A general and algorithmic method for computing the generating function of permutation classes and for their random generation

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Séminaire Algo du GREYC

Guideline for the talk

Data:

- B a finite set of permutations (the excluded patterns),
- $\mathcal{C} = Av(B)$ the class of permutations that avoid every pattern of B.

Problem:

Describe an algorithm to obtain automatically from B

- some enumerative results on $\mathcal{C}$, in terms of generating function $C(z) = \sum |Av_n(B)| z^n$,
- a random sampler of permutations in $\mathcal{C}$, that is uniform on $Av_n(B)$ for each n.

Result:

Such an algorithm . . . that works under some hypothesis on $\mathcal{C}$, also tested algorithmically.

Outline

Outline

Representation of permutations

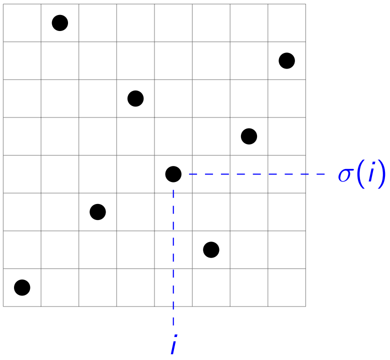
Permutation: Bijection from $[1..n]$ to itself. Set $\mathfrak{S}_n$.

- Linear representation:
  $\sigma = 1\ 8\ 3\ 6\ 4\ 2\ 5\ 7$

- Two lines representation:
  $$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 8 & 3 & 6 & 4 & 2 & 5 & 7 \end{pmatrix}$$

- Representation as a product of cycles:
  $\sigma = (1)\ (2\ 8\ 7\ 5\ 4\ 6)\ (3)$

- Graphical representation:



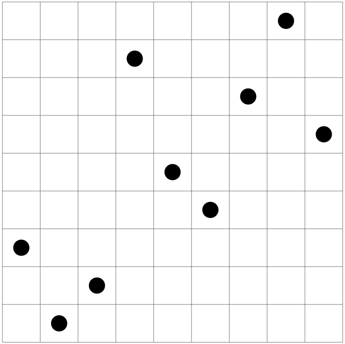
Patterns in permutations

Pattern (order) relation $\preccurlyeq$:
$\pi \in \mathfrak{S}_k$ is a pattern of $\sigma \in \mathfrak{S}_n$ if $\exists\ 1 \leq i_1 < \ldots < i_k \leq n$ such that $\sigma_{i_1} \ldots \sigma_{i_k}$ is order isomorphic ($\equiv$) to π.

Notation: $\pi \preccurlyeq \sigma$.

Equivalently:
The normalization of $\sigma_{i_1} \ldots \sigma_{i_k}$ on $[1..k]$ yields π.

Example: $2\,1\,3\,4 \preccurlyeq \mathbf{3}\,\mathbf{1}\,2\,8\,\mathbf{5}\,4\,\mathbf{7}\,9\,6$
since $3\,1\,5\,7 \equiv 2\,1\,3\,4$.



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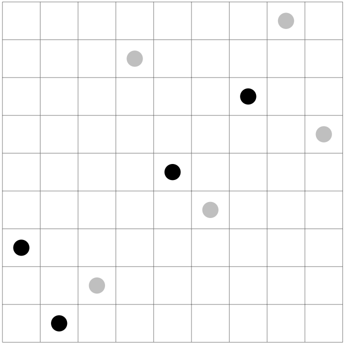

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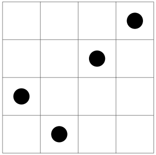

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Permutation classes

Permutation class : set of permutations downward-closed for $\preccurlyeq$.

$Av(B)$: the class of permutations that avoid every pattern of B.
If B is an antichain then B is the basis of $Av(B)$.

Conversely : Every class $\mathcal{C}$ can be characterized by its basis:

$$\mathcal{C} = Av(B) \text{ for } B = \{\sigma \notin \mathcal{C} : \forall \pi \preccurlyeq \sigma \text{ such that } \pi \neq \sigma, \pi \in \mathcal{C}\}$$

A class has a unique basis.
A basis can be either finite or infinite.

