Tri bulle et classes de permutations

Mathilde Bouvel

LaBRI, CNRS

Travail en collaboration avec
M.H. Albert, M.D. Atkinson, A. Claesson et M. Dukes

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The Bubble Sort Operator \( B \)

\( B = \) one pass of \textbf{bubble sort}.

On sequences that are \textbf{permutations}.

\textbf{Definition(s):}

- Algorithmically:
  \[ B \text{ processes a permutation } \sigma \text{ from left to right, and modifies } \sigma \text{ dynamically exchanging } \sigma(i) \text{ and } \sigma(i+1) \text{ when } \sigma(i) > \sigma(i+1). \]

- Recursively:
  \[
  \begin{align*}
  B(\sigma_1 n \sigma_2) &= B(\sigma_1)\sigma_2 n \text{ if } \sigma = \sigma_1 n \sigma_2 \in S_n \\
  B(\varepsilon) &= \varepsilon
  \end{align*}
  \]

- Explicitely:
  If \( \sigma = n_1 \lambda_1 n_2 \lambda_2 \cdots n_k \lambda_k \) where \( n_1, \ldots, n_k \) are the left to right maxima of \( \sigma \) then \( B(\sigma) = \lambda_1 n_1 \lambda_2 n_2 \cdots \lambda_k n_k \).

\textbf{NB Stack-sorting operator} \( S \)

\[
S(\sigma_1 n \sigma_2) = S(\sigma_1)S(\sigma_2)n
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$B = \text{one pass of bubble sort.}$

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  - $B(\sigma_1 n \sigma_2) = B(\sigma_1) \sigma_2 n$ if $\sigma = \sigma_1 n \sigma_2 \in S_n$
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- Explicitely:
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Permutation Classes

Permutations

- \( S_n = \) permutations \( \sigma \) of \( \{1, 2, \ldots, n\} \)
- Representation by a word: \( \sigma(1)\sigma(2) \cdots \sigma(n) \), by its diagram, . . .

Patterns

- Subpermutation of \( \sigma \)
- Subword or subset of points of the diagram that is normalized

Example: \( 2134 \preccurlyeq 312854796 \) since \( 3279 \equiv 2134 \)

\[
\begin{array}{cccccccc}
\sigma(1) & \sigma(2) & \sigma(3) & \sigma(4) & \sigma(5) & \sigma(6) & \sigma(7) & \sigma(8) \\
\end{array}
\]

\( \sigma = 312854796 \)
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Example: $2 1 3 4 \preceq 3 1 2 8 5 4 7 9 6$ since $3 2 7 9 \equiv 2 1 3 4$

$$\sigma = 3 1 2 8 5 4 7 9 6$$
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Example: $2 \ 1 \ 3 \ 4 \ < \ 3 \ 1 \ 2 \ 8 \ 5 \ 4 \ 7 \ 9 \ 6$ since $3 \ 2 \ 7 \ 9 \ \equiv \ 2 \ 1 \ 3 \ 4$

![Diagram showing patterns and permutations](image-url)
Permutation Classes

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- $S_n =$ permutations $\sigma$ of $\{1, 2, \ldots, n\}$
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Patterns

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Example: $2134 \preceq 312854796$ since $3279 \equiv 2134$

Occurrence of a pattern

- Occurrence = subpermutation without normalization

Example: $3279 \subseteq 312854796$

Classes

- Subset of $\mathbb{S} = \cup_n S_n$ downward closed for $\preceq$
- Characterization by a basis of excluded patterns: $C = \text{Av}(B)$
- Principal classes: $C = \text{Av}(\pi)$
Permutation Classes

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Classes
- Subset of $S = \bigcup_n S_n$ downward closed for $\preccurlyeq$
- Characterization by a basis of excluded patterns: $C = Av(B)$
- Principal classes: $C = Av(\pi)$
Proposition

The permutations that are sorted by $B$ are a class.
Namely: $B(\sigma) = \text{Id}$ iff $\sigma \in \text{Av}(231, 321)$.

