A general theory of Wilf-equivalence for Catalan structures

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joint work with Michael Albert (University of Otago)

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Let $\mathcal{C}$ be any combinatorial class, i.e.

- $\mathcal{C}$ is equipped with a notion of size
- such that for any $n$ there are finitely many objects of size $n$ in $\mathcal{C}$.
- The number of objects of size $n$ in $\mathcal{C}$ is denoted $c_n$.

To $\mathcal{C}$, we associate:

- its enumeration sequence $(c_n)$,
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To $C$, we associate:

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- its generating function $\sum c_n t^n$.

Sometimes (or very often!), two classes have the same enumeration sequences (or equivalently generating function).

Such enumeration coincidences are called Wilf-equivalences (terminology from the Permutation Patterns literature).
Motivation: from pattern-avoiding permutations

$S_n$ = set of permutations of \{1, 2, \ldots, n\}, seen as words $\sigma(1)\sigma(2)\ldots\sigma(n)$
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\( \mathcal{S}_n = \) set of permutations of \( \{1, 2, \ldots, n\} \), seen as words \( \sigma(1)\sigma(2)\ldots\sigma(n) \)

\( \pi \in \mathcal{S}_k \) is a **pattern** of \( \sigma \in \mathcal{S}_n \) if \( \exists \)

\( 1 \leq i_1 < \ldots < i_k \leq n \) such that the sequence \( \sigma(i_1)\ldots\sigma(i_k) \) is in the **same** relative order as \( \pi \).
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Example:
2 1 3 4 is a pattern of 3 1 2 8 5 4 7 9 6.

Notation: \( \text{Av}(\pi_1, \pi_2, \ldots) \) is the class of all permutations that do not contain \( \pi_1 \), nor \( \pi_2 \), \ldots as a pattern.
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$\pi$ and $\tau$ (or $Av(\pi)$ and $Av(\tau)$) are Wilf-equivalent if $Av(\pi)$ and $Av(\tau)$ have the same enumeration.
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For \( R \) and \( S \) sets of permutations, \( R \) and \( S \) (or \( \text{Av}(R) \) and \( \text{Av}(S) \)) are Wilf-equivalent if \( \text{Av}(R) \) and \( \text{Av}(S) \) have the same enumeration.
Some Wilf-equivalences for pattern-avoiding permutations

Small excluded patterns:

- $Av(123)$ and $Av(231)$ are Wilf-equivalent, and enumerated by the Catalan numbers $Cat_n = \frac{1}{n+1} \binom{2n}{n}$
- There are three Wilf-equivalence classes for permutation classes $Av(\pi)$ with $\pi$ of size 4, the enumeration of $Av(1324)$ being open.
- Check all Wilf-equivalences between $Av(\pi, \tau)$ when $\pi$ and $\tau$ have size 3 or 4 on Wikipedia.

Some results for arbitrary long patterns:

- $Av(231 \oplus \pi)$ and $Av(312 \oplus \pi)$ [West & Stankova 02]

First unbalanced Wilf-equivalences:

- $Av(1324, 3416725)$ and $Av(1234)$;
- $Av(2143, 3142, 246135)$ and $Av(2413, 3142)$ [Burstein & Pantone 14+]
**Novelty of our work: a global look**

**Our goal:** find all Wilf-equivalences between classes $\text{Av}(231, \pi)$.

**Harmless assumption:** In $\text{Av}(231, \pi)$, throughout the talk, $\pi$ avoids 231. (or we are just studying $\text{Av}(231)$...)
Novelty of our work: a global look

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Most important remark: Classes \( \text{Av}(231, \pi) \) are families of Catalan objects (\( \text{Av}(231) \)) with an additional avoidance restriction.
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Most important remark: Classes $\text{Av}(231, \pi)$ are families of Catalan objects ($\text{Av}(231)$) with an additional avoidance restriction.

So, equivalently but somehow more generally, our goal rephrases as:
find all Wilf-equivalences between “pattern-avoiding Catalan objects”.
Substructures in Catalan objects
Some Catalan structures, and their substructures

- 231-avoiding permutations
- Dyck paths
- Plane forests
- Arch systems
- Complete binary trees

41327658 =

M. H. Albert, M. Bouvel
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\[ 31254 = \]

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$$31254 = \text{Dyck path}$$
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- Arch systems

Essential fact: The usual bijections relating our quartet of Catalan structures preserve the substructure relation.
For any Catalan family in our quartet, we are interested in classes defined by the avoidance of one Catalan object.

