# Tri bulle et classes de permutations

Mathilde Bouvel

LaBRI, CNRS

Travail en collaboration avec
M.H. Albert, M.D. Atkinson, A. Claesson et M. Dukes

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# The Bubble Sort Operator $B$

$B$ = one pass of **bubble sort**.
On sequences that are **permutations**.

**Definition(s):**

- Algorithmically:

$\hookrightarrow$ $B$ processes a permutation $\sigma$ from left to right, and modifies $\sigma$ dynamically exchanging $\sigma(i)$ and $\sigma(i+1)$ when $\sigma(i) > \sigma(i+1)$.

- Recursively:

$$\hookrightarrow \begin{cases} B(\sigma_1 n \sigma_2) = B(\sigma_1)\sigma_2 n \text{ if } \sigma = \sigma_1 n \sigma_2 \in S_n \\ B(\varepsilon) = \varepsilon \end{cases}$$

**NB** *Stack-sorting operator S*
$S(\sigma_1 n \sigma_2) = S(\sigma_1)S(\sigma_2)n$

- Explicitely:

$\hookrightarrow$ If $\sigma = n_1\lambda_1 n_2\lambda_2 \cdots n_k\lambda_k$ where $n_1, \ldots, n_k$ are the left to right maxima of $\sigma$ then $B(\sigma) = \lambda_1 n_1 \lambda_2 n_2 \cdots \lambda_k n_k$.

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# Permutation Classes

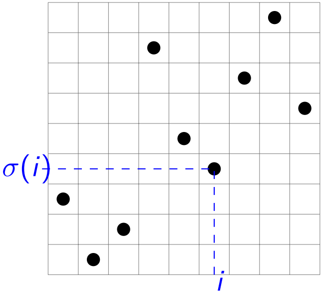
## Permutations

- $S_n$ = permutations $\sigma$ of $\{1, 2, \ldots, n\}$
- Representation by a word: $\sigma(1)\sigma(2)\cdots\sigma(n)$, by its diagram, ...

## Patterns

- Subpermutation of $\sigma$
- Subword or subset of points of the diagram that is normalized

Example: $2\,1\,3\,4 \preccurlyeq 3\,1\,2\,8\,5\,4\,7\,9\,6$ since $3\,2\,7\,9 \equiv 2\,1\,3\,4$

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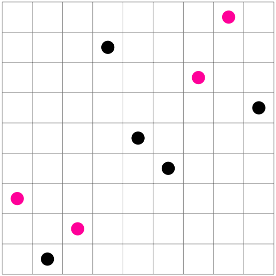

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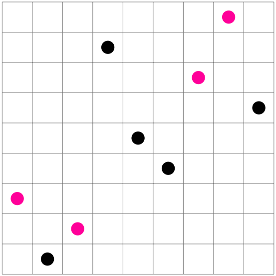

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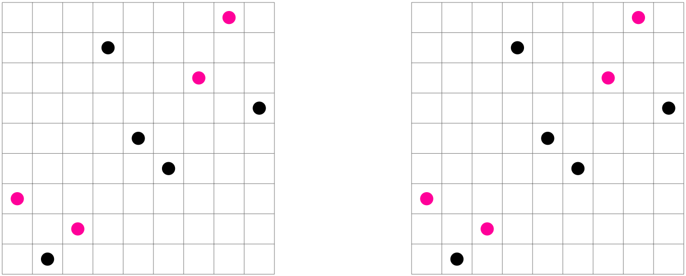

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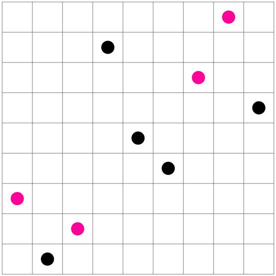

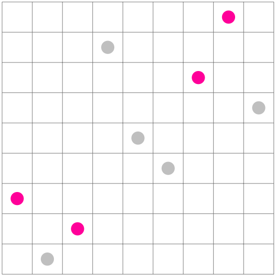

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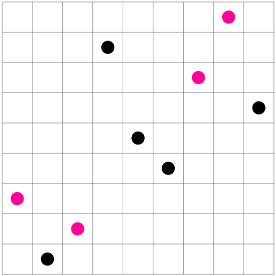

