Permutation patterns and permutation classes in (enumerative) combinatorics

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Several ways to describe permutations

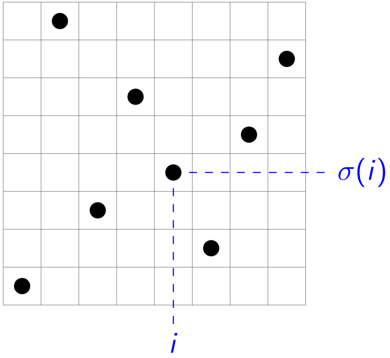
Permutation: Bijection from $[1..n]$ to itself. Set $\mathfrak{S}_n$.

- Two lines notation:
  $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 8 & 3 & 6 & 4 & 2 & 5 & 7 \end{pmatrix}$

- Linear notation:
  $\sigma = 1\ 8\ 3\ 6\ 4\ 2\ 5\ 7$

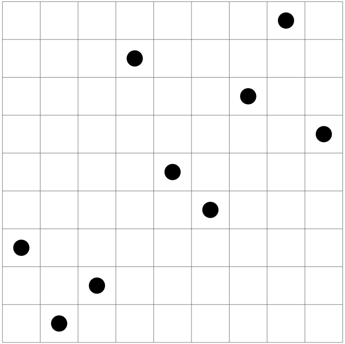
- Description as a product of cycles:
  $\sigma = (1)\ (2\ 8\ 7\ 5\ 4\ 6)\ (3)$

- Graphical description, or diagram:



Patterns in permutations

Pattern relation $\preccurlyeq$:
$\pi \in \mathfrak{S}_k$ is a pattern of $\sigma \in \mathfrak{S}_n$ if $\exists\ 1 \leq i_1 < \ldots < i_k \leq n$ such that $\sigma_{i_1} \ldots \sigma_{i_k}$ is order isomorphic ($\equiv$) to π.



Notation: $\pi \preccurlyeq \sigma$.

Equivalently:
The normalization of $\sigma_{i_1} \ldots \sigma_{i_k}$ on $[1..k]$ yields π.

Example: $2\,1\,3\,4 \preccurlyeq \mathbf{3}\,\mathbf{1}\,2\,8\,\mathbf{5}\,4\,\mathbf{7}\,9\,6$
since $3\,1\,5\,7 \equiv 2\,1\,3\,4$.

Remark: $\preccurlyeq$ is a partial order on $\mathfrak{S} = \bigcup_n \mathfrak{S}_n$

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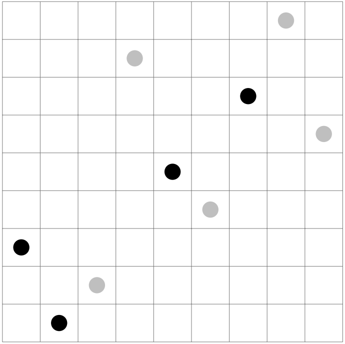

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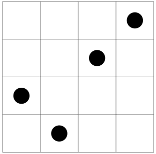

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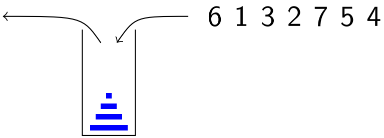
Sort (or try to do so) using a stack satisfying the Hanoi condition.



Stack sorting and pattern avoidance

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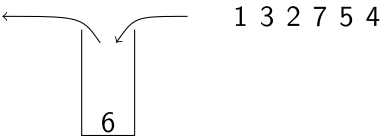

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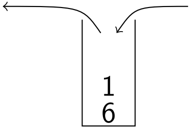

3 2 7 5 4

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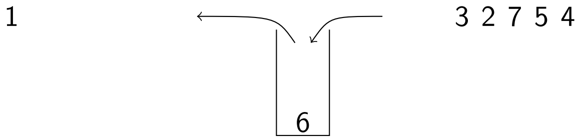

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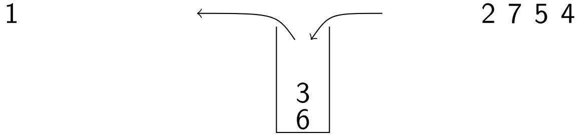

