Bubble Sort and Permutation Classes

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Joint work with
M.H. Albert, M.D. Atkinson, A. Claesson and M. Dukes
The Bubble Sort Operator $B$

$B = \text{one pass of bubble sort.}$
On sequences that are permutations.

Definition(s):

- Algorithmically:

  $B$ processes a permutation $\sigma$ from left to right, and modifies $\sigma$ dynamically exchanging $\sigma(i)$ and $\sigma(i+1)$ when $\sigma(i) > \sigma(i+1)$.

- Recursively:

  \[ B(\sigma_1 n \sigma_2) = B(\sigma_1)\sigma_2 n \quad \text{if} \quad \sigma = \sigma_1 n \sigma_2 \in S_n \]
  \[ B(\varepsilon) = \varepsilon \]

- Explicitely:

  If $\sigma = n_1 \lambda_1 n_2 \lambda_2 \cdots n_k \lambda_k$ where $n_1, \ldots, n_k$ are the left to right maxima of $\sigma$ then $B(\sigma) = \lambda_1 n_1 \lambda_2 n_2 \cdots \lambda_k n_k$.

NB Stack-sorting operator $S$

$S(\sigma_1 n \sigma_2) = S(\sigma_1)S(\sigma_2)n$
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Permutation Classes

Permutations
- \( S_n = \) permutations \( \sigma \) of \( \{1, 2, \ldots, n\} \)
- Representation by a word: \( \sigma(1)\sigma(2)\cdots\sigma(n) \), by its diagram, . . .

Patterns
- Subpermutation of \( \sigma \)
- Subword or subset of points of the diagram that is normalized

Example: \( 2134 \preceq 312854796 \) since \( 3279 \equiv 2134 \)

\[
\sigma = 312854796
\]
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Example: $2 1 3 4 \preceq 3 1 2 8 5 4 7 9 6$ since $3 2 7 9 \equiv 2 1 3 4$

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[Diagram of permutations with examples]
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\[ \text{Diagram of } 2 \ 1 \ 3 \ 4 \ \prec \ \text{Diagram of } 3 \ 1 \ 2 \ 8 \ 5 \ 4 \ 7 \ 9 \ 6 \]
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Occurrence of a pattern

- Occurrence = subpermutation without normalization

Example: $3 2 7 9 \subseteq 3 1 2 8 5 4 7 9 6$

Classes

- Subset of $S = \bigcup_n S_n$ downward closed for $\lessdot$
- Characterization by a basis of excluded patterns: $C = Av(B)$
- Principal classes: $C = Av(\pi)$
Permutation Classes

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Example: $2\,1\,3\,4 \preceq 3\,1\,2\,8\,5\,4\,7\,9\,6$ since $3\,2\,7\,9 \equiv 2\,1\,3\,4$

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Classes

- Subset of $S = \bigcup_n S_n$ downward closed for $\preceq$
- Characterization by a basis of excluded patterns: $\mathcal{C} = \text{Av}(\mathcal{B})$
- Principal classes: $\mathcal{C} = \text{Av}(\pi)$
**Proposition**

The permutations that are sorted by B are a class. Namely: $B(\sigma) = Id$ iff $\sigma \in Av(231, 321)$.

**Proof**: by induction.

Decompose $\sigma = \sigma_1 n \sigma_2$ around its maximum $n$.

Recall that $B(\sigma) = B(\sigma_1)\sigma_2 n$.

- $\sigma$ is sorted by $B$
- $\Leftrightarrow \sigma_1$ is sorted by $B$, $\sigma_2$ is increasing, and $\sigma_1 < \sigma_2$
- $\Leftrightarrow \sigma_1 \in Av(231, 321)$, $\sigma_2$ is increasing, and $\sigma_1 < \sigma_2$
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Motivation and main result

- $B$-sortable permutations

$x \mapsto B^{-1}(Av(21)) = Av(231, 321)$

- $SB$-sortable permutations?

$x \mapsto (SB)^{-1}(Av(21)) = B^{-1}(Av(231))$

- $B^2$-sortable permutations?

