478, 3290, 7390, 16709 \ldots$

For any size $n$, upper bounds on the number of Wilf-equivalence classes of classes $Av(A)$, where $A$ is an arch system with $n$ arches are:

- $Cat_n =$ number of plane forests of size $n$: $\sim \frac{1}{\sqrt{\pi}} \cdot 4^n \cdot n^{-3/2}$
Enumeration of \(\sim\)-equivalence classes

Up to size 15, there are as many Wilf-equivalence as \(\sim\)-equivalence classes: 1, 1, 2, 4, 8, 16, 32, 67, 142, 307, 669, 1478, 3290, 7390, 16709\ldots

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- \(Cat_n =\) number of plane forests of size \(n\): \(\sim \frac{1}{\sqrt{\pi}} \cdot 4^n \cdot n^{-3/2}\)
- Number of non-plane forests of size \(n\): \(\sim 0.440 \cdot 2.9558^n \cdot n^{-3/2}\)

\(\hookrightarrow\) because rules (1), (2) and (3) encode non-plane isomorphism.

- (1) \(A \sim B \implies \overline{A} \sim \overline{B}\)
- (2) \(a \sim b \implies PaQ \sim PbQ\)
- (3) \(PabQ \sim PbaQ\)
Enumeration of $\sim$-equivalence classes

Up to size 15, there are as many Wilf-equivalence as $\sim$-equivalence classes:
1, 1, 2, 4, 8, 16, 32, 67, 142, 307, 669, 1 478, 3 290, 7 390, 16 709…

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- $Cat_n =$ number of plane forests of size $n$: $\sim \frac{1}{\sqrt{\pi}} \cdot 4^n \cdot n^{-3/2}$
- Number of non-plane forests of size $n$: $\sim 0.440 \cdot 2.9558^n \cdot n^{-3/2}$
- Number of $\sim$-equivalence classes for excluded arch systems of size $n$: $\sim 0.455 \cdot 2.4975^n \cdot n^{-3/2}$

$\hookrightarrow$ take rule (4) into account, and use [Harary, Robinson & Schwenk 75] to study the asymptotics of the coefficients of $A(t)$ defined by

$$A = t + tA + \frac{1}{t}MSet_{\geq 2}(t^2MSet_{\geq 3}(A)) + tMSet_{\geq 3}(A)$$
Enumeration of $\sim$-equivalence classes

Up to size 15, there are as many Wilf-equivalence as $\sim$-equivalence classes:

$1, 1, 2, 4, 8, 16, 32, 67, 142, 307, 669, 1478, 3290, 7390, 16709 \ldots$

For any size $n$, upper bounds on the number of Wilf-equivalence classes of classes $Av(A)$, where $A$ is an arch system with $n$ arches are:

- $Cat_n =$ number of plane forests of size $n$: $\sim \frac{1}{\sqrt{\pi}} \cdot 4^n \cdot n^{-3/2}$
- Number of non-plane forests of size $n$: $\sim 0.440 \cdot 2.9558^n \cdot n^{-3/2}$
- Number of $\sim$-equivalence classes for excluded arch systems of size $n$:
  $\sim 0.455 \cdot 2.4975^n \cdot n^{-3/2}$

Moral of the story:
Many Wilf-equivalences between classes $Av(A)$ avoiding an arch system $A$
(or equivalently permutation classes $Av(231, \pi)$)!
Summary of results and open questions

- **Main theorem**: \( \sim \) refines Wilf-equivalence between classes of Catalan objects with one excluded substructure.

- **Open**: Find a completely bijective proof of main theorem.
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- **Extension to other contexts** (e.g. Schröder objects and separable permutations [Albert, Homberger, Pantone], ...).