Substitution decomposition of permutations in enumerative combinatorics

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Outline

1. Introduction to substitution decomposition of permutations
   - Context
   - Definition
2. Applications in combinatorics
   - Enumerative combinatorics
     - ↪ Enumeration of simple permutations
     - ↪ Enumeration of permutation classes
   - Analytic combinatorics
     - ↪ Analysis of algorithms: Perfect sorting by reversals
3. Perspectives

Substitution decomposition

- General framework of [Möhring & Radermacher 84]
- **Modular** decomposition of graphs
- **Substitution** decomposition or **strong interval** decomposition of permutations

Relies on:

- a principle for building objects (permutations, graphs) from smaller objects: the substitution.
- some "basic objects" for this construction: **simple** permutations, **prime** graphs.

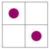
Required properties:

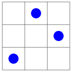
- every object can be decomposed using only "basic objects".
- this decomposition is unique.

Substitution for permutations

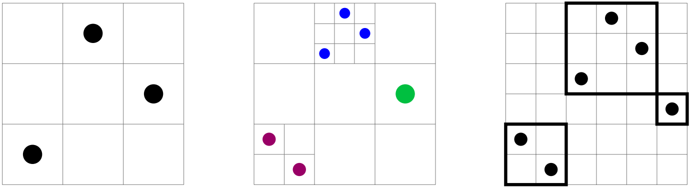
Substitution or inflation : $\sigma = \pi[\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)}]$.

Example : Here, $\pi = 1\,3\,2$, and

$\alpha^{(1)} = 2\,1 =$ 

$\alpha^{(2)} = 1\,3\,2 =$ 

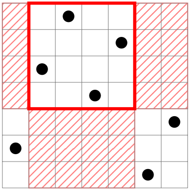
$\alpha^{(3)} = 1 =$  .



Hence $\sigma = 1\,3\,2[2\,1, 1\,3\,2, 1] = 2\,1\,4\,6\,5\,3$.

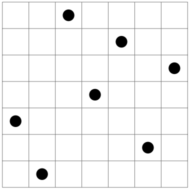
Simple permutations

Interval (or block) = set of elements of σ whose positions **and** values form intervals of integers
Example: 5 7 4 6 is an interval of 2 5 7 4 6 1 3



Simple permutation = permutation that has no interval, except the trivial intervals: $1, 2, \ldots, n$ and σ
Example: 3 1 7 4 6 2 5 is simple.

The smallest simple: 1 2, 2 1, 2 4 1 3, 3 1 4 2



Substitution decomposition of permutations

Theorem: Every σ ($\neq 1$) is uniquely decomposed as

- $12\dots k[\alpha^{(1)},\dots,\alpha^{(k)}]$, where the $\alpha^{(i)}$ are $\oplus$-indecomposable
- $k\dots 21[\alpha^{(1)},\dots,\alpha^{(k)}]$, where the $\alpha^{(i)}$ are $\ominus$-indecomposable
- $\pi[\alpha^{(1)},\dots,\alpha^{(k)}]$, where π is simple of size $k \geq 4$

Remarks:

- $\oplus$-indecomposable : that cannot be written as $12[\alpha^{(1)},\alpha^{(2)}]$
- First appeared in combinatorics in [Albert & Atkinson 05]
- The $\alpha^{(i)}$ are the maximal strong intervals of σ

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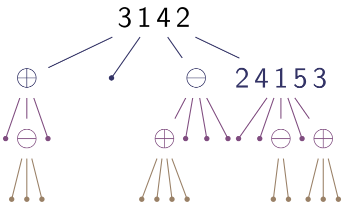
Decomposing recursively inside the $\alpha^{(i)}$ $\Rightarrow$ **decomposition tree**



- $\oplus = 12\ldots k$ and $\ominus = k\ldots 21$ = linear nodes.
- π simple of size ≥ 4 = prime node.
- No edge $\oplus - \oplus$ nor $\ominus - \ominus$.
- Ordered trees.

