Refined enumeration of permutations sorted with two stacks and a D_8 symmetry

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Permutation Patterns 2012, University of Strathclyde

The little story of the problem, with many characters!

- Questions of Anders, Einar and Mark:
  What are the permutations sorted by the composition of two operators of the form $\mathbf{S} \circ \alpha$ for $\alpha \in D_8$?
  How are they enumerated?
- Answer to the 1st question, with Mike and Michael also:
  Characterization of permutations sorted by $\mathbf{S} \circ \alpha \circ \mathbf{S}$ (a set we denote $\mathrm{Id}(\mathbf{S} \circ \alpha \circ \mathbf{S})$) by (generalized) excluded patterns.
- Conjectures of Anders, Einar and Mark for the 2nd question:
  - $\mathrm{Id}(\mathbf{S} \circ \mathbf{r} \circ \mathbf{S})$ and $\mathrm{Id}(\mathbf{S} \circ \mathbf{S})$ are enumerated by the same sequence, and a tuple of 15 statistics is equidistributed.
  - $\mathrm{Id}(\mathbf{S} \circ \mathbf{i} \circ \mathbf{S})$ and Bax are enumerated by the same sequence, and a tuple of 3 statistics is equidistributed.
- Answer to the 2nd question, by Olivier and myself:
  The conjectures are true, and a few more statistics can be added to the first one.

Definitions

Representation of permutations

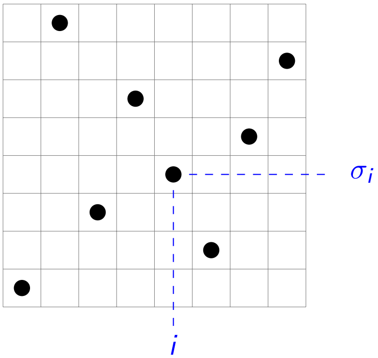
Permutation: Bijection from $[1..n]$ to itself. Set $\mathfrak{S}_n$.

- Linear representation:
  $\sigma = 1\ 8\ 3\ 6\ 4\ 2\ 5\ 7$

- Two lines representation:
  $$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 8 & 3 & 6 & 4 & 2 & 5 & 7 \end{pmatrix}$$

- Representation as a product of cycles:
  $\sigma = (1)\ (2\ 8\ 7\ 5\ 4\ 6)\ (3)$

- Representation by diagram :



Classical patterns in permutations

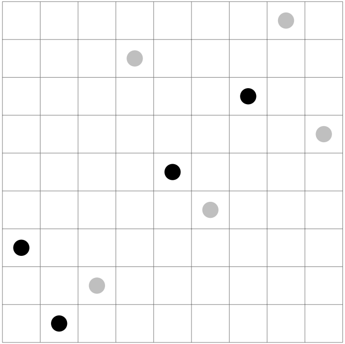
Occurrence of a pattern: $\pi \in \mathfrak{S}_k$ is a pattern of $\sigma \in \mathfrak{S}_n$ if $\exists\ i_1 < \ldots < i_k$ such that $\sigma_{i_1} \ldots \sigma_{i_k}$ is order isomorphic ($\equiv$) to π.

Notation: $\pi \preccurlyeq \sigma$.

Equivalently: The normalization of $\sigma_{i_1} \ldots \sigma_{i_k}$ on $[1..k]$ yields π.

Example: $2\,1\,3\,4 \preccurlyeq \mathbf{3}\,\mathbf{1}\,2\,8\,\mathbf{5}\,4\,\mathbf{7}\,9\,6$ since $3\,1\,5\,7 \equiv 2\,1\,3\,4$.

Avoidance: $\mathrm{Av}(\pi, \tau, \ldots) =$ set of permutations that do not contain any occurrence of π or τ or ...