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- In general, what can we say about $B^{-1}(C)$?

For $C = Av(\pi)$ a principal permutation class, we can determine

- when $B^{-1}(Av(\pi))$ is a class,

  - and in this case give its basis.

This result is proved by considering the LtoR-maxima of $\pi$. 
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### Summary of results

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<tr>
<th>$\pi$</th>
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<tbody>
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Remarks: $n, (n - 1), (n - 2), a, b$ and $c$ are LtoR-maxima.

If $\pi = \begin{array}{c} \alpha \\ \hline \beta \end{array}$, then $R(\pi)$ is the set of permutations $\begin{array}{c} \alpha \\ \hline \beta \end{array}$. 
Proposition

There are no permutations $\sigma$ of length $n \geq 1$ such that $B(\sigma)$ avoids 1. Hence $B^{-1}(\text{Av}(1)) = \{\varepsilon\} = \text{Av}(1)$.

Proposition

The only permutations $\sigma$ such that $B(\sigma)$ avoids 12 are $\varepsilon$ and 1. Hence $B^{-1}(\text{Av}(12)) = \{\varepsilon, 1\} = \text{Av}(12, 21)$.

Proof: $B(\sigma)$ always ends with its maximum.

Proposition

The permutations $\sigma$ such that $B(\sigma)$ avoids 21 are the $B$-sortable permutations. Hence $B^{-1}(\text{Av}(21)) = \text{Av}(231, 321)$. 

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Patterns \( \pi \in S_n \) ending with \( n \) but not with \( (n - 1)n \)

**Lemma**

If \( \pi \in S_n \) with \( n \geq 3 \) is such that \( \pi(n) = n \) but \( \pi(n - 1) \neq n - 1 \), then setting \( \pi' = \pi(1)\pi(2)\ldots\pi(n - 1) \) we have \( B^{-1}(Av(\pi)) = B^{-1}(Av(\pi')) \).

**Proof:**

- \( \sigma \in B^{-1}(Av(\pi')) \Rightarrow B(\sigma) \) avoids \( \pi' \)
  \( \Rightarrow B(\sigma) \) avoids \( \pi \Rightarrow \sigma \in B^{-1}(Av(\pi)) \)
- \( \sigma \in B^{-1}(Av(\pi)) \Rightarrow B(\sigma) \) avoids \( \pi = \pi'n \)
  But \( B(\sigma) = B(\sigma_1)\sigma_2 m \) ends with its maximum \( m \).
  Hence \( B(\sigma_1)\sigma_2 \) avoids \( \pi' \).
  But \( \pi' \) does not end with its maximum.
  Hence \( B(\sigma) = B(\sigma_1)\sigma_2 m \) avoids \( \pi' \) and \( \sigma \in B^{-1}(Av(\pi')) \).

This lemma applies in particular for \( \pi = (n - 1)\alpha n \) with \( \alpha \neq \varepsilon \) and \( \pi = a\alpha b\beta n \) with \( \beta \neq \varepsilon \).
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But $B(\sigma) = B(\sigma_1)\sigma_2 m$ ends with its maximum $m$.

Hence $B(\sigma_1)\sigma_2$ avoids $\pi'$.

But $\pi'$ does not end with its maximum.

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<td>231, 321</td>
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Patterns $\pi \in S_n$ with at least three LtoR-maxima $\neq \pi(n)$

**Proposition**

If $\pi = a\alpha b\beta c\gamma$, with $a$, $b$ and $c$ the first three LtoR-maxima of $\pi$ and $\gamma \neq \varepsilon$, then $B^{-1}(\text{Av}(\pi))$ is not a class.

**Proof:**

By the previous lemma, we may assume that $\pi = a\alpha b\beta c\gamma n$.

Set $\theta_1 = ba\alpha n\beta c\gamma$ and $\theta_2 = (n + 1)\theta_1$. Notice that $\theta_1 \preceq \theta_2$.