- **Motivation**: permutation classes $\text{Av}(231, \pi)$
- **In practice**: classes $\text{Av}(A)$ of arch systems avoiding some subsystem $A$

But all four contexts are equivalent!
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- **Motivation:** permutation classes $\text{Av}(231, \pi)$
- **In practice:** classes $\text{Av}(A)$ of arch systems avoiding some subsystem $A$

But all four contexts are equivalent!

- Which arch systems $A$ are Wilf-equivalent?
  - *i.e.* which classes $\text{Av}(A)$ have the same enumeration?
- **Bijections** between $\text{Av}(A)$ and $\text{Av}(B)$ for Wilf-equivalent arch systems $A$ and $B$?
- How many Wilf-equivalence classes of arch systems are there?
- The special case of the Wilf-equivalence class of $N_n = \ldots \bowtie \ldots$.
- **Comparison** between the enumeration sequences of $\text{Av}(A)$ and $\text{Av}(B)$ for some $A$ and $B$ that are not equivalent.
Quick detour: What about other Catalan structures?

Other Catalan objects having a natural notion of substructure:

- 123-avoiding permutations

\[
\begin{array}{ccccccc}
\bullet & & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
\end{array}
\]

\[
86421753 =
\begin{array}{ccccccc}
\bullet & & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
\end{array}
\]
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```plaintext
86421753 =
```

![Diagram of a 123-avoiding permutation with a grid and marked points]
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\[
54213 = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\]
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$54213 = \begin{array}{c}
\begin{array}{c}
\text{grid}
\end{array}
\end{array}$
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\[ 54213 = \]

\[ \begin{array}{|c|c|c|c|}
\hline
& & & \\
\hline
& & & \\
\hline
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However the “usual” or “canonical” bijections (if any...) with Catalan objects of our quartet do not preserve the substructure relation.
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Indeed, the associated posets are not isomorphic:

\[ \text{Av}(231) \text{ is } \]

\[ \text{but } \text{Av}(123) \text{ is } \]
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However the “usual” or “canonical” bijections (if any...) with Catalan objects of our quartet do not preserve the substructure relation.

Indeed, the associated posets are not isomorphic:

⇒ These Catalan objects are not part of our study. (Future work maybe?)
An equivalence relation strongly related to Wilf-equivalence
An equivalence relation on arch systems

Observation and terminology:
An arch system is a concatenation of atoms, i.e. (non-empty) arch systems having a single outermost arch.
An equivalence relation on arch systems

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An arch system is a concatenation of atoms, i.e. (non-empty) arch systems having a single outermost arch.

The binary relation, \( \sim \), is the finest equivalence relation that satisfies:

1. \( A \sim A \)
2. \( A \sim B \implies \bar{A} \sim \bar{B} \)
3. \( a \sim b \implies PaQ \sim PbQ \)
4. \( PabQ \sim PbaQ \)
5. \( a \bar{bc} \sim \bar{ab}c \)

where \( A, B, P \) and \( Q \) denote arbitrary arch systems and \( a, b \) and \( c \) denote atoms or empty arch systems.
Main theorem: If $A$ and $B$ are arch systems such that $A \sim B$ then $Av(A)$ and $Av(B)$ have the same enumeration, i.e. are Wilf-equivalent.

In other words, $\sim$ refines Wilf-equivalence.
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Conjecture: $\sim$ coincides with Wilf-equivalence.

