Some Statistics on Permutations avoiding Generalized Patterns

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Outline

1. Introduction
2. $S(1−23)$ and the symmetry class {$1−23, 32−1, 3−21, 12−3$}
3. The two other symmetry classes
4. Permutations avoiding a pair of generalized patterns
5. Conclusion and perspectives
Outline

1. Introduction
   - Some definitions and previous results
   - Graphical representation of permutations and ECO construction

2. \( S(1-23) \) and the symmetry class \( \{1-23, 32-1, 3-21, 12-3\} \)

3. The two other symmetry classes

4. Permutations avoiding a pair of generalized patterns

5. Conclusion and perspectives
Classical Pattern Avoidance

\[ \pi \in S_n, \tau \in S_k \text{ with } k \leq n \]

- The permutation \( \pi \) contains the pattern \( \tau \) iff \( \exists \)
  \( 1 \leq i_1 < i_2 < \ldots < i_k \leq n \) such that \( \pi_{i_1}\pi_{i_2}\ldots\pi_{i_k} \) is order-isomorphic to \( \tau : \pi_{i_p} < \pi_{i_q} \) iff \( \tau_p < \tau_q \)

- Otherwise, \( \pi \) avoids \( \tau \)

- For example, \( 135624 \) contains 132 and avoids 321

Notation :
\[ S_n(\tau) = \text{the set of } \tau\text{-avoiding permutations of length } n \]
\[ S(\tau) = \text{the set of } \tau\text{-avoiding permutations} \]
Generalized Pattern Avoidance

Generalized pattern = classical pattern + dashes

- Example: $\tau = 13 - 26 - 574$ is a generalized pattern

Generalized pattern avoidance: classical pattern avoidance + the elements that are adjacent in the pattern must correspond to adjacent elements in the permutation.

- Example: $7256134$ contains $13 - 2$ ($7256134$) but avoids $1 - 32$
Three Symmetry Classes

- Reverse of a pattern $p$: $p^r = p$ read from right to left
- Complement of $p$: $p^c_i = n + 1 - p_i$ (dashes unchanged)

Generalized patterns of length 3 are organised in 3 symmetry classes $\{p, p^r, p^c, p^{rc}\}$:

- $\{1 - 23, 32 - 1, 3 - 21, 12 - 3\}$, $|S_n(p)| = B_n$ (Bell)
- $\{3 - 12, 21 - 3, 1 - 32, 23 - 1\}$, $|S_n(p)| = B_n$ (Bell)
- $\{2 - 13, 31 - 2, 2 - 31, 13 - 2\}$, $|S_n(p)| = C_n$ (Catalan)
Staff Representation of permutations

Example of 632514

\[ \text{Staff = portée pentagramma} \]
Staff Representation of permutations

Example of 632514
Staff Representation of permutations

Example of 632514

\[
\begin{align*}
&\text{Staff} = \\
&\text{portée pentagramma}
\end{align*}
\]
Staff Representation of permutations

Example of 632514
Introductions
- $S(1-23)$ and the symmetry class $\{1-23, 32-1, 3-21, 12-3\}$
- The two other symmetry classes:
- Permutations avoiding a pair of generalized patterns
- Conclusion and perspectives

Some definitions and previous results
- Graphical representation of permutations and ECO construction

**ECO construction on staff representation**

Active sites $= n + 1$ regions on the right
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ECO construction on staff representation

7426153 is obtained from 632514
In this ECO construction, starting from a $\tau$-avoiding permutation, the pattern $\tau$ can appear only if it uses the new element inserted.

It allows us to determine which of the $n+1$ regions are active sites.
Our results

- Enumeration of $S(\tau)$ according to the length and the value of the last (or the first) element for every generalized pattern $\tau$ of length 3
- Two examples of extension to permutations avoiding 2 or 3 generalized patterns
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1. Introduction

2. \(S(1−23)\) and the symmetry class \(\{1−23, 32−1, 3−21, 12−3\}\)
   - ECO construction and generating tree for \(S(1−23)\)
   - Distribution according to the length and the last value
   - The remaining patterns in the symmetry class of \(1−23\)

3. The two other symmetry classes

4. Permutations avoiding a pair of generalized patterns

5. Conclusion and perspectives
Active sites: first case

$\pi \in S_n(1 - 23)$ a permutation that ends with 1

$\pi$ generates $n + 1$ permutations of $S_{n+1}(1 - 23)$
π ∈ S_n(1−23) a permutation that ends with k ≠ 1