Proof: by induction.

Decompose $\sigma = \sigma_1 n \sigma_2$ around its maximum $n$.

Recall that $B(\sigma) = B(\sigma_1)\sigma_2 n$.

- $\sigma$ is sorted by $B$

  $\iff \sigma_1$ is sorted by $B$, $\sigma_2$ is increasing, and $\sigma_1 < \sigma_2$

  $\iff \sigma_1 \in \text{Av}(231, 321)$, $\sigma_2$ is increasing, and $\sigma_1 < \sigma_2$

  $\iff \sigma \in \text{Av}(231, 321)$
**Proposition**

The permutations that are sorted by $B$ are a class. Namely: $B(\sigma) = Id$ iff $\sigma \in Av(231, 321)$.

**Proof**: by induction.

Decompose $\sigma = \sigma_1 n \sigma_2$ around its maximum $n$.

Recall that $B(\sigma) = B(\sigma_1)\sigma_2 n$.

- $\sigma$ is sorted by $B$ if and only if
  - $\sigma_1$ is sorted by $B$, $\sigma_2$ is increasing, and $\sigma_1 < \sigma_2$
  - $\sigma_1 \in Av(231, 321)$, $\sigma_2$ is increasing, and $\sigma_1 < \sigma_2$
  - $\sigma \in Av(231, 321)$
Motivation and main result

- $B$-sortable permutations

$\leftarrow B^{-1}(Av(21)) = Av(231, 321)$

- $SB$-sortable permutations?

$\leftarrow (SB)^{-1}(Av(21)) = B^{-1}(Av(231))$

- $B^2$-sortable permutations?

$\leftarrow (BB)^{-1}(Av(21)) = B^{-1}(Av(231, 321))$

- In general, what can we say about $B^{-1}(C)$?

For $C = Av(\pi)$ a principal permutation class, we can determine

- when $B^{-1}(Av(\pi))$ is a class,

- and in this case give its basis.

This result is proved by considering the LtoR-maxima of $\pi$. 
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- when $B^{-1}(\text{Av}(\pi))$ is a class,
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Summary of results

<table>
<thead>
<tr>
<th>$\pi$</th>
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<tbody>
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<td>1</td>
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Remarks: $n, (n - 1), (n - 2), a, b$ and $c$ are LtoR-maxima.

If $\pi = [\alpha \beta]$, then $R(\pi)$ is the set of permutations $[\alpha \beta]$. 
**Proposition**

There are no permutations $\sigma$ of length $n \geq 1$ such that $B(\sigma)$ avoids 1. Hence $B^{-1}(Av(1)) = \{\varepsilon\} = Av(1)$.

**Proposition**

The only permutations $\sigma$ such that $B(\sigma)$ avoids 12 are $\varepsilon$ and 1. Hence $B^{-1}(Av(12)) = \{\varepsilon, 1\} = Av(12, 21)$.

**Proof**: $B(\sigma)$ always ends with its maximum.

**Proposition**

The permutations $\sigma$ such that $B(\sigma)$ avoids 21 are the $B$-sortable permutations. Hence $B^{-1}(Av(21)) = Av(231, 321)$. 
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The permutations $\sigma$ such that $B(\sigma)$ avoids 21 are the $B$-sortable permutations.
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Patterns \( \pi \in S_n \) ending with \( n \) but not with \((n-1)n\)

**Lemma**

If \( \pi \in S_n \) with \( n \geq 3 \) is such that \( \pi(n) = n \) but \( \pi(n-1) \neq n-1 \), then setting \( \pi' = \pi(1)\pi(2)\ldots\pi(n-1) \) we have \( B^{-1}(Av(\pi)) = B^{-1}(Av(\pi')) \).