- Clearly, $B(\theta_1) = \pi$ and $\theta_1 \not\in B^{-1}(\text{Av}(\pi))$.
- $B(\theta_2) = ba\alpha n\beta c\gamma(n + 1)$
  Since $B(\theta_2)$ is only one term longer than $\pi$, we easily check that $B(\theta_2)$ avoids $\pi$. Hence $\theta_2 \in B^{-1}(\text{Av}(\pi))$.

We have $B^{-1}(\text{Av}(\pi)) \not\ni \theta_1 \preceq \theta_2 \in B^{-1}(\text{Av}(\pi))$. Consequently, $B^{-1}(\text{Av}(\pi))$ is not a class.
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Common framework for the remaining cases

**Lemma**

*For any pattern \( \pi \), if there exists a set \( \mathcal{R} \) of permutations such that \( \forall \sigma, \pi \preceq B(\sigma) \iff \rho \preceq \sigma \) for some \( \rho \in \mathcal{R} \), then \( B^{-1}(\text{Av}(\pi)) \) is a class. Furthermore, if \( \mathcal{R} \) is minimal, it is the basis of \( B^{-1}(\text{Av}(\pi)) \).*

**Proof:** Show that \( B^{-1}(\text{Av}(\pi)) \) is downward closed for \( \preceq \).

\[
\sigma \not\in B^{-1}(\text{Av}(\pi)) \iff B(\sigma) \not\in \text{Av}(\pi) \iff \pi \preceq B(\sigma) \iff \exists \rho \in \mathcal{R}, \rho \preceq \sigma
\]

so that \( \sigma \in B^{-1}(\text{Av}(\pi)) \iff \forall \rho \in \mathcal{R}, \rho \not\preceq \sigma \).

This show that \( B^{-1}(\text{Av}(\pi)) \) is a downset, hence a class.

This also shows that the minimal \( \mathcal{R} \) is its basis.
Lemma

For any pattern $\pi$, if there exists a set $\mathcal{R}$ of permutations such that
\[ \forall \sigma, \pi \preccurlyeq B(\sigma) \iff \rho \preccurlyeq \sigma \text{ for some } \rho \in \mathcal{R}, \text{ then } B^{-1}(\text{Av}(\pi)) \text{ is a class.} \]
Furthermore, if $\mathcal{R}$ is minimal, it is the basis of $B^{-1}(\text{Av}(\pi))$.

Proof: Show that $B^{-1}(\text{Av}(\pi))$ is downward closed for $\preccurlyeq$.

\[ \sigma \not\in B^{-1}(\text{Av}(\pi)) \iff B(\sigma) \not\in \text{Av}(\pi) \iff \pi \preccurlyeq B(\sigma) \iff \exists \rho \in \mathcal{R}, \rho \preccurlyeq \sigma \]

so that $\sigma \in B^{-1}(\text{Av}(\pi)) \iff \forall \rho \in \mathcal{R}, \rho \not\preccurlyeq \sigma$.

This show that $B^{-1}(\text{Av}(\pi))$ is a downset, hence a class.

This also shows that the minimal $\mathcal{R}$ is its basis.
**Patterns** $\pi \in S_n$ starting with $n$

**Proposition**

If $\pi \in S_n$ is such that $\pi = n\alpha$ for $\alpha \neq \varepsilon$, then $B^{-1}(Av(\pi))$ is a class whose basis is $\{n(n + 1)\alpha, (n + 1)n\alpha\}$.

**Lemma**

If $\pi \preceq B(\sigma)$, consider an occurrence $p\lambda \subseteq B(\sigma)$.
Then there exists $q > p > \lambda$ such that $pq\lambda \subseteq \sigma$ or $qp\lambda \subseteq \sigma$.
Hence $n(n + 1)\alpha$ or $(n + 1)n\alpha \preceq \sigma$.