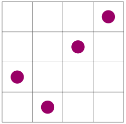

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- Occurrence = subpermutation without normalization

Example : $3\,2\,7\,9 \subseteq 3\,1\,2\,8\,5\,4\,7\,9\,6$

## Classes

- Subset of $S = \cup_n S_n$ downward closed for $\preccurlyeq$
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- Principal classes: $\mathcal{C} = Av(\pi)$

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# $B$-sortable permutations

## Proposition

*The permutations that are sorted by $B$ are a class.*
*Namely: $B(\sigma) = Id$ iff $\sigma \in Av(231, 321)$.*

**Proof**: by induction.

Decompose $\sigma = \sigma_1 n \sigma_2$ around its maximum $n$.
Recall that $B(\sigma) = B(\sigma_1)\sigma_2 n$.

$\sigma$ is sorted by $B$
$\Leftrightarrow$ $\sigma_1$ is sorted by $B$, $\sigma_2$ is increasing, and $\sigma_1 < \sigma_2$
$\Leftrightarrow$ $\sigma_1 \in Av(231, 321)$, $\sigma_2$ is increasing, and $\sigma_1 < \sigma_2$
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# Motivation and main result

- $B$-sortable permutations

$\hookrightarrow$ $B^{-1}(Av(21)) = Av(231, 321)$

- $SB$-sortable permutations?

$\hookrightarrow$ $(SB)^{-1}(Av(21)) = B^{-1}(Av(231))$

- $B^2$-sortable permutations?

$\hookrightarrow$ $(BB)^{-1}(Av(21)) = B^{-1}(Av(231, 321))$

- In general, what can we say about $B^{-1}(\mathcal{C})$?

For $\mathcal{C} = Av(\pi)$ a principal permutation class, we can determine

- when $B^{-1}(Av(\pi))$ is a class,
- and in this case give its basis.

This result is proved by considering the LtoR-maxima of $\pi$.

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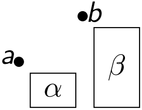
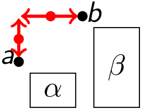
# Summary of results

| $\pi$ | $B^{-1}(Av(\pi))$ | Basis |
|---|---|---|
| $1$ | is a class | $1$ |
| $12$ | is a class | $12, 21$ |
| $21$ | is a class | $231, 321$ |
| $n\alpha$, $\alpha \neq \varepsilon$ | is a class | $n(n+1)\alpha, (n+1)n\alpha$ |
| $(n-1)\alpha n$, $\alpha \neq \varepsilon$ | is a class | $(n-1)n\alpha, n(n-1)\alpha$ |
| $a\alpha b\beta$, $\beta \neq \varepsilon$ | is a class | $R(\pi)$ |
| $a\alpha b\beta n$, $\beta \neq \varepsilon$ | is a class | $R(a\alpha b\beta)$ |
| $(n-2)\alpha(n-1)n$ | is a class | $(n-2)(n-1)\alpha\ n, (n-1)(n-2)\alpha n,$ $(n-2)n\ \alpha(n-1), n(n-2)\alpha\ (n-1)$ |
| $a\alpha b\beta c\gamma$, $\gamma \neq \varepsilon$ | is not a class | |

Remarks: $n$, $(n-1)$, $(n-2)$, $a$, $b$ and $c$ are LtoR-maxima.

If $\pi =$ 

, then $R(\pi)$ is the set of permutations 

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# Short patterns $\pi$

## Proposition

*There are no permutations $\sigma$ of length $n \geq 1$ such that $B(\sigma)$ avoids $1$.*
*Hence $B^{-1}(Av(1)) = \{\varepsilon\} = Av(1)$.*

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*The only permutations $\sigma$ such that $B(\sigma)$ avoids $12$ are $\varepsilon$ and $1$.*
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**Proof**: $B(\sigma)$ always ends with its maximum.

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# Patterns $\pi \in S_n$ ending with $n$ but not with $(n-1)n$

**Lemma**

*If $\pi \in S_n$ with $n \geq 3$ is such that $\pi(n) = n$ but $\pi(n-1) \neq n-1$, then setting $\pi' = \pi(1)\pi(2)\ldots\pi(n-1)$ we have $B^{-1}(Av(\pi)) = B^{-1}(Av(\pi'))$.*

**Proof**:

- $\sigma \in B^{-1}(Av(\pi')) \Rightarrow B(\sigma)$ avoids $\pi'$
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This lemmas applies in particular for $\pi = (n-1)\alpha n$ with $\alpha \neq \varepsilon$ and $\pi = a\alpha b\beta n$ with $\beta \neq \varepsilon$.