Origin : [Knuth 73] with stack-sortable permutations $= Av(231)$

Enumeration[Stanley & Wilf 92][Marcus & Tardos 04] : $|\mathcal{C} \cap \mathfrak{S}_n| \leq c^n$

Problematics

- **Combinatorics**: study of classes defined by their basis.

↪ Enumeration.

↪ Exhaustive generation.

- **Algorithmics**: problematics from text algorithmics.

↪ Pattern matching, longest common pattern.

↪ Linked with testing the membership of σ to a class.

- **Combinatorics (and algorithms)**: study families of classes.

↪ A class is not always described by its basis.

↪ Obtain general results on the structure of a class...

↪ ...and do it automatically (with algorithms).

Outline

Substitution decomposition: main ideas

Analogous to the decomposition of integers as products of primes.

- [Möhring & Radermacher 84]: general framework.
- Specialization: Modular decomposition of graphs.

Relies on:

- a principle for building objects (permutations, graphs) from smaller objects: the substitution.
- some "basic objects" for this construction: **simple** permutations, **prime** graphs.

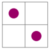
Required properties:

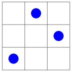
- every object can be decomposed using only "basic objects".
- this decomposition is unique.

Substitution for permutations

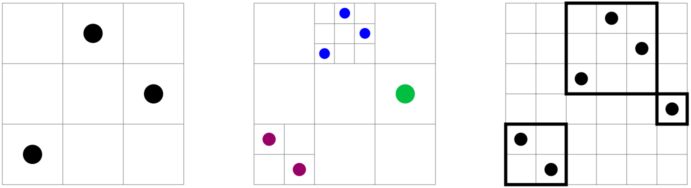
Substitution or inflation : $\sigma = \pi[\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)}]$.

Example : Here, $\pi = 1\,3\,2$, and

$\alpha^{(1)} = 2\,1 =$ 

$\alpha^{(2)} = 1\,3\,2 =$ 

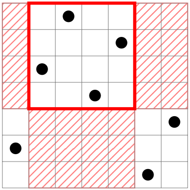
$\alpha^{(3)} = 1 =$  .



Hence $\sigma = 1\,3\,2[2\,1, 1\,3\,2, 1] = 2\,1\,4\,6\,5\,3$.

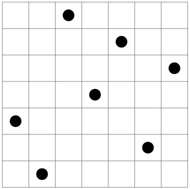
Simple permutations

Interval (or block) = set of elements of σ whose positions **and** values form intervals of integers
Example: 5 7 4 6 is an interval of 2 5 7 4 6 1 3



Simple permutation = permutation that has no interval, except the trivial intervals: $1, 2, \ldots, n$ and σ
Example: 3 1 7 4 6 2 5 is simple.

The smallest simple: 1 2, 2 1, 2 4 1 3, 3 1 4 2



Substitution decomposition of permutations

Theorem: Every σ ($\neq 1$) is uniquely decomposed as

- $12[\alpha^{(1)}, \alpha^{(2)}]$, where $\alpha^{(1)}$ is $\oplus$-indecomposable
- $21[\alpha^{(1)}, \alpha^{(2)}]$, where $\alpha^{(1)}$ is $\ominus$-indecomposable
- $\pi[\alpha^{(1)}, \ldots, \alpha^{(k)}]$, where π is simple of size $k \geq 4$

Remarks:

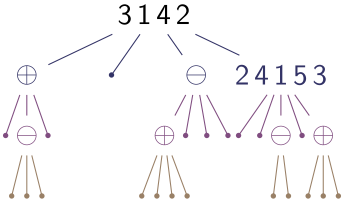
- $\oplus$-indecomposable : that cannot be written as $12[\alpha^{(1)}, \alpha^{(2)}]$
- Result stated as in [Albert & Atkinson 05]
- Can be rephrased changing the first two items into:
  - $12 \ldots k[\alpha^{(1)}, \ldots, \alpha^{(k)}]$, where the $\alpha^{(i)}$ are $\oplus$-indecomposable
  - $k \ldots 21[\alpha^{(1)}, \ldots, \alpha^{(k)}]$, where the $\alpha^{(i)}$ are $\ominus$-indecomposable