π generates k permutations of S_{n+1}(1−23)
Succession rule

- Each permutation of $S_n(1 - 23)$ with $k$ active sites is labelled $(k, n)$.
- Succession rule:

\[
\begin{cases}
(2, 1) \\
(k, n) \leadsto (2, n + 1)(3, n + 1) \cdots (k, n + 1)(n + 2, n + 1)
\end{cases}
\]
Introdution

The two other symmetry classes

Permutations avoiding a pair of generalized patterns

Conclusion and perspectives

Generating tree

Levels

1

2

3

4

5

2

3

4

5

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ECO construction and generating tree for $S(1-23)$

Distribution according to the length and the last value

The remaining patterns in the symmetry class of 1-23
Matrix $M$

$M = (m_{i,j})_{i,j \geq 1}$

- $m_{i,j}$ is the number of labels $j + 1$ at level $i$ in the generating tree.
- i.e. $m_{i,j}$ is the number of permutations of $S_i(1-23)$ with $j+1$ active sites.

$$
M = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \vdots \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & \vdots \\
2 & 1 & 2 & 0 & 0 & 0 & 0 & \vdots \\
5 & 3 & 2 & 5 & 0 & 0 & 0 & \vdots \\
15 & 10 & 7 & 5 & 15 & 0 & 0 & \vdots \\
52 & 37 & 27 & 20 & 15 & 52 & 0 & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$
Matrix $A$, known as the *Bell triangle*

\[
A = (a_{i,j})_{i,j \geq 1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \\
1 & 1 & 0 & 0 & 0 & 0 & \\
2 & 2 & 1 & 0 & 0 & 0 & \\
5 & 5 & 3 & 2 & 0 & 0 & \\
15 & 15 & 10 & 7 & 5 & 0 & \\
52 & 52 & 37 & 27 & 20 & 15 & \\
& & & & & & \\
& & & & & & \\
\end{pmatrix}
\]

$a_{i,j}$ is the number of $1-23$-avoiding permutations of length $i$ ending with $j$. 

...
Introducing the \textit{backward difference operator} : $\nabla$

\begin{align*}
\text{for } k \geq 3, \quad a_{n,k} &= a_{n,k-1} - a_{n-1,k-1} = \nabla a_{n,k-1} \\
\text{So recursively :} & \\
\text{for } k \geq 3, \quad a_{n,k} &= \nabla a_{n,k-1} \\
&= \nabla^2 a_{n,k-2} \\
&= \ldots \\
&= \nabla^{k-2} a_{n,2} = \nabla^{k-2} B_{n-1} \quad (\text{which holds also for } k = 2)
\end{align*}
Stating our first result

The distribution of $1-23$-avoiding permutations according to their length and to the value of their last entry is given by:

$$|\{\pi \in S_n(1-23) : \pi_n = 1\}| = B_{n-1}, \ n \geq 1;$$

$$|\{\pi \in S_n(1-23) : \pi_n = k\}| = \nabla^{k-2}(B_{n-1}), \ 2 \leq k \leq n.$$
If $\pi \in S_n(1−23)$ ends with $k$, then $\pi^r \in S_n(32−1)$, and $\pi_1^r = k$. Consequently:

$$|\{\pi \in S_n(32−1): \pi_1 = 1\}| = B_{n−1}, \quad n \geq 2$$

$$|\{\pi \in S_n(32−1): \pi_1 = k\}| = \nabla^{k−2}(B_{n−1}), \quad 2 \leq k \leq n$$
**S(3 – 21) and S(12 – 3)**

- **Complement:**
  
  \[ |\{\pi \in S_n(3-21) : \pi_n = n\}| = B_{n-1}, \ n \geq 1 \]

  \[ |\{\pi \in S_n(3-21) : \pi_n = k\}| = \nabla^{n-k-1}(B_{n-1}), \ 1 \leq k \leq n-1 \]

- **Reverse-complement:**
  
  \[ |\{\pi \in S(12-3) : \pi_1 = n\}| = B_{n-1}, \ n \geq 1 \]

  \[ |\{\pi \in S(12-3) : \pi_1 = k\}| = \nabla^{n-k-1}(B_{n-1}), \ 1 \leq k \leq n-1 \]
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3. The two other symmetry classes
   - The symmetry class $\{3 - 12, 21 - 3, 1 - 32, 23 - 1\}$
   - The symmetry class $\{2 - 13, 31 - 2, 2 - 31, 13 - 2\}$
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Same ideas