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  Hence $B(\sigma_1)\sigma_2$ avoids $\pi'$.
  But $\pi'$ does not end with its maximum.
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This lemmas applies in particular for $\pi = (n-1)\alpha n$ with $\alpha \neq \varepsilon$ and $\pi = a\alpha b\beta n$ with $\beta \neq \varepsilon$.

# Summary of results

| $\pi$ | $B^{-1}(Av(\pi))$ | Basis | Proof |
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| 1 | is a class | 1 | ✓ |
| 12 | is a class | $12, 21$ | ✓ |
| 21 | is a class | $231, 321$ | ✓ |
| $n\alpha$, $\alpha \neq \varepsilon$ | is a class | $n(n+1)\alpha, (n+1)n\alpha$ | |
| $(n-1)\alpha n$, $\alpha \neq \varepsilon$ | is a class | $(n-1)n\alpha, n(n-1)\alpha$ | ✓ |
| $a\alpha b\beta$, $\beta \neq \varepsilon$ | is a class | $R(\pi)$ | |
| $a\alpha b\beta n$, $\beta \neq \varepsilon$ | is a class | $R(a\alpha b\beta)$ | ✓ |
| $(n-2)\alpha(n-1)n$ | is a class | $(n-2)(n-1)\alpha\ n$,<br>$(n-1)(n-2)\alpha n$,<br>$(n-2)n\ \alpha(n-1)$,<br>$n(n-2)\alpha\ (n-1)$ | |
| $a\alpha b\beta c\gamma$, $\gamma \neq \varepsilon$ | is not a class | | |

# Patterns $\pi \in S_n$ with at least three LtoR-maxima $\neq \pi(n)$

## Proposition

*If $\pi = a\alpha b\beta c\gamma$, with a, b and c the first three LtoR-maxima of $\pi$ and $\gamma \neq \varepsilon$, then $B^{-1}(Av(\pi))$ is not a class.*

**Proof**:

By the previous lemma, we may assume that $\pi = a\alpha b\beta c\gamma n$.

Set $\theta_1 = ba\alpha n\beta c\gamma$ and $\theta_2 = (n+1)\theta_1$. Notice that $\theta_1 \preccurlyeq \theta_2$.

- Clearly, $B(\theta_1) = \pi$ and $\theta_1 \notin B^{-1}(Av(\pi))$.
- $B(\theta_2) = ba\alpha n\beta c\gamma(n+1)$
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We have $B^{-1}(Av(\pi)) \not\ni \theta_1 \preccurlyeq \theta_2 \in B^{-1}(Av(\pi))$. Consequently, $B^{-1}(Av(\pi))$ is not a class.

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# Common framework for the remaining cases

**Lemma**

*For any pattern $\pi$, if there exists a set $\mathcal{R}$ of permutations such that $\forall \sigma, \pi \preccurlyeq B(\sigma) \Leftrightarrow \rho \preccurlyeq \sigma$ for some $\rho \in \mathcal{R}$, then $B^{-1}(Av(\pi))$ is a class. Furthermore, if $\mathcal{R}$ is minimal, it is the basis of $B^{-1}(Av(\pi))$.*

**Proof**: Show that $B^{-1}(Av(\pi))$ is downward closed for $\preccurlyeq$.

$$\begin{aligned} & \sigma \notin B^{-1}(Av(\pi)) \\ \Leftrightarrow\ & B(\sigma) \notin Av(\pi) \\ \Leftrightarrow\ & \pi \preccurlyeq B(\sigma) \\ \Leftrightarrow\ & \exists \rho \in \mathcal{R}, \rho \preccurlyeq \sigma \end{aligned}$$

so that $\sigma \in B^{-1}(Av(\pi)) \Leftrightarrow \forall \rho \in \mathcal{R}, \rho \not\preccurlyeq \sigma$.

This shows that $B^{-1}(Av(\pi))$ is a downset, hence a class.