Decomposing recursively inside the $\alpha^{(i)}$ $\Rightarrow$ **decomposition tree**

Decomposition tree: witness of this decomposition

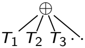
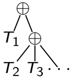
Example: Decomposition tree of $\sigma =$

10 13 12 11 14 1 18 19 20 21 17 16 15 4 8 3 2 9 5 6 7



Notations and properties:

- $\oplus = 12\ldots k$ and $\ominus = k\ldots 21$ = linear nodes.
- π simple of size ≥ 4 = prime node.
- No edge $\oplus - \oplus$ nor $\ominus - \ominus$.
- Ordered trees.

Expansion of  into  and recursively, for the version of the trees of [AA05]

$\sigma = 3\,1\,4\,2[\oplus[1, \ominus[1,1,1],1],1,\ominus[\oplus[1,1,1,1],1,1,1],2\,4\,1\,5\,3[1,1,\ominus[1,1],1,\oplus[1,1,1]]]$

Bijection between permutations and their decomposition trees.

Computation and examples of application

Computation: in linear time. [Uno & Yagiura 00] [Bui Xuan, Habib & Paul 05] [Bergeron, Chauve, Montgolfier & Raffinot 08]

In algorithms:

- Pattern matching [Bose, Buss & Lubiw 98] [Ibarra 97]
- Algorithms for bio-informatics [Bérard, Bergeron, Chauve & Paul 07] [Bérard, Chateau, Chauve, Paul & Tannier 08]

In combinatorics:

- Simple permutations [Albert, Atkinson & Klazar 03]
- Classes closed by substitution product [Atkinson & Stitt 02] [Brignall 07] [Atkinson, Ruškuc & Smith 09]
- Exhibit the structure of classes [Albert & Atkinson 05] [Brignall, Huczynska & Vatter 08a,08b] [Brignall, Ruškuc & Vatter 08]

Outline

Combinatorial classes and generating functions

Notations:

- $\mathcal{C} = \cup_{n\geq 0}\mathcal{C}_n$ with finite number $c_n = |\mathcal{C}_n|$ of objects of size n
- Generating function $C(z) = \sum c_n z^n$

Recursive description with constructors $\Rightarrow$ Equation on the g.f.:

Constructor	Notation	$C(z)$
Atom	$\mathcal{Z}$	z
Disjoint Union	$\mathcal{A}+\mathcal{B}$	$A(z)+B(z)$
Cartesian Product	$\mathcal{A}\times\mathcal{B}$	$A(z)B(z)$
Sequence	$\textsc{Seq}(\mathcal{A})$	$\dfrac{1}{1-A(z)}$
Restricted Seq.	$\textsc{Seq}_{=k}(\mathcal{A})$	$A(z)^k$

[Flajolet & Sedgewick 09]

Combinatorial classes and random samplers

Uniform sampling: objects of size n have the same probability

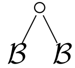
Two methods based on the recursive description of objects:

- Recursive method [Flajolet, Zimmerman & Van Cutsem 94]:
  size n chosen in advance. Requires to know the c_k for $k \leq n$.
- **Boltzmann method** [Duchon, Flajolet, Louchard & Schaeffer 04]:
  size n not fixed. Needs the evaluation of $C(z)$ at one point x.