Computation and examples of application

Computation: in linear time. [Uno & Yagiura 00] [Bui Xuan, Habib & Paul 05] [Bergeron, Chauve, Montgolfier & Raffinot 08]

In algorithms:

- Computation of modular decomposition trees through factorizing permutations [Habib, Paul & Viennot 98] [Habib, Montgolfier & Paul 04] [Tedder, Corneil, Habib & Paul 08] [Capelle, Habib & Montgolfier 02] [Bui Xuan, Habib & Paul 05] [Bergeron, Chauve, Montgolfier & Raffinot 08]
- Pattern matching [Bose, Buss & Lubiw 98] [Ibarra 97] [B. & Rossin 06] [B., Rossin & Vialette 07]
- Algorithms for bio-informatics [Bérard, Bergeron, Chauve & Paul 07] [Bérard, Chateau, Chauve, Paul & Tannier 08] [B., Chauve, Mishna & Rossin 09]

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In combinatorics:

- Simple permutations [Albert, Atkinson & Klazar 03]
- Classes closed by substitution product [Atkinson & Stitt 02] [Brignall 07] [Atkinson, Ruškuc & Smith 09]
- Exhibit the structure of classes [Albert & Atkinson 05] [Brignall, Huczynska & Vatter 08a,08b] [Brignall, Ruškuc & Vatter 08] [Bassino, B. & Rossin 08] [Bassino, B., Pierrot & Rossin 09,10]

Substitution Decomposition in Enumerative Combinatorics

- Quick reminder on enumerative combinatorics
- Enumeration of simple permutations
- General results for the enumeration of permutation classes

Enumerative combinatorics: main ideas

$\mathcal{C}$ a family of combinatorial objects *(permutations)*

- Notion of size *($|\sigma| = n$ for σ permutation on $\{1, \ldots, n\}$)*
- For any n, finite number of objects of size n
- c_n = number of objects of size n in $\mathcal{C}$

Many ways of providing the enumeration of $\mathcal{C}$:

- Closed formula of c_n
- Recurrence satisfied by c_n
- Asymptotic equivalent of c_n
- Explicit expression of the generating function $C(z) = \sum c_n z^n$
- Equations or properties satisfied by the generating function

Enumeration of simple permutations

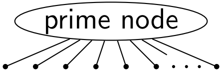
[Albert, Atkinson & Klazar 03]

- The enumeration sequence of simple permutations is not P-recursive

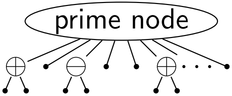
$\hookrightarrow$ No hope for a closed formula

- Asymptotic equivalent: $\frac{n!}{e^2}(1 - \frac{4}{n} + \mathcal{O}(\frac{1}{n^2}))$

$\Rightarrow$ Asymptotically, a proportion $\frac{1}{e^2}$ of decom- -position trees are reduced to one prime node.



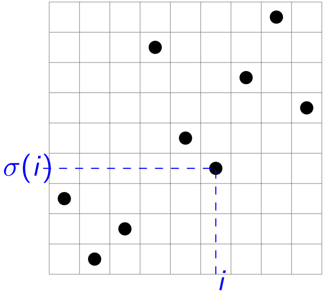
Rmk: Asymptotically, the proportion of decomposition trees of this shape is 1 [B., Chauve, Mishna & Rossin 09]



Permutations, patterns, and permutation classes

Permutation Represented by $\sigma(1)\sigma(2)\cdots\sigma(n)$ or on a grid

Pattern [Knuth 73] Sub-permutation with normalization
Example : $2\,1\,3\,4 \preccurlyeq 3\,1\,2\,8\,5\,4\,7\,9\,6$ since $3\,2\,7\,9 \equiv 2\,1\,3\,4$



$\sigma = 3\,1\,2\,8\,5\,4\,7\,9\,6$

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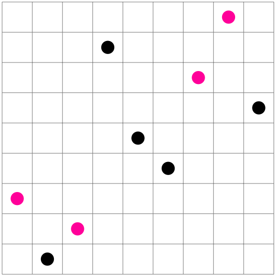

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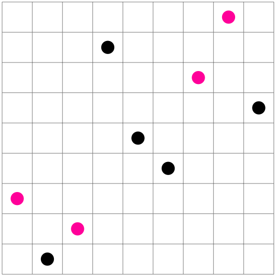

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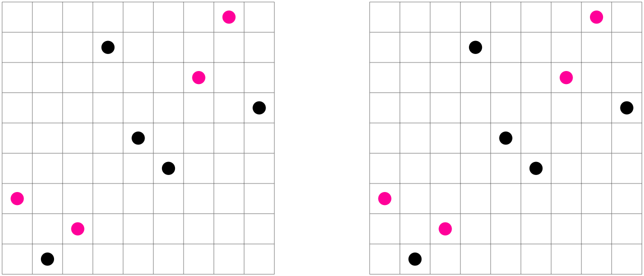