**Proof:**

- \( \sigma \in B^{-1}(Av(\pi')) \Rightarrow B(\sigma) \) avoids \( \pi' \)
  - \( \Rightarrow B(\sigma) \) avoids \( \pi \Rightarrow \sigma \in B^{-1}(Av(\pi)) \)

- \( \sigma \in B^{-1}(Av(\pi)) \Rightarrow B(\sigma) \) avoids \( \pi = \pi'n \)
  - But \( B(\sigma) = B(\sigma_1)\sigma_2m \) ends with its maximum \( m \).
  - Hence \( B(\sigma_1)\sigma_2 \) avoids \( \pi' \).
  - But \( \pi' \) does not end with its maximum.
  - Hence \( B(\sigma) = B(\sigma_1)\sigma_2m \) avoids \( \pi' \) and \( \sigma \in B^{-1}(Av(\pi')) \).

This lemmas applies in particular for \( \pi = (n-1)\alpha n \) with \( \alpha \neq \varepsilon \) and \( \pi = a\alpha b\beta n \) with \( \beta \neq \varepsilon \).
**Lemma**

If $\pi \in S_n$ with $n \geq 3$ is such that $\pi(n) = n$ but $\pi(n-1) \neq n-1$, then setting $\pi' = \pi(1)\pi(2)\ldots\pi(n-1)$ we have $B^{-1}(\mathrm{Av}(\pi)) = B^{-1}(\mathrm{Av}(\pi'))$.

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- $\sigma \in B^{-1}(\mathrm{Av}(\pi')) \Rightarrow B(\sigma)$ avoids $\pi'$
  $\Rightarrow B(\sigma)$ avoids $\pi$ $\Rightarrow \sigma \in B^{-1}(\mathrm{Av}(\pi))$

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Patterns $\pi \in S_n$ ending with $n$ but not with $(n - 1)n$

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If $\pi \in S_n$ with $n \geq 3$ is such that $\pi(n) = n$ but $\pi(n - 1) \neq n - 1$, then setting $\pi' = \pi(1)\pi(2)\ldots\pi(n - 1)$ we have $B^{-1}(Av(\pi)) = B^{-1}(Av(\pi'))$.

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Patterns $\pi \in S_n$ with at least three LtoR-maxima $\neq \pi(n)$

**Proposition**

If $\pi = a\alpha b\beta c\gamma$, with $a$, $b$ and $c$ the first three LtoR-maxima of $\pi$ and $\gamma \neq \varepsilon$, then $B^{-1}(Av(\pi))$ is not a class.

**Proof**:

By the previous lemma, we may assume that $\pi = a\alpha b\beta c\gamma n$.

Set $\theta_1 = ba\alpha n\beta c\gamma$ and $\theta_2 = (n+1)\theta_1$. Notice that $\theta_1 \preceq \theta_2$.

- Clearly, $B(\theta_1) = \pi$ and $\theta_1 \notin B^{-1}(Av(\pi))$.
- $B(\theta_2) = ba\alpha n\beta c\gamma(n+1)$
  
  Since $B(\theta_2)$ is only one term longer than $\pi$, we easily check that $B(\theta_2)$ avoids $\pi$. Hence $\theta_2 \in B^{-1}(Av(\pi))$.

We have $B^{-1}(Av(\pi)) \not\ni \theta_1 \preceq \theta_2 \in B^{-1}(Av(\pi))$. Consequently, $B^{-1}(Av(\pi))$ is not a class.
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Common framework for the remaining cases

Lemma
For any pattern $\pi$, if there exists a set $\mathcal{R}$ of permutations such that
\[ \forall \sigma, \pi \preceq B(\sigma) \iff \rho \preceq \sigma \text{ for some } \rho \in \mathcal{R}, \text{ then } B^{-1}(\text{Av}(\pi)) \text{ is a class}. \]
Furthermore, if $\mathcal{R}$ is minimal, it is the basis of $B^{-1}(\text{Av}(\pi))$.

