Enumeration of Pin-Permutations

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LIAFA

Main result of the talk

Conjecture[Brignall, Ruškuc, Vatter]:
The pin-permutation class has a rational generating function.

Theorem: The generating function of the pin-permutation class is

$$P(z) = z\frac{8z^6 - 20z^5 - 4z^4 + 12z^3 - 9z^2 + 6z - 1}{8z^8 - 20z^7 + 8z^6 + 12z^5 - 14z^4 + 26z^3 - 19z^2 + 8z - 1}$$

Technique for the proof:

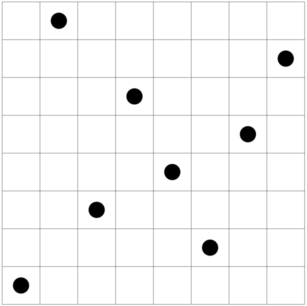
- Characterize the decomposition trees of pin-permutations
- Compute the generating function of *simple* pin-permutations
- Put things together to compute the generating function of pin-permutations

Outline of the talk

Representations of permutations

Permutation: Bijective map from $[1..n]$ to itself

- One-line representation:
  $\sigma = 1\ 8\ 3\ 6\ 4\ 2\ 5\ 7$
- Two-line representation:
  $$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 8 & 3 & 6 & 4 & 2 & 5 & 7 \end{pmatrix}$$
- Cyclic representation:
  $\sigma = (1)\ (2\ 8\ 7\ 5\ 4\ 6)\ (3)$
- Graphical representation:

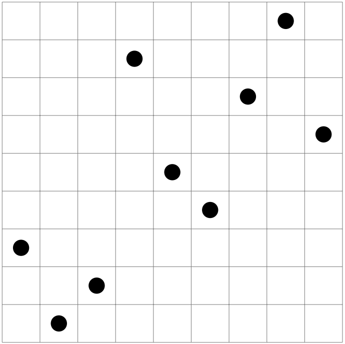


Patterns in permutations

Pattern relation $\preccurlyeq$:
$\pi \in S_k$ is a pattern of $\sigma \in S_n$ when $\exists\ 1 \leq i_1 < \ldots < i_k \leq n$ such that $\sigma_{i_1} \ldots \sigma_{i_k}$ is order-isomorphic to π.
We write $\pi \preccurlyeq \sigma$.

Equivalently: Normalizing $\sigma_{i_1} \ldots \sigma_{i_k}$ on $[1..k]$ yields π.

Example: $1\,2\,3\,4 \preccurlyeq 3\,\mathbf{1}\,\mathbf{2}\,8\,\mathbf{5}\,4\,\mathbf{7}\,9\,6$
since $1\,2\,5\,7 \equiv 1\,2\,3\,4$.



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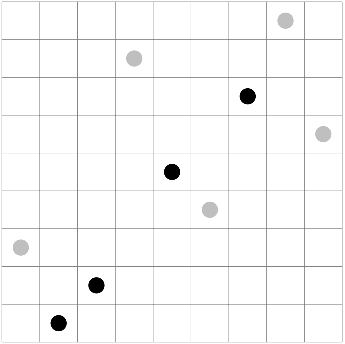

Classes of permutations

Class of permutations: set downward closed for $\preccurlyeq$
Equivalently: $\sigma \in \mathcal{C}$ and $\pi \preccurlyeq \sigma \Rightarrow \pi \in \mathcal{C}$

$S(B)$: the class of perm. avoiding all the patterns in the basis B.

Prop.: Every class $\mathcal{C}$ is characterized by its basis:

$$\mathcal{C} = S(B) \text{ for } B = \{\sigma \notin \mathcal{C} : \forall \pi \preccurlyeq \sigma \text{ with } \pi \neq \sigma, \pi \in \mathcal{C}\}$$

Basis may be finite or infinite.

