Operators of equivalent sorting power and related Wilf-equivalences

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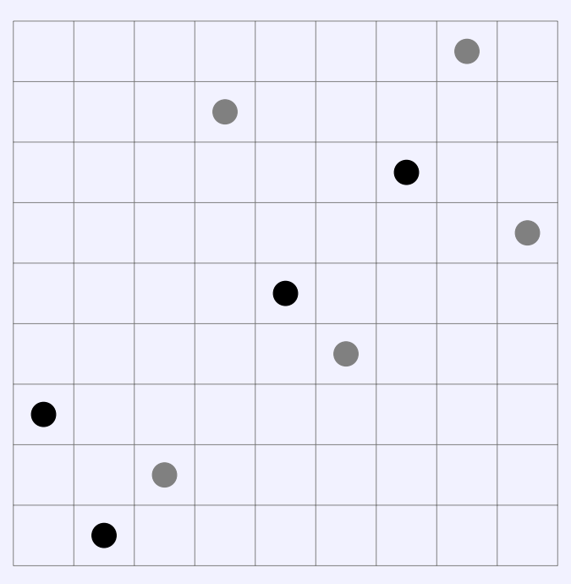
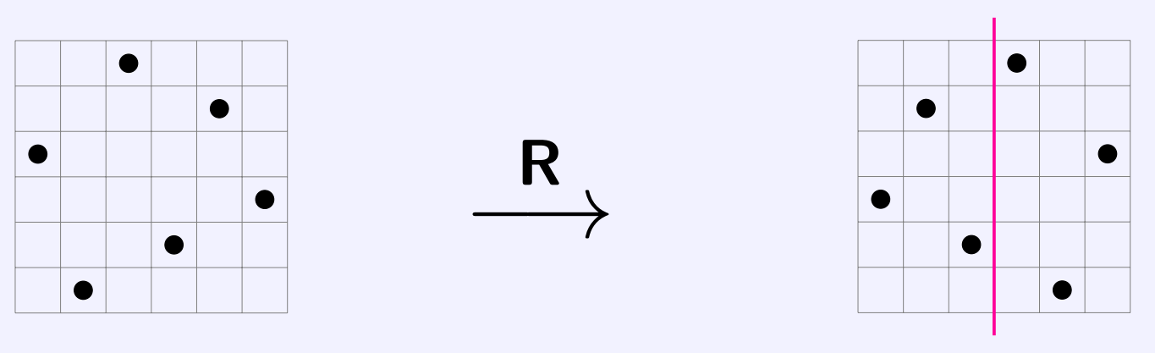
Definitions: Permutation Patterns, Reverse, Stack Sorting

Permutation patterns

$\pi \in \mathfrak{S}_k$ is a pattern of $\sigma \in \mathfrak{S}_n$ if $\exists$ $i_1 < \ldots < i_k$ such that $\sigma_{i_1} \ldots \sigma_{i_k}$ is order isomorphic to π.



Reverse

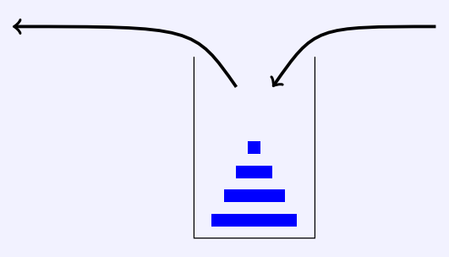


$\sigma = \sigma_1\sigma_2 \ldots \sigma_n$ $\qquad$ $\mathbf{R}(\sigma) = \sigma_n \ldots \sigma_2\sigma_1$

Stack sorting

Algorithmic description: Try to sort with a stack satisfying the Hanoi condition.