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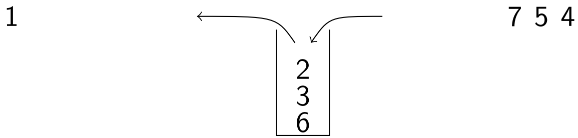

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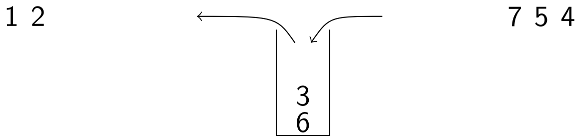

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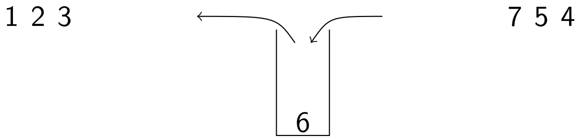

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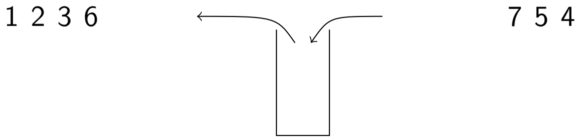

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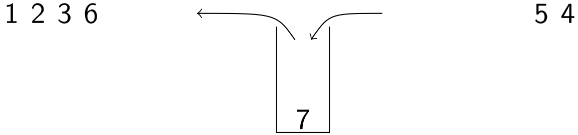

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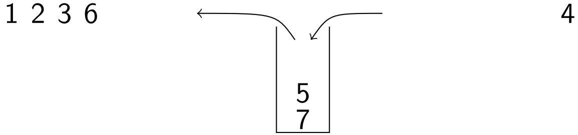

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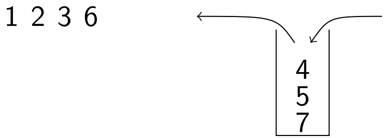

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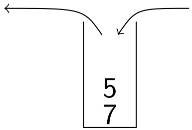

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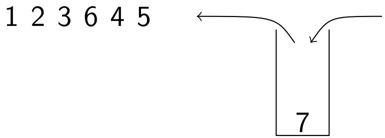

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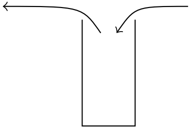

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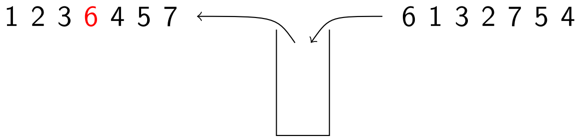

Stack sorting and pattern avoidance

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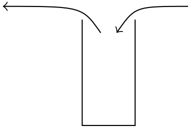
Sort (or try to do so) using a stack satisfying the Hanoi condition.

$\mathbf{S}(\sigma) = 1\ 2\ 3\ 6\ 4\ 5\ 7$



$6\ 1\ 3\ 2\ 7\ 5\ 4 = \sigma$

Equivalently, $\mathbf{S}(\varepsilon) = \varepsilon$ and $\mathbf{S}(LnR) = \mathbf{S}(L)\mathbf{S}(R)n$, $n = \max(LnR)$

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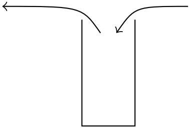
$\mathbf{S}(\sigma) = 1\ 2\ 3\ 6\ 4\ 5\ 7$  $6\ 1\ 3\ 2\ 7\ 5\ 4 = \sigma$

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First result on permutation patterns [Knuth 68] :
Stack-sortable permutations are those avoiding the pattern 231

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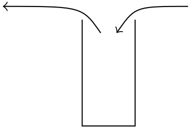
$\mathbf{S}(\sigma) = 1\ 2\ 3\ 6\ 4\ 5\ 7$  $6\ 1\ 3\ 2\ 7\ 5\ 4 = \sigma$

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First result on permutation patterns [Knuth 68] :
Stack-sortable permutations are those avoiding the pattern 231

- $\mathbf{S}(231) = 213$, and σ containing a pattern 231 is not sortable
- 231 patterns are the only ones that prevent a permutation from being sortable

Counted by the Catalan numbers: there are $c_n = \frac{1}{n+1}\binom{2n}{n}$ permutations of size n that avoid 231, *i.e.* that are sorted by $\mathbf{S}$

Pattern avoiding permutations

Permutations avoiding classical patterns

- $Av(\pi)$ = the set of permutations that avoid the pattern π
- $Av(B) = \bigcap_{\pi \in B} Av(\pi)$

Generalizations of excluded patterns

- Dashed and bivincular patterns [Babson, Steingrímsson 00]:
  Add adjacency constraints
- Barred patterns [West 90]:
  To describe $\mathbf{S} \circ \mathbf{S}$-sortable permutations
- Mesh, decorated patterns [Úlfarsson, Brändén, Claesson 11]:
  To describe $\mathbf{S} \circ \mathbf{S} \circ \mathbf{S}$-sortable permutations

Often motivated by the study of sorting operators.