Enumeration[Stanley-Wilf, Marcus-Tardos]: $|S_n(B)| \leq c_B^n$

Two points of view:
class given by its basis or by a (graphical) property stable for $\preccurlyeq$

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Simple permutations

Interval = window of elements of σ whose values form a range
Example: 5 7 4 6 is an interval of 2 5 7 4 6 1 3

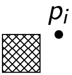
Simple permutation = has no interval except $1, 2, \ldots, n$ and σ
Example: 3 1 7 4 6 2 5 is simple. *Smallest ones*: 1 2, 2 1, 2 4 1 3, 3 1 4 2

Decomposition trees: formalize the idea that simple permutations are "building blocks" for all permutations

Thm[Albert Atkinson]: $\mathcal{C}$ contains finitely many simple permutations
$\Rightarrow \mathcal{C}$ has an algebraic generating function
Pin-permutations: used for deciding whether $\mathcal{C}$ contains finitely many simple permutations

Pin representations

Pin representation of σ = sequence $(p_1, \ldots, p_n)$ such that each p_i satisfies

- the externality condition 

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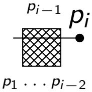
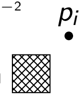
- the separation condition 

- or the independence condition 

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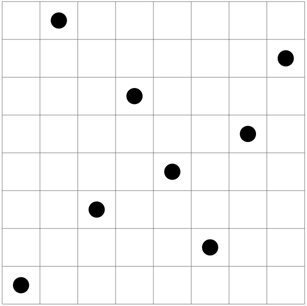

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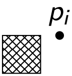
- the externality condition 

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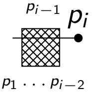
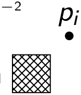
• the separation condition 

• or the independence condition 

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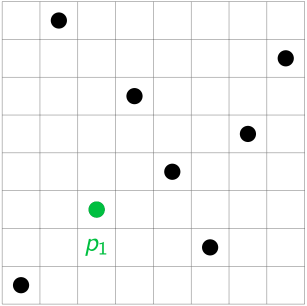

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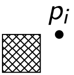
- the externality condition 

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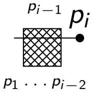
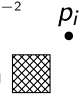
• the separation condition 

• or the independence condition 

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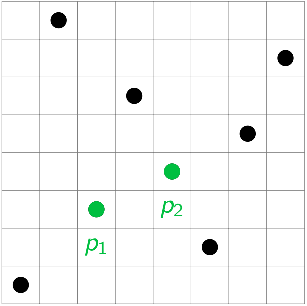

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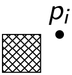
- the externality condition 

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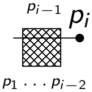
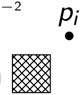
- the separation condition 

- or the independence condition 

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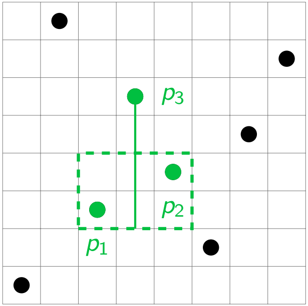

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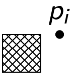
- the externality condition 

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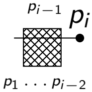
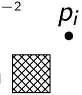
• the separation condition 

• or the independence condition 

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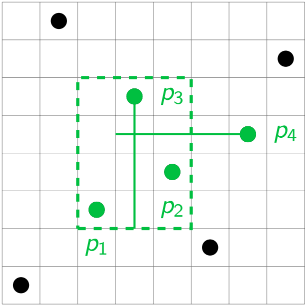

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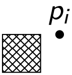
- the externality condition 

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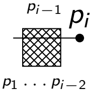
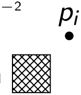
• the separation condition 

• or the independence condition 

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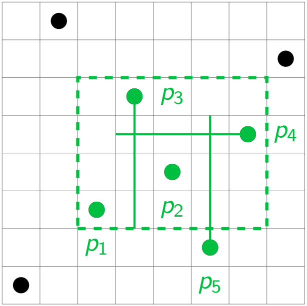

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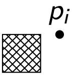
- the externality condition 

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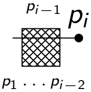
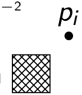
- the separation condition 

- or the independence condition 

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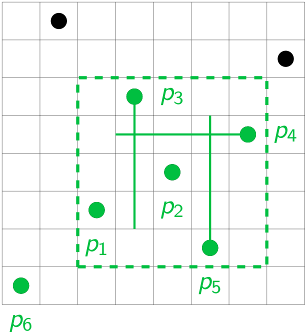