$\mathcal{Z}$	return an atom
$\mathcal{A}+\mathcal{B}$	call $\Gamma A(x)$ with proba. $\frac{A(x)}{A(x)+B(x)}$, else $\Gamma B(x)$
$\mathcal{A}\times\mathcal{B}$	call $\Gamma A(x)$ and $\Gamma B(x)$
$\mathrm{SEQ}(\mathcal{A})$	choose k according to a geometric law of parameter $A(x)$ and call $\Gamma A(x)$ k times
$\mathrm{SEQ}_{=k}(\mathcal{A})$	call the sampler $\Gamma A(x)$ k times

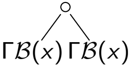
Example: binary trees

$\mathcal{B} = \cup_{n \geq 1} \mathcal{B}_n$
where $\mathcal{B}_n$ denotes the set of binary trees with n leaves.

Recursive description (also called specification): $\mathcal{B} = \bullet \quad + $ 

Equation for the g.f.: $B(z) = z + B(z)^2$, hence $B(z) = \frac{1-\sqrt{1-4z}}{2}$.

Boltzmann random sampler $\Gamma\mathcal{B}(x)$ for $\mathcal{B}$:

- Data: x, $B(x)$
- Result: a random binary tree
- Procedure:
  - Choose r uniformly at random on $[0, 1]$
  - If $\frac{x}{B(x)} < r$ then return $\bullet$
  - Else return 

Specifications for permutation classes

For all permutations, with $\mathcal{S}$ the set of all simple permutations:

$$\begin{cases} \mathfrak{S} = \bullet + \oplus(\mathfrak{S}^+, \mathfrak{S}) + \ominus(\mathfrak{S}^-, \mathfrak{S}) + \sum_{\pi \in \mathcal{S}} \pi(\mathfrak{S}, \mathfrak{S}, \dots, \mathfrak{S}) \\ \mathfrak{S}^+ = \bullet + \ominus(\mathfrak{S}^-, \mathfrak{S}) + \sum_{\pi \in \mathcal{S}} \pi(\mathfrak{S}, \mathfrak{S}, \dots, \mathfrak{S}) \\ \mathfrak{S}^- = \bullet + \oplus(\mathfrak{S}^+, \mathfrak{S}) + \sum_{\pi \in \mathcal{S}} \pi(\mathfrak{S}, \mathfrak{S}, \dots, \mathfrak{S}) \end{cases}$$

$\Rightarrow$The generating functions of $\mathfrak{S}$ and $\mathcal{S}$ are related [Albert, Atkinson & Klazar 03].

This can be adapted to (substitution-closed and arbitrary) permutation classes [Albert & Atkinson 05].

The simpler case of substitution-closed classes

A permutation class $\mathcal{C}$ is **substitution-closed** when $\pi[\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)}] \in \mathcal{C}$ for all $\pi, \alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)} \in \mathcal{C}$.

Hence, with $\mathcal{S}_{\mathcal{C}} = \mathcal{C} \cap \mathcal{S}$ the set of simple permutations in $\mathcal{C}$:

$$\begin{cases} \mathcal{C} = \bullet + \oplus[\mathcal{C}^{+}, \mathcal{C}] + \ominus[\mathcal{C}^{-}, \mathcal{C}] + \sum_{\pi \in \mathcal{S}_{\mathcal{C}}} \pi[\mathcal{C}, \mathcal{C}, \ldots, \mathcal{C}] \\ \ldots \end{cases}$$

When $\mathcal{S}_{\mathcal{C}}$ is finite, this is a simple family of trees in the sense of [Flajolet & Sedgewick 09].

$\Rightarrow$Enumerative results and random samplers can be obtained by efficient algorithms.

For general permutation classes

For non substitution-closed classes, we have only a strict inclusion:

$$\mathcal{C} \subsetneq \bullet + \oplus[\mathcal{C}^+, \mathcal{C}] + \ominus[\mathcal{C}^-, \mathcal{C}] + \sum_{\pi \in \mathcal{S}_{\mathcal{C}}} \pi[\mathcal{C}, \mathcal{C}, \cdots, \mathcal{C}]$$