Proof: Show that $B^{-1}(\text{Av}(\pi))$ is downward closed for $\preceq$.

\[
\begin{align*}
\sigma &\not\in B^{-1}(\text{Av}(\pi)) \\
\iff &\ B(\sigma) \not\in \text{Av}(\pi) \\
\iff &\ \pi \preceq B(\sigma) \\
\iff &\ \exists \rho \in \mathcal{R}, \rho \preceq \sigma
\end{align*}
\]
so that $\sigma \in B^{-1}(\text{Av}(\pi)) \iff \forall \rho \in \mathcal{R}, \rho \not\preceq \sigma$.
This shows that $B^{-1}(\text{Av}(\pi))$ is a downset, hence a class.
This also shows that the minimal $\mathcal{R}$ is its basis.
Common framework for the remaining cases

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For any pattern $\pi$, if there exists a set $\mathcal{R}$ of permutations such that

$$\forall \sigma, \pi \preceq B(\sigma) \iff \rho \preceq \sigma \text{ for some } \rho \in \mathcal{R},$$

then $B^{-1}(\text{Av}(\pi))$ is a class. Furthermore, if $\mathcal{R}$ is minimal, it is the basis of $B^{-1}(\text{Av}(\pi))$.

**Proof:** Show that $B^{-1}(\text{Av}(\pi))$ is downward closed for $\preceq$.

$$\sigma \notin B^{-1}(\text{Av}(\pi))$$

$$\iff B(\sigma) \notin \text{Av}(\pi)$$

$$\iff \pi \preceq B(\sigma)$$

$$\iff \exists \rho \in \mathcal{R}, \rho \preceq \sigma$$

so that $\sigma \in B^{-1}(\text{Av}(\pi)) \iff \forall \rho \in \mathcal{R}, \rho \not\preceq \sigma$.

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Patterns $\pi \in S_n$ starting with $n$

**Proposition**

If $\pi \in S_n$ is such that $\pi = n\alpha$ for $\alpha \neq \varepsilon$, then $B^{-1}(\text{Av}(\pi))$ is a class whose basis is $\{n(n + 1)\alpha, (n + 1)n\alpha\}$.

**Lemma**

If $\pi \preceq B(\sigma)$, consider an occurrence $p\lambda \subseteq B(\sigma)$. Then there exists $q > p > \lambda$ such that $pq\lambda \subseteq \sigma$ or $qp\lambda \subseteq \sigma$. Hence $n(n + 1)\alpha$ or $(n + 1)n\alpha \preceq \sigma$.

**Lemma**

If $n(n + 1)\alpha$ or $(n + 1)n\alpha \preceq \sigma$, consider an occurrence $pq\lambda$ or $qp\lambda \subseteq \sigma$. Then $p\lambda \subseteq B(\sigma)$. Hence $\pi \preceq B(\sigma)$. 
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If $\pi \preceq B(\sigma)$, consider an occurrence $p\lambda \subseteq B(\sigma)$.
Then there exists $q > p > \lambda$ such that $pq\lambda \subseteq \sigma$ or $qp\lambda \subseteq \sigma$.
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**Lemma**

If $n(n+1)\alpha$ or $(n+1)n\alpha \preceq \sigma$, consider an occurrence $pq\lambda$ or $qp\lambda \subseteq \sigma$.
Then $p\lambda \subseteq B(\sigma)$.
Hence $\pi \preceq B(\sigma)$. 
Proof of the first lemma for $\pi = n\alpha$ with $\alpha \neq \varepsilon$

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Then there exists $q > p > \lambda$ such that $pq\lambda \subseteq \sigma$ or $qp\lambda \subseteq \sigma$.

Hence $n(n + 1)\alpha$ or $(n + 1)n\alpha \preceq \sigma$.

