Limit shapes of pattern-avoiding permutations

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talk based on joint works with
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Including additional pictures by Jacopo Borga and Carine Pivoteau

Journées combinatoires de Bordeaux, Février 2019
What are permutations? (in this talk)

A permutation of size $n$ is a bijection from $\{1, 2, \ldots, n\}$ to itself.

We often write a permutation $\sigma$ of size $n$ as the word $\sigma(1)\sigma(2)\ldots\sigma(n)$.

For the purpose of this talk, we represent permutations by their permutation matrices, or rather their diagram.

Example: the diagram of $\sigma = 596741283$ is

![Permutation diagram]

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What are random permutations (and their limit shapes)?

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- Size 10
- Size 100
- Size 1000
- Size 10,000
- Size 100,000
- In the limit
What are random permutations (and their limit shapes)?

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- Size 10
- Size 100
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- Size 10,000
- Size 100,000
- In the limit

**Goal of the talk:** Describe limit shapes of (the diagrams of) pattern-avoiding permutations.
A permutation $\pi$ of size $k$ is a pattern of a permutation $\sigma$ of size $n$ if there exist $1 \leq i_1 < \ldots < i_k \leq n$ such that $\sigma(i_1) \ldots \sigma(i_k)$ is in the same relative order ($\equiv$) as $\pi$.

Example: $2 1 3 4$ is a pattern of $3 1 2 8 5 4 7 9 6$ since $3 1 5 7 \equiv 2 1 3 4$. 
Patterns in permutations

A permutation $\pi$ of size $k$ is a pattern of a permutation $\sigma$ of size $n$ if there exist $1 \leq i_1 < \ldots < i_k \leq n$ such that $\sigma(i_1) \ldots \sigma(i_k)$ is in the same relative order ($\equiv$) as $\pi$.

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Permutation classes are sets of permutations defined by the avoidance of patterns. They are denoted $Av(B)$ for $B$ a set of excluded patterns.
Uniform random permutations in $\mathcal{A}_\nu(\tau)$ for $\tau$ of size 3

$\mathcal{A}_\nu(231)$

$\mathcal{A}_\nu(321)$

Uniform random permutations in $\text{Av}(\tau)$ for $\tau$ of size 3

$\text{Av}(231)$


- Miner-Pak, also Madras with Atapour, Liu and Pehlivan: very precise local description of the average asymptotic shape

- Hoffman-Rizzolo-Slivken: scaling limits and link with the Brownian excursion (for the fluctuations around the main diagonal)
At first order, the limit of the diagram of a uniform random permutation in \( \text{Av}(\tau) \) for \( \tau = 231 \) or 321 is just \[
\begin{array}{c}
\end{array}
\].

So, is first order interesting?
Diagrams of uniform random permutations in classes

At **first order**, the limit of the diagram of a uniform random permutation in $\text{Av}(\tau)$ for $\tau = 231$ or $321$ is just \[
\begin{array}{c}
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So, is first order interesting? **Yes!**

Typical large permutations in $\text{Av}(2413, 3142)$, the class of separable permutations, also described as the substitution-closed class with set of simple permutations $\emptyset$:
At first order, the limit of the diagram of a uniform random permutation in \( Av(\tau) \) for \( \tau = 231 \) or 321 is just

\[
\begin{array}{c}
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So, is first order interesting? Yes!

Typical large permutations in the substitution-closed class with set of simple permutations \( \{2413, 3142, 24153\} \):
At first order, the limit of the diagram of a uniform random permutation in $\text{Av}(\tau)$ for $\tau = 231$ or 321 is just .

So, is first order interesting? Yes!

Typical large permutation in the substitution-closed class with (infinite) set of simple permutations $\text{Av}(321) \cap \{\text{Simples}\}$, i.e. in the substitution closure of $\text{Av}(321)$:
Diagrams of uniform random permutations in classes

At first order, the limit of the diagram of a uniform random permutation in $Av(\tau)$ for $\tau = 231$ or 321 is just .

So, is first order interesting? **Yes!**

Typical (large) permutations in $Av(2413, 1243, 2341, 41352, 531642)$:
At first order, the limit of the diagram of a uniform random permutation in $Av(\tau)$ for $\tau = 231$ or 321 is just

So, is first order interesting? Yes!

Typical (large) permutation in $Av(2413, 3142, 2143, 3412)$, called the X-class and denoted $\mathcal{X}$ later:
Diagrams of uniform random permutations in classes

At first order, the limit of the diagram of a uniform random permutation in $Av(\tau)$ for $\tau = 231$ or 321 is just $\square$.

So, is first order interesting? Yes!

Typical (large) permutation in $Av(2413, 3142, 2143, 34512)$:
At first order, the limit of the diagram of a uniform random permutation in $Av(\tau)$ for $\tau = 231$ or 321 is just \[
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Typical (large) permutation in the downward closure of $\oplus[\mathcal{X}, \mathcal{X}]$:
At **first order**, the limit of the diagram of a uniform random permutation in $Av(\tau)$ for $\tau = 231$ or $321$ is just 
![Diagonals](image)

So, is first order interesting? **Yes!**

Typical (large) permutation in the downward closure of $\oplus[\mathcal{X}, \mathcal{X}]$:

![Permutation Diagram](image)

How can we explain these pictures?
A **permuton** is a probability measure on the unit square with uniform marginals, 

*i.e.* the total mass on any vertical or horizontal strip of width $x$ is $x$. 
What type of objects are the limiting diagrams?

A permuton is a probability measure on the unit square with uniform marginals, i.e. the total mass on any vertical or horizontal strip of width $x$ is $x$.

