First-order logic for permutations

Mathilde Bouvel

talk based on joint work with M. Albert and V. Féray

Enumerative Combinatorics meeting at Oberwolfach, May 2018.
What is a permutation (of size $n$)?

- A bijection from $\{1, 2, \ldots, n\}$ to itself,
- or more generally from $X$ to $X$, for $|X| = n$.

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- A word containing exactly once each letter from $\{1, 2, \ldots, n\}$,
- or more visually a diagram.

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- Very few results consider both points of view simultaneously.
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**Goal:** Give a “proof” that the two points of view are hardly reconciled.
Formalize each point of view as a logic for permutations. More precisely, we consider two first-order (logical) theories.
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- (logical) formulas express properties of the permutations.

To prove that the two points of view are essentially different, we study the expressivity of the theories:

- describe properties expressible in each theory,
- show that the properties expressible in both theories are trivial.
Two logics for permutations
TOOB: models

TOOB: the Theory Of One Bijection  (already appeared in the literature)
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- its formulas = what the theory can say about its models  \textit{syntax}
- its models = the objects the theory talks about  \textit{interpretation}
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Pairs \((X, R_X)\) where \(X\) is a finite set and \(R_X\) a binary relation on \(X\).
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- Surjectivity: $\forall x \exists y \ y R x$
- Injectivity: $\neg \exists x, y, z (x \neq y \land x R z \land y R z)$
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\textbf{Permutations} are models, and every model is a permutation.
(Possibly, up to a conjugating by a bijection between \(X\) and \(\{1, 2, \ldots, n\}\).)

The relation \(R_{\sigma}\) associated to \(\sigma\) of size \(n\) is given by:

\[ i \ R_{\sigma} \ \sigma(i) \ \text{for all} \ i \leq n \]
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Sentences \((\psi)\) are formulas where all variables are quantified (no free variable).

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Ex.: $\phi(x) := xRx$ and $\psi := \exists x xRx$.

A model of a sentence $\psi$ is a model which in addition satisfies $\psi$.

Ex.: The models of $\exists x xRx$ are the permutations having a fixed point.
TOOB: expressivity

A property of permutations is **expressible in a theory** (here, TOOB) if it can be described by a sentence, *i.e.*, there is a sentence whose models are exactly the permutations for which this property holds.

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Definition-by-example of $|=\$: we write $\sigma |= \psi$ when $\sigma$ has a fixed point.
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But not all such! For instance, being a full cycle is not expressible.
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But not all such! For instance, being a full cycle is not expressible.

**Thm.** If \( \sigma \models \psi \), then for any \( \tau \) in the conjugacy class of \( \sigma \), \( \tau \models \psi \).

In other words, TOOB does not distinguish between conjugate permutations.
TOTO: the Theory Of Two Orders  (new as a logic for permutations)
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*(new as a logic for permutations)*

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- Axioms: ensure that $<_P$ and $<_V$ represent total orders.
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- Axioms: ensure that $<_P$ and $<_V$ represent total orders.
- Models: permutations as pairs of total orders on a finite set:
  - $<_P$ represents the position order between the elements;
  - $<_V$ represents their value order.

Ex.: $\sigma = \begin{array}{c}
\text{2} & \text{5} & \text{1} & \text{4} & \text{3}
\end{array}$ is represented for instance by $(\{a, b, c, d, e\}, \triangleleft, \bowtie)$
where $a \triangleleft b \triangleleft c \triangleleft d \triangleleft e$ and $c \bowtie a \bowtie e \bowtie d \bowtie b$. 
TOTO: the Theory Of Two Orders \((\text{new as a logic for permutations})\)

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Ex.: \(\sigma = \begin{array}{ccc}
2 & 5 & 1 \\
4 & 3 & 2 \\
\end{array}\)

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where \(a < b < c < d < e\) and \(c \bowtie a \bowtie e \bowtie d \bowtie b\).

Summary of differences:

- TOOB speaks about the cycle structure but the total order on \(\{1, 2, \ldots, n\}\) is lost.
- TOTO speaks about the relative order of the elements, but the cycle structure is lost.
Unlike TOOB, TOTO does distinguish between any two different permutations.

In other words, for any permutation $\sigma$, there exists a sentence whose only model is $\sigma$ (up to isomorphism on the ground set).
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Some concepts \textbf{expressible in TOTO}:

Containment/avoidance of a classical \textbf{pattern};

\textbf{Ex.}: Avoidance of 231 is expressed by the sentence

$$\phi_{Av(231)} := \neg \exists x \exists y \exists z \ (x <_P y <_P z) \land (z <_V x <_V y)$$
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- $\oplus$- (resp. $\ominus$-) decomposability;
- Generalization to being an inflation of $\pi$ for any $\pi$;
- Being simple;
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- Being simple;
- Being West-$k$-stack sortable, for any $k$ (+ construction of the corresponding sentences)
Inexpressibility results in TOTO
Inexpressibility of fixed points

**Thm.** There is no sentence $\psi$ in TOTO such that $\sigma \models \psi$ if and only if $\sigma$ has a fixed point.
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**Intermezzo:** Expressing properties of elements of permutations.

- A formula $\phi(x)$ with one (or several) free variable(s) expresses properties of one (or several) element(s) of a permutation.

- **Ex:** $xRx$ expresses the property that a given element is a fixed point: For $\pi$ a permutation and $a$ an element of $\pi$, we write $(\pi, a) \models \phi(x)$ when $a$ is a fixed point of $\pi$. 