**Proof:** by induction on $|\sigma|$.

- If $|\sigma| \leq 2$, result vacuously true (since $B(\sigma)$ ends with its maximum).
- If $\sigma = \sigma_1m\sigma_2$ with $m = |\sigma| > 2$, then $p\lambda \subseteq B(\sigma_1)\sigma_2m$.
  Because $p\lambda$ does not end with its maximum, $p\lambda \subseteq B(\sigma_1)\sigma_2$.
  - If $\lambda = \lambda_1\lambda_2$ with $\lambda_1 \neq \varepsilon$, $p\lambda_1 \subseteq B(\sigma_1)$ and $\lambda_2 \subseteq \sigma_2$, then by induction $p\lambda_1 \subseteq B(\sigma_1)$ implies that $\exists q > p$ such that $pq\lambda_1 \subseteq \sigma_1$ or $qp\lambda_1 \subseteq \sigma_1$.
    Hence $\sigma = \sigma_1m\sigma_2$ contains an occurrence of $pq\lambda_1\lambda_2$ or of $qp\lambda_1\lambda_2$.
  - If $p \subseteq B(\sigma_1)$ and $\lambda \subseteq \sigma_2$, then $p \subseteq \sigma_1$ and $pm\lambda \subseteq \sigma_1m\sigma_2 = \sigma$.
  - If $p\lambda \subseteq \sigma_2$, then $mp\lambda \subseteq m\sigma_2 \subseteq \sigma$. 

Mathilde Bouvel (LaBRI, CNRS)
Proof of the second lemma for $\pi = n\alpha$ with $\alpha \neq \varepsilon$

**Lemma**

If $n(n + 1)\alpha$ or $(n + 1)n\alpha \preceq \sigma$, consider an occurrence $pq\lambda$ or $qp\lambda \subseteq \sigma$. Then $p\lambda \subseteq B(\sigma)$. Hence $\pi \preceq B(\sigma)$.

**Proof:**

Recall that if $\sigma = n_1\lambda_1 n_2\lambda_2 \cdots n_k\lambda_k$ where $n_1, \ldots, n_k$ are the left to right maxima of $\sigma$ then $B(\sigma) = \lambda_1 n_1\lambda_2 n_2 \cdots \lambda_k n_k$. Hence, the order of the elements not LtoR-maxima is preserved by $B$.

- If $qp\lambda \subseteq \sigma$, $p\lambda$ are not LtoR-maxima. Hence $p\lambda \subseteq B(\sigma)$.
- This also holds when $pq\lambda \subseteq \sigma$ and $p$ is not a LtoR-maximum.
- If $pq\lambda \subseteq \sigma$ and $p$ is a LtoR-maximum, then there exists some $r$ between $p$ and $q$ (possibly $r = q$) in $\sigma$ that is a LtoR-maximum. This implies that $p$ still precedes $\lambda$ in $B(\sigma)$, hence $p\lambda \subseteq B(\sigma)$.
## Summary of results

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A set of points in the plane that are pairwise neither horizontally nor vertically aligned represents a permutation.

When some points are horizontally or vertically aligned, sets of permutations are represented (considering all possible disambiguations).

**Example:**

\{a, b, c, d\} represents 3142.

\{a, b, c, d, x, y\} represents the set \{241563, 241653, 341562, 341652\}.
Introducing ambiguity in diagram representations

- A set of points in the plane that are pairwise neither horizontally nor vertically aligned represents a permutation.

- When some points are horizontally or vertically aligned, **sets of permutations** are represented (considering all possible disambiguations).

**Example:**

\[
\begin{array}{c}
\bullet \quad c \\
\bullet \quad x \\
\bullet \quad d \\
\end{array}
\begin{array}{c}
a \\
y \\
b \\
\end{array}
\]

- \{a, b, c, d\} represents 3142.
- \{a, b, c, d, x, y\} represents the set \{241563, 241653, 341562, 341652\}.
Definition of $R(\pi)$ for $\pi = a\alpha b\beta$ with $\beta \neq \varepsilon$

$R(\pi)$ is the set of \textbf{minimal} permutations in the set $\alpha a \beta b$.

When $x$ is above $\beta$ and $y$ is to the left of $\alpha$, $x$ and $y$ coalesce into a unique point.