With its diagram, every permutation $\sigma$ can be viewed as a permuton $\mu_\sigma$. 

![Diagram of a permutation](image)

![Diagram of a permuton](image)
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With its diagram, every permutation $\sigma$ can be viewed as a permuton $\mu_\sigma$.

Informally, permuton can represent permutations of finite size, but also “permutations of infinite size”.

Permuton convergence

We say that a sequence of permutations \((\sigma_n)\) converges to a permuton \(\mu\) when the sequence of permutons \((\mu_{\sigma_n})\) converges to \(\mu\) (for the weak convergence of measures).

This extends to sequences of random permutations \((\sigma_n)\), converging to a (a priori random) permuton \(\mu\).
Permuton convergence

We say that a sequence of permutations $\sigma_n$ converges to a permuton $\mu$ when the sequence of permutons $\mu_{\sigma_n}$ converges to $\mu$ (for the weak convergence of measures).

This extends to sequences of random permutations $\sigma_n$, converging to a (a priori random) permuton $\mu$.

Permuton convergence is characterized by the convergence of probabilities of all patterns:

- Define $\tilde{\text{occ}}(\pi, \sigma)$ as the probability of occurrence of the pattern $\pi$ in $\sigma$:
  $$\tilde{\text{occ}}(\pi, \sigma) = \frac{\text{number of occurrences of } \pi \text{ in } \sigma}{|\sigma|}.$$  
- $\tilde{\text{occ}}(\pi, \mu)$ is similarly defined as the probability that $|\pi|$ points of the unit square picked at random according to $\mu$ induce the pattern $\pi$.
- $(\sigma_n)$ converges to $\mu \iff (\tilde{\text{occ}}(\pi, \sigma_n))_\pi$ converges to $(\tilde{\text{occ}}(\pi, \mu))_\pi$ in distribution (jointly for all patterns $\pi$).
Goal: Prove limit shape results for the diagrams of uniform random permutations in permutation classes.

Framework: Express these limit shape results in the framework of permutons.
Summary so far, and what comes next

**Goal:** Prove limit shape results for the diagrams of uniform random permutations in permutation classes.

**Framework:** Express these limit shape results in the framework of permutons.

**Key tools:**
- Permuton convergence is the convergence of all pattern probabilities;
- Thanks to their *substitution decomposition*, permutations are trees and their patterns are subtrees;
Summary so far, and what comes next

**Goal:** Prove limit shape results for the diagrams of uniform random permutations in permutation classes.

**Framework:** Express these limit shape results in the framework of permutons.

**Key tools:**
- Permuton convergence is the convergence of all pattern probabilities;
- Thanks to their *substitution decomposition*, permutations are trees and their patterns are subtrees;
- Limit shape results on random trees
  OR
- Singularity analysis of generating functions for trees.
Substitution decomposition

Ingredients:

- A way of building bigger permutations from smaller ones
  \(\leadsto\) substitution or inflation;

- “Building blocks” allowing to build all permutations
  \(\leadsto\) simple permutations.

Essential property:

- For every permutation \(\sigma\), there exists a unique way of obtaining it recursively using inflations of simple permutations.
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Essential property:

- For every permutation \(\sigma\), there exists a unique way of obtaining it recursively using inflations of simple permutations.

Outcome:

- Bijection between permutations and decomposition trees.
- (Some) permutation classes are (nice) families of trees:
  - easiest case: substitution-closed classes;
  - beyond those: classes with a finite combinatorial specification.
Separable permutations
and the Brownian separable permutohedral lattice
Separable permutations

They are equivalently described as

- $\text{Av}(2413, 3142)$;
- the substitution-closed class with set of simple permutations $\emptyset$.

Theorem:
Uniform random separable permutations converge to a genuinely random permuton: the Brownian separable permuton.
A Schröder tree of size $n$ is a rooted plane tree with $n$ leaves whose internal vertices have at least two children.
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Decomposition trees of separable permutations are signed Schröder trees, as above with additional signs $\oplus$ and $\ominus$ on the internal vertices, which alternate on any path from the root to a leaf (i.e. the sign of the root determines all others).
The combinatorial specification of separable permutations

Up to the “right binarization”

\[ T_1 \lor T_2 \land \cdots \land T_k \mapsto T_1 \lor T_2 \land \cdots \land T_k \]

(and same with $\oplus$),

the decomposition trees of separable permutations are generated by the following combinatorial specification:

\[
\begin{align*}
\mathcal{T}_{\text{sep}} &= \{\bullet\} \lor \mathcal{T}_{\text{not} \oplus} \lor \mathcal{T}_{\text{not} \ominus} ; \\
\mathcal{T}_{\text{not} \oplus} &= \{\bullet\} \lor \mathcal{T}_{\text{sep}} ; \\
\mathcal{T}_{\text{not} \ominus} &= \{\bullet\} \lor \mathcal{T}_{\text{sep}} .
\end{align*}
\]

Starting point of the “analytic combinatorics” proof, discussed later.
The combinatorial specification of separable permutations

Up to the “right binarization”

\[ T_1 \oplus T_2 \oplus \cdots \oplus T_k \mapsto T_1 \oplus T_2 \oplus \cdots \oplus T_k \] (and same with \( \ominus \)),

the decomposition trees of separable permutations are generated by the following combinatorial specification:

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\begin{align*}
\mathcal{T}_{sep} &= \{ \bullet \} \cup \mathcal{T}_{not} \oplus \mathcal{T}_{sep} \\
\mathcal{T}_{not} \oplus \mathcal{T}_{sep} &= \{ \bullet \} \cup \mathcal{T}_{not} \oplus \mathcal{T}_{sep} \\
\mathcal{T}_{not} \ominus \mathcal{T}_{sep} &= \{ \bullet \} \cup \mathcal{T}_{not} \ominus \mathcal{T}_{sep}
\end{align*}
\]

Starting point of the “analytic combinatorics” proof, discussed later. For now, we present the “random trees” proof.
The contour of a uniform random Schröder tree converges to the Brownian excursion.
Contours and their limits

The contour of a uniform random Schröder tree converges to the Brownian excursion.