Mostly studied for enumeration:
Sequence $|Av_n(B)|$? Generating function $\sum_n |Av_n(B)| z^n$?

First enumeration results (for classical patterns)

One excluded pattern:

- of size 3: [MacMahon 1915] for 123 and [Knuth 68] for 231
  $Av(\pi)$ is enumerated by the Catalan numbers
- of size 4: Only three different enumerations.
  Representatives are:
  - 1342 [Bóna 97], algebraic generating function
  - 1234 [Gessel 90], holonomic (or D-finite) generating function
  - 1324 ... remains an open problem

Two excluded patterns:

- both of size 3: all are known [Simion & Schmidt 85]
- one of size 3 and one of size 4: all are known
- both of size 4: most are known

See

`http://en.wikipedia.org/wiki/Enumerations_of_specific_permutation_classes`

From pattern avoidance to permutation classes

Permutation class: set of permutations downward-closed for $\preccurlyeq$.
$\mathcal{C}$ is a class when $\sigma \in \mathcal{C}$ and $\pi \preccurlyeq \sigma \Rightarrow \pi \in \mathcal{C}$

- Permutations avoiding (classical) patterns form permutation classes: $Av(B)$ is stable by $\preccurlyeq$, hence is a class.

  If B is an antichain then B is called the basis of $Av(B)$.

- Conversely: Classes can be described by excluded (classical) patterns. Every class $\mathcal{C}$ can be characterized by:
  $\mathcal{C} = Av(B)$ for $B = \{\sigma \notin \mathcal{C} : \forall \pi \preccurlyeq \sigma \text{ such that } \pi \neq \sigma, \pi \in \mathcal{C}\}$

Basis: A class has a unique basis.
A basis may be either finite or infinite.

Remark: First example of infinite basis for the class of permutations sortable by a double-ended queue [Pratt 73]

The Stanley-Wilf (ex-)conjecture

Theorem:

For every permutation π, $\sqrt[n]{|Av_n(\pi)|}$ converges to a constant c_π.

Conjectured by [Stanley & Wilf 92]; proved by [Marcus & Tardos 04].

What can be said about the growth rate c_π of $Av(\pi)$?

- First bound on c_π: $c_\pi \leq 15^{2k^4\binom{k^2}{k}}$, where $k = |\pi|$
- Improved to $c_\pi \leq 2^{\mathcal{O}(k \log k)}$ [Cibulka 09]
- Conjecture: $c_\pi \leq P(k)$ for some polynomial P
- The Arratia conjecture ($c_\pi \leq (k-1)^2$) has been disproved by [Albert, Elder, Rechnitzer, Westcott & Zabrocki 06]: $c_{1324} \geq 9.47$

Growth rate of a class $\mathcal{C} = \limsup_{n\to\infty} \sqrt[n]{|\mathcal{C}_n|}$.

Does $\lim_{n\to\infty} \sqrt[n]{|\mathcal{C}_n|}$ exists?

What growth rates can occur? What can be said about classes of particular growth rates? [Vatter and co authors 10-12+]

The specific and the general perspective

Detailed study of particular classes

Motivations:
- Systematic study of small patterns,
- or classes arising from other problems (sorting devices,...)

Results:
- Description by excluded patterns
- Enumeration, distribution of statistics

Methods:
- Mostly *ad hoc* constructions
- Bijections, recursive descriptions, ...

Results on families of classes

General idea:
- Description of the structure of permutation classes
- *Nice enumeration is a consequence of nice structure*

Typical result:
- Under some sufficient conditions, $\mathcal{C}$ has some nice property

Possible methods:
- Geometric conditions
- Substitution decomposition
- Encodings ...