Data, obtained with PermLab:
The conjecture holds for arch systems of size up to 15 (where $\sim$ has 16,709 equivalence classes on the $Cat_{15} = 9,694,845$ arch systems).
~ is (a refinement of?) Wilf-equivalence

Main theorem: If $A$ and $B$ are arch systems such that $A \sim B$ then $\mathcal{A}_V(A)$ and $\mathcal{A}_V(B)$ have the same enumeration, i.e. are Wilf-equivalent.

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Terminology: The equivalence classes of $\sim$ are called cohorts.
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**Main theorem:** If \( A \) and \( B \) are arch systems such that \( A \sim B \) then \( \Av(A) \) and \( \Av(B) \) have the same enumeration, i.e. are Wilf-equivalent.

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**Terminology:** The equivalence classes of \( \sim \) are called **cohorts**.

To any arch system \( A \), we can associate:

- its \( \sim \)-equivalence class, i.e., its cohort;
- its avoidance class \( \Av(A) \);
- the enumeration sequence, or generating function \( F_A \), of \( \Av(A) \).
Overview of the proof

**Main theorem:** If $A$ and $B$ are arch systems such that $A \sim B$ then $A_v(A)$ and $A_v(B)$ have the same enumeration, *i.e.* are Wilf-equivalent.
Overview of the proof... by induction!

Main theorem: If $A$ and $B$ are arch systems such that $A \sim B$ then $A \vee(A)$ and $A \vee(B)$ have the same enumeration, i.e. are Wilf-equivalent.

Base case: If $A = B$ then $A \vee(A)$ and $A \vee(B)$ are Wilf-equivalent...

Inductive case: One case for each rule defining $\sim$.

<table>
<thead>
<tr>
<th>Rule</th>
<th>$A \sim B \Rightarrow [A] \sim [B]$</th>
<th>bijective proof</th>
<th>analytic proof</th>
</tr>
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<tbody>
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<td>yes</td>
<td>_</td>
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<tr>
<td>(2)</td>
<td>$a \sim b \Rightarrow PaQ \sim PbQ$</td>
<td>yes</td>
<td>_</td>
</tr>
<tr>
<td>(3)</td>
<td>$PabQ \sim PbaQ$</td>
<td>yes</td>
<td>_</td>
</tr>
<tr>
<td>(4)</td>
<td>$abc \sim abc$</td>
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<td>(4)</td>
<td>$a(bc) \sim (ab)c$</td>
<td>no</td>
<td>yes</td>
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<tr>
<td>(4 weak)</td>
<td>$a[b] \sim [ba]$</td>
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Overview of the proof... by induction!

Main theorem: If $A$ and $B$ are arch systems such that $A \sim B$ then $A^\vee(A)$ and $A^\vee(B)$ have the same enumeration, i.e. are Wilf-equivalent.

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Having only bijective proofs would allow to “unfold” the induction into a bijective proof that $A^\vee(A)$ and $A^\vee(B)$ are Wilf-equivalent, for all $A \sim B$. 
Bijective proof in case (2)

\[(2) \quad a \sim b \implies PaQ \sim PbQ\]

Take \(a \sim b\) and suppose that \(\Av(a)\) and \(\Av(b)\) are Wilf-equivalent. Take a size-preserving bijection \(\sigma : X \mapsto X^\sigma\) from \(\Av(a)\) to \(\Av(b)\). Build a size-preserving bijection \(\tau\) from \(\Av(PaQ)\) to \(\Av(PbQ)\) as follows:
Bijective proof in case (2)

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Take \(a \sim b\) and suppose that \(A_{\nu}(a)\) and \(A_{\nu}(b)\) are Wilf-equivalent.

Take a size-preserving bijection \(\sigma : X \mapsto X^\sigma\) from \(A_{\nu}(a)\) to \(A_{\nu}(b)\).

Build a size-preserving bijection \(\tau\) from \(A_{\nu}(PaQ)\) to \(A_{\nu}(PbQ)\) as follows:

- If \(X\) avoids \(PQ\), then take \(X^\tau = X\).
- Otherwise, apply \(\sigma\) to all intervals determined by the arches having one extremity between the leftmost \(P\) and the rightmost \(Q\):

\[X = \begin{array}{c}
\vdots \\
| l_1 | l_2 | \cdots | l_k |
\end{array}
\quad \mapsto \quad X^\tau = \begin{array}{c}
\vdots \\
| l_1^\sigma | l_2^\sigma | \cdots | l_k^\sigma |
\end{array}\]

- \(X^\tau\) avoids \(PbQ\) if and only if \(X\) avoids \(PaQ\).
Analytic proof in case (4)

(4) $a\overbrace{bc} \sim \overbrace{ab}c$

Notations: $a = \overline{A}$, $b = \overline{B}$ and $c = \overline{C}$.

$F_X$ = the generating function of $A_v(X)$.

We want that $F_{a\overbrace{bc}} = F_{\overbrace{ab}c}$.
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Notations: \( a = \overline{A} \), \( b = \overline{B} \) and \( c = \overline{C} \).

\( F_X \) = the generating function of \( \text{Av}(X) \).

We want that \( F_{a\overline{bc}} = F_{\overline{ab}c} \).

- Compute a system for \( F_{a\overline{bc}} \):

\[
F_{a\overline{bc}} = 1 + tF_A F_{a\overline{bc}} + t(F_{a\overline{bc}} - F_A) F_{\overline{bc}}
\]

\( \text{Av}(a\overline{bc}) = \varepsilon + \overline{X\mid Y} + \overline{Z\mid T} \)

\( X \) avoids \( A \) \hspace{1cm} \( Z \) contains \( A \)
Analytic proof in case (4)

\[
(4) \quad \overline{ab \times c} \sim \overline{abc}
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\]
\[
F_{\overline{bc}} = 1 + tF_{bc} F_{\overline{bc}}
\]
\[
F_{bc} = 1 + tF_B F_{bc} + t(F_{bc} - F_B)F_c
\]
\[
F_c = 1 + tF_C F_c
\]
Analytic proof in case (4)

\[(4) \quad a|bc \sim ab|c\]

Notations: \(a = \begin{array}{c}A \end{array}\), \(b = \begin{array}{c}B \end{array}\) and \(c = \begin{array}{c}C \end{array}\).

\(F_X = \) the generating function of \(A\nu(X)\).

We want that \(F_{a|bc} = F_{ab|c}\).

- Compute a system for \(F_{a|bc}\):
- The solution \(F_{a|bc}\) is a terrible mess depending on \(F_A, F_B\) and \(F_C\)
Analytic proof in case (4)

\[ a \overline{bc} \sim \overline{ab} c \]

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\( F_X = \) the generating function of \( A v(X) \).

We want that \( F_{a \overline{bc}} = F_{\overline{ab} c} \).

- Compute a system for \( F_{a \overline{bc}} \):
- The solution \( F_{a \overline{bc}} \) is a terrible mess depending on \( F_A, F_B \) and \( F_C \)
  ... but symmetric in \( F_A, F_B \) and \( F_C \)!
- Consequently, \( F_{a \overline{bc}} = F_{c \overline{ab}} = F_{\overline{ab} c} \).
Analytic proof in case (4)

\[(4) \quad a \overline{bc} \sim \overline{ab}c\]

Notations: \(a = \overline{A}, b = \overline{B}\) and \(c = \overline{C}\).

\(F_X\) = the generating function of \(Av(X)\).

We want that \(F_{a\overline{bc}} = F_{\overline{ab}c}\).

- Compute a system for \(F_{a\overline{bc}}\):
- The solution \(F_{a\overline{bc}}\) is a terrible mess depending on \(F_A, F_B\) and \(F_C\) … but symmetric in \(F_A, F_B\) and \(F_C\)!
- Consequently, \(F_{a\overline{bc}} = F_{c\overline{ab}} = F_{\overline{ab}c}\).
- Using \(F_X = 1/(1 - tF_X)\), we can write:

\[
F_{a\overline{bc}} = \frac{1 - t(F_aF_b + F_bF_c + F_cF_a - F_aF_bF_c)}{1 - t(F_a + F_b + F_c - F_aF_bF_c)}
\]
How many cohorts?

How many Wilf-equivalence classes?
Up to size 15, there are as many Wilf-equivalence as cohorts: 1, 1, 2, 4, 8, 16, 32, 67, 142, 307, 669, 1478, 3290, 7390, 16709...
Number of Wilf-equivalence classes: upper bounds

Up to size 15, there are as many Wilf-equivalence as cohorts: 
1, 1, 2, 4, 8, 16, 32, 67, 142, 307, 669, 1478, 3290, 7390, 16709\ldots