We can define signed contours of signed Schröder trees:

- Peaks ↔ leaves.
- Local minima with signs ↔ signed internal nodes.
The contour of a uniform random Schröder tree converges to the Brownian excursion.

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We can define a signed version of the Brownian excursion:

- Local minima carry balanced independent signs.
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Not known: Do signed contours of signed Schröder trees converge to the signed Brownian excursion? But...
Convergence of the extracted patterns/subtrees

On finite objects:
Extracting a pattern $\pi$ of size $k$ from a separable permutation $\sigma$

$$\sigma = 3214576 \rightarrow \pi = 123$$

$\equiv$ Extracting a signed subtree (induced by $k$ leaves) in a signed Schröder tree of $\sigma$

$\equiv$ Extracting a signed tree from a set of $k$ peaks in a signed contour of $\sigma$

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Convergence of the extracted patterns/subtrees

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In the **limit**:
Extracting a signed tree from a set of $k$ uniformly chosen points in the signed Brownian excursion $e_{\pm}$
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In the limit:
This characterizes the probabilities of all subtrees extracted from $e_\pm$,

Extracting a signed tree from a set of $k$ uniformly chosen points in the signed Brownian excursion $e_\pm$

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Convergence of the extracted patterns/subtrees

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In the limit:
This characterizes the probabilities of all subtrees extracted from $e_{\pm}$, hence of all patterns extracted from the Brownian separable permuton (so defined).

Extracting a signed tree from a set of $k$ uniformly chosen points in the signed Brownian excursion $e_{\pm}$
Substitution-closed classes and universality of the Brownian separable permuton
The class of **separable permutations** is the one whose decomposition trees are described by the combinatorial specification

\[
\begin{align*}
\mathcal{T}_{\text{sep}} &= \{\bullet\} \uplus \mathcal{T}_{\text{not}\oplus} \mathcal{T}_{\text{sep}} \uplus \mathcal{T}_{\text{not}\ominus} \mathcal{T}_{\text{sep}} \\
\mathcal{T}_{\text{not}\oplus} &= \{\bullet\} \uplus \mathcal{T}_{\text{not}\ominus} \mathcal{T}_{\text{sep}} \uplus \mathcal{T}_{\text{sep}} \\
\mathcal{T}_{\text{not}\ominus} &= \{\bullet\} \uplus \mathcal{T}_{\text{not}\ominus} \mathcal{T}_{\text{sep}} \uplus \mathcal{T}_{\text{sep}}
\end{align*}
\]
Substitution-closed classes are those whose decomposition trees described by a combinatorial specification of the form

\[ \mathcal{T} = \{\bullet\} \cup \mathcal{T}_{\text{not} \oplus} \cup \mathcal{T}_{\text{not} \ominus} \cup \mathcal{T}_{\ominus} \cup \mathcal{T}_{\oplus} \cup \cdots \cup \mathcal{T} ; \]

\[ \mathcal{T}_{\text{not} \oplus} = \{\bullet\} \cup \mathcal{T}_{\text{not} \ominus} \cup \cdots \cup \mathcal{T} ; \]

\[ \mathcal{T}_{\text{not} \ominus} = \{\bullet\} \cup \mathcal{T}_{\text{not} \oplus} \cup \cdots \cup \mathcal{T} ; \]

where \( S \) is the set of simple permutations in the class \( \mathcal{C}_S \) considered.
Separable permutations and substitution-closed classes

Substitution-closed classes are those whose decomposition trees described by a combinatorial specification of the form

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\begin{align*}
\mathcal{T} &= \{\bullet\} \cup \\
\mathcal{T}_{\text{not} \oplus} &= \{\bullet\} \cup \\
\mathcal{T}_{\text{not} \ominus} &= \{\bullet\} \cup \\
\end{align*}
\]

where \( S \) is the set of simple permutations in the class \( \mathcal{C}_S \) considered.

**Theorem:**
Under an analytic condition on the generating function \( S(z) \) of \( S \), uniform random permutations in the substitution-closed class \( \mathcal{C}_S \) converge to a biased Brownian separable permutoh.
The biased Brownian separable permuton of parameter $p$

- **Biased** version of the signed Brownian excursion (for $p \in [0,1]$): in $e_{\pm,p}$, local minima carry independent signs, but not balanced; instead, $+$ with probability $p$, $-$ with probability $1 - p$.

- The **biased** Brownian separable permuton $\mu_p$ of parameter $p$ is characterized as above but **starting from** $e_{\pm,p}$ instead of $e_{\pm}$. 
The biased Brownian separable permuton of parameter $p$

- **Biased** version of the signed Brownian excursion (for $p \in [0, 1]$): in $e_{\pm,p}$, local minima carry independent signs, but not balanced; instead, $+$ with probability $p$, $-$ with probability $1 - p$.

- The **biased Brownian separable permuton** $\mu_p$ of parameter $p$ is characterized as above but starting from $e_{\pm,p}$ instead of $e_{\pm}$.

The higher $p$ is, the more drift there is towards the direction of the main diagonal in $\mu_p$.