\textbf{Remark}

$R(\pi)$ contains exactly

- 4 one-point extensions of $\pi$
- $4|\alpha|(n - a - 1)$ two-points extensions of $\pi$
Definition of $R(\pi)$ for $\pi = a\alpha b\beta$ with $\beta \neq \varepsilon$

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**Proposition**

If $\pi \in S_n$ is such that $\pi = a\alpha b\beta$ for $\beta \neq \varepsilon$, then $B^{-1}(Av(\pi))$ is a class whose basis is $R(\pi)$.

**Lemma**

If $\pi \preceq B(\sigma)$, consider an occurrence $p\lambda q\mu \subseteq B(\sigma)$. Then there exists a subsequence of $\sigma$ which is an occurrence of some pattern in $R(\pi)$.

**Lemma**

If $\sigma$ contains an occurrence of some pattern in $R(\pi)$, then there exists a subsequence of $B(\sigma)$ which is an occurrence of $\pi$.
Patterns $\pi \in S_n$ with two LtoR-maxima $\neq \pi(n)$

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If $\pi \in S_n$ is such that $\pi = a\alpha b\beta$ for $\beta \neq \varepsilon$, then $B^{-1}(\text{Av}(\pi))$ is a class whose basis is $R(\pi)$.

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**Lemma**

If $\sigma$ contains an occurrence of some pattern in $R(\pi)$, then there exists a subsequence of $B(\sigma)$ which is an occurrence of $\pi$. 
Lemma

If $\pi \not\preceq B(\sigma)$, consider an occurrence $p\lambda q\mu \subseteq B(\sigma)$. Then there exists a subsequence of $\sigma$ which is an occurrence of some pattern in $R(\pi)$.

Proof: We prove that $px\lambda_1 y\lambda_2 z\mu$ or $xp\lambda_1 y\lambda_2 z\mu \subseteq \sigma$ with

\[
\begin{aligned}
\lambda &= \lambda_1 \lambda_2, \quad p < x \\
y \text{ and } z \text{ are the two largest terms of this sequence} \\
\text{if } \lambda_1 = \varepsilon \text{ and } x > \mu, \text{ then } x \text{ and } y \text{ coalesce}
\end{aligned}
\]

Such a sequence is a permutation in $R(\pi)$.

The proof follows by induction on $|\sigma|$.

- If $|\sigma| \leq 3$, result vacuously true (since $B(\sigma)$ ends with its maximum).
- If $\sigma = \sigma_1 m \sigma_2$ with $m = |\sigma| > 3$, then $p\lambda q\mu \subseteq B(\sigma_1)\sigma_2 m$.
  Because $p\lambda q\mu$ does not end with its maximum, $p\lambda q\mu \subseteq B(\sigma_1)\sigma_2$. 
Proof of the first lemma

Lemma

If \( \pi \preceq B(\sigma) \), consider an occurrence \( p\lambda q\mu \subseteq B(\sigma) \).
Then there exists a subsequence of \( \sigma \) which is an occurrence of some pattern in \( R(\pi) \).

Proof: We prove that \( px_1y_1z_1 \mu \) or \( xp_1y_1z_1 \mu \subseteq \sigma \) with

\[
\begin{cases}
\lambda = \lambda_1\lambda_2, & p < x \\
y \text{ and } z \text{ are the two largest terms of this sequence} \\
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- If \( \sigma = \sigma_1m\sigma_2 \) with \( m = |\sigma| > 3 \), then \( p\lambda q\mu \subseteq B(\sigma_1)\sigma_2m \).
  Because \( p\lambda q\mu \) does not end with its maximum, \( p\lambda q\mu \subseteq B(\sigma_1)\sigma_2 \).
Proof of the first lemma, continued

As before, distinguish how $p\lambda q\mu$ can lie across $B(\sigma_1)\sigma_2$.

* If $\mu = \mu_1\mu_2$ with $\mu_1 \neq \varepsilon$, $p\lambda q\mu_1 \subseteq B(\sigma_1)$ and $\mu_2 \subseteq \sigma_2$ then by induction $\sigma_1$ contains a subsequence of the form $px\lambda_1y\lambda_2z\mu_1$ or $xp\lambda_1y\lambda_2z\mu_1$ to which $\mu_2$ can be appended.