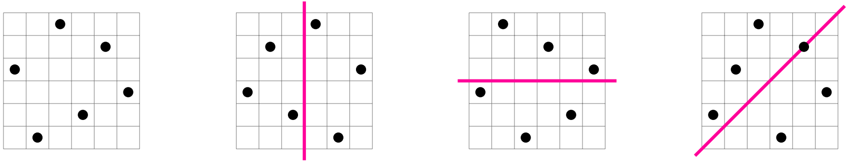
Specific results $\Leftarrow$ General method

Study of a particular class $\Rightarrow$ Generalization to a family of classes

Permutations sorted by $\mathbf{S} \circ \alpha \circ \mathbf{S}$ for $\alpha \in D_8$

Symmetries of the square act on permutations:
$D_8 = \{\mathbf{id}, \mathbf{r}, \mathbf{c}, \mathbf{i}, \mathbf{r} \circ \mathbf{c}, \mathbf{i} \circ \mathbf{r}, \mathbf{i} \circ \mathbf{c}, \mathbf{i} \circ \mathbf{c} \circ \mathbf{r}\}$

Reverse — Complement — Inverse



σ — $\mathbf{r}(\sigma)$ — $\mathbf{c}(\sigma)$ — $\mathbf{i}(\sigma)$

For any $\alpha \in D_8$, study permutations sorted by $\mathbf{S} \circ \alpha \circ \mathbf{S}$

- Characterization with (generalized) excluded patterns
  [West 93] [Albert, Atkinson, B., Claesson & Dukes 11]
- Enumeration and distribution of statistics
  [Zeilberger 92] [B. & Guibert 12]

Permutations sorted by $\mathbf{S} \circ \mathbf{r} \circ \mathbf{S}$

Theorem [Zeilberger 92] [West 93]:
Permutations sorted by $\mathbf{S} \circ \mathbf{S}$ are $Av(2341, 3\bar{5}241)$, and are counted by $\frac{2(3n)!}{(n+1)!(2n+1)!}$

Theorem [Albert, Atkinson, B., Claesson & Dukes 11]:
Permutations sorted by $\mathbf{S} \circ \mathbf{r} \circ \mathbf{S}$ are $Av(1342, 31\text{-}4\text{-}2)$

Theorem [B. & Guibert 12]:
Permutations sorted by $\mathbf{S} \circ \mathbf{r} \circ \mathbf{S}$ and those sorted by $\mathbf{S} \circ \mathbf{S}$ are enumerated by the same sequence.
Furthermore the tuple of statistics (udword, rmax, lmax, zeil, indmax, slmax, slmax $\circ\mathbf{r}$) has the same distribution of both sets.
Hence the statistics asc, des, maj, maj $\circ\mathbf{r}$, maj $\circ\mathbf{c}$, maj $\circ\mathbf{rc}$, valley, peak, ddes, dasc, rir, rdr, lir, ldr are also (and jointly) equidistributed.

Tool: Generating tree

Structure in permutation classes

It is not easy to define *structure*...

Example of results about structure:

- $\sqrt[n]{|Av_n(\pi)|}$ converges to a constant c_π
- Permutation classes of growth rate less than $\kappa \approx 2.20557$ have rational generating function
  (κ is the unique positive solution of $x^3 - 2x^2 - 1 = 0$)
- Under a sufficient condition $(\star)$, permutation classes have finite bases and algebraic generating functions

Algorithmic counterpart to each statement:

- Given π, compute the growth rate c_π
- Given $\mathcal{C}$ of growth rate $< \kappa$, compute its generating function
- Given $\mathcal{C}$ satisfying condition $(\star)$, compute its basis and its generating function

Structure through substitution decomposition and trees

With substitution decomposition, permutations are trees.

[Flajolet & Sedgewick 09]: Trees are easy to study and enumerate.