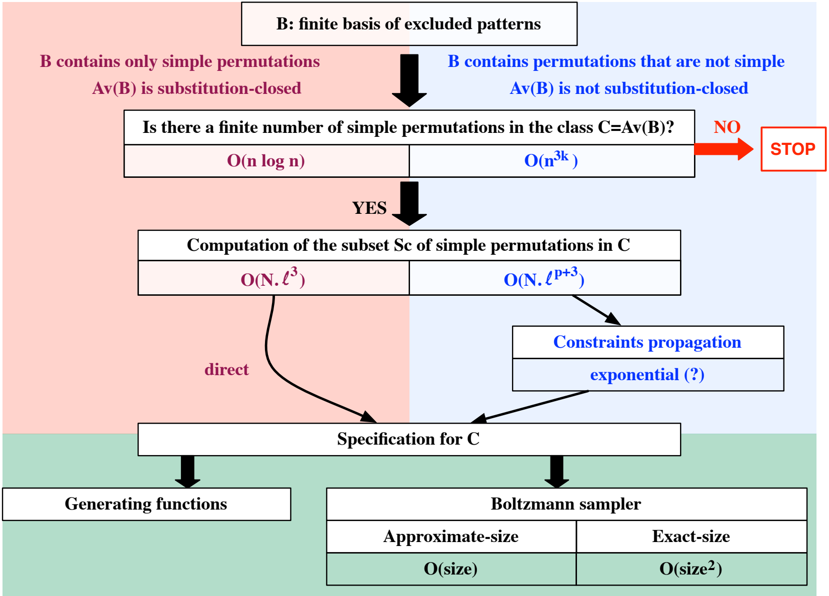
Example: $\ominus[12, 1] \notin Av(231)$ whereas $12, 1 \in Av(231)$.

The system describing $\mathcal{C}$ has to be refined with new equations for these constraints. The system can be computed by an algorithm.

$\Rightarrow$Enumerative results and random samplers can be obtained algorithmically, but this is less efficient.

Outline

Summary of results



First semi-decision procedure

Theorem [Albert & Atkinson 05]: If $\mathcal{C}$ contains a finite number of simple permutations, then $\mathcal{C}$ has a finite basis and an algebraic g.f..

Constructive proof: compute, for each given class,

- the specification for decomposition trees of $\mathcal{C}$
- a system of equations satisfied by the g.f.

from the finite set of simple permutations in $\mathcal{C}$

Testing the precondition:

- Semi-decision procedure

$\hookrightarrow$ Find simples of size $4, 5, 6, \ldots$ until k and $k+1$ for which there are 0 simples [Schmerl & Trotter 93]

- "Very exponential" ($\sim n!$) computation of the simples in $\mathcal{C}$

Step 1: Is there a finite number of simple permutations in $\mathcal{C}$? A first decision result

Theorem [Brignall, Ruškuc & Vatter 08]: It is decidable whether $\mathcal{C}$ given by its finite basis contains a finite number of simples.

Prop: $\mathcal{C} = Av(B)$ contains infinitely many simples iff $\mathcal{C}$ contains:

1. either infinitely many parallel permutations
2. or infinitely many simple wedge permutations
3. or infinitely many proper pin-permutations

	Decision procedure	Complexity
1. and 2. :	pattern matching of patterns of size 3 or 4 in the $\beta \in B$.	Polynomial $\mathcal{O}(n \log n)$
3. :	Decidability with automata techniques	Decidable 2ExpTime

Polynomial algorithms for the finite number of simples

Points similar to [BRV 08] :

- Encoding by pin words on $\{1, 2, 3, 4, L, R, U, D\}$
- Construction of automata

Study of pin-permutations [BBR 09] $\Rightarrow$ better understanding of the relationship between pin words and patterns in permutations

Points specific to [BBPR 10 & 11] :

- Polynomial construction of a **(deterministic, complete)** automaton for the language $\mathcal{L}$ = pin words of proper pin-permutations containing some $\beta \in B$
- Is this language co-finite ? Polynomial.

$\hookrightarrow$ Yes iff the class contains finitely many simples.

Polynomial w.r.t. $n = \sum_{\beta \in B} |\beta|$, but $k = |B|$ is an exponent.