* If $p\lambda q \subseteq B(\sigma_1)$ and $\mu \subseteq \sigma_2$, then by a previous lemma $\exists t > p$ such that $tp\lambda$ or $pt\lambda \subseteq \sigma_1$. If $q$ is to the left of $\lambda$ in $\sigma$, then $pq\lambda m\mu$ or $qp\lambda m\mu \subseteq \sigma$ is of the required form. Otherwise, $q$ and $t$ can play the rôle of $y$ and $x$, and appending $m\mu$ gives the desired subsequence.

* If $\lambda = \lambda_1\lambda_2$ with $\lambda_1 \neq \varepsilon$, $p\lambda_1 \subseteq B(\sigma_1)$ and $\lambda_2 q\mu \subseteq \sigma_2$, then as before $\exists x > p$ such that $xp\lambda_1$ or $px\lambda_1 \subseteq \sigma_1$. Appending $m\lambda_2 q\mu$ gives the desired subsequence.

* If $p \subseteq B(\sigma_1)$ and $\lambda q\mu \subseteq \sigma_2$, then $pm\lambda q\mu \subseteq \sigma_1 m\sigma_2 = \sigma$ is of the desired form, with $x$ and $y$ coalescing in $m$.

* If $p\lambda q\mu \subseteq \sigma_2$, then $mp\lambda q\mu \subseteq \sigma$. Again, $x$ and $y$ coalesce in $m$. 
Proof of the second lemma

Lemma

If \( \sigma \) contains an occurrence \( px\lambda_1y\lambda_2q\mu \) or \( xp\lambda_1y\lambda_2q\mu \) of some pattern in \( R(\pi) \), then there exists a subsequence of \( B(\sigma) \) which is an occurrence of \( \pi \).

Proof: Recall that if \( \sigma = n_1\lambda_1n_2\lambda_2 \cdots n_k\lambda_k \) where \( n_1, \ldots, n_k \) are the left to right maxima of \( \sigma \) then \( B(\sigma) = \lambda_1n_1\lambda_2n_2 \cdots \lambda_kn_k \).

Hence \( \lambda\mu \subseteq B(\sigma) \). Notice also that \( p\lambda q\mu \) is an occurrence of \( \pi \) in \( \sigma \).

1. We show that \( p \) is to the left of \( \lambda \) in \( B(\sigma) \).
   - If \( p \) is not a LtoR-maximum, this is true.
   - If \( p \) is a LtoR-maximum, then \( px\lambda_1y\lambda_2q\mu \subseteq \sigma \) and there exists some \( t \) between \( p \) and \( x \) (possibly \( t = x \)) in \( \sigma \) that is a LtoR-maximum.
     This implies that \( p \) still precedes \( \lambda \) in \( B(\sigma) \).

2. We show that there exists \( r \) in \( B(\sigma) \) between \( \lambda \) and \( \mu \) with \( r > p\lambda\mu \) (to be continued).
Proof of the second lemma

Lemma

If $\sigma$ contains an occurrence $px\lambda_1y\lambda_2q\mu$ or $xp\lambda_1y\lambda_2q\mu$ of some pattern in $R(\pi)$, then there exists a subsequence of $B(\sigma)$ which is an occurrence of $\pi$.

Proof: Recall that if $\sigma = n_1\lambda_1 n_2\lambda_2 \cdots n_k\lambda_k$ where $n_1, \ldots, n_k$ are the left to right maxima of $\sigma$ then $B(\sigma) = \lambda_1 n_1 \lambda_2 n_2 \cdots \lambda_k n_k$.

Hence $\lambda\mu \subseteq B(\sigma)$. Notice also that $p\lambda q\mu$ is an occurrence of $\pi$ in $\sigma$.