**Lemma**

If $n(n + 1)\alpha$ or $(n + 1)n\alpha \preceq \sigma$, consider an occurrence $pq\lambda$ or $qp\lambda \subseteq \sigma$.
Then $p\lambda \subseteq B(\sigma)$.
Hence $\pi \preceq B(\sigma)$.
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**Lemma**

If $\pi \not\succeq B(\sigma)$, consider an occurrence $p\lambda \subseteq B(\sigma)$. Then there exists $q > p > \lambda$ such that $pq\lambda \subseteq \sigma$ or $qp\lambda \subseteq \sigma$. Hence $n(n + 1)\alpha$ or $(n + 1)n\alpha \not\succeq \sigma$.

**Lemma**

If $n(n + 1)\alpha$ or $(n + 1)n\alpha \not\succeq \sigma$, consider an occurrence $pq\lambda$ or $qp\lambda \subseteq \sigma$. Then $p\lambda \subseteq B(\sigma)$. Hence $\pi \not\succeq B(\sigma)$. 
Proof of the first lemma for $\pi = n\alpha$ with $\alpha \neq \varepsilon$

**Lemma**

If $\pi \preceq B(\sigma)$, consider an occurrence $p\lambda \subseteq B(\sigma)$. Then there exists $q > p > \lambda$ such that $pq\lambda \subseteq \sigma$ or $qp\lambda \subseteq \sigma$. Hence $n(n + 1)\alpha$ or $(n + 1)n\alpha \preceq \sigma$.

**Proof:** by induction on $|\sigma|$.

- If $|\sigma| \leq 2$, result vacuously true (since $B(\sigma)$ ends with its maximum).
- If $\sigma = \sigma_1 m\sigma_2$ with $m = |\sigma| > 2$, then $p\lambda \subseteq B(\sigma_1)\sigma_2 m$. Because $p\lambda$ does not end with its maximum, $p\lambda \subseteq B(\sigma_1)\sigma_2$.
  - If $\lambda = \lambda_1 \lambda_2$ with $\lambda_1 \neq \varepsilon$, $p\lambda_1 \subseteq B(\sigma_1)$ and $\lambda_2 \subseteq \sigma_2$, then by induction $p\lambda_1 \subseteq B(\sigma_1)$ implies that $\exists q > p$ such that $pq\lambda_1 \subseteq \sigma_1$ or $qp\lambda_1 \subseteq \sigma_1$. Hence $\sigma = \sigma_1 m\sigma_2$ contains an occurrence of $pq\lambda_1 \lambda_2$ or of $qp\lambda_1 \lambda_2$.
  - If $p \subseteq B(\sigma_1)$ and $\lambda \subseteq \sigma_2$, then $p \subseteq \sigma_1$ and $pm\lambda \subseteq \sigma_1 m\sigma_2 = \sigma$.
  - If $p\lambda \subseteq \sigma_2$, then $mp\lambda \subseteq m\sigma_2 \subseteq \sigma$. 
Lemma

If \( n(n+1)\alpha \) or \( (n+1)n\alpha \preceq \sigma \), consider an occurrence \( pq\lambda \) or \( qp\lambda \subseteq \sigma \). Then \( p\lambda \subseteq B(\sigma) \).
Hence \( \pi \preceq B(\sigma) \).

Proof:

Recall that if \( \sigma = n_1\lambda_1 n_2\lambda_2 \cdots n_k\lambda_k \) where \( n_1, \ldots, n_k \) are the left to right maxima of \( \sigma \) then \( B(\sigma) = \lambda_1 n_1\lambda_2 n_2 \cdots \lambda_k n_k \).
Hence, the order of the elements not LtoR-maxima is preserved by \( B \).

- If \( qp\lambda \subseteq \sigma \), \( p\lambda \) are not LtoR-maxima. Hence \( p\lambda \subseteq B(\sigma) \).
- This also holds when \( pq\lambda \subseteq \sigma \) and \( p \) is not a LtoR-maximum.
- If \( pq\lambda \subseteq \sigma \) and \( p \) is a LtoR-maximum, then there exists some \( r \) between \( p \) and \( q \) (possibly \( r = q \)) in \( \sigma \) that is a LtoR-maximum. This implies that \( p \) still precedes \( \lambda \) in \( B(\sigma) \), hence \( p\lambda \subseteq B(\sigma) \).
### Summary of results

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Introducing ambiguity in diagram representations

- A set of points in the plane that are pairwise neither horizontally nor vertically aligned represents a permutation.
- When some points are horizontally or vertically aligned, sets of permutations are represented (considering all possible disambiguations).