From a combinatorial specification for a simple variety of trees (*i.e.* unambiguous tree grammar), we obtain in a systematic way

- a polynomial system for the generating function
- efficient random samplers

Specific results: Enumeration of pin-permutations by a rational generating function [Bassino, B. & Rossin 11]

General results: Classes with finitely many simples are nicely structured [Albert & Atkinson 05] and its developments

Substitution decomposition: main ideas

Analogous to the decomposition of integers as products of primes

- [Möhring & Radermacher 84]: general framework
- Applies to relations, graphs, posets, boolean functions, set systems, ...
- Permutations (almost) fit into this framework

Relies on:

- a principle for building objects (permutations, graphs) from smaller objects: the substitution
- some "basic objects" for this construction: **simple** permutations, **prime** graphs

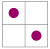
Required properties:

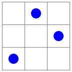
- every object can be decomposed using only "basic objects"
- this decomposition is unique

Substitution for permutations

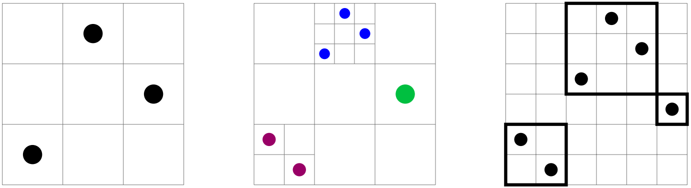
Substitution or inflation : $\sigma = \pi[\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(k)}]$.

Example : Here, $\pi = 1\,3\,2$, and

$\alpha^{(1)} = 2\,1 =$ 

$\alpha^{(2)} = 1\,3\,2 =$ 

$\alpha^{(3)} = 1 =$  .



Hence $\sigma = 1\,3\,2[2\,1, 1\,3\,2, 1] = 2\,1\,4\,6\,5\,3$.

Simple permutations

Interval (or block) = set of elements of σ whose positions **and** values form intervals of integers
Example: 5 7 4 6 is an interval of 2 5 7 4 6 1 3

Simple permutation = permutation with no interval, except the trivial ones: $1, 2, \ldots, n$ and σ
Example: 3 1 7 4 6 2 5 is simple

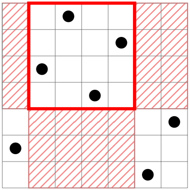
The smallest simples: 12, 21, **2413**, **3142**

Some fact about simples:

- Asymptotically $\frac{n!}{e^2}$ simples of size n
- Generating function not D-finite
- In many (conjecturally all) permutation classes they have density 0

[Albert, Atkinson & Klazar 03] [Brignall 08]

Not simple:



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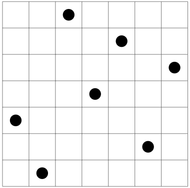


Substitution decomposition of permutations

Theorem: [Albert, Atkinson & Klazar 03]
Every σ ($\neq 1$) is uniquely decomposed as

- $12\dots k[\alpha^{(1)},\dots,\alpha^{(k)}]$, where the $\alpha^{(i)}$ are $\oplus$-indecomposable
- $k\dots 21[\alpha^{(1)},\dots,\alpha^{(k)}]$, where the $\alpha^{(i)}$ are $\ominus$-indecomposable
- $\pi[\alpha^{(1)},\dots,\alpha^{(k)}]$, where π is simple of size $k \geq 4$

Remarks:

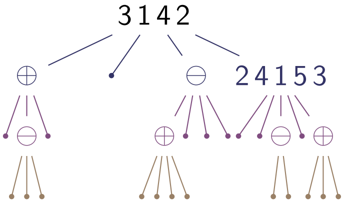
- $\oplus$-indecomposable: that cannot be written as $12[\alpha^{(1)},\alpha^{(2)}]$
- Allows to relate the generating function for simples with that of all permutations

Decomposing recursively inside the $\alpha^{(i)}$ $\Rightarrow$ **decomposition tree**

Decomposition tree: witness of this decomposition

Example: Decomposition tree of

$\sigma = 10\,13\,12\,11\,14\,1\,18\,19\,20\,21\,17\,16\,15\,4\,8\,3\,2\,9\,5\,6\,7$



$\sigma = 3\,1\,4\,2[\oplus[1, \ominus[1,1,1],1],1,\ominus[\oplus[1,1,1,1],1,1,1],2\,4\,1\,5\,3[1,1,\ominus[1,1],1,\oplus[1,1,1]]]$

Notations and properties:

- $\oplus = 12\ldots k$, $\ominus = k\ldots 21$ $=$ linear nodes.
- π simple of size ≥ 4 $=$ prime node.
- No edge $\oplus - \oplus$ nor $\ominus - \ominus$.
- Ordered trees.
- These conditions characterize decomposition trees.