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# Patterns $\pi \in S_n$ starting with $n$

**Proposition**

*If $\pi \in S_n$ is such that $\pi = n\alpha$ for $\alpha \neq \varepsilon$, then $B^{-1}(Av(\pi))$ is a class whose basis is $\{n(n+1)\alpha, (n+1)n\alpha\}$.*

**Lemma**

*If $\pi \preccurlyeq B(\sigma)$, consider an occurrence $p\lambda \subseteq B(\sigma)$.*
*Then there exists $q > p > \lambda$ such that $pq\lambda \subseteq \sigma$ or $qp\lambda \subseteq \sigma$.*
*Hence $n(n+1)\alpha$ or $(n+1)n\alpha \preccurlyeq \sigma$.*

**Lemma**

*If $n(n+1)\alpha$ or $(n+1)n\alpha \preccurlyeq \sigma$, consider an occurrence $pq\lambda$ or $qp\lambda \subseteq \sigma$.*
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# Proof of the first lemma for $\pi = n\alpha$ with $\alpha \neq \varepsilon$

**Lemma**

*If $\pi \preccurlyeq B(\sigma)$, consider an occurrence $p\lambda \subseteq B(\sigma)$. Then there exists $q > p > \lambda$ such that $pq\lambda \subseteq \sigma$ or $qp\lambda \subseteq \sigma$. Hence $n(n+1)\alpha$ or $(n+1)n\alpha \preccurlyeq \sigma$.*

**Proof**: by induction on $|\sigma|$.

- If $|\sigma| \leq 2$, result vacuously true (since $B(\sigma)$ ends with its maximum).
- If $\sigma = \sigma_1 m \sigma_2$ with $m = |\sigma| > 2$, then $p\lambda \subseteq B(\sigma_1)\sigma_2 m$.
  Because $p\lambda$ does not end with its maximum, $p\lambda \subseteq B(\sigma_1)\sigma_2$.

⋆ If $\lambda = \lambda_1\lambda_2$ with $\lambda_1 \neq \varepsilon$, $p\lambda_1 \subseteq B(\sigma_1)$ and $\lambda_2 \subseteq \sigma_2$, then by induction $p\lambda_1 \subseteq B(\sigma_1)$ implies that $\exists q > p$ such that $pq\lambda_1 \subseteq \sigma_1$ or $qp\lambda_1 \subseteq \sigma_1$.
Hence $\sigma = \sigma_1 m \sigma_2$ contains an occurrence of $pq\lambda_1\lambda_2$ or of $qp\lambda_1\lambda_2$.

⋆ If $p \subseteq B(\sigma_1)$ and $\lambda \subseteq \sigma_2$, then $p \subseteq \sigma_1$ and $pm\lambda \subseteq \sigma_1 m \sigma_2 = \sigma$.

⋆ If $p\lambda \subseteq \sigma_2$, then $mp\lambda \subseteq m\sigma_2 \subseteq \sigma$.

# Proof of the second lemma for $\pi = n\alpha$ with $\alpha \neq \varepsilon$

**Lemma**

*If $n(n+1)\alpha$ or $(n+1)n\alpha \preccurlyeq \sigma$, consider an occurrence $pq\lambda$ or $qp\lambda \subseteq \sigma$.*
*Then $p\lambda \subseteq B(\sigma)$.*
*Hence $\pi \preccurlyeq B(\sigma)$.*

**Proof**:

Recall that if $\sigma = n_1\lambda_1 n_2\lambda_2 \cdots n_k\lambda_k$ where $n_1, \ldots, n_k$ are the left to right maxima of $\sigma$ then $B(\sigma) = \lambda_1 n_1 \lambda_2 n_2 \cdots \lambda_k n_k$.
Hence, the order of the elements not LtoR-maxima is preserved by $B$.

- If $qp\lambda \subseteq \sigma$, $p\lambda$ are not LtoR-maxima. Hence $p\lambda \subseteq B(\sigma)$.
- This also holds when $pq\lambda \subseteq \sigma$ and $p$ is not a LtoR-maximum.
- If $pq\lambda \subseteq \sigma$ and $p$ is a LtoR-maximum, then there exists some $r$ between $p$ and $q$ (possibly $r = q$) in $\sigma$ that is a LtoR-maximum. This implies that $p$ still precedes $\lambda$ in $B(\sigma)$, hence $p\lambda \subseteq B(\sigma)$.