$\mathbf{S}(\sigma)$=1 2 3 6 4 5 7 $\qquad$ 6 1 3 2 7 5 4 $=\sigma$



Equivalent recursive description:

$$\begin{cases} \mathbf{S}(\varepsilon) = \varepsilon \\ \mathbf{S}(LnR) = \mathbf{S}(L)\mathbf{S}(R)n \text{ where } n = \max(LnR) \end{cases}$$

Avoidance

$\mathrm{Av}(\pi, \tau, \ldots)$ is the set of permutations that do not contain any occurrence of the patterns $\pi, \tau, \ldots$

Some previous results: about Stack Sorting, Permutation Patterns and Enumeration

Sorting with one stack

Permutations sortable by $\mathbf{S}$ $= \mathrm{Av}(231)$

Enumeration by Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$ [Knuth 1973]

With two stacks

Permutations sortable by $\mathbf{S}\circ\mathbf{S}$ $= \mathrm{Av}(2341, 3\bar{5}241)$

Enumeration by $\frac{2(3n)!}{(n+1)!(2n+1)!}$

[West 1993, Zeilberger 1992,. . .]

With three stacks

Permutations sortable by $\mathbf{S}\circ\mathbf{S}\circ\mathbf{S}$ are characterized by the avoidance of (generalized) patterns. [Claesson, Úlfarsson 2012]

But no enumeration result is known.

With stacks and reverse

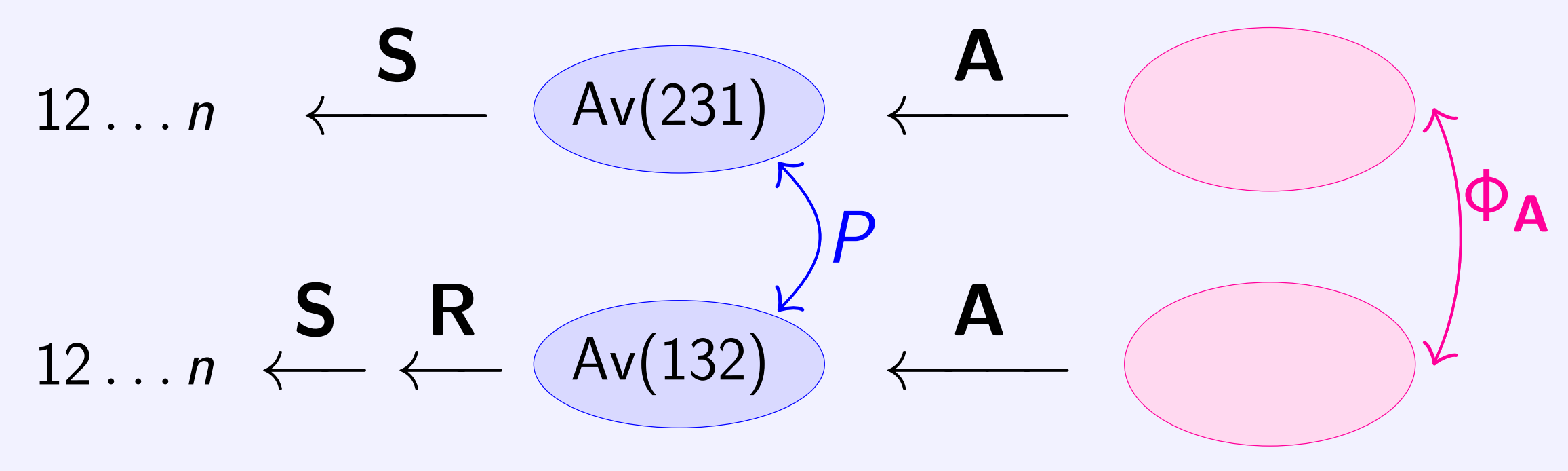
Bijection between permutations sortable by $\mathbf{S}\circ\mathbf{R}\circ\mathbf{S}$ and those sortable by $\mathbf{S}\circ\mathbf{S}$. [Bouvel, Guibert 2012]

Similar enumeration results with other symmetries instead of $\mathbf{R}$ (inverse, complement, . . .).

Main result: A Bijection Between Two Sets of Permutations Sortable by Stacks and Reverse

Main theorem

For any operator $\mathbf{A}$ which is a composition of operators $\mathbf{S}$ and $\mathbf{R}$, there are as many permutations of $\mathfrak{S}_n$ sortable by $\mathbf{S}\circ\mathbf{A}$ as permutations of $\mathfrak{S}_n$ sortable by $\mathbf{S}\circ\mathbf{R}\circ\mathbf{A}$.