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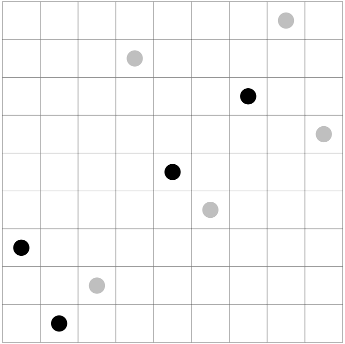
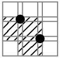
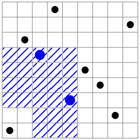


Generalizations of excluded patterns

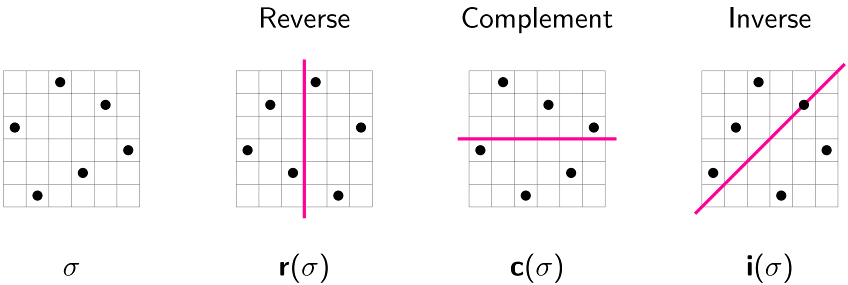
- **Dashed** pattern [Babson, Steingrímsson 2000]:
  Add adjacency constraints between some elements $\sigma_{i_1}, \dots, \sigma_{i_k}$.
  Example: $\sigma_{i_1}\sigma_{i_2}\sigma_{i_3}\sigma_{i_4}$ occurrence of 2-41-3 $\Rightarrow i_3 = i_2 + 1$.
- **Barred** pattern [West 1990]: Add some absence constraints
  Example: Occurrence of $3\bar{5}241$ = occurrence of 3241 that cannot be extended to an occurrence of 35241
- **Mesh** pattern [Úlfarsson, Brändén, Claesson 2011]:
  Stretched diagram with shaded cells.

An occurrence of a mesh pattern is a set of points matching the diagram while leaving zones empty.

Example: $\mu =$  is a pattern of $\sigma =$  .

D_8 symmetries

Symmetries of the square transform permutations *via* their diagrams

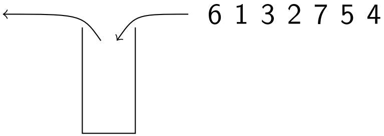


These operators generate an 8-element group:

$$D_8 = \{\mathbf{id}, \mathbf{r}, \mathbf{c}, \mathbf{i}, \mathbf{r} \circ \mathbf{c}, \mathbf{i} \circ \mathbf{r}, \mathbf{i} \circ \mathbf{c}, \mathbf{i} \circ \mathbf{c} \circ \mathbf{r}\}$$

The stack sorting operator **S**

Sort (or try to do so) using a stack satisfying the Hanoi condition.



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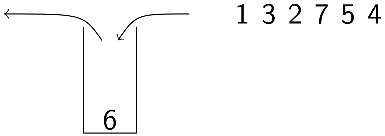

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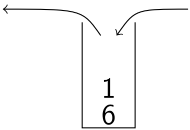

3 2 7 5 4

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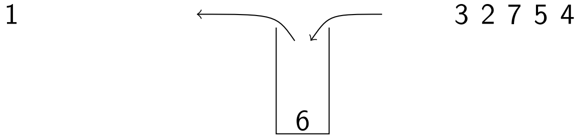

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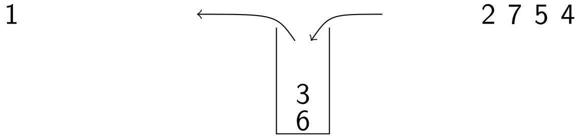
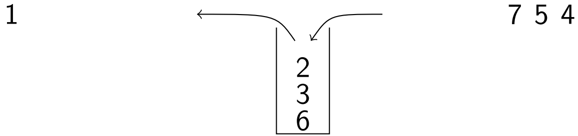


The stack sorting operator $\mathbf{S}$

Sort (or try to do so) using a stack satisfying the Hanoi condition.