# Summary of results

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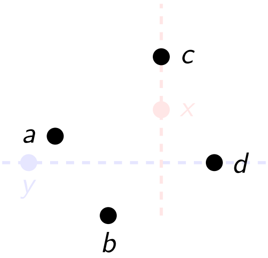
# Introducing ambiguity in diagram representations

- A set of points in the plane that are pairwise neither horizontally nor vertically aligned represents a permutation.
- When some points are horizontally or vertically aligned, **sets of permutations** are represented (considering all possible disambiguations).

**Example:**




- $\{a, b, c, d\}$ represents 3142.
- $\{a, b, c, d, x, y\}$ represents the set $\{241563, 241653, 341562, 341652\}$.

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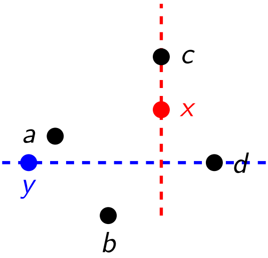


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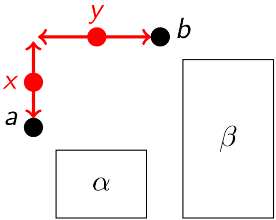
# Definition of $R(\pi)$ for $\pi = a\alpha b\beta$ with $\beta \neq \varepsilon$

$R(\pi)$ is the set of minimal permutations in the set




When $x$ is above $\beta$ and $y$ is to the left of $\alpha$, $x$ and $y$ coalesce into a unique point.




**Remark**

*$R(\pi)$ contains exactly*

- 4 *one-point extensions of* $\pi$
- $4|\alpha|(n-a-1)$ *two-points extensions of* $\pi$

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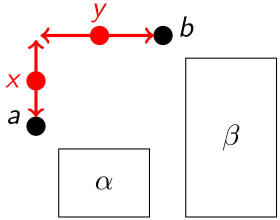


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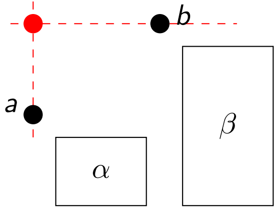


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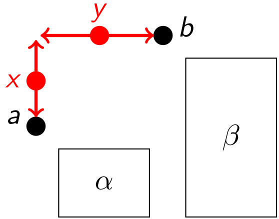


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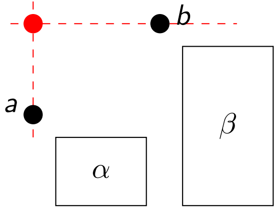


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# Patterns $\pi \in S_n$ with two LtoR-maxima $\neq \pi(n)$

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*If $\pi \in S_n$ is such that $\pi = a\alpha b\beta$ for $\beta \neq \varepsilon$, then $B^{-1}(Av(\pi))$ is a class whose basis is $R(\pi)$.*

**Lemma**

*If $\pi \preccurlyeq B(\sigma)$, consider an occurrence $p\lambda q\mu \subseteq B(\sigma)$. Then there exists a subsequence of $\sigma$ which is an occurrence of some pattern in $R(\pi)$.*

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*If $\sigma$ contains an occurrence of some pattern in $R(\pi)$, then there exists a subsequence of $B(\sigma)$ which is an occurrence of $\pi$.*

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**Proof**: We prove that $px\lambda_1 y\lambda_2 z\mu$ or $xp\lambda_1 y\lambda_2 z\mu \subseteq \sigma$ with

$$\begin{cases} \lambda = \lambda_1\lambda_2, \quad p < x \\ y \text{ and } z \text{ are the two largest terms of this sequence} \\ \text{if } \lambda_1 = \varepsilon \text{ and } x > \mu, \text{ then } x \text{ and } y \text{ coalesce} \end{cases}$$

Such a sequence is a permutation in $R(\pi)$.

The proof follows by induction on $|\sigma|$.

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  Because $p\lambda q\mu$ does not end with its maximum, $p\lambda q\mu \subseteq B(\sigma_1)\sigma_2$.

# Proof of the first lemma, continued

As before, distinguish how $p\lambda q\mu$ can lie across $B(\sigma_1)\sigma_2$.