Moreover, many permutation statistics are equidistributed across these two sets.

Equivalent statement

For any operator $\mathbf{A}$ which is a composition of operators $\mathbf{S}$ and $\mathbf{R}$, there is a size-preserving bijection between

- permutations of $\mathrm{Av}(231)$ that belong to the image of $\mathbf{A}$,

and

- permutations of $\mathrm{Av}(132)$ that belong to the image of $\mathbf{A}$,

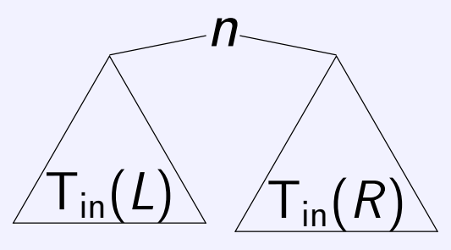
that preserves the number of preimages under $\mathbf{A}$.

Proof: Some Ingredients; and Sketch of Proof

Preimages under S and trees [Bousquet-Mélou 2000]

In-order trees are recursively defined by:

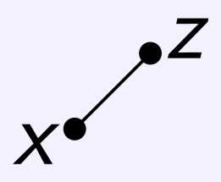
$\mathrm{T_{in}}(LnR) =$



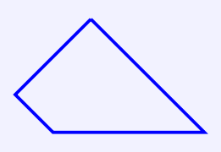
where $n = \max(LnR)$, and $\mathrm{T_{in}}(\varepsilon) = \emptyset$

Stack sorting on trees: Stack sorting θ is equivalent to post-order reading $\mathrm{T_{in}}(\theta)$, *i.e.* $\mathbf{S}(\theta) = \mathbf{Post}(\mathrm{T_{in}}(\theta))$

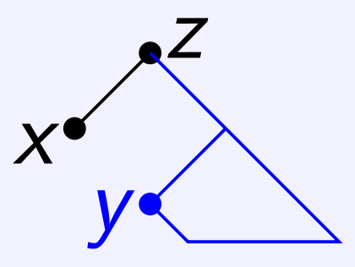
Canonical trees: a tree is canonical when for all x, z such that



in the tree, there exists



$\neq \emptyset$ and y such that



and $y < x$.

Lemma: For π in the image of $\mathbf{S}$, there is a unique canonical tree $\mathcal{T}_\pi$ such that $\mathbf{Post}(\mathcal{T}_\pi) = \pi$.

Theorem: All trees $\mathrm{T_{in}}(\theta)$ for θ such that $\mathbf{S}(\theta) = \pi$ may be recovered from $\mathcal{T}_\pi$ by re-rootings.

Consequence: $\mathcal{T}_\pi$ determines $\mathbf{S}^{-1}(\pi)$. Moreover $|\mathbf{S}^{-1}(\pi)|$ is determined only by the shape of $\mathcal{T}_\pi$.

The bijections $\Phi_{\mathbf{A}}$

For $\pi \in \mathrm{Av}(231)$, write $P(\pi) \in \mathrm{Av}(132)$ as $P(\pi) = \lambda_\pi \circ \pi$.

For θ sortable by $\mathbf{S}\circ\mathbf{A}$, set $\pi = \mathbf{A}(\theta)$. Because $\pi \in \mathrm{Av}(231)$, we may define $\Phi_{\mathbf{A}}(\theta) = \lambda_\pi \circ \theta$.

Theorem: $\Phi_{\mathbf{A}}$ is a bijection between permutation sortable by $\mathbf{S}\circ\mathbf{A}$ and those sortable by $\mathbf{S}\circ\mathbf{R}\circ\mathbf{A}$.