Simulations of $\mu_p$ for $p = 0.2, 0.45, 0.5$. 
Statement and examples

Theorem:
Let $\mathcal{C}_S$ be the substitution-closed class with set of simple permutations $S$. Let $S(z)$ be the generating function of $S$, and let $R_S$ be its (positive) radius of convergence.
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Assuming that $\lim_{r \to R_S} S'(r) > \frac{2}{(1+R_S)^2} - 1$, uniform random permutations in $C_S$ converge to the biased Brownian separable permuton $\mu_p$. 
Theorem:
Let \( C_S \) be the substitution-closed class with set of simple permutations \( S \). Let \( S(z) \) be the generating function of \( S \), and let \( R_S \) be its (positive) radius of convergence. Assuming that \( \lim_{r \to R_S} S'(r) > \frac{2}{(1+R_S)^2} - 1 \), uniform random permutations in \( C_S \) converge to the biased Brownian separable permuton \( \mu_p \), where \( p \) is explicit in terms of (the generating function of) the number of occurrences of patterns 12 and 21 in permutations of \( S \).
Theorem:
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Example 1: Separable permutations, i.e. $C_{\emptyset}$, $\Rightarrow p = 0.5$
Theorem:
Let $\mathcal{C}_S$ be the substitution-closed class with set of simple permutations $S$. Let $S(z)$ be the generating function of $S$, and let $R_S$ be its (positive) radius of convergence.
Assuming that $\lim_{r \to R_S} S'(r) > \frac{2}{(1+R_S)^2} - 1$, uniform random permutations in $\mathcal{C}_S$ converge to the biased Brownian separable permuton $\mu_p$, where $p$ is explicit in terms of (the generating function of) the number of occurrences of patterns 12 and 21 in permutations of $S$.

Example 2: $\mathcal{C}_S$ with $S = \{2413, 3142, 24153\}$, $\Rightarrow p = 0.5$
**Theorem:**
Let $C_S$ be the substitution-closed class with set of simple permutations $S$. Let $S(z)$ be the generating function of $S$, and let $R_S$ be its (positive) radius of convergence. Assuming that $\lim_{r \to R_S, r < R_S} S'(r) > \frac{2}{(1+R_S)^2} - 1$, uniform random permutations in $C_S$ converge to the biased Brownian separable permuton $\mu_p$, where $p$ is explicit in terms of (the generating function of) the number of occurrences of patterns 12 and 21 in permutations of $S$.

**Example 3:** $C_S$ with $S = Av(321) \cap \{\text{Simples}\}$, $\Rightarrow p \in [0.577, 0.622]$
Statement and examples

Theorem:
Let $C_S$ be the substitution-closed class with set of simple permutations $S$. Let $S(z)$ be the generating function of $S$, and let $R_S$ be its (positive) radius of convergence.
Assuming that $\lim_{r \rightarrow R_S} S'(r) > \frac{2}{(1+R_S)^2} - 1$, uniform random permutations in $C_S$ converge to the biased Brownian separable permuton $\mu_p$, where $p$ is explicit in terms of (the generating function of) the number of occurrences of patterns 12 and 21 in permutations of $S$.

Further examples: All substitution-closed classes
- with finitely many simple permutations,
- or more generally such that $R_S = 1$,
- or such that $S'$ diverges at $R_S$ (rational, square root singularities, . . . ).

This covers all substitution-closed classes whose simple permutations have been enumerated.
Theorem: Let $C_S$ be the substitution-closed class with set of simple permutations $S$. Let $S(z)$ be the generating function of $S$, and let $R_S$ be its (positive) radius of convergence. Assuming that $\lim_{r \to R_S} S'(r) > \frac{2}{(1+R_S)^2} - 1$, uniform random permutations in $C_S$ converge to the biased Brownian separable permuton $\mu_p$, where $p$ is explicit in terms of (the generating function of) the number of occurrences of patterns 12 and 21 in permutations of $S$.

Non-example: $Av(2413)$

Limit not known.

We believe it is “degenerate”: typical permutations in $Av(2413)$ look like typical simple permutations in $Av(2413)$.
\( \sigma_n = \) uniform random permutation of size \( n \) in \( \mathcal{C}_S \).

- The law of \((\tilde{\text{occ}}(\pi, \sigma_n))_\pi\) is determined by its moments.
- These moments are all determined by \((\mathbb{E}[\tilde{\text{occ}}(\pi, \sigma_n)])_\pi\).

\[ \Rightarrow \] It is enough to prove the convergence of \( (\mathbb{E}[\tilde{\text{occ}}(\pi, \sigma_n)])_n \) for all \( \pi \).
\( \sigma_n = \) uniform random permutation of size \( n \) in \( C_S \).

- The law of \( (\widetilde{\text{occ}}(\pi, \sigma_n))_\pi \) is determined by its moments.
- These moments are all determined by \( (\mathbb{E} [\widetilde{\text{occ}}(\pi, \sigma_n)])_\pi \).

\[ \Rightarrow \] It is enough to prove the convergence of \( \left( \mathbb{E} [\widetilde{\text{occ}}(\pi, \sigma_n)] \right)_n \) for all \( \pi \).

By definition, \( \mathbb{E} [\widetilde{\text{occ}}(\pi, \sigma_n)] = \frac{\text{total number of occ. of } \pi \text{ in } \sigma \text{ of size } n \text{ in } C_S \left(\frac{n}{|\pi|}\right) \times \text{number of } \sigma \text{ of size } n \text{ in } C_S}{\left(\frac{n}{|\pi|}\right) \times \text{number of } \sigma \text{ of size } n \text{ in } C_S} \).
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By definition, \( \mathbb{E}[\tilde{\text{occ}}(\pi, \sigma_n)] = \frac{\text{total number of occ. of } \pi \text{ in } \sigma \text{ of size } n \text{ in } C_S}{\binom{n}{|\pi|} \times \text{number of } \sigma \text{ of size } n \text{ in } C_S} \).