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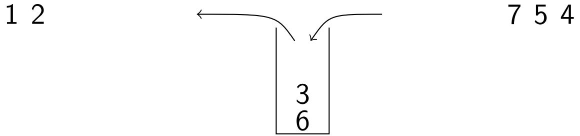

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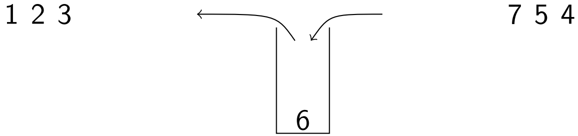

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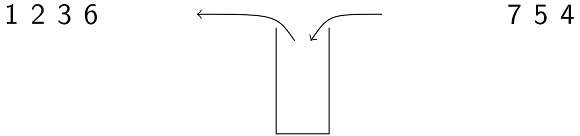

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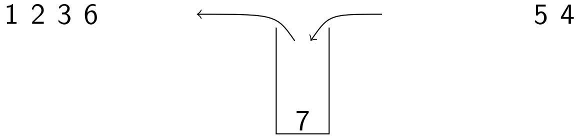

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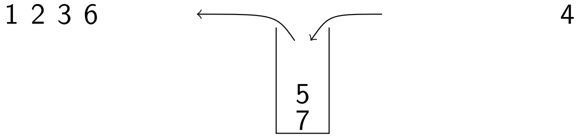

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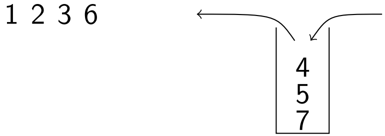

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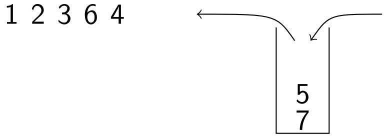

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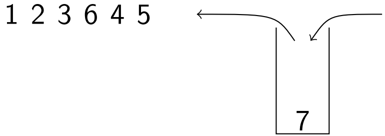

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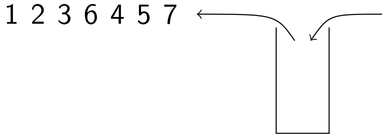
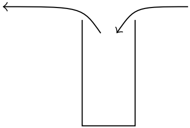


The stack sorting operator **S**

Sort (or try to do so) using a stack satisfying the Hanoi condition.

$\mathbf{S}(\sigma) = 1\ 2\ 3\ 6\ 4\ 5\ 7$



$6\ 1\ 3\ 2\ 7\ 5\ 4 = \sigma$

Equivalently, $\mathbf{S}(\varepsilon) = \varepsilon$ and $\mathbf{S}(LnR) = \mathbf{S}(L)\mathbf{S}(R)n$, $n = \max(LnR)$

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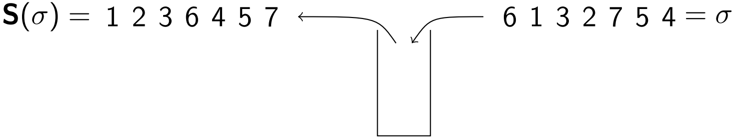


Equivalently, $\mathbf{S}(\varepsilon) = \varepsilon$ and $\mathbf{S}(LnR) = \mathbf{S}(L)\mathbf{S}(R)n$, $n = \max(LnR)$

- Stack sortable permutations: $\mathrm{Id}(\mathbf{S}) = \mathrm{Av}(231)$, enumeration by Catalan numbers [Knuth]
- Two-stack sortable: $\mathrm{Id}(\mathbf{S} \circ \mathbf{S}) = \mathrm{Av}(2341, 3\bar{5}241)$, enumeration by $\frac{2(3n)!}{(n+1)!(2n+1)!}$ [West, Zeilberger,...]
- Many other variants studied, in connection with excluded patterns [Bóna, Bousquet-Mélou, Rossin, ...]