**Example:**

- \( \{a, b, c, d\} \) represents 3142.
- \( \{a, b, c, d, x, y\} \) represents the set \( \{241563, 241653, 341562, 341652\} \).
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\]
Definition of $R(\pi)$ for $\pi = a\alpha b\beta$ with $\beta \neq \varepsilon$

$R(\pi)$ is the set of minimal permutations in the set

![Diagram showing permutations and coalescence.]

When $x$ is above $\beta$ and $y$ is to the left of $\alpha$, $x$ and $y$ coalesce into a unique point.

Remark

$R(\pi)$ contains exactly

- 4 one-point extensions of $\pi$
- $4|\alpha|(n - a - 1)$ two-points extensions of $\pi$
Definition of $R(\pi)$ for $\pi = a\alpha b\beta$ with $\beta \neq \varepsilon$

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![Diagram](image)

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**Lemma**

If $\pi \preceq B(\sigma)$, consider an occurrence $p\lambda q\mu \subseteq B(\sigma)$. Then there exists a subsequence of $\sigma$ which is an occurrence of some pattern in $R(\pi)$.

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If $\sigma$ contains an occurrence of some pattern in $R(\pi)$, then there exists a subsequence of $B(\sigma)$ which is an occurrence of $\pi$. 
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Proof of the first lemma

Lemma

If \( \pi \not\preceq B(\sigma) \), consider an occurrence \( p\lambda q\mu \subseteq B(\sigma) \).
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Proof: We prove that \( px\lambda_1y\lambda_2z\mu \) or \( xp\lambda_1y\lambda_2z\mu \subseteq \sigma \) with

\[
\begin{align*}
\lambda &= \lambda_1\lambda_2, \quad p < x \\
y \text{ and } z \text{ are the two largest terms of this sequence} \\
\text{if } \lambda_1 = \varepsilon \text{ and } x > \mu, \text{ then } x \text{ and } y \text{ coalesce}
\end{align*}
\]

Such a sequence is a permutation in \( R(\pi) \).

The proof follows by induction on \(|\sigma|\).

- If \(|\sigma| \leq 3\), result vacuously true (since \( B(\sigma) \) ends with its maximum).
- If \( \sigma = \sigma_1m\sigma_2 \) with \( m = |\sigma| > 3 \), then \( p\lambda q\mu \subseteq B(\sigma_1)\sigma_2m \).
  Because \( p\lambda q\mu \) does not end with its maximum, \( p\lambda q\mu \subseteq B(\sigma_1)\sigma_2 \).
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If $\pi \preceq B(\sigma)$, consider an occurrence $p\lambda q\mu \subseteq B(\sigma)$. Then there exists a subsequence of $\sigma$ which is an occurrence of some pattern in $R(\pi)$.

Proof: We prove that $px\lambda_1 y\lambda_2 z\mu$ or $xp\lambda_1 y\lambda_2 z\mu \subseteq \sigma$ with

\[
\begin{cases}
\lambda = \lambda_1 \lambda_2, & p < x \\
y \text{ and } z \text{ are the two largest terms of this sequence} \\
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Proof of the first lemma, continued

As before, distinguish how $p\lambda q\mu$ can lie across $B(\sigma_1)\sigma_2$.

- If $\mu = \mu_1\mu_2$ with $\mu_1 \neq \varepsilon$, $p\lambda q\mu_1 \subseteq B(\sigma_1)$ and $\mu_2 \subseteq \sigma_2$ then by induction $\sigma_1$ contains a subsequence of the form $px\lambda_1y\lambda_2z\mu_1$ or $xp\lambda_1y\lambda_2z\mu_1$ to which $\mu_2$ can be appended.