- Numerator and denominator are expressed as coefficients of generating series of trees (possibly with marked leaves).
- The combinatorial specification for \( C_S \) yields equations for these tree series.
Proof schema 1/2

\( \sigma_n = \) uniform random permutation of size \( n \) in \( \mathcal{C}_S \).

- The law of \( (\widetilde{\text{occ}}(\pi, \sigma_n))_\pi \) is determined by its moments.
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By definition, \( \mathbb{E}[\widetilde{\text{occ}}(\pi, \sigma_n)] = \frac{\text{total number of occ. of } \pi \text{ in } \sigma \text{ of size } n \text{ in } \mathcal{C}_S}{n_{|\pi|} \times \text{number of } \sigma \text{ of size } n \text{ in } \mathcal{C}_S} \).

- Numerator and denominator are expressed as coefficients of generating series of trees (possibly with marked leaves).
- The combinatorial specification for \( \mathcal{C}_S \) yields equations for these tree series.
- Estimate their coefficients with analytic combinatorics.
Determine the singular behavior of the trees series.

They all depend on the one of $T_{\text{not} \oplus}$ satisfying

$$T_{\text{not} \oplus}(z) = z + \Lambda(T_{\text{not} \oplus}(z))$$

where $\Lambda$ is explicit, involving $S$ and rational series.
Determine the singular behavior of the trees series. They all depend on the one of \( T_{\text{not} \oplus} \) satisfying

\[
T_{\text{not} \oplus}(z) = z + \Lambda(T_{\text{not} \oplus}(z))
\]

where \( \Lambda \) is explicit, involving \( S \) and rational series.

**Rk:** If \( \Lambda' > 1 \) at its radius of convergence (equivalent to the condition of our theorem), then \( T_{\text{not} \oplus}(z) \) has a square root singularity.
Determine the **singular behavior** of the trees series.

They all depend on the one of \( T_{\text{not} \oplus} \) satisfying

\[
T_{\text{not} \oplus}(z) = z + \Lambda(T_{\text{not} \oplus}(z))
\]

where \( \Lambda \) is explicit, involving \( S \) and rational series.

**Rk:** If \( \Lambda' > 1 \) at its radius of convergence (equivalent to the condition of our theorem), then \( T_{\text{not} \oplus}(z) \) has a **square root singularity**.

All generating series have the same radius of convergence \( \rho \), and we can compute their **expansions at** \( \rho \).
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All generating series have the same radius of convergence \( \rho \), and we can compute their expansions at \( \rho \).

**Transfer theorem** gives an asymptotic estimate of their coefficients, and hence of \( \mathbb{E} [\widehat{\text{occ}}(\pi, \sigma_n)] \).
Determine the **singular behavior** of the trees series.

They all depend on the one of $T_{\text{not} \oplus}$ satisfying

$$T_{\text{not} \oplus}(z) = z + \Lambda(T_{\text{not} \oplus}(z))$$

where $\Lambda$ is explicit, involving $S$ and rational series.

**Rk:** If $\Lambda'$ is $>1$ at its radius of convergence (equivalent to the condition of our theorem), then $T_{\text{not} \oplus}(z)$ has a **square root singularity**.

All generating series have the same radius of convergence $\rho$, and we can compute their expansions at $\rho$.

**Transfer theorem** gives an asymptotic estimate of their coefficients, and hence of $E[\widetilde{\text{occ}}(\pi, \sigma_n)]$.

**Rk:** The limit of $E[\widetilde{\text{occ}}(\pi, \sigma_n)]$ is non-zero if and only if $\pi$ is separable.
Finitely generated classes, not necessarily substitution-closed
Combinatorial specifications of classes

- For substitution-closed classes: there is always a combinatorial specification for the associated decomposition trees.
- For other classes, there is sometimes a combinatorial specification for the associated decomposition trees.
- It is always the case when the number of simple permutations in the class is finite. Moreover, in this case, the specification is automatically produced.
For substitution-closed classes: there is always a combinatorial specification for the associated decomposition trees.

For other classes, there is sometimes a combinatorial specification for the associated decomposition trees.

It is always the case when the number of simple permutations in the class is finite.

Moreover, in this case, the specification is automatically produced.

**Example:** $Av(132)$

\[
\begin{align*}
\mathcal{T} &= \{\bullet\} \uplus \mathcal{T}_{\text{not}\oplus} \oplus \mathcal{T}_{\langle 21 \rangle} \uplus \mathcal{T}_{\text{not}\oplus} \\
\mathcal{T}_{\text{not}\oplus} &= \{\bullet\} \uplus \mathcal{T}_{\text{not}\oplus} \\
\mathcal{T}_{\text{not}\ominus} &= \{\bullet\} \uplus \mathcal{T}_{\text{not}\ominus} \\
\mathcal{T}_{\langle 21 \rangle} &= \{\bullet\} \uplus \mathcal{T}_{\langle 21 \rangle} \\
\mathcal{T}_{\text{not}\ominus}^{\langle 21 \rangle} &= \{\bullet\}. 
\end{align*}
\]
Consider a specification for the decomposition trees of permutations in a class $\mathcal{C} = \mathcal{T}_0$ where the families $(\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_k)$ appear.
Relevant properties of the specifications

Consider a specification for the decomposition trees of permutations in a class $C = \mathcal{T}_0$ where the families $\{\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_k\}$ appear.