Stating the main results

Characterization by generalized excluded patterns

Theorem

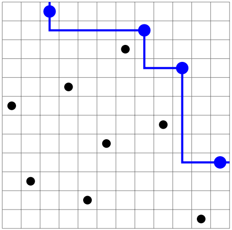
Permutations that are sorted by $\mathbf{S} \circ \alpha \circ \mathbf{S}$*, for* α *in* D_8*, are:*

- $\mathrm{Id}(\mathbf{S} \circ \mathbf{S}) = \mathrm{Id}(\mathbf{S} \circ \mathbf{i} \circ \mathbf{c} \circ \mathbf{r} \circ \mathbf{S}) = \mathrm{Av}(2341, 3\bar{5}241)$;
- $\mathrm{Id}(\mathbf{S} \circ \mathbf{c} \circ \mathbf{S}) = \mathrm{Id}(\mathbf{S} \circ \mathbf{i} \circ \mathbf{r} \circ \mathbf{S}) = \mathrm{Av}(231)$;
- $\mathrm{Id}(\mathbf{S} \circ \mathbf{r} \circ \mathbf{S}) = \mathrm{Id}(\mathbf{S} \circ \mathbf{i} \circ \mathbf{c} \circ \mathbf{S}) = \mathrm{Av}(1342, 31\text{-}4\text{-}2) = \mathrm{Av}(1342, 3\bar{5}142)$;
- $\mathrm{Id}(\mathbf{S} \circ \mathbf{i} \circ \mathbf{S}) = \mathrm{Id}(\mathbf{S} \circ \mathbf{r} \circ \mathbf{c} \circ \mathbf{S}) = \mathrm{Av}(3412, 3\text{-}4\text{-}21)$.

- $\mathrm{Av}(231) = \mathrm{Id}(\mathbf{S})$ is enumerated by Catalan numbers
- $\mathrm{Av}(2341, 3\bar{5}241) = \mathrm{Id}(\mathbf{S} \circ \mathbf{S})$ is enumerated by $\frac{2(3n)!}{(n+1)!(2n+1)!}$

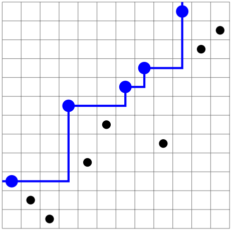
Some permutation statistics. . . and many more

Number of RtoL-max



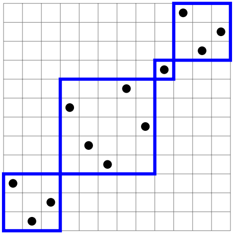
$\text{rmax}(\sigma) = 4$

Number of LtoR-max



$\text{lmax}(\sigma) = 5$

Number of components



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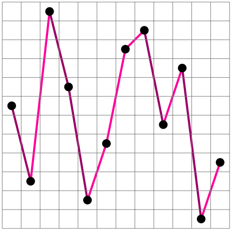

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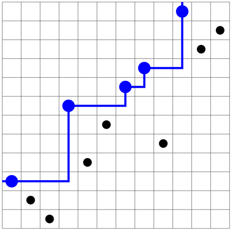

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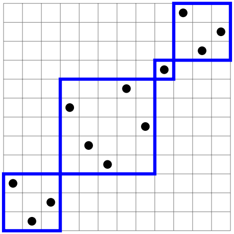


$\text{comp}(\sigma) = 4$

Up-down word of $\sigma \in \mathfrak{S}_n$: $w \in \{u, d\}^{n-1}$, $w_i = \begin{cases} u \text{ if } \sigma_i < \sigma_{i+1} \\ d \text{ if } \sigma_i > \sigma_{i+1} \end{cases}$

$\text{udword}(\sigma) = dudduuudud u$

Refined enumeration of Id(**S** ∘ **r** ∘ **S**)

Theorem

The two sets Id(**S** ∘ **S**) *and* Id(**S** ∘ **r** ∘ **S**) *are enumerated by the same sequence. Moreover, the tuple of statistics* (udword, rmax, lmax, zeil, indmax, slmax, slmax ∘**r**) *has the same distribution.*

The underlying bijection actually preserves the joint distribution of these statistics.