For any size $n$, an upper bound on the number of Wilf-equivalence classes of classes $A \nabla (A)$, where $A$ is an arch system with $n$ arches is:

- $Cat_n = \text{number of arch systems with } n \text{ arches}$
- $= \text{number of plane forests of size } n$: $\sim \frac{1}{\sqrt{\pi}} \cdot 4^n \cdot n^{-3/2}$
Number of Wilf-equivalence classes: upper bounds

Up to size 15, there are as many Wilf-equivalence as cohorts:
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For any size $n$, an upper bound on the number of Wilf-equivalence classes of classes $\text{Av}(A)$, where $A$ is an arch system with $n$ arches is:

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Improved upper bounds can also be obtained:
- Number of non-plane forests of size $n$: $\sim 0.440 \cdot 2.9558^n \cdot n^{-3/2}$
- Number of cohorts of arch systems of size $n$: $\sim 0.455 \cdot 2.4975^n \cdot n^{-3/2}$
Up to size 15, there are as many Wilf-equivalence as cohorts:
1, 1, 2, 4, 8, 16, 32, 67, 142, 307, 669, 1478, 3290, 7390, 16709…

For any size $n$, an upper bound on the number of Wilf-equivalence classes of classes $Av(A)$, where $A$ is an arch system with $n$ arches is:

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Moral of the story:
Many Wilf-equivalences between classes $Av(A)$ avoiding an arch system $A$!
Fewer cohorts than non-plane forests

Arch systems are in bijection with plane forests:

and atoms correspond to (plane) trees.
Fewer cohorts than non-plane forests

Arch systems are in bijection with plane forests:

\[ \phi \leftrightarrow \]

and atoms correspond to (plane) trees.

**Proposition:** If \( \phi(A) = \phi(B) \) as non-plane forests, then \( A \sim B \).

**Sketch of proof:**

- **(3)** \( PabQ \sim PbaQ \): The order of the trees does not affect the cohort.
- **(1)** \( A \sim B \implies \overline{A} \sim \overline{B} \) and **(2)** \( a \sim b \implies PaQ \sim PbQ \): This also holds in context, *i.e.* for siblings.
Arch systems are in bijection with plane forests:

\[ \phi \leftarrow\rightarrow \]

and atoms correspond to (plane) trees.

**Proposition:** If \( \phi(A) = \phi(B) \) as non-plane forests, then \( A \sim B \).

**Sketch of proof:**

1. \( a \sim b \Rightarrow PaQ \sim PbQ \): This also holds in context, *i.e.* for siblings.

**Corollary:** There are fewer cohorts than non-plane forests, hence fewer Wilf-equivalence classes than non-plane forests.
Asymptotic estimate of the number of cohorts

Interpretation of (4') $a\overline{bc} \sim \overline{ab}c$ on forests:
Asymptotic estimate of the number of cohorts

Interpretation of \((4')\) \(a \overline{bc} \sim \overline{ab} c\) on trees:

\[
\begin{array}{c}
\text{\(T_a\)} & \sim & \text{\(T_c\)} \\
T_b & T_c & T_a & T_b
\end{array}
\]
Asymptotic estimate of the number of cohorts

Interpretation of $(4') \; [a \overbrace{bc} \sim \overbrace{ab}c]$ on trees:

\[
\begin{align*}
T_a & \sim T_c \sim T_a \sim T_b \sim T_a \\
T_b & T_c T_a T_b & T_a & T_a T_b T_b & T_a \\
\end{align*}
\]
Asymptotic estimate of the number of cohorts

Interpretation of \((4')\) \(a\overline{bc} \sim \overline{ab}c\) on trees:

\[
\begin{array}{cccc}
T_a & \sim & T_c & ; \\
T_b & & T_a & T_b & T_a \\
T_a & & T_a & T_b & T_b \\
& & & & T_a
\end{array}
\]

**Proposition:** The generating function of cohorts is \(A(t)/t\) where

\[
A = t + tA + \frac{1}{t} \text{MSet}_{\geq 2}(t^2 \text{MSet}_{\geq 3}(A)) + t \text{MSet}_{\geq 3}(A)
\]

where \(\text{MSet}(Z) = \exp\left(\frac{Z(t)}{1} + \frac{Z(t^2)}{2} + \frac{Z(t^3)}{3} + \frac{Z(t^4)}{4} + \ldots\right)\)

\[
\text{MSet}_{\geq 2}(Z) = \text{MSet}(Z) - 1 - Z(t)
\]

\[