- ⋆ If $\mu = \mu_1\mu_2$ with $\mu_1 \neq \varepsilon$, $p\lambda q\mu_1 \subseteq B(\sigma_1)$ and $\mu_2 \subseteq \sigma_2$ then by induction $\sigma_1$ contains a subsequence of the form $px\lambda_1 y\lambda_2 z\mu_1$ or $xp\lambda_1 y\lambda_2 z\mu_1$ to which $\mu_2$ can be appended.
- ⋆ If $p\lambda q \subseteq B(\sigma_1)$ and $\mu \subseteq \sigma_2$, then by a previous lemma $\exists t > p$ such that $tp\lambda$ or $pt\lambda \subseteq \sigma_1$. If $q$ is to the left of $\lambda$ in $\sigma$, then $pq\lambda m\mu$ or $qp\lambda m\mu \subseteq \sigma$ is of the required form. Otherwise, $q$ and $t$ can play the rôle of $y$ and $x$, and appending $m\mu$ gives the desired subsequence.
- ⋆ If $\lambda = \lambda_1\lambda_2$ with $\lambda_1 \neq \varepsilon$, $p\lambda_1 \subseteq B(\sigma_1)$ and $\lambda_2 q\mu \subseteq \sigma_2$, then as before $\exists x > p$ such that $xp\lambda_1$ or $px\lambda_1 \subseteq \sigma_1$. Appending $m\lambda_2 q\mu$ gives the desired subsequence.
- ⋆ If $p \subseteq B(\sigma_1)$ and $\lambda q\mu \subseteq \sigma_2$, then $pm\lambda q\mu \subseteq \sigma_1 m\sigma_2 = \sigma$ is of the desired form, with $x$ and $y$ coalesing in $m$.
- ⋆ If $p\lambda q\mu \subseteq \sigma_2$, then $mp\lambda q\mu \subseteq \sigma$. Again, $x$ and $y$ coalese in $m$.

# Proof of the second lemma

**Lemma**

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**Proof**: Recall that if $\sigma = n_1\lambda_1 n_2\lambda_2 \cdots n_k\lambda_k$ where $n_1, \dots, n_k$ are the left to right maxima of $\sigma$ then $B(\sigma) = \lambda_1 n_1 \lambda_2 n_2 \cdots \lambda_k n_k$.

Hence $\lambda\mu \subseteq B(\sigma)$. Notice also that $p\lambda q\mu$ is an occurrence of $\pi$ in $\sigma$.

1. We show that $p$ is to the left of $\lambda$ in $B(\sigma)$.
    - If $p$ is not a LtoR-maximum, this is true.
    - If $p$ is a LtoR-maximum, then $px\lambda_1 y\lambda_2 q\mu \subseteq \sigma$ and there exists some $t$ between $p$ and $x$ (possibly $t = x$) in $\sigma$ that is a LtoR-maximum. This implies that $p$ still precedes $\lambda$ in $B(\sigma)$.
2. We show that there exists $r$ in $B(\sigma)$ between $\lambda$ and $\mu$ with $r > p\lambda\mu$ (to be continued).

# Proof of the second lemma

## Lemma

*If $\sigma$ contains an occurrence $px\lambda_1 y\lambda_2 q\mu$ or $xp\lambda_1 y\lambda_2 q\mu$ of some pattern in $R(\pi)$, then there exists a subsequence of $B(\sigma)$ which is an occurrence of $\pi$.*

**Proof**: Recall that if $\sigma = n_1\lambda_1 n_2\lambda_2 \cdots n_k\lambda_k$ where $n_1, \ldots, n_k$ are the left to right maxima of $\sigma$ then $B(\sigma) = \lambda_1 n_1 \lambda_2 n_2 \cdots \lambda_k n_k$.

Hence $\lambda\mu \subseteq B(\sigma)$. Notice also that $p\lambda q\mu$ is an occurrence of $\pi$ in $\sigma$.

1. We show that $p$ is to the left of $\lambda$ in $B(\sigma)$. ✓
2. We show that there exists $r$ in $B(\sigma)$ between $\lambda$ and $\mu$ with $r > p\lambda\mu$.
   - If $q$ is not a LtoR-maximum, choose $r = q$.
   - If $q$ is a LtoR-maximum, choose $r =$ the LtoR-maximum of $\sigma$ immediately to the left of $q$. Then $p\lambda r\mu \subseteq B(\sigma)$.
     By contradiction, assume that $r < y$ then in $\sigma$ we have
     - ⋆ either $\cdots y \cdots r \cdots q \cdots$, and $r$ is not a LtoR-maximum,
     - ⋆ or $\cdots r \cdots y \cdots q \cdots$, and there is a LtoR-maximum between $r$ and $q$.