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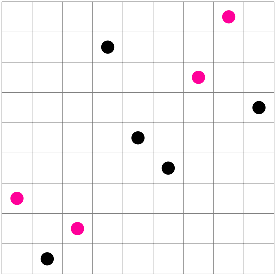

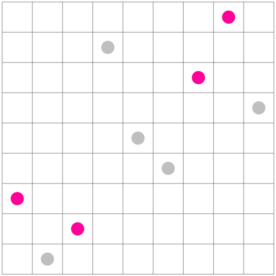

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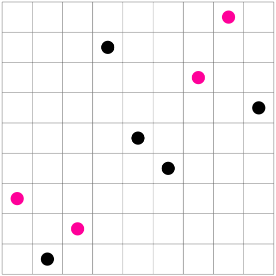

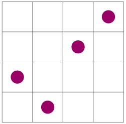

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Class Set downward-closed for $\preccurlyeq$. Characterized by a *(finite or infinite)* basis B of excluded patterns: $\mathcal{C} = Av(B)$

Some results

- Enumeration for many specific B's
- General enumerative result: Stanley-Wilf ex-conjecture [Marcus & Tardos 04] $\exists c$ s.t. $|\mathcal{C} \cap S_n| \leq c^n$
- Since 2005: Finding general properties of permutation classes

Substitution decomposition for general enumerative results

Theorem [Albert & Atkinson 05]: If $\mathcal{C}$ contains a finite number of simple permutations, then

- $\mathcal{C}$ has a finite basis
- $\mathcal{C}$ has an algebraic generating function

Proof: relies on the substitution decomposition.
Construction: compute the gen. fun. from the simples in $\mathcal{C}$

Algorithmically :

- Semi-decision procedure

$\hookrightarrow$ Find simples of size $4, 5, 6, \dots$ until k and $k+1$ for which there are 0 simples [Schmerl & Trotter 93]

- Disastrous complexity ($\sim n!$) for computing the simples in $\mathcal{C}$

Substitution decomposition for general enumerative results

Theorem [Brignall, Ruškuc & Vatter 08] : It is decidable whether $\mathcal{C}$ given by its finite basis contains a finite number of simples.

Complexity of this procedure: 2ExpTime

Improvement of the complexity:
[Bassino, B., Pierrot & Rossin 09,10]

- For substitution-closed classes: $\mathcal{O}(n \log n)$ with $n = \sum_{\pi \in B} |\pi|$
- In general: $\mathcal{O}(n^{3k})$ with $n = \max_{\pi \in B} |\pi|$ and $k = |B|$

Proof: relies on the substitution decomposition of pin-permutations

Open questions

- Efficient algorithm for finding the simples in a class instead of the $\mathcal{O}(n!)$ procedure
- Compute algorithmically the generating function automatizing the proof of [Albert & Atkinson 05]
- Perform random generation in permutation classes starting with substitution-closed classes

Substitution Decomposition in Analytic Combinatorics

- Analytic combinatorics for analysis of algorithms
- Example of the perfect sorting by reversals

Perfect sorting by reversals

Perfect reversal on a signed permutation $\sigma = \sigma(1) \dots \sigma(n)$
$=$ Reverse the orientation and the signs of a contiguous fragment of the permutation, *without breaking any interval*

Problem:

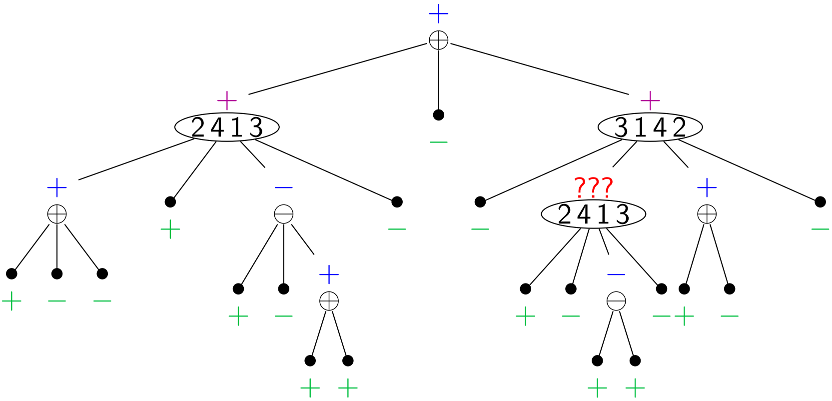
- Input: A signed permutation σ
- Output: A parsimonious perfect scenario from σ to Id or $\overline{Id}$