Consequence

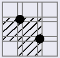
The statistics (asc, des, maj, maj ∘**r**, maj ∘**c**, maj ∘**rc**, valley, peak, ddes, dasc, rir, rdr, lir, ldr) *are also (and jointly) equidistributed.*

Refined enumeration of Id($\mathbf{S} \circ \mathbf{i} \circ \mathbf{S}$)

Theorem

The set Id($\mathbf{S} \circ \mathbf{i} \circ \mathbf{S}$) *is enumerated by the Baxter numbers, and the triple of statistics* $(\mathrm{lmax}, \mathrm{des}, \mathrm{comp})$ *has the same joint distribution on* Id($\mathbf{S} \circ \mathbf{i} \circ \mathbf{S}$) *and on* Bax.

Lemma

It also has the same distribution than the triple of statistics $(\mathrm{lmax}, \mathrm{occ}_\mu, \mathrm{comp})$ *on* TBax, *where* $\mu =$  .

- Baxter permutations: Bax = Av(2-41-3, 3-14-2)
- Twisted Baxter permutations: TBax = Av(2-41-3, 3-41-2)

Both are enumerated by $Bax_n = \frac{2}{n(n+1)^2} \sum_{k=1}^{n} \binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}$

(A few) elements of proof

- Characterization of $\mathsf{Id}(\mathbf{S} \circ \alpha \circ \mathbf{S})$ with excluded patterns
- ✓ Bijection between $\mathsf{Id}(\mathbf{S} \circ \mathbf{S})$ and $\mathsf{Id}(\mathbf{S} \circ \mathbf{r} \circ \mathbf{S})$
- Bijection between $\mathsf{Id}(\mathbf{S} \circ \mathbf{i} \circ \mathbf{S})$ and TBax
- ✓ Bijection between TBax and Bax

Enumeration of Id(**S** ∘ **r** ∘ **S**)

Theorem (partial statement)

The two sets Id(**S** ∘ **S**) *and* Id(**S** ∘ **r** ∘ **S**) *are enumerated by the same sequence.*

Method of proof:

- Id(**S** ∘ **S**) $=$ Av$(2341, 3\bar{5}241)$
- Id(**S** ∘ **r** ∘ **S**) $=$ Av$(1342, 3\bar{5}142)$
- Provide a common rewriting system (encoding isomorphic **generating trees**) for Id(**S** ∘ **S**) and Id(**S** ∘ **r** ∘ **S**)

Refinement according to statistics: introduce each statistics into the rewriting system

Generating trees

- Generating tree for $\text{Av}(\pi, \tau, \ldots)$: an infinite tree where vertices at level n are permutations of $\mathfrak{S}_n$ avoiding $\pi, \tau, \ldots$.
- The children σ' of σ are obtained by insertion of a new element in the active sites of σ.
  - Sites are often on one of the four sides of the diagram (e.g. on the right).
  - Sites are active when $\sigma' \in \text{Av}(\pi, \tau, \ldots)$ *i.e.*, when the insertion does not create a pattern π or τ ...

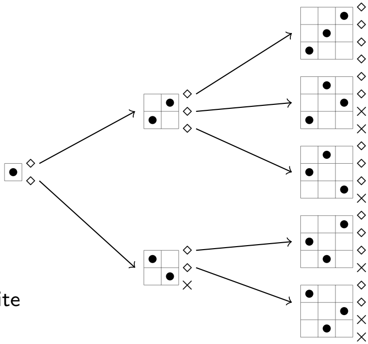
Theorem

Two classes having isomorphic generating trees are in bijection.

...eventhough the bijection is not explicit.