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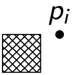
- the externality condition 

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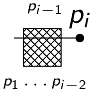
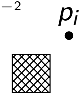
- the separation condition 

- or the independence condition 

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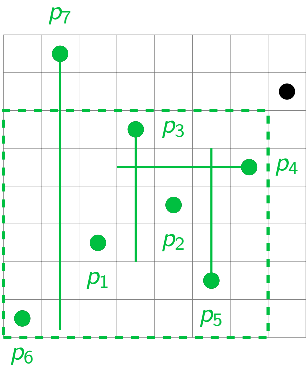

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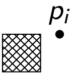
- the externality condition 

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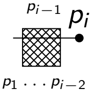
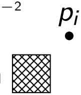
• the separation condition 

• or the independence condition 

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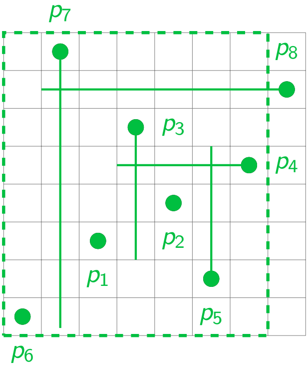
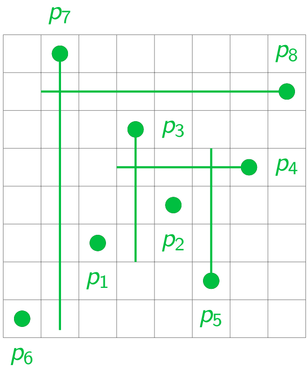
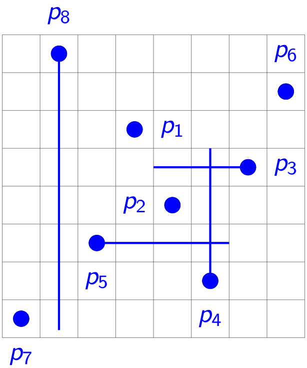


Non-uniqueness of pin representation





Active points

Active point of σ:
p_1 for some pin representation p of σ

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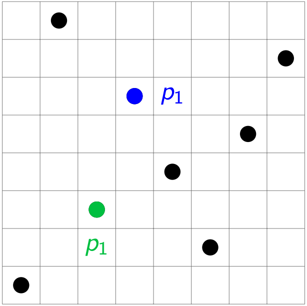

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Remark:
Not every point is an active point.

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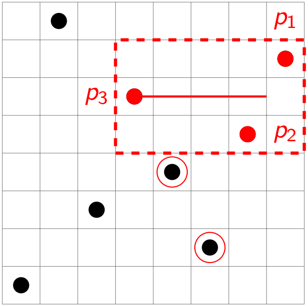

The class of pin-permutations

Fact: Not every permutation admits pin representations.

Def: Pin-permutation = that has a pin representation.

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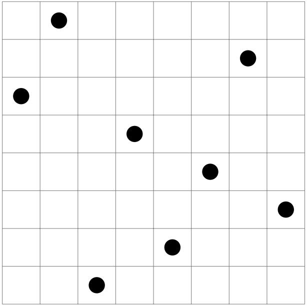

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Idea of the proof: σ has a pin representation $p \Rightarrow$ for $\tau \prec \sigma$ remove the same points in p.

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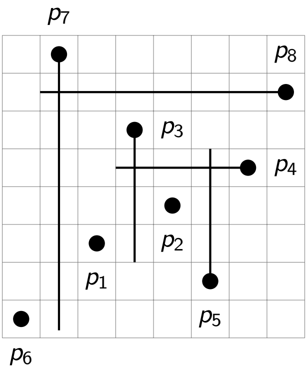

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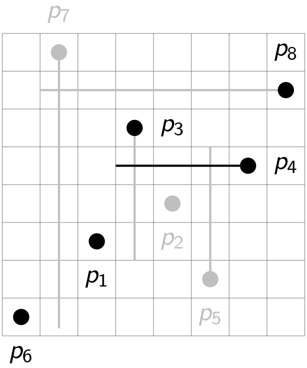
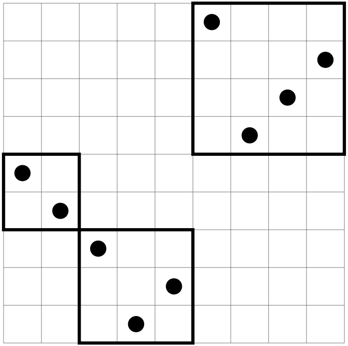