Define that $\mathcal{T}_i$ is critical when its generating function has minimal radius of convergence among all those of the $(\mathcal{T}_j)_j$ (which is that of $\mathcal{T}_0$).
Consider a specification for the decomposition trees of permutations in a class $\mathcal{C} = \mathcal{T}_0$ where the families $(\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_k)$ appear.

Define that $\mathcal{T}_i$ is critical when its generating function has minimal radius of convergence among all those of the $(\mathcal{T}_j)_j$ (which is that of $\mathcal{T}_0$).

The limiting behavior of uniform permutations in $\mathcal{C}$ is determined by:

- the strongly connected components of the specification restricted to critical families;
- whether the restriction of the specification to each strongly connected component is linear or branching.
Essentially branching specifications

The restriction of the specification to critical families is strongly connected, and contains a product.

⇒ The limiting permuton of the class is a biased Brownian separable permuton (of parameter $p$ possibly 0 or 1).
Essentially branching specifications

The restriction of the specification to critical families is strongly connected, and contains a **product**.

⇒ The limiting permuton of the class is a biased **Brownian separable permuton** (of parameter $p$ possibly 0 or 1).

**Example 1:** $\text{Av}(132)$, with critical families in **blue**.

\[
\begin{align*}
\mathcal{T} & = \{\bullet\} \cup \mathcal{T}_{\not\oplus} \bigoplus \mathcal{T}_{\langle 21 \rangle} \cup \mathcal{T}_{\not\ominus} \\
\mathcal{T}_{\not\oplus} & = \{\bullet\} \cup \mathcal{T}_{\not\ominus} \bigoplus \mathcal{T} \\
\mathcal{T}_{\not\ominus} & = \{\bullet\} \cup \mathcal{T}_{\not\oplus} \bigoplus \mathcal{T}_{\langle 21 \rangle} \\
\mathcal{T}_{\langle 21 \rangle} & = \{\bullet\} \cup \mathcal{T}_{\not\oplus} \bigoplus \mathcal{T}_{\langle 21 \rangle} \\
\mathcal{T}_{\not\oplus}^{\langle 21 \rangle} & = \{\bullet\}.
\end{align*}
\]

The limit is the Brownian separable permuton of parameter $p = 0$. 
Essentially branching specifications

The restriction of the specification to critical families is strongly connected, and contains a product.

⇒ The limiting permuton of the class is a biased Brownian separable permuton (of parameter $p$ possibly 0 or 1).

Example 2: $Av(2413, 31452, 41253, 41352, 531246)$, with critical families in blue.

\[
\begin{align*}
\mathcal{T}_0 &= \{\bullet\} \uplus \ominus[\mathcal{T}_1, \mathcal{T}_0] \uplus \ominus[\mathcal{T}_2, \mathcal{T}_0] \uplus 3142[\mathcal{T}_0, \mathcal{T}_3, \mathcal{T}_3, \mathcal{T}_0] \\
\mathcal{T}_1 &= \{\bullet\} \uplus \ominus[\mathcal{T}_2, \mathcal{T}_0] \uplus 3142[\mathcal{T}_0, \mathcal{T}_3, \mathcal{T}_3, \mathcal{T}_0] \\
\mathcal{T}_2 &= \{\bullet\} \uplus \ominus[\mathcal{T}_1, \mathcal{T}_0] \uplus 3142[\mathcal{T}_0, \mathcal{T}_3, \mathcal{T}_3, \mathcal{T}_0] \\
\mathcal{T}_3 &= \{\bullet\} \uplus \ominus[\mathcal{T}_4, \mathcal{T}_3] \\
\mathcal{T}_4 &= \{\bullet\}
\end{align*}
\]

The limit is the Brownian separable permuton of parameter $p \approx 0.4748692376...$ (only real root of a certain polynomial of degree 9).
Essentially linear specifications

The restriction of the specification to critical families is strongly connected, and contains no product.

⇒ The limiting permuton of the class is the \( X \)-permuton of parameter \( p = (p_1, p_2, p_3, p_4) \) (not necessarily centered, possibly degenerate).
Essentially linear specifications

The restriction of the specification to critical families is strongly connected, and contains no product.

\[ \Rightarrow \text{The limiting permuton of the class is the } X\text{-permuton of parameter } \mathbf{p} = (p_1, p_2, p_3, p_4) \text{ (not necessarily centered, possibly degenerate).} \]

Example 1: \( \text{Av}(2413, 3142, 2143, 3412) \), a.k.a. the X-class, with critical families in red and blue (for two strongly connected components).