- If $p\lambda q \subseteq B(\sigma_1)$ and $\mu \subseteq \sigma_2$, then by a previous lemma $\exists t > p$ such that $tp\lambda$ or $pt\lambda \subseteq \sigma_1$. If $q$ is to the left of $\lambda$ in $\sigma$, then $pq\lambda m\mu$ or $qp\lambda m\mu \subseteq \sigma$ is of the required form. Otherwise, $q$ and $t$ can play the rôle of $y$ and $x$, and appending $m\mu$ gives the desired subsequence.

- If $\lambda = \lambda_1\lambda_2$ with $\lambda_1 \neq \varepsilon$, $p\lambda_1 \subseteq B(\sigma_1)$ and $\lambda_2 q\mu \subseteq \sigma_2$, then as before $\exists x > p$ such that $xp\lambda_1$ or $px\lambda_1 \subseteq \sigma_1$. Appending $m\lambda_2 q\mu$ gives the desired subsequence.

- If $p \subseteq B(\sigma_1)$ and $\lambda q\mu \subseteq \sigma_2$, then $pm\lambda q\mu \subseteq \sigma_1 m\sigma_2 = \sigma$ is of the desired form, with $x$ and $y$ coalesing in $m$.

- If $p\lambda q\mu \subseteq \sigma_2$, then $mp\lambda q\mu \subseteq \sigma$. Again, $x$ and $y$ coalesce in $m$. 
Proof of the second lemma

Lemma

If \( \sigma \) contains an occurrence \( px\lambda_1 y\lambda_2 q\mu \) or \( xp\lambda_1 y\lambda_2 q\mu \) of some pattern in \( R(\pi) \), then there exists a subsequence of \( B(\sigma) \) which is an occurrence of \( \pi \).

Proof: Recall that if \( \sigma = n_1\lambda_1 n_2\lambda_2 \cdots n_k\lambda_k \) where \( n_1, \ldots, n_k \) are the left to right maxima of \( \sigma \) then \( B(\sigma) = \lambda_1 n_1\lambda_2 n_2 \cdots \lambda_k n_k \).

Hence \( \lambda\mu \subseteq B(\sigma) \). Notice also that \( p\lambda q\mu \) is an occurrence of \( \pi \) in \( \sigma \).

1. We show that \( p \) is to the left of \( \lambda \) in \( B(\sigma) \).
   - If \( p \) is not a LtoR-maximum, this is true.
   - If \( p \) is a LtoR-maximum, then \( px\lambda_1 y\lambda_2 q\mu \subseteq \sigma \) and there exists some \( t \) between \( p \) and \( x \) (possibly \( t = x \)) in \( \sigma \) that is a LtoR-maximum. This implies that \( p \) still precedes \( \lambda \) in \( B(\sigma) \).

2. We show that there exists \( r \) in \( B(\sigma) \) between \( \lambda \) and \( \mu \) with \( r > p\lambda\mu \) (to be continued).
Proof of the second lemma

Lemma

If $\sigma$ contains an occurrence $px\lambda_1 y\lambda_2 q\mu$ or $xp\lambda_1 y\lambda_2 q\mu$ of some pattern in $R(\pi)$, then there exists a subsequence of $B(\sigma)$ which is an occurrence of $\pi$.

Proof: Recall that if $\sigma = n_1\lambda_1 n_2\lambda_2 \cdots n_k\lambda_k$ where $n_1, \ldots, n_k$ are the left to right maxima of $\sigma$ then $B(\sigma) = \lambda_1 n_1\lambda_2 n_2 \cdots \lambda_k n_k$. Hence $\lambda\mu \subseteq B(\sigma)$. Notice also that $p\lambda q\mu$ is an occurrence of $\pi$ in $\sigma$.