Substitution decomposition



Definitions

Inflation:
$\pi[\alpha_1, \alpha_2, \dots, \alpha_k]$

Example:
$213[21, 312, 4123] =$
$54\,312\,9678$

Substitution decomposition

Results

Prop.[Albert Atkinson]: $\forall \sigma, \exists$ a unique simple permutation π and unique α_i such that $\sigma = \pi[\alpha_1, \ldots, \alpha_k]$.
If $\pi = 12$ (21), for unicity, α_1 is *plus (minus) -indecomposable*.

Thm [Albert Atkinson]: (Wreath-closed) class $\mathcal{C}$ containing finitely many simple permutations $\Rightarrow$

- $\mathcal{C}$ is finitely based.
- $\mathcal{C}$ has an algebraic generating function.

Strong interval decomposition

Thm: Every σ can be uniquely decomposed as

- $12\dots k[\alpha_1,\dots,\alpha_k]$, with the α_i plus-indecomposable
- $k\dots 21[\alpha_1,\dots,\alpha_k]$, with the α_i minus-indecomposable
- $\pi[\alpha_1,\dots,\alpha_k]$, with π simple of size ≥ 4

Remarks:

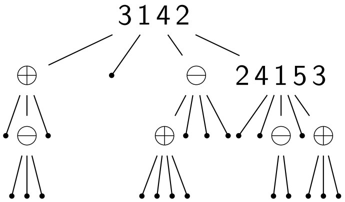
- This decomposition is unique without any further restriction.
- The α_i are the maximal strong intervals of σ.

Decompose the α_i recursively to get the decomposition tree.

Decomposition tree

Example: The substitution decomposition tree of $\sigma =$

10 13 12 11 14 1 18 19 20 21 17 16 15 4 8 3 2 9 5 6 7



Notations and properties:

- $\oplus = 12 \dots k$ and $\ominus = k \dots 21$ = linear nodes.
- π simple of size ≥ 4 = prime nodes.
- No $\oplus - \oplus$ or $\ominus - \ominus$ egde.
- Decomposition trees of permutations are ordered.
- N.B.: Modular decomposition trees are unordered.

$\boxed{\text{Bijection between decomposition trees and permutations.}}$

Theorem

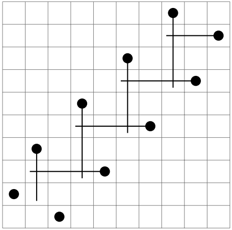
σ **is a pin-permutation iff its decomposition tree satifies:**

- Any linear node $\oplus$ ($\ominus$) has at most one child that is not an ascending (descending) weaving permutation
- For any prime node labelled by π, π is a simple pin-permutation and
  - all of its children are leaves
  - it has exactly one child that is not a leaf, and it inflates one active point of π
  - π is an ascending (descending) quasi-weaving permutation and exactly two children are not leaves
    - $\hookrightarrow$ one is 12 (21) inflating the auxiliary substitution point of π
    - $\hookrightarrow$ the other one inflates the main substitution point of π

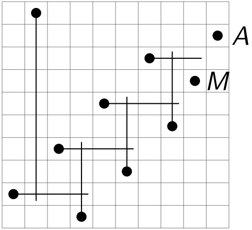
Definitions

Active point σ: there is a pin representation of σ starting with it.