\[
\begin{align*}
\mathcal{T}_0 &= \{\bullet\} \uplus \uplus [\mathcal{T}_1, \mathcal{T}_2] \uplus \uplus [\mathcal{T}_1, \mathcal{T}_3] \uplus \uplus [\mathcal{T}_4, \mathcal{T}_2] \uplus \ominus [\mathcal{T}_1, \mathcal{T}_5] \uplus \ominus [\mathcal{T}_1, \mathcal{T}_6] \uplus \ominus [\mathcal{T}_7, \mathcal{T}_5] \\
\mathcal{T}_1 &= \{\bullet\} \\
\mathcal{T}_2 &= \{\bullet\} \uplus \uplus [\mathcal{T}_1, \mathcal{T}_2] \\
\mathcal{T}_3 &= \uplus [\mathcal{T}_1, \mathcal{T}_3] \uplus \uplus [\mathcal{T}_4, \mathcal{T}_2] \uplus \ominus [\mathcal{T}_1, \mathcal{T}_5] \uplus \ominus [\mathcal{T}_1, \mathcal{T}_6] \uplus \ominus [\mathcal{T}_7, \mathcal{T}_5] \\
\mathcal{T}_4 &= \ominus [\mathcal{T}_1, \mathcal{T}_5] \uplus \ominus [\mathcal{T}_1, \mathcal{T}_6] \uplus \ominus [\mathcal{T}_7, \mathcal{T}_5] \\
\mathcal{T}_5 &= \{\bullet\} \uplus \ominus [\mathcal{T}_1, \mathcal{T}_5] \\
\mathcal{T}_6 &= \uplus [\mathcal{T}_1, \mathcal{T}_2] \uplus \uplus [\mathcal{T}_1, \mathcal{T}_3] \uplus \uplus [\mathcal{T}_4, \mathcal{T}_2] \uplus \ominus [\mathcal{T}_1, \mathcal{T}_6] \uplus \ominus [\mathcal{T}_7, \mathcal{T}_5] \\
\mathcal{T}_7 &= \uplus [\mathcal{T}_1, \mathcal{T}_2] \uplus \uplus [\mathcal{T}_1, \mathcal{T}_3] \uplus \uplus [\mathcal{T}_4, \mathcal{T}_2].
\end{align*}
\]
Essentially linear specifications

The restriction of the specification to critical families is strongly connected, and contains no product.

⇒ The limiting permuton of the class is the X-permuton of parameter $p = (p_1, p_2, p_3, p_4)$ (not necessarily centered, possibly degenerate).

Example 1: $\text{Av}(2413, 3142, 2143, 3412)$, a.k.a. the X-class, with critical families in red and blue (for two strongly connected components).

The limit is the centered X-permuton (of parameter $(1/4, 1/4, 1/4, 1/4)$).
Essentially linear specifications

The restriction of the specification to critical families is strongly connected, and contains no product.

⇒ The limiting permuton of the class is the X-permuton of parameter \( p = (p_1, p_2, p_3, p_4) \) (not necessarily centered, possibly degenerate).

**Example 2:** \( Av(2413, 3142, 2143, 34512) \), with critical families in red and blue (for two strongly connected components).

\[
\begin{align*}
\mathcal{T}_0 &= \{•\} \uplus \mathcal{T}_1, \mathcal{T}_2 \uplus \mathcal{T}_4, \mathcal{T}_2 \uplus \mathcal{T}_5, \mathcal{T}_6 \uplus \mathcal{T}_8, \mathcal{T}_6 \\
\mathcal{T}_1 &= \{•\} \\
\mathcal{T}_2 &= \{•\} \uplus \mathcal{T}_1, \mathcal{T}_2 \\
\mathcal{T}_3 &= \uplus \mathcal{T}_1, \mathcal{T}_3 \uplus \mathcal{T}_4, \mathcal{T}_2 \uplus \mathcal{T}_5, \mathcal{T}_6 \uplus \mathcal{T}_5, \mathcal{T}_7 \uplus \mathcal{T}_8, \mathcal{T}_6 \\
\mathcal{T}_4 &= \uplus \mathcal{T}_5, \mathcal{T}_6 \uplus \mathcal{T}_5, \mathcal{T}_7 \uplus \mathcal{T}_8, \mathcal{T}_6 \\
\mathcal{T}_5 &= \{•\} \uplus \mathcal{T}_1, \mathcal{T}_1 \uplus \mathcal{T}_9, \mathcal{T}_9 \uplus \mathcal{T}_9, \mathcal{T}_1 \\
\mathcal{T}_6 &= \{•\} \uplus \mathcal{T}_1, \mathcal{T}_6 \\
\mathcal{T}_7 &= \uplus \mathcal{T}_1, \mathcal{T}_2 \uplus \mathcal{T}_1, \mathcal{T}_3 \uplus \mathcal{T}_4, \mathcal{T}_2 \uplus \mathcal{T}_10, \mathcal{T}_6 \uplus \mathcal{T}_10, \mathcal{T}_7 \uplus \mathcal{T}_1, \mathcal{T}_7 \uplus \mathcal{T}_8, \mathcal{T}_6 \\
\mathcal{T}_8 &= \uplus \mathcal{T}_1, \mathcal{T}_{11} \uplus \mathcal{T}_1, \mathcal{T}_{12} \uplus \mathcal{T}_1, \mathcal{T}_{13} \uplus \mathcal{T}_9, \mathcal{T}_{11} \uplus \mathcal{T}_9, \mathcal{T}_{11} \uplus \mathcal{T}_9, \mathcal{T}_1 \\
\mathcal{T}_9 &= \uplus \mathcal{T}_1, \mathcal{T}_6 \\
\mathcal{T}_{10} &= \uplus \mathcal{T}_1, \mathcal{T}_1 \uplus \mathcal{T}_9, \mathcal{T}_9 \uplus \mathcal{T}_9, \mathcal{T}_1 \\
\mathcal{T}_{11} &= \uplus \mathcal{T}_1, \mathcal{T}_2 \\
\mathcal{T}_{12} &= \uplus \mathcal{T}_1, \mathcal{T}_3 \uplus \mathcal{T}_4, \mathcal{T}_2 \uplus \mathcal{T}_10, \mathcal{T}_6 \uplus \mathcal{T}_10, \mathcal{T}_7 \uplus \mathcal{T}_1, \mathcal{T}_7 \uplus \mathcal{T}_8, \mathcal{T}_6 \\
\mathcal{T}_{13} &= \uplus \mathcal{T}_10, \mathcal{T}_6 \uplus \mathcal{T}_10, \mathcal{T}_7 \uplus \mathcal{T}_9, \mathcal{T}_7 \uplus \mathcal{T}_9, \mathcal{T}_6.
\end{align*}
\]
Essentially linear specifications

The restriction of the specification to critical families is strongly connected, and contains no product.

