Enumeration of Pin-Permutations

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Main result of the talk

**Conjecture** [Brignall, Ruškuc, Vatter]: The pin-permutation class has a rational generating function.

**Theorem**: The generating function of the pin-permutation class is

\[
P(z) = z\frac{8z^6 - 20z^5 - 4z^4 + 12z^3 - 9z^2 + 6z - 1}{8z^8 - 20z^7 + 8z^6 + 12z^5 - 14z^4 + 26z^3 - 19z^2 + 8z - 1}
\]

**Technique for the proof**:
- Characterize the decomposition trees of pin-permutations
- Compute the generating function of *simple* pin-permutations
- Put things together to compute the generating function of pin-permutations
Outline of the talk

1. Introduction: permutation classes
2. Definition of pin-permutations
3. Substitution decomposition and decomposition trees
4. Characterization of the decomposition trees of pin-permutations
5. Generating function of the pin-permutation class
6. Conclusion and discussion on the basis
**Permutation**: Bijective map from \([1..n]\) to itself

- **One-line representation**:
  \[ \sigma = 1\ 8\ 3\ 6\ 4\ 2\ 5\ 7 \]

- **Two-line representation**:
  \[ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 8 & 3 & 6 & 4 & 2 & 5 & 7 \end{pmatrix} \]

- **Cyclic representation**:
  \[ \sigma = (1)\ (2\ 8\ 7\ 5\ 4\ 6)\ (3) \]
Pattern relation ≼:

$\pi \in S_k$ is a pattern of $\sigma \in S_n$ when

$\exists \ 1 \leq i_1 < \ldots < i_k \leq n$ such that

$\sigma_{i_1} \ldots \sigma_{i_k}$ is order-isomorphic to $\pi$.

We write $\pi \lessdot \sigma$.

Equivalently: Normalizing $\sigma_{i_1} \ldots \sigma_{i_k}$
on [1..k] yields $\pi$.

Example: $1234 \lessdot 312854796$

since $1257 \equiv 1234$. 
Patterns in permutations

Pattern relation $\preceq$:

$\pi \in S_k$ is a pattern of $\sigma \in S_n$ when

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$\sigma_{i_1} \ldots \sigma_{i_k}$ is order-isomorphic to $\pi$.

We write $\pi \preceq \sigma$.

Equivalently: Normalizing $\sigma_{i_1} \ldots \sigma_{i_k}$ on $[1..k]$ yields $\pi$.

Example: $1234 \preceq 312854796$

since $1257 \equiv 1234$. 
Classes of permutations

Class of permutations: set downward closed for $\preceq$
Equivalently: $\sigma \in C$ and $\pi \preceq \sigma \Rightarrow \pi \in C$

$S(B)$: the class of perm. avoiding all the patterns in the basis $B$.

Prop.: Every class $C$ is characterized by its basis:

$$C = S(B) \text{ for } B = \{\sigma \notin C : \forall \pi \preceq \sigma \text{ with } \pi \neq \sigma, \pi \in C\}$$

Basis may be finite or infinite.

Enumeration[Stanley-Wilf, Marcus-Tardos]: $|S_n(B)| \leq c_B^n$

Two points of view:
class given by its basis or by a (graphical) property stable for $\preceq$
Classes of permutations

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Two points of view:
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Simple permutations

**Interval** = window of elements of $\sigma$ whose values form a range

**Example**: $5\ 7\ 4\ 6$ is an interval of $2\ 5\ 7\ 4\ 6\ 1\ 3$

**Simple permutation** = has no interval except 1, 2, ..., $n$ and $\sigma$

**Example**: $3\ 1\ 7\ 4\ 6\ 2\ 5$ is simple. *Smallest ones*: 1 2, 2 1, 2 4 1 3, 3 1 4 2

**Decomposition trees**: formalize the idea that simple permutations are “building blocks” for all permutations

**Thm** [Albert Atkinson]: $C$ contains finitely many simple permutations

$\Rightarrow$ $C$ has an algebraic generating function

**Pin-permutations**: used for deciding whether $C$ contains finitely many simple permutations
**Pin representation** of \( \sigma = \text{sequence} (p_1, \ldots, p_n) \) such that each \( p_i \) satisfies