Weaving permutation $\mathcal{W}$



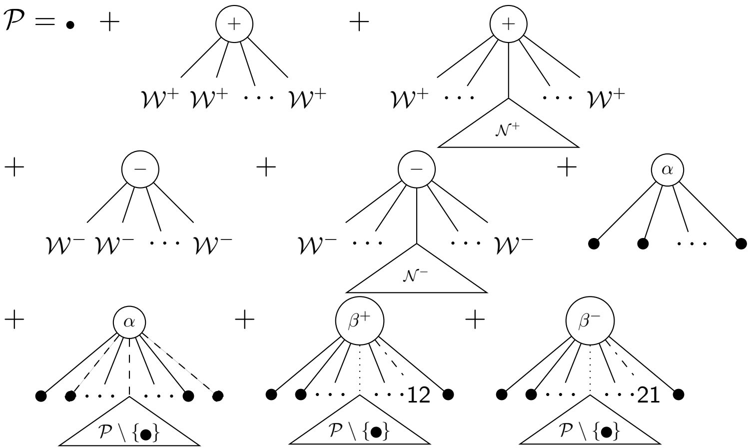
Quasi-weaving permutation β



Both are ascending $(+)$. Other are obtained by symmetry.

Enumeration: 4 $(= 2 + 2)$ weaving and 8 $(= 4 + 4)$ quasi-weaving permutations of size n, except for small n.

Back to the characterization



Basic generating functions involved

Weaving permutations: $W^{+}(z) = W^{-}(z) = \mathbf{W(z)} = \frac{z+z^3}{1-z}$.
Remark: $\mathcal{W}^{+} \cap \mathcal{W}^{-} = \{1, 2431, 3142\}$

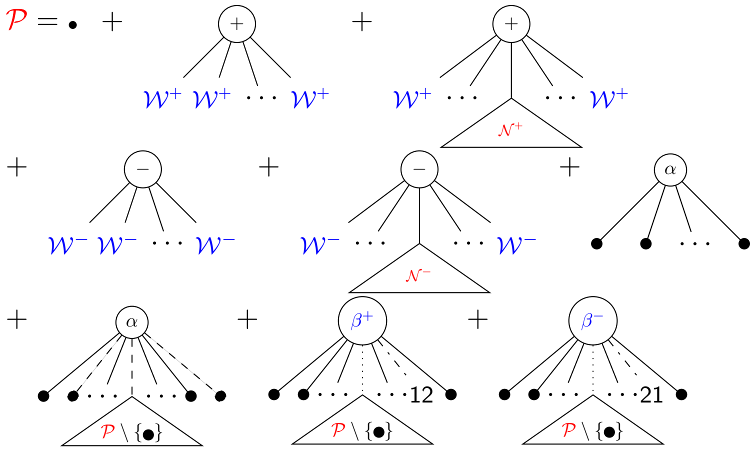
Quasi-weaving permutations:
$QW^{+}(z) = QW^{-}(z) = \mathbf{QW(z)} = \frac{4z^4}{1-z}$.

Trees $\mathcal{N}^{+}$ and $\mathcal{N}^{-}$: pin-permutations except ascending (descending) weaving permutations and those whose root is $\oplus$ ($\ominus$).

$$N^{+}(z) = N^{-}(z) = \mathbf{N(z)} = \frac{(z^3+2z-1)(z^3+P(z)z^3+2P(z)z+z-P(z))}{1-2z+z^2}$$

$P(z)$ = generating function of pin-permutations.

From characterization to generating function (1)



Generating functions of **simple** pin-permutations

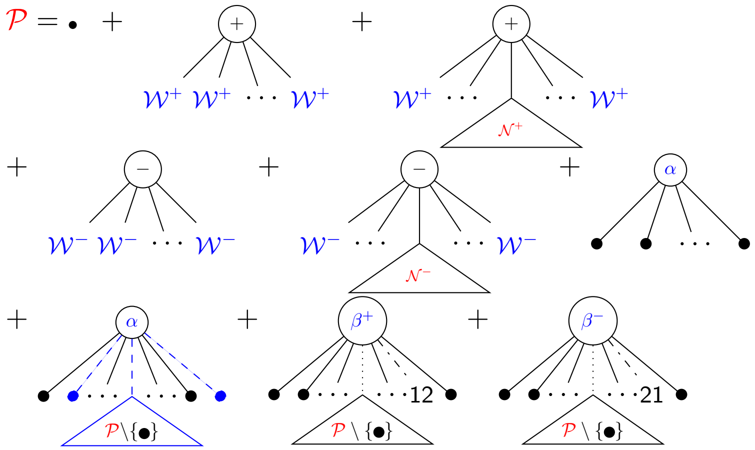
- Enumerate pin representations encoding simple pin-permutations.
- Characterize how many pin representations for a simple pin-permutation.
- Describe number of active points in simple pin-permutations.