⇒ The limiting permuton of the class is the X-permuton of parameter $\mathbf{p} = (p_1, p_2, p_3, p_4)$ (not necessarily centered, possibly degenerate).

Example 2: $\text{Av}(2413, 3142, 2143, 34512)$, with critical families in red and blue (for two strongly connected components).

The limit is the X-permuton of parameter $\approx (0.2003, 0.2003, 0.4313, 0.1681)$. 
Essentially linear specifications

The restriction of the specification to critical families is strongly connected, and contains no product.

⇒ The limiting permuton of the class is the \textbf{X-permuton} of parameter \( p = (p_1, p_2, p_3, p_4) \) (not necessarily centered, possibly degenerate).

\textbf{Example 3:} \( \text{Av}(2413, 1243, 2341, 531642, 41352) \), with critical families in red and blue (for two strongly connected components).

\begin{align*}
\mathcal{T}_0 &= \{•\} \uplus [\mathcal{T}_1, \mathcal{T}_2] \uplus \uplus [\mathcal{T}_1, \mathcal{T}_3] \uplus \uplus [\mathcal{T}_4, \mathcal{T}_2] \uplus \uplus [\mathcal{T}_5, \mathcal{T}_0] \uplus \uplus \uplus \uplus 3142[\mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_6] \\
\mathcal{T}_1 &= \{•\} \uplus \uplus [\mathcal{T}_7, \mathcal{T}_1] \\
\mathcal{T}_2 &= \{•\} \uplus \uplus [\mathcal{T}_7, \mathcal{T}_2] \\
\mathcal{T}_3 &= \uplus [\mathcal{T}_8, \mathcal{T}_2] \uplus \uplus [\mathcal{T}_9, \mathcal{T}_6] \\
\mathcal{T}_4 &= \uplus [\mathcal{T}_{10}, \mathcal{T}_{11}] \uplus \uplus [\mathcal{T}_{10}, \mathcal{T}_1] \uplus \uplus [\mathcal{T}_7, \mathcal{T}_{11}] \uplus \uplus \uplus \uplus 3142[\mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_6] \\
\mathcal{T}_5 &= \{•\} \uplus \uplus [\mathcal{T}_1, \mathcal{T}_1] \uplus \uplus \uplus \uplus 3142[\mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_1] \\
\mathcal{T}_6 &= \{•\} \uplus \uplus [\mathcal{T}_{12}, \mathcal{T}_2] \uplus \uplus [\mathcal{T}_9, \mathcal{T}_6] \\
\mathcal{T}_7 &= \{•\} \\
\mathcal{T}_8 &= \uplus \uplus [\mathcal{T}_9, \mathcal{T}_6] \\
\mathcal{T}_9 &= \{•\} \uplus \uplus [\mathcal{T}_1, \mathcal{T}_7] \\
\mathcal{T}_{10} &= \uplus [\mathcal{T}_1, \mathcal{T}_1] \uplus \uplus \uplus \uplus 3142[\mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_1] \\
\mathcal{T}_{11} &= \uplus [\mathcal{T}_1, \mathcal{T}_2] \uplus \uplus [\mathcal{T}_1, \mathcal{T}_3] \uplus \uplus [\mathcal{T}_4, \mathcal{T}_2] \uplus \uplus [\mathcal{T}_{10}, \mathcal{T}_{11}] \uplus \uplus [\mathcal{T}_{10}, \mathcal{T}_1] \uplus \uplus [\mathcal{T}_7, \mathcal{T}_{11}] \uplus \uplus \uplus \uplus \uplus 3142[\mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_6] \\
\mathcal{T}_{12} &= \{•\} \uplus \uplus [\mathcal{T}_9, \mathcal{T}_6]
\end{align*}
Essentially linear specifications

The restriction of the specification to critical families is strongly connected, and contains no product.

\[ \Rightarrow \text{ The limiting permuton of the class is the X-permuton of parameter } p = (p_1, p_2, p_3, p_4) \text{ (not necessarily centered, possibly degenerate).} \]

**Example 3:** \( Av(2413, 1243, 2341, 531642, 41352) \), with critical families in red and blue (for two strongly connected components).

\begin{align*}
\text{The limit is the X-permuton of parameter } & (0, 0, 1 - p, p) \text{ with } p \approx 0.81863. \\
\end{align*}
Several strongly connected components

Assume the specification for $\mathcal{C}$ restricted to critical families has several strongly connected components.

- Limit shape in each strongly connected component: as above.
- Sometimes, the limit shape of $\mathcal{C}$ is a combination of those in each component.
Several strongly connected components

Assume the specification for $\mathcal{C}$ restricted to critical families has several strongly connected components.

- Limit shape in each strongly connected component: as above.
- Sometimes, the limit shape of $\mathcal{C}$ is a combination of those in each component.

Example: the downward closure of $\oplus[\mathcal{X}, \mathcal{X}]$: 

![Diagram of downward closure of $\oplus[\mathcal{X}, \mathcal{X}]$]
Summary of the described limit shapes

**(Biased) Brownian separable permuton:**
- Separable permutations;
- Substitution-closed class under an analytic condition on $S(z)$;
- Classes with a specification that is essentially strongly connected and essentially branching.

**(Parametrized) X-permuton:**
- Classes with a specification that is essentially strongly connected and essentially linear.

A mix of the above:
- Classes with a specification that is not essentially strongly connected.
Summary of the described limit shapes

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- Classes with a specification that is not essentially strongly connected.

Thank you!