- the externality condition

- and

- the separation condition

- or the independence condition

\[
\text{bounding box of } \{p_1, \ldots, p_{i-1}\}
\]
Pin representations

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Definition of pin-permutations

Non-uniqueness of pin representation

\[
p_7
\]

\[
p_8
\]

\[
p_6
\]

\[
p_5
\]

\[
p_4
\]

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Pin-Permutations
Active points

Active point of $\sigma$:

$p_1$ for some pin representation $p$ of $\sigma$

Example:
**Active points**

**Active point of** $\sigma$:

$p_1$ for some pin representation $p$ of $\sigma$

**Remark:**
Not every point is an active point.

Example:

```
  p_1
  p_3
  p_2
```
The class of pin-permutations

**Fact:** Not every permutation admits pin representations.

**Def:** Pin-permutation = that has a pin representation.

**Example 1:**
The class of pin-permutations

**Fact:** Not every permutation admits pin representations.

**Def:** Pin-permutation = that has a pin representation.

**Thm:** Pin-permutations are a permutation class.

**Idea of the proof:** $\sigma$ has a pin representation $p \Rightarrow$ for $\tau \prec \sigma$ remove the same points in $p$. 

**Example 2:**

\begin{align*}
p_7 & \quad p_8 \\
p_3 & \quad p_4 \\
p_2 & \quad p_5 \\
p_1 & \quad p_6
\end{align*}
Definition of pin-permutations

The class of pin-permutations

**Fact:** Not every permutation admits pin representations.

**Def:** Pin-permutation = that has a pin representation.

**Thm:** Pin-permutations are a permutation class.

**Idea of the proof:** $\sigma$ has a pin representation $p \Rightarrow$ for $\tau \prec \sigma$ remove the same points in $p$.

**Example 2:**

- $p_1$
- $p_2$
- $p_3$
- $p_4$
- $p_5$
- $p_6$
- $p_7$
- $p_8$
Substitution decomposition

Definitions

Inflation:
$\pi[\alpha_1, \alpha_2, \ldots, \alpha_k]$

Example:
$213[21, 312, 4123] = 543129678$
Substitution decomposition

**Results**

**Prop.** [Albert Atkinson]: \( \forall \sigma, \exists \) a unique simple permutation \( \pi \) and unique \( \alpha_i \) such that \( \sigma = \pi[\alpha_1, \ldots, \alpha_k] \).

If \( \pi = 12 (21) \), for unicity, \( \alpha_1 \) is \textit{plus (minus) -indecomposable}.

**Thm** [Albert Atkinson]: (Wreath-closed) class \( C \) containing finitely many simple permutations \( \Rightarrow \)

- \( C \) is finitely based.
- \( C \) has an algebraic generating function.
**Strong interval decomposition**

**Thm:** Every $\sigma$ can be uniquely decomposed as

- $12 \ldots k[\alpha_1, \ldots, \alpha_k]$, with the $\alpha_i$ plus-indecomposable
- $k \ldots 21[\alpha_1, \ldots, \alpha_k]$, with the $\alpha_i$ minus-indecomposable
- $\pi[\alpha_1, \ldots, \alpha_k]$, with $\pi$ simple of size $\geq 4$

**Remarks:**

- This decomposition is unique without any further restriction.
- The $\alpha_i$ are the maximal strong intervals of $\sigma$.

Decompose the $\alpha_i$ recursively to get the decomposition tree.
Substitution decomposition and decomposition trees

**Decomposition tree**

**Example:** The substitution decomposition tree of $\sigma = 10 \ 13 \ 12 \ 11 \ 14 \ 1 \ 18 \ 19 \ 20 \ 21 \ 17 \ 16 \ 15 \ 4 \ 8 \ 3 \ 2 \ 9 \ 5 \ 6 \ 7$

```
  3 1 4 2
  ⊕ ⊕ ⊕ ⊕
```

**Notations and properties:**
- $⊕ = 12 \ldots k$ and $⊖ = k \ldots 21 = \text{linear nodes}$.
- $π$ simple of size $≥ 4 = \text{prime nodes}$.
- No $⊕ − ⊕$ or $⊖ − ⊖$ edge.
- Decomposition trees of permutations are **ordered**.
- N.B.: Modular decomposition trees are unordered.