Generating trees

Example: Av(321) with insertion on the right:



Legend: ◇ for an active site; × for an inactive site

Generating trees

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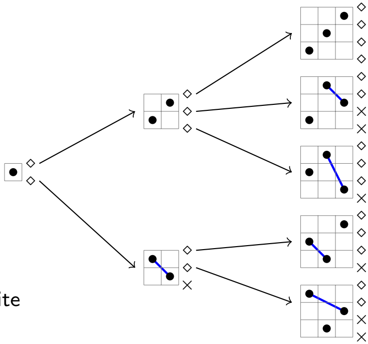


Legend: ◇ for an active site; × for an inactive site

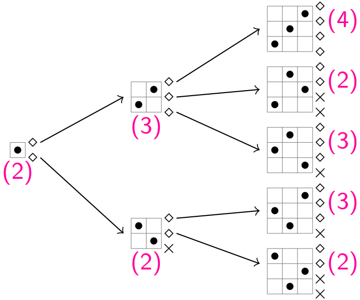
Remark: Active sites are the one above all the inversions of σ

Associated rewriting systems

Idea: Describe the tree compactly by a rewriting rule/systems

- Associate labels to permutations (e.g. number of active sites)
- From the label of σ, describe the labels of the children of σ

Example: The generating tree of Av(321) with labels



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- From the label of σ, describe the labels of the children of σ

Example: For Av(321), we obtained

$$\begin{cases} (2) \\ (k) \quad \rightsquigarrow \quad (k+1)(2)(3)\dots(k) \end{cases}$$

Proof:

- Labels record the number of sites above all the inversions.
- Insertion in the topmost site creates one new active site.
- Insertion in any other site creates an inversion with $\max(\sigma)$.

Rewriting system for Id($\mathbf{S} \circ \mathbf{S}$) and Id($\mathbf{S} \circ \mathbf{r} \circ \mathbf{S}$)

Lemma

A rewriting system for both Id($\mathbf{S} \circ \mathbf{S}$) *and* Id($\mathbf{S} \circ \mathbf{r} \circ \mathbf{S}$) *is*

$$\mathcal{R}_\Phi \begin{cases} (2,1,(1)) \\ (x,k,(p_1,\dots,p_k)) & \rightsquigarrow \quad (2+p_j, j, (p_1,\dots,p_{j-1},i)) \\ & \qquad \textit{for } 1 \le j \le k \textit{ and } p_{j-1} < i \le p_j \\ & \quad\;\; (x+1, k+1, (p_1,\dots,p_k,i)) \\ & \qquad \textit{for } p_k < i \le x \end{cases}$$

Adapted from [Dulucq, Gire, Guibert + West] by application of $\mathbf{c} \circ \mathbf{i}$.

Interpretation of the labels:

- x = the number of active sites of σ,
- k = the number of RtoL-max in σ
- p_ℓ = the number of active sites above the ℓ-th RtoL-max in σ

Refinement according to the statistics rmax

Recall the common rewriting system for Id($\mathbf{S} \circ \mathbf{S}$) and Id($\mathbf{S} \circ \mathbf{r} \circ \mathbf{S}$):

$$\mathcal{R}_\Phi \begin{cases} (2, 1, (1)) & \\ (x, k, (p_1, \ldots, p_k)) & \rightsquigarrow \quad (2 + p_j, j, (p_1, \ldots, p_{j-1}, i)) \\ & \qquad \text{for } 1 \leq j \leq k \text{ and } p_{j-1} < i \leq p_j \\ & \quad (x+1, k+1, (p_1, \ldots, p_k, i)) \\ & \qquad \text{for } p_k < i \leq x \end{cases}$$

- Id($\mathbf{S} \circ \mathbf{S}$) and Id($\mathbf{S} \circ \mathbf{r} \circ \mathbf{S}$) have isomorphic generating trees.

$\Rightarrow$ At any level n, there is the same number of vertices labeled $(x, k, (p_1, \ldots, p_k))$ in both trees.

- In the label $(x, k, (p_1, \ldots, p_k))$ of σ we have $k = \text{rmax}(\sigma)$.