1. We show that $p$ is to the left of $\lambda$ in $B(\sigma)$. ✓

2. We show that there exists $r$ in $B(\sigma)$ between $\lambda$ and $\mu$ with $r > p\lambda\mu$.

- If $q$ is not a LtoR-maximum, choose $r = q$.
- If $q$ is a LtoR-maximum, choose $r =$ the LtoR-maximum of $\sigma$ immediately to the left of $q$. Then $p\lambda r\mu \subseteq B(\sigma)$.

By contradiction, assume that $r < y$ then in $\sigma$ we have

★ either $\cdots y \cdots r \cdots q \cdots$, and $r$ is not a LtoR-maximum,

★ or $\cdots r \cdots y \cdots q \cdots$, and there is a LtoR-maximum between $r$ and $q$.

Hence $r \geq y$, and $r > p\lambda\mu$ as desired.
### Summary of results

<table>
<thead>
<tr>
<th>$\pi$</th>
<th>$B^{-1}(Av(\pi))$</th>
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</tr>
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<tbody>
<tr>
<td>1</td>
<td>is a class</td>
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</tr>
<tr>
<td>12</td>
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<td>12, 21</td>
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</tr>
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<td>$n\alpha$, $\alpha \neq \varepsilon$</td>
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Patterns $\pi \in S_n$ with 3 LtoR-max. $\pi(1)$, $\pi(n-1)$ and $\pi(n)$

**Proposition**

If $\pi \in S_n$ is such that $\pi = (n-2)\alpha(n-1)n$, then $B^{-1}(\text{Av}(\pi))$ is a class whose basis is

$$\{(n-2)(n-1)\alpha n, (n-1)(n-2)\alpha n, (n-2)n\alpha(n-1), n(n-2)\alpha(n-1)\}.$$ 

**Lemma**

If $\pi \not\preceq B(\sigma)$, consider an occurrence $p\lambda qr \subseteq B(\sigma)$. Then there exists a subsequence of $\sigma$ which is an occurrence of some pattern among the four above.

**Lemma**

If $\sigma$ contains an occurrence of some pattern among the four above, then there exists a subsequence of $B(\sigma)$ which is an occurrence of $\pi$. 
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If $\pi \preceq B(\sigma)$, consider an occurrence $p\lambda qr \subseteq B(\sigma)$. Then there exists a subsequence of $\sigma$ which is an occurrence of some pattern among the four above.

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Some open questions

Q1. When is $B^{-1}(\text{Av}(B))$ a class, for $|B| \geq 2$?

Partial answer: $B^{-1}(\text{Av}(B))$ is a class when $B^{-1}(\text{Av}(\pi))$ is a class for every $\pi \in B$, but not only.

- $B^{-1}(\text{Av}(B)) = \bigcap_{\pi \in B} B^{-1}(\text{Av}(\pi))$.
- An example is $\Gamma_3 =$ the set of permutations of length 4 ending with 1: $B^{-1}(\text{Av}(\Gamma_3))$ is a class, although $\Gamma_3$ contains 2341 and $B^{-1}(\text{Av}(2341))$ is not a class.

Q2. Are the growth rates of $C$ and $B^{-1}(C)$ related?

Growth rate of a permutation class $C = \limsup_{n \to \infty} \sqrt[n]{c_n}$ where $c_n$ is the number of permutations of length $n$ in $C$. 
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Composing sorting operators

- **SB-sortable permutations**:  
  \[(SB)^{-1}(Av(21)) = B^{-1}(Av(231)) = Av(3241, 2341, 4231, 2431)\]

- **B²-sortable permutations**:  
  \[(BB)^{-1}(Av(21)) = B^{-1}(Av(231, 321)) = Av(\Gamma_2)\]

- **B^k-sortable permutations**:  
  \[(B^k)^{-1}(Av(21)) = Av(\Gamma_k) \text{ with } \Gamma_k = \text{the set of permutations of length } k + 1 \text{ ending with } 1.\]

Other sorting operators:

- built from \(B, S, \ldots\) and symmetries of the permutations \((i, r, c)\)
- with a queue
- definition of abstract sorting operator
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