- One pattern in the class
- Succession rule
- Matrix of the distribution
- Recursive relation defining the entries of the matrix
- Extension to the remaining patterns in the symmetry class
The distribution of permutations avoiding $3 - 12$ according to their length (row index) and their last value (column index) is given by:

$$M = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \\
1 & 1 & 0 & 0 & 0 & 0 & \\
2 & 1 & 2 & 0 & 0 & 0 & \\
5 & 3 & 2 & 5 & 0 & 0 & \\
15 & 10 & 7 & 5 & 15 & 0 & \\
52 & 37 & 27 & 20 & 15 & 52 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\end{pmatrix}$$
The distribution of permutations avoiding $2-13$ according to their length (row index) and their last value (column index) is given by:

$$M' = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \vdots \\
1 & 1 & 0 & 0 & 0 & 0 & \vdots \\
2 & 2 & 1 & 0 & 0 & 0 & \vdots \\
5 & 5 & 3 & 1 & 0 & 0 & \vdots \\
14 & 14 & 9 & 4 & 1 & 0 & \vdots \\
42 & 42 & 28 & 14 & 5 & 1 & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$
Outline

1 Introduction

2 $S(1 - 23)$ and the symmetry class $\{1 - 23, 32 - 1, 3 - 21, 12 - 3\}$

3 The two other symmetry classes

4 Permutations avoiding a pair of generalized patterns
   - $S(1 - 23, 1 - 32)$: an easy case
   - $S(1 - 23, 21 - 3) = S(1 - 23, 21 - 3, 12 - 3)$: a not so easy case

5 Conclusion and perspectives
Avoiding more than one pattern

- Claesson and Mansour [2003]: enumeration of permutations avoiding any pair of generalized patterns of length 3, according to their length
- Bernini, Ferrari and Pinzani [2005]: enumeration of permutations avoiding any triple of generalized patterns of length 3, according to their length

Refine those enumerations according to the first or last entry? Two examples.
Labelling and succession rule

- \(|S_n(1-23, 1-32)| = I_n\) n-th involution number

\(\pi \in S(1-23, 1-32)\) is labelled \((k, n)\) where \(k\) is the number of active sites of \(\pi\).

- \(k = 1\) when \(\pi_n \neq 1\)
- \(k = n + 1\) when \(\pi_n = 1\)

Succession rule:

\[
\begin{align*}
(2, 1) & \mapsto (n + 2, n + 1) \\
(1, n) & \mapsto (1, n + 1) \cdot (n + 2, n + 1)
\end{align*}
\]
Subsequent matrix

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & \cdots \\
6 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & \cdots \\
16 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & \cdots \\
50 & 0 & 0 & 0 & 0 & 0 & 26 & 0 & \cdots \\
156 & 0 & 0 & 0 & 0 & 0 & 0 & 76 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
\]
Main steps

\[ |S_n(1-23, 21-3)| = |S_n(1-23, 21-3, 12-3)| = M_n \]

\[ n \text{-th Motzkin number} \]

- Succession rule with coloured labels.
- Generating tree.
- Matrix recording the number of labels at each level in the tree.
- Interpretation of this matrix as the distribution of 
  \( S(1-23, 21-3) \) according to the length and the last value
- Recursive description of the entries of the matrix.
- Generating function of each column of the matrix.
Distribution of $S(1 - 23, 21 - 3)$ according to the length and the last value

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
2 & 2 & 0 & 0 & 0 & 0 & 0 & \cdots \\
4 & 4 & 1 & 0 & 0 & 0 & 0 & \cdots \\
9 & 9 & 3 & 0 & 0 & 0 & 0 & \cdots \\
21 & 21 & 8 & 1 & 0 & 0 & 0 & \cdots \\
51 & 51 & 21 & 4 & 0 & 0 & 0 & \cdots \\
127 & 127 & 55 & 13 & 1 & 0 & 0 & \cdots \\
323 & 323 & 145 & 39 & 5 & 0 & 0 & \cdots \\
835 & 835 & 385 & 113 & 19 & 1 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]
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For any generalized pattern $p$ of length 3, distribution of the $p$-avoiding permutations according to the length and the value of the first or last element.

Similar distributions for two sets of patterns.

Can we get such a distribution for other sets of up to 3 patterns? For all of them?