\text{MSet}_{\geq 3}(Z) = \text{MSet}(Z) - 1 - Z(t) - \frac{1}{2} (Z(t^2) + Z(t^2))
\]
Asymptotic estimate of the number of cohorts

Interpretation of \((4')\) \(\mathcal{abc} \sim \mathcal{abc}\) on trees:

\[
\begin{align*}
T_a \sim T_c ; \quad T_b \sim T_c \sim T_a ; \quad T_a \sim T_a \sim T_b \sim T_a ; \quad T_a \sim T_a \sim T_a
\end{align*}
\]

Proposition: The generating function of cohorts is \(A(t)/t\) where

\[
A = t + tA + \frac{1}{t} MSet_{\geq 2}(t^2 MSet_{\geq 3}(A)) + t MSet_{\geq 3}(A)
\]

Proposition: The number of cohorts is asymptotically equivalent to \(c \cdot \gamma^n \cdot n^{-3/2}\) where \(c \approx 0.455\) and \(\gamma \approx 2.4975\).

Proof: Use the “twenty steps” of [Harary, Robinson & Schwenk 75].

This is an upper bound (conjecturally tight) on the number of Wilf-equivalence classes of classes \(\text{Av}(A)\) defined by the avoidance of an arch system \(A\) of size \(n\).
Further results: the “main” cohort, and comparison between cohorts
Define the sequence \((C^{(n)})\) of generating functions by

\[ C^{(0)} = 1 \text{ and } C^{(n)} = \frac{1}{1 - t C^{(n-1)}} \text{ for } n \geq 1. \]
Original motivation for our work

Define the sequence \((C^{(n)})\) of generating functions by
\[
C^{(0)} = 1 \quad \text{and} \quad C^{(n)} = \frac{1}{1-t C^{(n-1)}} \quad \text{for} \quad n \geq 1.
\]

**Proposition:** The generating function of \(Av(231, \pi)\) is \(C^{(n)}\) whenever:

1. \(\pi = k \ldots 21 \cdot n \ldots (k+2)(k+1)\) for any \(1 \leq k \leq n\)  
   \[\text{[Mansour & Vainshtein 01]}\]
2. \(\pi\) is a “wedge permutation” of size \(n\)  
   \[\text{[Mansour & Vainshtein 02]}\]
3. \(\pi = \lambda_k \oplus \lambda_{n-k}\) for any \(1 \leq k \leq n\), with e.g. \(\lambda_6 = \)
   \[\text{[A. & B. 13]}\]

These were proved independently (and analytically). Our original goal was a uniform (and possibly bijective) proof.
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**Proposition:** The generating function of \(Av(231, \pi)\) is \(C^{(n)}\) whenever:
- \(\pi = k \ldots 21 \cdot n \ldots (k + 2)(k + 1)\) for any \(1 \leq k \leq n\) \[\text{[Mansour & Vainshtein 01]}\]
- \(\pi\) is a “wedge permutation” of size \(n\) \[\text{[Mansour & Vainshtein 02]}\]
- \(\pi = \lambda_k \oplus \lambda_{n-k}\) for any \(1 \leq k \leq n\), with e.g. \(\lambda_6 = \)

These were proved independently (and analytically).

Our original goal was a uniform (and possibly bijective) proof.

**Remark:**
\(C^{(n)}\) is also the generating function of Dyck path of height at most \(n\).

**New results:**
We can explain these statements (and more) studying the “main” cohort.
The main cohort

Definition: $N_n = \ldots \begin{array}{c} \circ \end{array} \ldots$ is the nested arch system with $n$ arches. The main cohort (of size $n$) $\mathcal{M}_n$ is the cohort of $N_n$.

Theorem: The arch systems $A$ such that the generating function $F_A$ of $Av(A)$ is $C^{(n)}$ are exactly those of $\mathcal{M}_n$. 
The main cohort

**Definition:** $N_n = \ldots \text{arch} \ldots$ is the nested arch system with $n$ arches. The main cohort (of size $n$) $\mathcal{M}_n$ is the cohort of $N_n$.

**Theorem:** The arch systems $A$ such that the generating function $F_A$ of $Av(A)$ is $C^{(n)}$ are exactly those of $\mathcal{M}_n$.

**Remarks:**
- This encapsulates all results of previous slides.
The main cohort

**Definition:** $N_n = \cdots \bigcirc \cdots$ is the nested arch system with $n$ arches. The **main cohort** (of size $n$) $\mathcal{M}_n$ is the cohort of $N_n$.

**Theorem:** The arch systems $A$ such that the generating function $F_A$ of $\text{Av}(A)$ is $C^n$ are exactly those of $\mathcal{M}_n$.

**Remarks:**
- This encapsulates all results of previous slides.