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**Cor.**: There is no formula with one free variable in TOTO expressing the property that a given element is a fixed point.
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Proof strategy:

- Assume such a sentence $\psi$ exists.
  - Call $k$ its quantifier depth ($=\text{max. number of nested quantifiers in } \psi$).
- Exhibit two permutations $\sigma$ and $\sigma'$ such that
  - $\sigma$ has a fixed point but $\sigma'$ does not; and
  - $\sigma \models \psi$ if and only if $\sigma' \models \psi$.
  (Actually, $\sigma$ and $\sigma'$ satisfy the same sentences of quantifier depth at most $k$)
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To show that two permutations satisfy the same sentences, use the Ehrenfeucht-Fraïssé Theorem:

Two permutations $\sigma$ and $\sigma'$ satisfy the same sentences of quantifier depth at most $k$ if and only if Duplicator wins the EF-game with $k$ rounds on $\sigma$ and $\sigma'$. 
EF-games (a.k.a. Duplicator-Spoiler games)

The setting:

- Two players: Duplicator (D) and Spoiler (S).
- They play on a pair of permutations $\sigma$ and $\sigma'$.
- Goal of D: show that $\sigma$ and $\sigma'$ cannot be distinguish in $k$ rounds.
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At each round $i$:

- S picks an element $s_i$ in $\sigma$ or $s'_i$ in $\sigma'$;
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Winner of the EF-game with $k$ rounds:
- D if $s = (s_1, \ldots, s_k)$ and $s' = (s'_1, \ldots, s'_k)$ are isomorphic, i.e., if the position- and value-orders on $s$ and $s'$ are identical;
- S otherwise.
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Answer: $\sigma$ and $\sigma'$ are decreasing permutations of sizes $2^k - 1$ and $2^k$. 
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For $k = 3$:

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$S$ and $D$ alternate turns.
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<table>
<thead>
<tr>
<th></th>
<th>1</th>
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For $k = 3$:

$S$ and $D$ alternate turns. After 3 rounds, $D$ wins!
Intersection of TOTO and TOOB
Examples of properties expressible in one of TOOB and TOTO only:

- Having a fixed point: expressible in TOOB but not in TOTO;
- Avoiding a 231-pattern: expressible in TOTO but not in TOOB.

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- Being a transposition, i.e., being of cycle type $(2, 1^k)$ for some $k$:
  - in TOOB: $\exists x \exists y \ (x \neq y \land x Ry \land y Rx) \land (\forall z ((z \neq x \land z \neq y) \rightarrow z Rz))$
  - in TOTO:
    Express that the diagram of the permutation looks like
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Examples of properties expressible in both TOOB and TOTO:

- Being an identity permutation:
  - in TOOB: $\forall x \ xRx$
  - in TOTO: $\forall x \forall y (x <_P y \iff x <_V y)$

- Being a transposition, i.e., being of cycle type $(2, 1^k)$ for some $k$:
  - in TOOB: $\exists x \exists y (x \neq y \land xRy \land yRx) \land (\forall z ((z \neq x \land z \neq y) \rightarrow zRz))$
  - in TOTO:
    Express that the diagram of the permutation looks like

- Extension to larger cycle types $\lambda \cup (1^k)$
Which properties are expressible in both TOOB and TOTO?
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⇒ The intersection of TOOB and TOTO is trivial, so, as claimed, permutations-as-elts-of-the-symmetric-group \( \neq \) permutations-as-words.

In addition, we have a complete characterization of the properties expressible in both theories.
For any partition $\lambda$, define

- $C_\lambda$ the set of permutations of cycle-type $\lambda$;
- $D_\lambda = \bigcup_{k \geq 0} C_\lambda \cup (1^k)$. 

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**Thm.** A set $E$ of permutations is defined by a property expressible in both TOOB and TOTO if and only if it belongs to the Boolean algebra generated by all $C_\lambda$ and $D_\lambda$ (where $\lambda$ runs over all partitions).
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**Rk:** This is more precise than the previous theorem. Indeed:

- in $C_\lambda$ and $D_\lambda$ there is a bound on the size of the support.
- the property *either* $E$ contains all permutations of sufficiently large support, *or* there is a bound on the size of the support of permutations in $E$ is stable by union, intersection and complement.
For any partition $\lambda$, define

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**Rk:** This is more precise than the previous theorem.

**Tricks/tools in the proof:**

- expressing $D_\lambda$ in TOTO;
- use previous theorem to write $E$ as a finite union of $C_\lambda$’s and $D_\lambda$’s;
- and more EF games!
Some other things we know (or not)

- Characterization of the permutation classes $\mathcal{C}$ such that “having a fixed point” is expressible in the restriction of TOTO to $\mathcal{C}$. 
Characterization of the permutation classes $C$ such that “having a fixed point” is expressible in the restriction of TOTO to $C$.

The condition is: there exist $k$, $n$, $m$ such that $C$ does not contain $n$ nor $m$.

\[ \begin{array}{c}
\text{ nor } \\
\end{array} \]
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  \\
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- Formula-variant: Describe classes where TOTO can express (by $\phi(x)$) the property that a given element is a fixed point. The same as above!
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- **Formula-variant**: Describe classes where TOTO can express (by $\phi(x)$) the property that a given element is a fixed point. The same as above!

- **Extension** to description of classes where TOTO can express that two (resp. more) given elements form a transposition (resp. cycle).
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- But we don’t know in which classes the existence of a transposition (resp. cycle of a given size) is expressible in TOTO.
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- But we don’t know in which classes the existence of a transposition (resp. cycle of a given size) is expressible in TOTO.

- Further project with M. Noy: Prove convergence laws in permutation classes (for properties expressible in TOTO).