**Bijection** between decomposition trees and permutations.
Theorem

\( \sigma \) is a pin-permutation iff its decomposition tree satisfies:

1. Any linear node \( \oplus (\ominus) \) has at most one child that is not an ascending (descending) weaving permutation

2. For any prime node labelled by \( \pi \), \( \pi \) is a simple pin-permutation and
   - all of its children are leaves
   - it has exactly one child that is not a leaf, and it inflates one active point of \( \pi \)
   - \( \pi \) is an ascending (descending) quasi-weaving permutation and exactly two children are not leaves
     - one is 12 (21) inflating the auxiliary substitution point of \( \pi \)
     - the other one inflates the main substitution point of \( \pi \)
Definitions

**Active point** \( \sigma \): there is a pin representation of \( \sigma \) starting with it.

**Weaving permutation** \( \mathcal{W} \)

**Quasi-weaving permutation** \( \beta \)

Both are ascending (\(+\)). Other are obtained by symmetry.

**Enumeration**: 4 \((= 2 + 2)\) weaving and 8 \((= 4 + 4)\) quasi-weaving permutations of size \(n\), except for small \(n\).
Characterization of the decomposition trees of pin-permutations

Back to the characterization

\[ \mathcal{P} = \cdot + \mathcal{W}^+ \mathcal{W}^+ \cdots \mathcal{W}^+ + \mathcal{W}^+ \cdots \cdots \mathcal{W}^+ + \mathcal{N}^+ + \mathcal{W}^- \mathcal{W}^- \cdots \mathcal{W}^- + \mathcal{W}^- \cdots \cdots \mathcal{W}^- + \mathcal{N}^- + \alpha \mathcal{P} \setminus \{\cdot\} + \beta^+ \mathcal{P} \setminus \{\cdot\} + \beta^- \mathcal{P} \setminus \{\cdot\} + 12 + 21 \]
Basic generating functions involved

Weaving permutations: \( W^+(z) = W^-(z) = W(z) = \frac{z + z^3}{1 - z} \).

Remark: \( W^+ \cap W^- = \{1, 2431, 3142\} \)

Quasi-weaving permutations:
\( QW^+(z) = QW^-(z) = QW(z) = \frac{4z^4}{1 - z} \).

Trees \( N^+ \) and \( N^- \): pin-permutations except ascending (descending) weaving permutations and those whose root is \( \oplus \) (\( \ominus \)).

\( N^+(z) = N^-(z) = N(z) = \frac{(z^3 + 2z - 1)(z^3 + P(z)z^3 + 2P(z)z + z - P(z))}{1 - 2z + z^2} \)

\( P(z) = \) generating function of pin-permutations.
From characterization to generating function (1)

\[ \mathcal{P} = \cdot + \]

\[ \mathcal{W}^+ \mathcal{W}^+ \cdots \mathcal{W}^+ \]

\[ \mathcal{W}^+ \cdots \cdots \mathcal{W}^+ \]

\[ \mathcal{N}^+ \]

\[ \mathcal{W}^- \mathcal{W}^- \cdots \mathcal{W}^- \]

\[ \mathcal{W}^- \cdots \cdots \mathcal{W}^- \]

\[ \mathcal{N}^- \]

\[ \mathcal{\alpha} \]

\[ \mathcal{\beta}^+ \]

\[ \mathcal{\beta}^- \]

\[ \mathcal{P} \setminus \{\bullet\} \]

\[ \mathcal{P} \setminus \{\bullet\} \]

\[ \mathcal{P} \setminus \{\bullet\} \]

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Pin-Permutations
Generating functions of simple pin-permutations

- Enumerate pin representations encoding simple pin-permutations.
- Characterize how many pin representations for a simple pin-permutation.
- Describe number of active points in simple pin-permutations.