- It also generalizes them to more excluded patterns.

\[ \text{Motz}_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \text{Cat}_k \text{ objects in the main cohort.} \]
Definition: $N_n = \cdots \bigcirc \cdots$ is the nested arch system with $n$ arches.

The main cohort (of size $n$) $M_n$ is the cohort of $N_n$.

Theorem: The arch systems $A$ such that the generating function $F_A$ of $Av(A)$ is $C^{(n)}$ are exactly those of $M_n$.

Remarks:
- This encapsulates all results of previous slides.
- It also generalizes them to more excluded patterns.
- It provides a bijective explanation of all these Wilf-equivalences.

Because rule (4) defining $\sim$ is useless to explain $\sim$-equivalences inside the main cohort, the proof of our main theorem gives bijections between $Av(A)$ and $Av(B)$ for $A, B \in M_n$. 
The main cohort

Definition: \( N_n = \ldots \overbrace{\ldots} \ldots \) is the nested arch system with \( n \) arches. The main cohort (of size \( n \)) \( M_n \) is the cohort of \( N_n \).

Theorem: The arch systems \( A \) such that the generating function \( F_A \) of \( Av(A) \) is \( C^{(n)} \) are exactly those of \( M_n \).

Remarks:
- This encapsulates all results of previous slides.
- It also generalizes them to more excluded patterns.
- It provides a bijective explanation of all these Wilf-equivalences.

Proof:
- For \( A \in M_n \), \( F_A = C^{(n)} \) follows from main theorem and \( F_{N_n} = C^{(n)} \).
The main cohort

**Definition:** \( N_n = \ldots \underbrace{\cdots} \ldots \) is the nested arch system with \( n \) arches. The **main cohort** (of size \( n \)) \( \mathcal{M}_n \) is the cohort of \( N_n \).

**Theorem:** The arch systems \( A \) such that the generating function \( F_A \) of \( \text{Av}(A) \) is \( C^{(n)} \) are exactly those of \( \mathcal{M}_n \).

**Remarks:**
- This encapsulates all results of previous slides.
- It also generalizes them to more excluded patterns.
- It provides a bijective explanation of all these Wilf-equivalences.

**Proof:**
- For \( A \in \mathcal{M}_n \), \( F_A = C^{(n)} \) follows from main theorem and \( F_{N_n} = C^{(n)} \).
- Why not for other \( A \)? For \( A \) of size \( n \), if \( A \notin \mathcal{M}_n \) then \( C^{(n)} \) dominates \( F_A \) term by term (and eventually strictly).

\( \hookrightarrow \) This follows from results on the comparison of cohorts.
Comparison of cohorts

$A, B$ arch systems. Generating functions $F_A$ and $F_B$ for $\mathrm{Av}(A)$ and $\mathrm{Av}(B)$.

**Definition:** Write $A \leq B$ when $F_B$ dominates $F_A$ term by term, and $A < B$ when $F_B$ dominates $F_A$ term by term and eventually strictly.
Comparison of cohorts

$A, B$ arch systems. Generating functions $F_A$ and $F_B$ for $\Av(A)$ and $\Av(B)$.

**Definition:** Write $A \leq B$ when $F_B$ dominates $F_A$ term by term, and $A < B$ when $F_B$ dominates $F_A$ term by term and eventually strictly.

**Proposition:** If $A \leq B$ then $\overline{\Av(A)} \leq \overline{\Av(B)}$, and if $A < B$ then $\overline{\Av(A)} < \overline{\Av(B)}$.

**Proof:** Recall the bijective proof for case (1) of main theorem: from a bijection $\Av(A) \rightarrow \Av(B)$, build a bijection $\Av(\overline{A}) \rightarrow \Av(\overline{B})$. The same construction applies to injections instead of bijections (resp. injections which eventually fail to be surjective).
Comparison of cohorts

$A, B$ arch systems. Generating functions $F_A$ and $F_B$ for $\Lambda\nu(A)$ and $\Lambda\nu(B)$.