$\Rightarrow$ The statistics rmax is equidistributed in Id($\mathbf{S} \circ \mathbf{S}$) and Id($\mathbf{S} \circ \mathbf{r} \circ \mathbf{S}$)

Refinement according to the statistics lmax

Lemma

The rewriting system can be refined to account for the statistics lmax *as follows:*

$$\mathcal{R}_\Phi^{\mathrm{lmax}} \begin{cases} (2,1,(1),1) & \\ (x,k,(p_1,\dots,p_k),q) & \rightsquigarrow \quad (2+p_1,1,(1),q+1) \\ & \qquad (2+p_j,j,(p_1,\dots,p_{j-1},i),q) \\ & \qquad\qquad \text{for } 1 \le j \le k \text{ and } p_{j-1} < i \le p_j, i \ne 1 \\ & \qquad (x+1,k+1,(p_1,\dots,p_k,i),q) \\ & \qquad\qquad \text{for } p_k < i \le x \end{cases}$$

Proof: The number of LtoR-max does not change when inserting a new element on the right, except when inserting a maximal element ($+1$ in this case).

Refinement according to the statistics udword

Lemma

The rewriting system can be refined to account for the statistics udword *as follows:*

$$\mathcal{R}_{\Phi}^{\text{udword}} \begin{cases} (2, 1, (1), \varepsilon) & \\ (x, k, (p_1, \dots, p_k), w) & \rightsquigarrow \quad (2 + p_j, j, (p_1, \dots, p_{j-1}, i), w \cdot u) \\ & \qquad\qquad \textit{for } 1 \leq j \leq k \textit{ and } p_{j-1} < i \leq p_j \\ & \qquad (x + 1, k + 1, (p_1, \dots, p_k, i), w \cdot d) \\ & \qquad\qquad \textit{for } p_k < i \leq x \end{cases}$$

Proof: In the first (resp. second) case of the rewriting rule, a new element on the right is inserted above (resp. below) the rightmost one.

From Id(**S** ∘ **i** ∘ **S**) to Bax. . . via TBax and twin binary trees

				Pairs of twin binary trees		
Id(**S** ∘ **i** ∘ **S**)	⟷	TBax	⟷		⟷	Bax
lmax	⟷	lmax	⟷	rightmost branch	⟷	lmax
des	⟷	occ_μ	⟷	left edges	⟷	des
comp	⟷	comp	⟷	?	⟷	comp

From Id(**S** ∘ **i** ∘ **S**) to Bax... via TBax and twin binary trees

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Bijection between Id(**S** ∘ **i** ∘ **S**) and TBax: Rewriting system, refined according to the three statistics.

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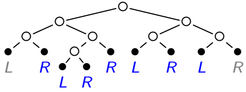
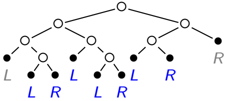
Bijection between Id(**S** ∘ **i** ∘ **S**) and TBax: Rewriting system, refined according to the three statistics.

Bijection between TBax and Bax: One recently described by S. Giraudo, that goes through pairs of twin binary trees *i.e.*, trees of complementary canopies

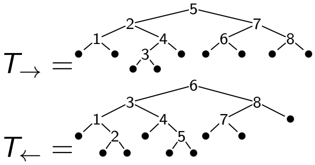
Example: 

and 

S. Giraudo's bijection between TBax and Bax

- To any $\sigma \in \mathfrak{S}_n$, associate $T_\rightarrow$ the (unlabelled) binary search tree obtained by insertion of $\sigma_1, \sigma_2, \ldots, \sigma_n$.
- Similarly for $T_\leftarrow$ by insertion of $\sigma_n, \ldots, \sigma_2, \sigma_1$.

Example: $\sigma = 5\,2\,4\,7\,1\,8\,3\,6$



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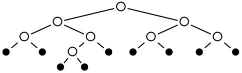
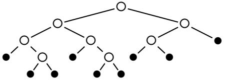
$T_{\rightarrow} =$ 

$T_{\leftarrow} =$ 

Lemma

$(T_{\rightarrow}, T_{\leftarrow})$ is a pair of twin binary trees (with $n+1$ leaves).

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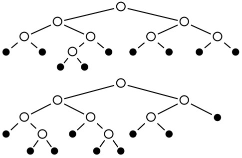

Lemma

$(T_{\rightarrow}, T_{\leftarrow})$ is a pair of twin binary trees (with $n+1$ leaves).