Improvements for substitution-closed classes

Prop: $\mathcal{C} = Av(B)$ is substitution-closed iff B contains only simple permutations.

For simple β, $\beta \preccurlyeq \pi$ translates into a factor relation on pin words.
$\Rightarrow$ B gives a set of factors F (whose lengths sum to $\mathcal{O}(n)$) such that w has a factor in F iff $\beta \preccurlyeq \pi_w$ for some $\beta \in B$

[Aho & Corasick 75]:
build in linear time a complete deterministic automaton $\mathcal{A}_F$ recognizing the language of words containing a factor F

$\mathcal{L}(\mathcal{A}_F)$ co-finite iff finite number of simples in $\mathcal{C}$
. . . and testing the co-finiteness of $\mathcal{L}(\mathcal{A}_F)$ is in linear time.

Theorem: Testing the finiteness of the number of simple permutations in a substitution-closed class is solved in $\mathcal{O}(n \log n)$

Polynomial algorithm for general classes

When β is not simple (but is a pin permutation), $\beta \preccurlyeq \pi$ translates into a piecewise factor relation on pin words.

Def: $f = (f_1, f_2, \ldots, f_k)$ is a piecewise factor of w iff $w = w_0 f_1 w_1 f_2 w_2 \ldots w_{k-1} f_k w_k$.

Piecewise factors F_β corresponding to $\beta \in B$ are computed inductively on the decomposition trees of β.
And similarly for the deterministic automaton $\mathcal{A}_\beta$ recognizing the language of words containing a piecewise factor in F_β.

Construction of $\mathcal{A}_\beta$ in $\mathcal{O}(|\beta|^3)$.
Then build the product of the $\mathcal{A}_\beta$ for $\beta \in B$ (deterministic union).

Theorem: Testing the finiteness of the number of simple permutations in a permutation class is solved in $\mathcal{O}(n^{3k})$

Step 2: Finding the set of simple permutations in $\mathcal{C}$

Starting point: Find simple permutations in $\mathcal{C}$ of size $4, 5, 6, \ldots$ until k and $k+1$ for which there are 0 simples

Problem: There are $\sim \frac{n!}{e^2}$ simple permutations of size n

Reduce the number of simples σ of size n that are candidate to the membership to $\mathcal{C}$ [Pierrot & Rossin, 11].

Prop: The simples of $\mathcal{C}_{n+1}$ can be described as one-point (or special two-points) extensions of the simples of $\mathcal{C}_n$
$\Rightarrow$ There are at most $\mathcal{O}(n^2.|\mathcal{S} \cap \mathcal{C}_n|)$ candidates of size $n+1$.

Test whether σ contains an occurrence of $\beta \in B$: in $\mathcal{O}(n^{|\beta|})$.

Theorem: Computing the finite set of simple permutations in $\mathcal{C}$ is done in $\mathcal{O}(N \cdot \ell^{p+3})$ with $N = |\mathcal{S} \cap \mathcal{C}|$, $p = \max\{|\beta| : \beta \in B\}$ and $\ell = \max\{|\pi| : \pi \in \mathcal{S} \cap \mathcal{C}\}$

Refinement for substitution-closed classes

Prop: $\mathcal{C} = Av(B)$ is substitution-closed iff B contains only simples.

Prop [Pierrot & Rossin, 11]: If $\beta \preccurlyeq \sigma$ for β and σ simples, then there are simples $\beta = \sigma_1 \preccurlyeq \sigma_2 \ldots \preccurlyeq \sigma_k = \sigma$ s.t. for all i, $|\sigma_i| - |\sigma_{i-1}| = 1$ (or 2 in special cases).

Improvement of the complexity:

- Avoid testing occurrences of $\beta \in B$ in σ candidate simple of $\mathcal{C}$.
- Instead, test whether for every one point (or special two points) deletion in σ resulting in σ' simple, then $\sigma' \in \mathcal{C}$.

$\Rightarrow$ It is more efficient for computing $\mathcal{S} \cap \mathcal{C}_{n+1}$ from $\mathcal{S} \cap \mathcal{C}_n$.