Proof idea:

$$\begin{tikzcd}
12\ldots n & & \pi \arrow[ll, "\mathbf{S}"'] & \tau \arrow[l, "\mathbf{S}"'] & \gamma \arrow[l, "\mathbf{S}\text{ or }\mathbf{R}"'] & \cdots\ \rho \arrow[l, "\mathbf{S}\text{ or }\mathbf{R}"'] & \theta \arrow[l, "\mathbf{S}\text{ or }\mathbf{R}"'] \arrow[llll, bend right, "\mathbf{A}"'] \\
12\ldots n & {} \arrow[l, "\mathbf{S}"'] & \begin{array}{c}\lambda_\pi \circ \pi \\ = P(\pi)\end{array} \arrow[l, "\mathbf{R}"'] & \lambda_\pi \circ \tau \arrow[l, "\mathbf{S}"'] & \lambda_\pi \circ \gamma \arrow[l, "\mathbf{S}\text{ or }\mathbf{R}"'] & \cdots\ \lambda_\pi \circ \rho \arrow[l, "\mathbf{S}\text{ or }\mathbf{R}"'] & \begin{array}{c}\lambda_\pi \circ \theta \\ = \Phi_{\mathbf{A}}(\theta)\end{array} \arrow[l, "\mathbf{S}\text{ or }\mathbf{R}"']
\end{tikzcd}$$

$\lambda_\pi(\mathcal{T}_\pi) = \mathcal{T}_{\lambda_\pi\circ\pi}$ $\qquad$ $\lambda_\pi(\mathcal{T}_\tau) = \mathcal{T}_{\lambda_\pi\circ\tau}$ $\qquad$ $\lambda_\pi(\mathcal{T}_\rho) = \mathcal{T}_{\lambda_\pi\circ\rho}$

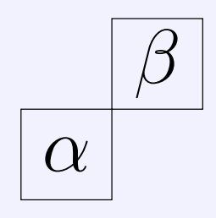
$\lambda_\pi(\mathrm{T_{in}}(\tau)) = \mathrm{T_{in}}(\lambda_\pi \circ \tau)$ $\quad$ $\lambda_\pi(\mathrm{T_{in}}(\gamma)) = \mathrm{T_{in}}(\lambda_\pi \circ \gamma)$ $\qquad$ $\lambda_\pi(\mathrm{T_{in}}(\theta)) = \mathrm{T_{in}}(\lambda_\pi \circ \theta)$

Bijection P between Av(231) and Av(132)

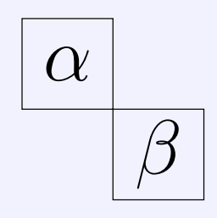
[Dokos, Dwyer, Johnson, Sagan, Selsor 2012]

Sum and skew sum of permutations, on their diagrams:

$\alpha \oplus \beta =$

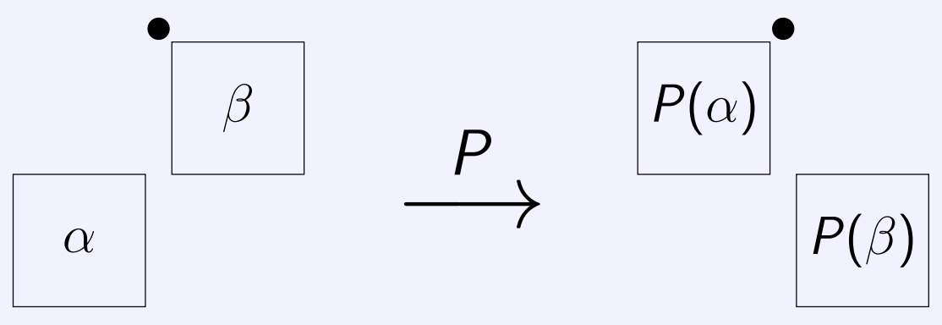


and $\alpha \ominus \beta =$



Bijection P from Av(231) to Av(132) is recursively defined as:

- if $\pi = \alpha \oplus (1 \ominus \beta)$ then $P(\pi) = (P(\alpha) \oplus 1) \ominus P(\beta)$
- or equivalently,



Some properties of P: P is the identity map on $\mathrm{Av}(231, 132)$

P preserves the shape of in-order trees

Statistics preserved by $\Phi_{\mathbf{A}}$

$\Phi_{\mathbf{A}}$ preserves the shape of in-order trees, hence

- the number and positions of the right-to-left maxima,
- the number and positions of the left-to-right maxima,
- the up-down word.