   Hence $r \geq y$, and $r > p\lambda\mu$ as desired.

# Summary of results

| $\pi$ | $B^{-1}(Av(\pi))$ | Basis | Proof |
|---|---|---|---|
| 1 | is a class | 1 | ✓ |
| 12 | is a class | $12, 21$ | ✓ |
| 21 | is a class | $231, 321$ | ✓ |
| $n\alpha$, $\alpha \neq \varepsilon$ | is a class | $n(n+1)\alpha, (n+1)n\alpha$ | ✓ |
| $(n-1)\alpha n$, $\alpha \neq \varepsilon$ | is a class | $(n-1)n\alpha, n(n-1)\alpha$ | ✓ |
| $a\alpha b\beta$, $\beta \neq \varepsilon$ | is a class | $R(\pi)$ | ✓ |
| $a\alpha b\beta n$, $\beta \neq \varepsilon$ | is a class | $R(a\alpha b\beta)$ | ✓ |
| $(n-2)\alpha(n-1)n$ | is a class | $(n-2)(n-1)\alpha\ n$,<br>$(n-1)(n-2)\alpha n$,<br>$(n-2)n\ \alpha(n-1)$,<br>$n(n-2)\alpha\ (n-1)$ | |
| $a\alpha b\beta c\gamma$, $\gamma \neq \varepsilon$ | is not a class | | ✓ |

# Patterns $\pi \in S_n$ with 3 LtoR-max. $\pi(1)$, $\pi(n-1)$ and $\pi(n)$

**Proposition**

*If $\pi \in S_n$ is such that $\pi = (n-2)\alpha(n-1)n$, then $B^{-1}(Av(\pi))$ is a class whose basis is*
$\{(n-2)(n-1)\alpha n, (n-1)(n-2)\alpha n, (n-2)n\alpha(n-1), n(n-2)\alpha(n-1)\}$.

**Lemma**

*If $\pi \preccurlyeq B(\sigma)$, consider an occurrence $p\lambda qr \subseteq B(\sigma)$. Then there exists a subsequence of $\sigma$ which is an occurrence of some pattern among the four above.*

**Lemma**

*If $\sigma$ contains an occurrence of some pattern among the four above, then there exists a subsequence of $B(\sigma)$ which is an occurrence of $\pi$.*

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# Some open questions

Q1. When is $B^{-1}(Av(\mathcal{B}))$ a class, for $|\mathcal{B}| \geq 2$?

Partial answer: $B^{-1}(Av(\mathcal{B}))$ is a class when $B^{-1}(Av(\pi))$ is a class for every $\pi \in \mathcal{B}$, but not only.

- $B^{-1}(Av(\mathcal{B})) = \cap_{\pi \in \mathcal{B}} B^{-1}(Av(\pi))$.
- An example is $\Gamma_3 =$ the set of permutations of length 4 ending with 1: $B^{-1}(Av(\Gamma_3))$ is a class, although $\Gamma_3$ contains 2341 and $B^{-1}(Av(2341))$ is not a class.

Q2. Are the growth rates of $\mathcal{C}$ and $B^{-1}(\mathcal{C})$ related?

Growth rate of a permutation class $\mathcal{C} = \limsup_{n \to \infty} \sqrt[n]{c_n}$ where $c_n$ is the number of permutations of length $n$ in $\mathcal{C}$

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# Composing sorting operators

- $SB$-sortable permutations:

$\hookrightarrow$ $(SB)^{-1}(Av(21)) = B^{-1}(Av(231)) = Av(3241, 2341, 4231, 2431)$

- $B^2$-sortable permutations:

$\hookrightarrow$ $(BB)^{-1}(Av(21)) = B^{-1}(Av(231, 321)) = Av(\Gamma_2)$

- $B^k$-sortable permutations:

$\hookrightarrow$ $(B^k)^{-1}(Av(21)) = Av(\Gamma_k)$ with
$\Gamma_k =$ the set of permutations of length $k+1$ ending with 1.

Other sorting operators:

- built from $B$, $S$, ... and symmetries of the permutations ($i$, $r$, $c$)
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