Theorem: Computing the finite set of simple permutations in $\mathcal{C}$ is done in $\mathcal{O}(N \cdot \ell^5)$ for substitution-closed classes.

Step 3: Compute the specification for $\mathcal{C}$

From the set $\mathcal{S}_\mathcal{C}$ of simple permutations in $\mathcal{C}$, the specification for the substitution closure $\hat{\mathcal{C}}$ of $\mathcal{C}$ is obtained immediately:

$$\begin{cases} \hat{\mathcal{C}} = \bullet + \oplus[\hat{\mathcal{C}}^+, \hat{\mathcal{C}}] + \ominus[\hat{\mathcal{C}}^-, \hat{\mathcal{C}}] + \sum_{\pi \in \mathcal{S}_\mathcal{C}} \pi[\hat{\mathcal{C}}, \hat{\mathcal{C}}, \dots, \hat{\mathcal{C}}] \\ \hat{\mathcal{C}}^+ = \bullet + \ominus[\hat{\mathcal{C}}^-, \hat{\mathcal{C}}] + \sum_{\pi \in \mathcal{S}_\mathcal{C}} \pi[\hat{\mathcal{C}}, \hat{\mathcal{C}}, \dots, \hat{\mathcal{C}}] \\ \hat{\mathcal{C}}^- = \bullet + \oplus[\hat{\mathcal{C}}^+, \hat{\mathcal{C}}] + \sum_{\pi \in \mathcal{S}_\mathcal{C}} \pi[\hat{\mathcal{C}}, \hat{\mathcal{C}}, \dots, \hat{\mathcal{C}}] \end{cases}$$

- If $\mathcal{C}$ is substitution-closed, $\mathcal{C} = \hat{\mathcal{C}}$ and we are done.
- Otherwise, $\mathcal{C} = \hat{\mathcal{C}}\langle B^\star \rangle$ and propagate the constraints from $B^\star = \{\beta \in B \ : \ \beta \text{ is not simple } \}$ into the subtrees.

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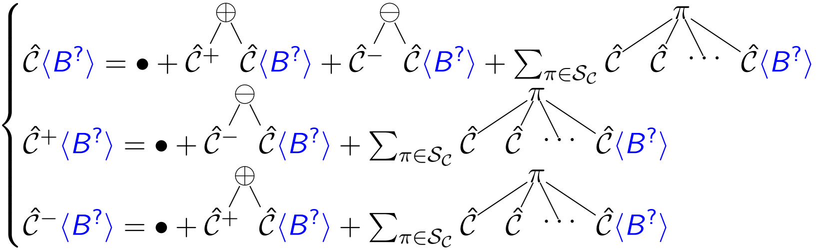
Step 3: Compute the specification for $\mathcal{C}$

From the set $\mathcal{S}_{\mathcal{C}}$ of simple permutations in $\mathcal{C}$, the specification for the substitution closure $\hat{\mathcal{C}}$ of $\mathcal{C}$ is obtained immediately:



- If $\mathcal{C}$ is substitution-closed, $\mathcal{C} = \hat{\mathcal{C}}$ and we are done.
- Otherwise, $\mathcal{C} = \hat{\mathcal{C}}\langle B^\star \rangle$ and propagate the constraints from $B^\star = \{\beta \in B \ : \ \beta \text{ is not simple } \}$ into the subtrees.