Bijection between permutations and their decomposition trees.

Computation: Linear time algorithm [Uno & Yagiura 00] [Bui Xuan, Habib & Paul 05] [Bergeron, Chauve, Montgolfier & Raffinot 08]

A result about structure in permutation classes

Theorem [Albert & Atkinson 05]: If $\mathcal{C}$ contains a finite number of simple permutations, then

- $\mathcal{C}$ has a finite basis
- $\mathcal{C}$ has an algebraic generating function ($= \sum_n |\mathcal{C}_n| z^n$)

Proof: relies on the substitution decomposition

- Easy for substitution-closed classes.
- Otherwise, in the decomposition trees of permutations of $\mathcal{C}$, propagate the excluded patterns in the subtrees.
- This gives a (possibly ambiguous) tree grammar describing $\mathcal{C}$.
- Inclusion-exclusion then gives a polynomial system for its generating function.

The proof is constructive: it should provide an algorithm to compute the generating function from the simples in $\mathcal{C}$

Related algorithmic questions

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Algorithmic questions

A. How to determine whether $\mathcal{C}$ contains finitely many simples?

B. How to compute these simples?

C. How to compute the tree grammar? the generating function?

First answers

A.&B. Semi-decision procedure from [Schmerl & Trotter 93]:
find simples of size $4, 5, 6, \ldots$ until k and $k+1$ for which there are 0 simples. Huge complexity...

C. For each class, go through the proof of [Albert & Atkinson 05]

Finite number of simple permutations: decision (A.)

Theorem [Brignall, Ruškuc & Vatter 08] : It is decidable whether $\mathcal{C}$ given by its finite basis contains a finite number of simples.

Prop. $\mathcal{C} = Av(B)$ contains infinitely many simples iff $\mathcal{C}$ contains:

1. either infinitely many parallel alternantions
2. or infinitely many simple wedge permutations
3. or infinitely many proper pin-permutations

	Decision procedure	Complexity
1. and 2.:	pattern matching of patterns of size 3 or 4 in the $\beta \in B$.	Polynomial
3.:	• Encode pin-permutations by words over a finite alphabet • Decidability with automata techniques	Decidable 2ExpTime

Finite number of simple permutations: improvements (A.&B.)

$\mathcal{C} = Av(B)$ with finite basis B

Test whether $\mathcal{C}$ contains a finite number of simples:

Method: detailed study of pin-permutations, of their encoding by words and optimized automata construction following decomposition trees

- If $\mathcal{C}$ is substitution-closed [Bassino, B., Pierrot & Rossin 10]
  Algorithm in $\mathcal{O}(n \log n)$ where $n = \sum_{\beta \in B} |\beta|$
- Otherwise [Bassino, B., Pierrot & Rossin 12+]
  Algorithm in $\mathcal{O}(p^{2k})$ where $p = \max_{\beta \in B} |\beta|$ and $k = |B|$

Compute the set $\mathcal{SI}_{\mathcal{C}}$ of simples in $\mathcal{C}$ [Pierrot & Rossin 12]:

Method: Analyzing the poset of simple permutations (for $\preccurlyeq$)

- Algorithm whose complexity depends on the size of the output

Computation of the tree grammar and consequence (c.)

$\mathcal{C} = Av(B)$ with finite basis B. Assume that $\mathcal{C}$ contains finitely many simples, and that their set $\mathcal{SI}_{\mathcal{C}}$ is given.

Algorithm to compute the **unambiguous** tree grammar associated with $\mathcal{C}$: [Bassino, B., Pierrot, Pivoteau & Rossin 12]

- Propagate pattern avoidance constraint in subtrees
- Replace inclusion-exclusion by disambiguation of the grammar, introducing pattern **containement** constraints

$\Rightarrow$ Algorithmic chain from B finite to unambiguous grammar for $\mathcal{C}$

Consequences:

- Computes a polynomial system for the generating function
- Provides a combinatorial specification for $\mathcal{C}$, hence efficient random samplers (recursive or Boltzmann method)

What next?

Study average properties of random permutations in permutation classes, in particular w.r.t. classical permutation statistics

Example:
30 000 permutations of size 500 in $Av(2413, 1243, 2341, 531642, 41352)$