Simple pin representations: **SiRep(z)** $= 8z^4 + \frac{32z^5}{1-2z} - \frac{16z^5}{1-z}$

Simple pin-permutations: **Si(z)** $= 2z^4 + 6z^5 + 32z^6 + \frac{128z^7}{1-2z} - \frac{28z^7}{1-z}$

Simple pin-permutations with multiplicity = number of active points: **SiMult(z)** $= 8z^4 + 26z^5 + 84z^6 + \frac{256z^7}{1-2z} - \frac{40z^7}{1-z}$

From characterization to generating function (2)



The rational generating function of pin-permutations

Equation on trees $\Rightarrow$ equation on generating functions:

$$\begin{aligned} P(z) &= z + \frac{W^+(z)^2}{1-W^+(z)} + \frac{2W^+(z) - W^+(z)^2}{(1-W^+(z))^2} N^+(z) \\ &+ \frac{W^-(z)^2}{1-W^-(z)} + \frac{2W^-(z) - W^-(z)^2}{(1-W^-(z))^2} N^-(z) + Si(z) \\ &+ SiMult(z)\left(\frac{P(z)-z}{z}\right) + QW^+(z)\left(z\frac{P(z)-z}{z}\right) + QW^-(z)\left(z\frac{P(z)-z}{z}\right) \end{aligned}$$

Generating function of pin-permutations:

$$\boxed{P(z) = z\frac{8z^6-20z^5-4z^4+12z^3-9z^2+6z-1}{8z^8-20z^7+8z^6+12z^5-14z^4+26z^3-19z^2+8z-1}}$$

First terms: $1, 2, 6, 24, 120, 664, 3596, 19004, 99596, 521420, \ldots$

The rational generating function of pin-permutations

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Generating function of pin-permutations:

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First terms: $1, 2, 6, 24, 120, 664, 3596, 19004, 99596, 521420, \ldots$

Conclusion and open question

Overview of the results:

- Class of pin-permutations define by a graphical property
- Characterization of the associated decomposition trees
- Enumeration of simple pin-permutations

$\Rightarrow$ Generating function of the pin-permutation class

- Rationality of the generating function

Characterization of the pin-permutation class:

✓ by a recursive description

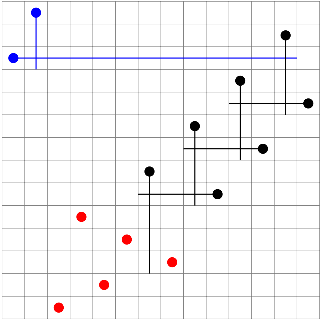
? by a (finite?) basis of excluded patterns

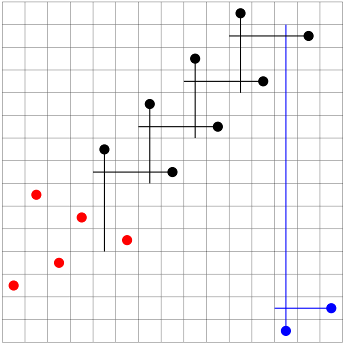
This basis is infinite, but yet unknown.

Infinite antichain in the basis

Prop. σ is in the basis $\Leftrightarrow$ σ is not a pin-permutation but any strict pattern of σ is.

We describe (σ_n) an infinite antichain in the basis:





Perspectives

Thm[Brignall et al.]: $\mathcal{C}$ a class given by its finite basis B. It is decidable whether $\mathcal{C}$ contains infinitely many simple permutations

Procedure: Check whether $\mathcal{C}$ contains arbitrarily long

- parallel alternations — Easy, Polynomial
- wedge simple permutations — Easy, Polynomial
- proper pin-permutations — Difficult, Complexity?

Analysis of the procedure for proper pin-permutations
$\Rightarrow$ Polynomial construction using automata techniques except last step *(Determinization of a transducer)*
$\Rightarrow$ makes the construction exponential

$\boxed{\text{Better knowlegde of pin-permutations} \Rightarrow \text{improve this complexity ?}}$