Constraint propagation 1/2

Embeddings of $\beta \in B^\star$ into $\pi \in \mathcal{S}_\mathcal{C}$

- Example: for $\ominus[\mathcal{C}^-, \mathcal{C}]\langle 231 \rangle$, and for the embedding $(23, 1) \hookrightarrow (2, 1)$, we get $\mathcal{C}^-\langle 12 \rangle$.
- additional restrictions α in $B^?$ that are blocks of $\beta \in B^\star$
- and do it inductively while new constraints α appear
- this terminates since each $\alpha \preccurlyeq \beta$ for some $\beta \in B^\star$

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Result: A system describing $\mathcal{C}$, that may be ambiguous
Example: For $2413[\mathcal{C}, \mathcal{C}, \mathcal{C}, \mathcal{C}]\langle 1234 \rangle$,
the embeddings $(1, 234) \hookrightarrow (2, 4)$ and $(1, 234) \hookrightarrow (1, 3)$
produce the terms $2413[\mathcal{C}, \mathcal{C}\langle 123 \rangle, \mathcal{C}, \mathcal{C}]$ and $2413[\mathcal{C}, \mathcal{C}, \mathcal{C}, \mathcal{C}\langle 123 \rangle]$
whose intersection is not empty.

Constraint propagation 2/2

Disambiguation of the system:

- Use formulas of the type $A \cup B = A \cap B \uplus \overline{A} \cap B \uplus A \cap \overline{B}$
- In complement set, excluded patterns become mandatory patterns: $\mathcal{C}_\gamma$ for $\gamma \preccurlyeq \beta \in B^\star$
- Propagate also mandatory restrictions

Constraint propagation 2/2

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Result: An unambiguous system describing $\mathcal{C}$, where the left-hand-sides are $\mathcal{C}^{\epsilon}_{\gamma_1,\ldots,\gamma_p}\langle\alpha_1,\ldots,\alpha_k\rangle$ with $\epsilon \in \{\ \ ,+,-\}$.

Termination: all α_i and γ_j are patterns of some $\beta \in B^\star$

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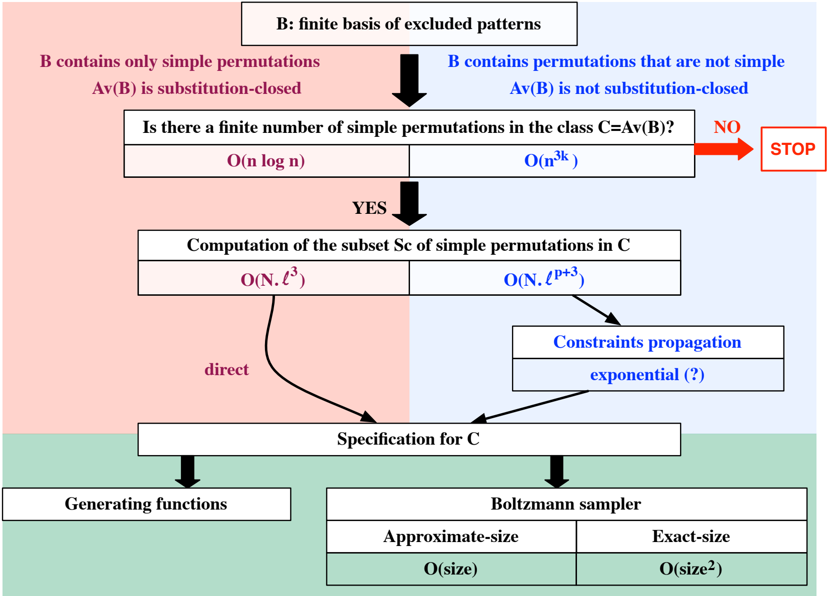
Theorem: The propagation of the constraints to obtain a specification for $\mathcal{C}$ is algorithmic, but there is an explosion of the number of equations in the system.

Putting things together



Outline

What next?

About the algorithm:

- Implementation in progress
- Complexity analysis of step 3 (explosion of the system)
- Dependency of the complexity of Boltzmann random samplers w.r.t. the size of the specification

With the algorithm:

- From the specifications, estimate growth rates of classes
- Are random permutations in $\mathcal{C}$ "like" in $\mathfrak{S}$?
- Compare statistics on $\mathcal{C}$ and $\mathfrak{S}$, or on $\mathcal{C}_1$ and $\mathcal{C}_2$

Related questions:

- How general is our algorithm?
- Classes with infinite set of simples, but finitely described?
- Use specification of a class to decide membership efficiently?