Enumeration of permutations sorted with two passes through a stack and $D_8$ symmetries

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Definitions: Permutation Patterns, Symmetries, Stack Sorting, Permutation Statistics

Classical patterns

- $\pi \in S_n$ is a pattern of $\sigma \in S_n$ if $\exists i < \ldots < i_k$ such that $\sigma_{i_1} \ldots \sigma_{i_k}$ is order isomorphic to $\pi$.

Generalized patterns

- Dashed: Add adjacency constraints between some elements $\sigma_{i_1}, \ldots, \sigma_{i_k}$.
  
  Example: $\sigma_i \sigma_j \sigma_k \sigma_l$ is an occurrence of 2-41-3 if $i = l + 1$.

- Barred: Add some absence constraints.
  
  Example: Occurrence of 35241 = occurrence of 3241 that cannot be extended to an occurrence of 35241.

- Mesh pattern: Stretched diagram with shaded cells $\blacksquare$. An occurrence of a mesh pattern is a set of points matching the diagram while leaving zones $\emptyset$ empty.

  Example: $\mu$ is a pattern of $\sigma$.

Symmetries

- Symmetries of the square transform permutations via their diagrams.

  - Reverse
  - Complement
  - Inverse

  These operators generate an 8-element group: $D_8 = \{i, r, c, i r, i o, c o, i c o, c o r\}$.

Pattern avoidance

$Av(\pi, \tau, \ldots)$ is the set of permutations that do not contain any occurrence of the (generalized) patterns $\pi, \tau, \ldots$.

Some permutation statistics

- Number of RtoL-max
- Number of LtoR-max
- Number of components
- Up-down word

\[
\begin{align*}
r_{\max}(\sigma) &= 4 \\
l_{\max}(\sigma) &= 5 \\
c(\sigma) &= 4 \\
udword(\sigma) &= d u d u d u d u a d
\end{align*}
\]

First results: Characterization and Enumeration of Permutations Sorted by $S \circ \alpha \circ S$ for any $\alpha \in D_8$

Characterization with excluded patterns

- $Id(S \circ S) = Id(S \circ i o \circ S) = Av(2341, 35241)$
- $Id(S \circ o \circ S) = Id(S \circ i o \circ S) = Av(231)$
- $Id(S \circ o \circ S) = Id(S \circ o \circ S) = Av(1342, 31-4-2)$
- $Id(S \circ o \circ S) = Id(S \circ o \circ S) = Av(3412, 3-4-21)$

Enumeration of $Id(S \circ o \circ S)$

- The sets $Id(S \circ S)$ and $Id(S \circ o \circ S)$ are enumerated by the same sequence $\frac{(2(n))}{((n+1)!)}$, $n \geq 1$.

- $Bax$ and $TBax$ are enumerated by the Baxter numbers.

Method of proof: Generating trees and rewriting systems

Generating tree for $Av(\pi, \tau, \ldots)$

- Infinite tree where
  - Vertices at level $n$ are permutations of $\pi$ avoiding $\tau, \ldots$.

- Children are obtained by insertion of a new element in an active site.

- Sites are on one of the four sides of the diagram.

Active site: when insertion does not create a pattern $\pi$ or $\tau$.

Fact: Two classes having isomorphic generating trees are in bijection.

Rewriting system for $Av(\pi, \tau, \ldots)$

- Associate labels to permutations (e.g., number of active sites).
- Find a rule that describes the labels of the children of $\sigma$ from the label of $\sigma$.

Rewriting system encoding the tree:

- Label of permutation 1.
- Succession rule(s) for the labels of the children.

Example: $Av(321)$ with insertion on the right

Generating tree

Rewriting system

- Active sites $\circ$ are the ones above all the inclusions.
- Insertion in the topmost site creates a new active site.
- Insertion in any other site creates an inversion with max($\sigma$).

Further results: Refined Enumeration According to Permutation Statistics

Statistics preserved by the bijection between $Id(S \circ o \circ S)$ and $Id(S \circ o \circ S)$

- Bijection $\Phi$ between $Id(S \circ o \circ S)$ and $Id(S \circ o \circ S)$ preserves the statistics $udword, r_{\max}, l_{\max}, zeil, ind, ndima, slmax, slmax or$. Consequently, asc, des, maj, maj or, maj or, major, major, peak, valley, peak, ddes, dasc, rir, rdr, lur, ldr are also preserved.

Hence, these statistics are all jointly equidistributed.

Proof: Plug each of these statistics in the common rewriting system.

Statistics preserved by the bijection between $Id(S \circ o \circ S)$ and $Bax$

- Pairs of twin binary trees $\leftrightarrow$ length of rightmost branch $\leftrightarrow$ number of left edges

Proof: $\Phi$ is a bijection and $\Phi$ preserves the statistics.

Evaluating the enumeration sorted with two passes through a stack and $D_8$ symmetries.