1. We show that $p$ is to the left of $\lambda$ in $B(\sigma)$. ✓

2. We show that there exists $r$ in $B(\sigma)$ between $\lambda$ and $\mu$ with $r > p\lambda\mu$.
   - If $q$ is not a LtoR-maximum, choose $r = q$.
   - If $q$ is a LtoR-maximum, choose $r =$ the LtoR-maximum of $\sigma$ immediately to the left of $q$. Then $p\lambda r\mu \subseteq B(\sigma)$.

By contradiction, assume that $r < y$ then in $\sigma$ we have

- either $\cdots y \cdots r \cdots q \cdots$, and $r$ is not a LtoR-maximum,
- or $\cdots r \cdots y \cdots q \cdots$, and there is a LtoR-maximum between $r$ and $q$.

Hence $r \geq y$, and $r > p\lambda\mu$ as desired.
<table>
<thead>
<tr>
<th>$\pi$</th>
<th>$B^{-1}(Av(\pi))$</th>
<th>Basis</th>
<th>Proof</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>is a class</td>
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</tr>
<tr>
<td>12</td>
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<td>12, 21</td>
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<td>231, 321</td>
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<td>$n\alpha$, $\alpha \neq \varepsilon$</td>
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<tr>
<td>$(n - 2)\alpha(n - 1)n$</td>
<td>is a class</td>
<td>$(n - 2)(n - 1)\alpha n$, $(n - 1)(n - 2)\alpha n$, $(n - 2)n \alpha(n - 1)$, $n(n - 2)\alpha (n - 1)$</td>
<td></td>
</tr>
<tr>
<td>$a\alpha b\beta c\gamma$, $\gamma \neq \varepsilon$</td>
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Patterns $\pi \in S_n$ with 3 LtoR-max. $\pi(1)$, $\pi(n-1)$ and $\pi(n)$

**Proposition**

If $\pi \in S_n$ is such that $\pi = (n-2)\alpha(n-1)n$, then $B^{-1}(Av(\pi))$ is a class whose basis is
\[
\{(n-2)(n-1)\alpha n, (n-1)(n-2)\alpha n, (n-2)n\alpha(n-1), n(n-2)\alpha(n-1)\}.
\]

**Lemma**

If $\pi \preceq B(\sigma)$, consider an occurrence $p\lambdaqr \subseteq B(\sigma)$. Then there exists a subsequence of $\sigma$ which is an occurrence of some pattern among the four above.

**Lemma**

If $\sigma$ contains an occurrence of some pattern among the four above, then there exists a subsequence of $B(\sigma)$ which is an occurrence of $\pi$. 
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Mathilde Bouvel (LaBRI, CNRS) 
Bubble Sort and Permutation Classes
### Summary of results

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Some open questions

Q1. When is $B^{-1}(Av(B))$ a class, for $|B| \geq 2$?

Partial answer: $B^{-1}(Av(B))$ is a class when $B^{-1}(Av(\pi))$ is a class for every $\pi \in B$, but not only.

- $B^{-1}(Av(B)) = \cap_{\pi \in B} B^{-1}(Av(\pi))$.
- An example is $\Gamma_2 = \{\text{the set of permutations of length 4 ending with 1: } B^{-1}(Av(\Gamma_2)) \text{ is a class, although } \Gamma_2 \text{ contains 2341 and } B^{-1}(Av(2341)) \text{ is not a class.}\}$

Q2. Are the growth rates of $C$ and $B^{-1}(C)$ related?

Growth rate of a permutation class $C = \limsup_{n \to \infty} \sqrt[n]{c_n}$

where $c_n$ is the number of permutations of length $n$ in $C$
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Composing sorting operators

- $SB$-sortable permutations:
  \[(SB)^{-1}(Av(21)) = B^{-1}(Av(231)) = Av(3241, 2341, 4231, 2431)\]

- $B^2$-sortable permutations:
  \[(BB)^{-1}(Av(21)) = B^{-1}(Av(231, 321)) = Av(\Gamma_2)\]

- $B^k$-sortable permutations:
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Other sorting operators:
- built from $B$, $S$, \ldots and symmetries of the permutations $(i, r, c)$
- with a queue
- definition of abstract sorting operator
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