Theorem ([Giraudo])

A pair $(T_{\rightarrow}, T_{\leftarrow})$ corresponds to a set of permutations containing exactly one Baxter and exactly one Twisted Baxter permutation. This provides a bijection between Bax and TBax.

The statistics lmax into S. Giraudo's bijection

Lemma

The elements on the rightmost branch from the root of $T_{\rightarrow}$ *are the LtoR-max of* σ.

This holds in particular when $\sigma \in$ Bax or TBax.

Theorem (partial statement)

The bijection of S. Giraudo between Bax *and* TBax *preserves the number of LtoR-max.*

The statistics comp into S. Giraudo's bijection

Lemma ([Giraudo])

If $\sigma \in \text{Bax}$ *has exactly one component, then so does every* τ *sharing* $(T_{\rightarrow}, T_{\leftarrow})$ *with* σ*.*

This holds in particular for $\tau \in \text{TBax}$.

The statistics comp into S. Giraudo's bijection

Lemma ([Giraudo])

If $\sigma \in \mathrm{Bax}$ *has exactly one component, then so does every* τ *sharing* $(T_{\rightarrow}, T_{\leftarrow})$ *with* σ*.*

This holds in particular for $\tau \in \mathrm{TBax}$.

Lemma

If $\sigma \in \mathrm{Bax}$ *and* $\tau \in \mathrm{TBax}$ *are in correspondance by S. Giraudo's bijection, then* $\mathrm{comp}(\sigma) = \mathrm{comp}(\tau)$*.*

This does not hold in general, but only for $\tau \in \mathrm{TBax}$!

Proof: The above lemma and $\mathrm{TBax} = \mathrm{Av}(2\text{-}41\text{-}3, 3\text{-}41\text{-}2)$.
In particular, no interpretation of comp on $(T_{\rightarrow}, T_{\leftarrow})$...

From computer experiments to open questions

- For any $\alpha, \beta \in D_8$, describe the permutations sorted by $\mathbf{S} \circ \alpha \circ \mathbf{S} \circ \beta \circ \mathbf{S}$ by excluded patterns.

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Or when computers provide conjectures (Schröder numbers)...

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- Count such permutations.
- Refine enumeration according to statistics.

Or when computers provide conjectures (Schröder numbers)...

- And keep composing: $\mathbf{S} \circ \alpha \circ \mathbf{S} \circ \beta \circ \mathbf{S} \circ \gamma \circ \mathbf{S} \ldots$

And new conjectures...

Conjecture

Fix $k \geq 1$. For any $(k-1)$-tuple $(\alpha_2, \dots, \alpha_k) \in \{\mathbf{id}, \mathbf{r}\}^{k-1}$, permutations sorted by $\mathbf{S} \circ \mathbf{id} \circ \mathbf{S} \circ \alpha_2 \circ \dots \circ \mathbf{S} \circ \alpha_k \circ \mathbf{S}$ and by $\mathbf{S} \circ \mathbf{r} \circ \mathbf{S} \circ \alpha_2 \circ \dots \circ \mathbf{S} \circ \alpha_k \circ \mathbf{S}$ are enumerated by the same sequence.

$\Rightarrow$ New approach to the study of k-stack sortable permutations?

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$\Rightarrow$ New approach to the study of k-stack sortable permutations?

Stronger conjecture

For any $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $(\beta_1, \beta_2, \ldots, \beta_k)$, we have either

- $\mathrm{Id}(\mathbf{S} \circ \alpha_1 \circ \mathbf{S} \circ \alpha_2 \circ \ldots \circ \mathbf{S} \circ \alpha_k \circ \mathbf{S}) = \mathrm{Id}(\mathbf{S} \circ \beta_1 \circ \mathbf{S} \circ \beta_2 \circ \ldots \circ \mathbf{S} \circ \beta_k \circ \mathbf{S})$;
- *or these sets are not enumerated by the same sequence;*
- *or they fall into the first conjecture.*