Complexity:

- *NP*-hard problem [Figeac & Varré 04]
- FPT algorithm [Bérard, Bergeron, Chauve & Paul 07]: uses the decomposition tree, in time $\mathcal{O}\left(2^{p} \cdot n^{\mathcal{O}(1)}\right)$
- Complexity parametrized by
  $p =$ number of prime nodes (with a prime parent)

Idea of the algorithm on an example

$$\sigma = 5\ \overline{6}\ \overline{7}\ 9\ 4\ \overline{3}\ 1\ 2\ \overline{8}\ \overline{10}\ \overline{17}\ 13\ \overline{15}\ 12\ 11\ \overline{14}\ 18\ \overline{19}\ \overline{16}$$



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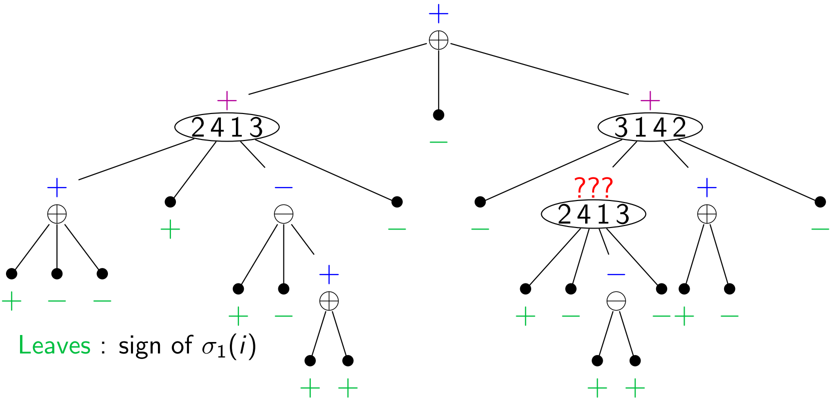

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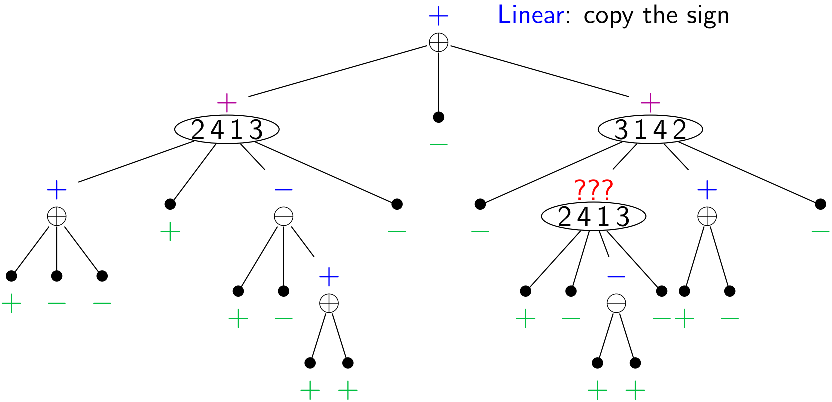

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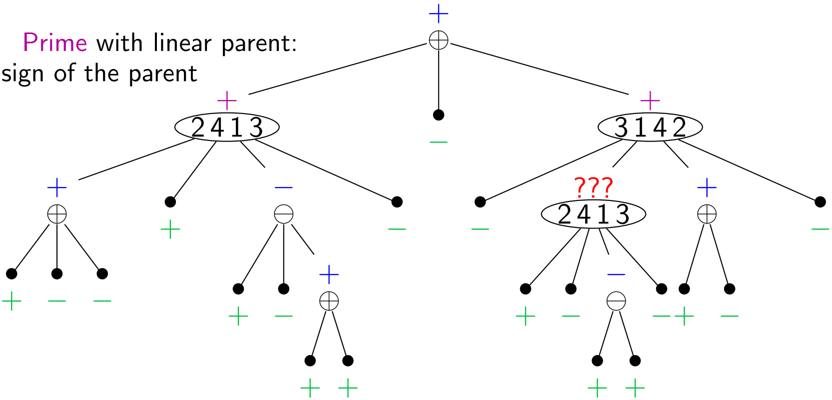
Prime with linear parent:
sign of the parent