**Definition:** Write $A \leq B$ when $F_B$ dominates $F_A$ term by term, and $A < B$ when $F_B$ dominates $F_A$ term by term and eventually strictly.

**Proposition:** If $A \leq B$ then $\overline{\Lambda\nu(A)} \leq \overline{\Lambda\nu(B)}$, and if $A < B$ then $\overline{\Lambda\nu(A)} < \overline{\Lambda\nu(B)}$.

**Proof:** Recall the bijective proof for case (1) of main theorem: from a bijection $\Lambda\nu(A) \to \Lambda\nu(B)$, build a bijection $\Lambda\nu(\overline{\Lambda\nu(A)}) \to \Lambda\nu(\overline{\Lambda\nu(B)})$.

The same construction applies to *injections* instead of bijections (resp. injections which eventually fail to be surjective).

Similar results and proofs for rules (2) and (4 weak).

**Corollary:** For $A$ of size $n$, either $F_A = C^{(n)}$ or $C^{(n)}$ dominates $F_A$ term by term (and eventually strictly).
Main theorem: $\sim$ refines Wilf-equivalence between classes of Catalan objects with one excluded substructure.

Open: Find a completely bijective proof of main theorem.
Main theorem: \(\sim\) refines Wilf-equivalence between classes of Catalan objects with one excluded substructure.

Open: Find a completely bijective proof of main theorem.

From the proof: Comparison between the enumeration of Av\((A)\) and Av\((B)\). More comparisons to be found from more bijective proofs?
Main theorem: $\sim$ refines Wilf-equivalence between classes of Catalan objects with one excluded substructure.

Open: Find a completely bijective proof of main theorem.

From the proof: Comparison between the enumeration of $\text{Av}_n(A)$ and $\text{Av}_n(B)$. More comparisons to be found from more bijective proofs?

Conjecture: $\sim$ and Wilf-equivalence coincide.

Stronger conjecture: Given two arch systems $A$ and $B$ both with $n$ arches, either $A \sim B$ or $|\text{Av}_{2n-2}(A)| \neq |\text{Av}_{2n-2}(B)|$. 

Asymptotic enumeration of cohorts. It is an upper bound (conjecturally tight) on the number of Wilf-classes.
Summary of results and open questions

- **Asymptotic enumeration** of cohorts. It is an upper bound (conjecturally tight) on the number of Wilf-classes.

- Study of the **main cohort**: unifies and generalizes previous results on classes $\mathcal{A}_V(231, \pi)$, and provides the first bijective proofs.
Asymptotic enumeration of cohorts. It is an upper bound (conjecturally tight) on the number of Wilf-classes.

Study of the main cohort: unifies and generalizes previous results on classes $\mathcal{A}_V(231, \pi)$, and provides the first bijective proofs.

Maximum cardinality of a cohort: We know the main cohort $\mathcal{M}_n$ contains $\text{Motz}_n$ arch systems. Is this the largest possible cardinality of a cohort?
Asymptotic enumeration of cohorts. It is an upper bound (conjecturally tight) on the number of Wilf-classes.

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Extension to other contexts (e.g. Schröder objects and separable permutations [Albert, Homberger, Pantone], ...).
• **Asymptotic enumeration** of cohorts. It is an upper bound (conjecturally tight) on the number of Wilf-classes.

• Study of the main cohort: unifies and generalizes previous results on classes $Av(231, \pi)$, and provides the first bijective proofs.

• **Maximum cardinality** of a cohort: We know the main cohort $M_n$ contains $Motz_n$ arch systems. Is this the largest possible cardinality of a cohort?

• Extension to other contexts (e.g. Schröder objects and separable permutations [Albert, Homberger, Pantone], . . .).

• What about other Catalan posets?