Simple pin representations: \( \text{SiRep}(z) = 8z^4 + \frac{32z^5}{1-2z} - \frac{16z^5}{1-z} \)

Simple pin-permutations: \( \text{Si}(z) = 2z^4 + 6z^5 + 32z^6 + \frac{128z^7}{1-2z} - \frac{28z^7}{1-z} \)

Simple pin-permutations with multiplicity = number of active points: \( \text{SiMult}(z) = 8z^4 + 26z^5 + 84z^6 + \frac{256z^7}{1-2z} - \frac{40z^7}{1-z} \)
From characterization to generating function (2)

\[ P = \, . \, + \]

\[ W^+ \, W^+ \, \ldots \, W^+ \]

\[ W^+ \, \ldots \, \ldots \, W^+ \]

\[ N^+ \]

\[ + \]

\[ W^- \, W^- \, \ldots \, W^- \]

\[ W^- \, \ldots \, \ldots \, W^- \]

\[ N^- \]

\[ + \]

\[ \alpha \]

\[ + \]

\[ \beta^+ \]

\[ + \]

\[ \beta^- \]

\[ + \]

\[ P \setminus \{ \bullet \} \]

\[ P \setminus \{ \bullet \} \]

\[ P \setminus \{ \bullet \} \]

\[ + \]

\[ + \]

\[ + \]
The rational generating function of pin-permutations

Equation on trees $\Rightarrow$ equation on generating functions:

$$P(z) = z + \frac{W^+(z)^2}{1 - W^+(z)} + \frac{2W^+(z) - W^+(z)^2}{(1 - W^+(z))^2} N^+(z)$$

$$+ \frac{W^-(z)^2}{1 - W^-(z)} + \frac{2W^-(z) - W^-(z)^2}{(1 - W^-(z))^2} N^-(z) + Si(z)$$

$$+ \text{SiMult}(z) \left( \frac{P(z) - z}{z} \right) + QW^+(z) \left( z \frac{P(z) - z}{z} \right) + QW^-(z) \left( z \frac{P(z) - z}{z} \right)$$

Generating function of pin-permutations:

$$P(z) = z \frac{8z^6 - 20z^5 - 4z^4 + 12z^3 - 9z^2 + 6z - 1}{8z^8 - 20z^7 + 8z^6 + 12z^5 - 14z^4 + 26z^3 - 19z^2 + 8z - 1}$$

First terms: 1, 2, 6, 24, 120, 664, 3596, 19004, 99596, 521420, …
The rational generating function of pin-permutations

Equation on trees ⇒ equation on generating functions:

\[ P(z) = z + \frac{W^+(z)^2}{1 - W^+(z)} + \frac{2W^+(z) - W^+(z)^2}{(1 - W^+(z))^2} N^+(z) \]

\[ + \frac{W^-(z)^2}{1 - W^-(z)} + \frac{2W^-(z) - W^-(z)^2}{(1 - W^-(z))^2} N^-(z) + Si(z) \]

\[ + \text{Simult}(z) \left( \frac{P(z) - z}{z} \right) + QW^+(z) \left( z \frac{P(z) - z}{z} \right) + QW^-(z) \left( z \frac{P(z) - z}{z} \right) \]

Generating function of pin-permutations:

\[ P(z) = z \frac{8z^6 - 20z^5 - 4z^4 + 12z^3 - 9z^2 + 6z - 1}{8z^8 - 20z^7 + 8z^6 + 12z^5 - 14z^4 + 26z^3 - 19z^2 + 8z - 1} \]

First terms: 1, 2, 6, 24, 120, 664, 3596, 19004, 99596, 521420, ...
Conclusion and open question

**Overview of the results:**

- Class of pin-permutations define by a graphical property
- Characterization of the associated decomposition trees
- Enumeration of simple pin-permutations
  ⇒ Generating function of the pin-permutation class
- Rationality of the generating function

**Characterization of the pin-permutation class:**

✓ by a recursive description

? by a (finite?) basis of excluded patterns

This basis is infinite, but yet unknown.
Prop. $\sigma$ is in the basis $\iff$ $\sigma$ is not a pin-permutation but any strict pattern of $\sigma$ is.

We describe $(\sigma_n)$ an infinite antichain in the basis:
**Perspectives**

**Thm** [Brignall et al.]: $C$ a class given by its finite basis $B$. It is **decidable** whether $C$ contains infinitely many simple permutations.

**Procedure**: Check whether $C$ contains arbitrarily long
- parallel alternations Easy, Polynomial
- wedge simple permutations Easy, Polynomial
- proper pin-permutations Difficult, Complexity?

**Analysis** of the procedure for proper pin-permutations
⇒ Polynomial construction using automata techniques except last step (*Determinization of a transducer*)
⇒ makes the construction exponential

Better knowledge of pin-permutations ⇒ improve this complexity?