Other statistics preserved:

- Zeilberger's statistics when $\mathbf{A} = \mathbf{A}_0 \circ \mathbf{S}$:

$\mathrm{zeil}(\theta) = \max\{k \mid n(n-1)\ldots(n-k+1) \text{ is a subword of } \theta\}$

- the reversed Zeilberger's statistics when $\mathbf{A} = \mathbf{A}_0 \circ \mathbf{S}$ and $\mathbf{A}_0$ contains at least a composition $\mathbf{S}\circ\mathbf{R}$:

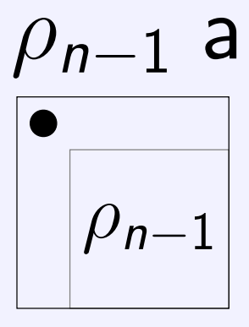
$\mathrm{Rzeil}(\theta) = \max\{k \mid (n-k+1)\ldots(n-1)n \text{ is a subword of } \theta\}$

More properties of P: Related Wilf-equivalences

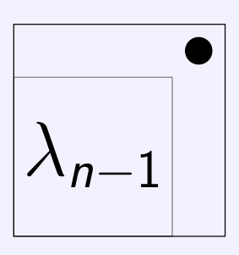
The families (λ_n) and (ρ_n)

$\lambda_1 = \rho_1 = 1$, $\lambda_n = 1 \ominus \rho_{n-1}$ and $\rho_n = \lambda_{n-1} \oplus 1$

i.e. $\lambda_1 = \rho_1 = \bullet$, $\lambda_n =$



and $\rho_n =$



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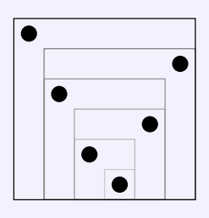
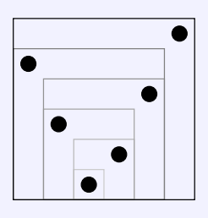
Examples: $\lambda_6 =$

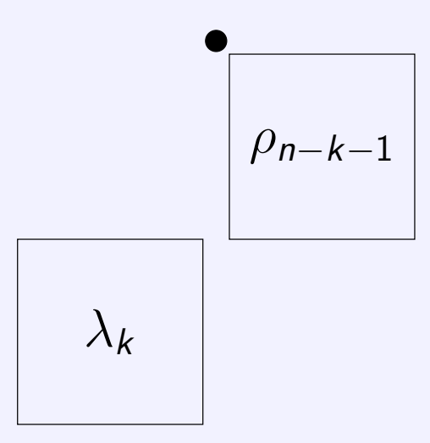


and $\rho_6 =$



Patterns π such that $\mathrm{Av}(231, \pi) \xleftrightarrow{P} \mathrm{Av}(132, P(\pi))$

Theorem: $\mathrm{Av}(231, \pi)$ and $\mathrm{Av}(132, P(\pi))$ are in bijection *via* P if and only if $\pi = \lambda_k \oplus (1 \ominus \rho_{n-k-1})$. $\quad \pi =$



In this case, $\{231, \pi\}$ and $\{132, P(\pi)\}$ are Wilf-equivalent.

Generating function of $\mathrm{Av}(231, \pi)$

Define $F_1(t) = 1$ and $F_{n+1}(t) = \frac{1}{1-tF_n(t)}$

Theorem: When $\mathrm{Av}(231, \pi) \xleftrightarrow{P} \mathrm{Av}(132, P(\pi))$, denoting $n = |\pi|$, the GF of $\mathrm{Av}(231, \pi)$ is F_n.

Consequence: For all such π of the same size, $\{231, \pi\}$ and $\{132, P(\pi)\}$ are all Wilf-equivalent.