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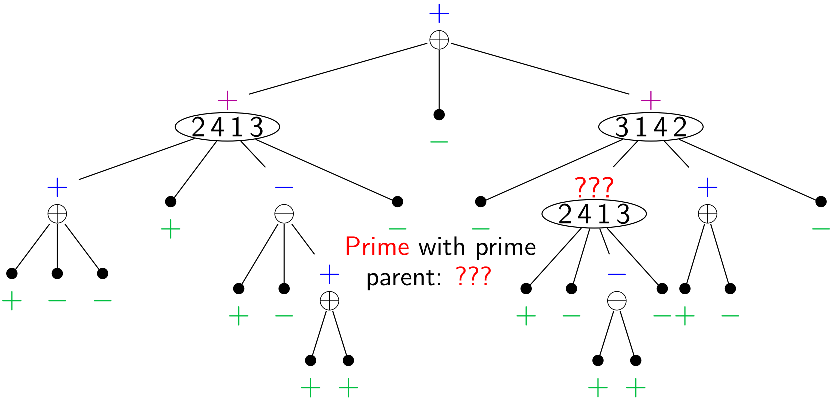

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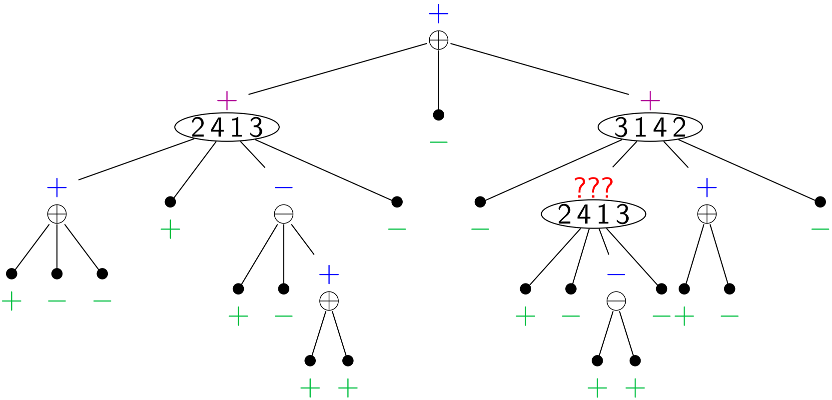

Complexity results

Previous results [Bérard, Bergeron, Chauve & Paul 07]:

- $\mathcal{O}(2^p n \sqrt{n \log n})$, where $p =$ number of prime nodes
- polynomial on separable permutations ($p = 0$)

Complexity analysis [B., Chauve, Mishna & Rossin 09] :

- polynomial on average

↪ with lemmas on the number of trees with p prime nodes

- in a parsimonious scenario for separable permutations
  - average number of reversals $\sim 1.2n$
  - average size of a reversal $\sim 1.02\sqrt{n}$

↪ with bivariate generating functions and analytic combinatorics

Probability distribution: always uniform

Average value of parameters with [Flajolet & Sedgewick 09]

Average number of reversals for separable permutations

$$= \begin{cases} \text{average number of internal nodes (except root)} \\ + \text{ average number of leaves with label different from its parent} \end{cases}$$

$$= \text{average number of internal nodes } -1 + n/2$$

Bivariate generating function: $S(x,y) = \sum s_{n,k} x^n y^k$ where $s_{n,k}$ = number of trees with n leaves and k internal nodes

Equation on $S(x,y)$ giving $S(x,y) = \frac{(x+1)-\sqrt{(x+1)^2-4x(y+1)}}{2(y+1)}$

Average number of internal nodes $= \frac{\sum_k k s_{n,k}}{\sum_k s_{n,k}} = \frac{[x^n]\frac{\partial S(x,y)}{\partial y}|_{y=1}}{[x^n]S(x,1)}$

Asymptotic equivalent $\frac{[x^n]\frac{\partial S(x,y)}{\partial y}|_{y=1}}{[x^n]S(x,1)} \sim \frac{n}{\sqrt{2}}$

Conclusion Asymptotically $\frac{1+\sqrt{2}}{2}n$ reversals on average

Open questions

- Not only the average of the parameters
  but also variance and distribution
- Extend this analysis to decomposition trees containing some (constrained) prime nodes
- Apply similar methods to other algorithms involving decomposition trees (ex: Double-Cut and Join)

From graphs to permutations and vice-versa

- Further results in combinatorics of permutation classes with substitution decomposition, making use of the many concepts and results on graph decomposition

- Investigate combinatorial or enumerative questions about graphs by adapting the methods that have been developed for permutations