Operators of equivalent sorting power and related Wilf-equivalences

Mathilde Bouvel
joint work with Michael Albert

29 mars 2013

Previously, on *groupe de travail CÉA...*

We study permutations sortable by sorting operators which are compositions of stack sorting operators **S** and reverse operators **R**.

From our previous work with O. Guibert, we have:

Theorem

There are as many permutations of $\mathfrak{S}_n$ sortable by $\mathbf{S} \circ \mathbf{S}$ as permutations of $\mathfrak{S}_n$ sortable by $\mathbf{S} \circ \mathbf{R} \circ \mathbf{S}$, and many permutation statistics are equidistributed across these two sets.

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Computer experiments then suggest that:

Conjecture (*The* $(id, \mathbf{R})$ *conjecture*)

*For any operator **A** which is a composition of operators **S** and **R**, there are as many permutations of $\mathfrak{S}_n$ sortable by $\mathbf{S} \circ id \circ \mathbf{A}$ as permutations of $\mathfrak{S}_n$ sortable by $\mathbf{S} \circ \mathbf{R} \circ \mathbf{A}$. Moreover, many permutation statistics are equidistributed across these two sets.*

In this episode...

Our primary purpose is to prove the $(id, \mathbf{R})$ conjecture.

Theorem

The $(id, \mathbf{R})$ conjecture holds.

The proof uses:

- The characterization of preimages of permutations by $\mathbf{S}$
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The proof uses:

- The characterization of preimages of permutations by **S**
- A new bijection (denoted P) between Av(231) and Av(132)

The bijection P has nice properties, which allow us to derive unexpected enumerative results (Wilf-equivalences). For instance:

Theorem

Av(231, 31254) *and* Av(132, 42351) *have the same enumerative sequence, and their common generating function is*

$$F_5(t) = \frac{t^3 - t^2 - 2t + 1}{2t^3 - 3t + 1}.$$

Definitions

Permutations and patterns

Permutation: Bijection from $[1..n]$ to itself. Set $\mathfrak{S}_n$.

We view permutations as words, $\sigma = \sigma_1 \sigma_2 \dots \sigma_n$
Example: $\sigma = 1\ 8\ 3\ 6\ 4\ 2\ 5\ 7$.

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Occurrence of a pattern: $\pi \in \mathfrak{S}_k$ is a pattern of $\sigma \in \mathfrak{S}_n$ if $\exists\ i_1 < \dots < i_k$ such that $\sigma_{i_1} \dots \sigma_{i_k}$ is order isomorphic ($\equiv$) to π.

Notation: $\pi \preccurlyeq \sigma$.

Equivalently: The normalization of $\sigma_{i_1} \dots \sigma_{i_k}$ on $[1..k]$ yields π.

Example: $2134 \preccurlyeq \mathbf{3} \mathbf{1} 28 \mathbf{5} 4 \mathbf{7} 96$ since $3157 \equiv 2134$.

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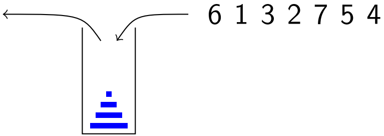
Avoidance: $\text{Av}(\pi, \tau, \ldots)$ = set of permutations that do not contain any occurrence of π or τ or ...

The stack sorting operator **S**

Sort (or try to do so) using a stack satisfying the Hanoi condition.



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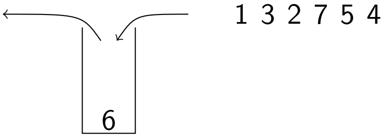

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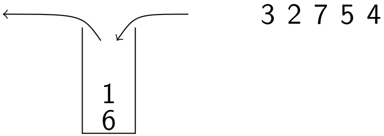

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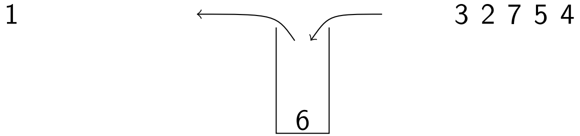

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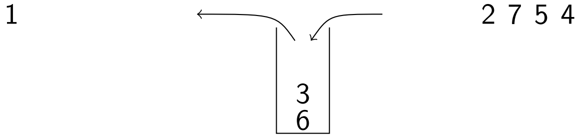

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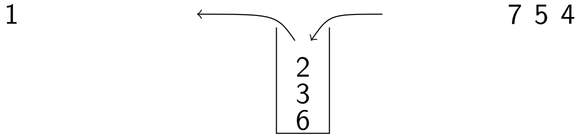

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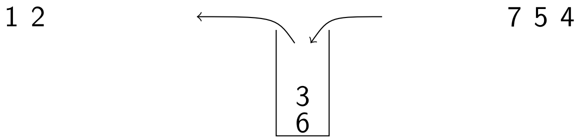

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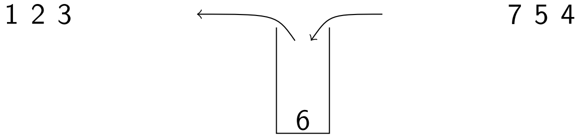

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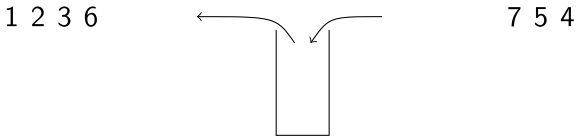

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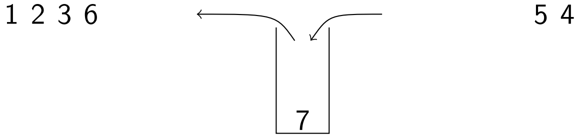

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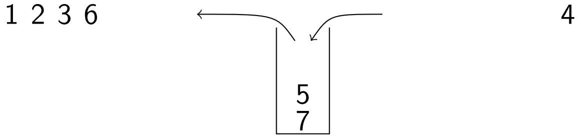

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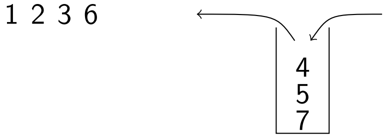

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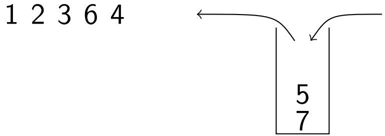

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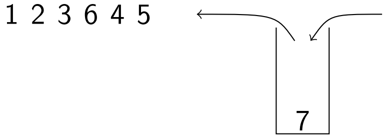

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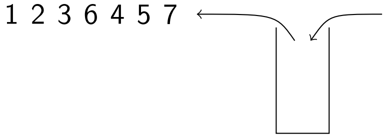

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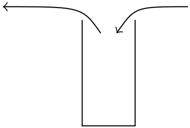
$\mathbf{S}(\sigma) = 1\ 2\ 3\ 6\ 4\ 5\ 7$  $6\ 1\ 3\ 2\ 7\ 5\ 4 = \sigma$

Equivalently, $\mathbf{S}(\varepsilon) = \varepsilon$ and $\mathbf{S}(LnR) = \mathbf{S}(L)\mathbf{S}(R)n$, $n = \max(LnR)$

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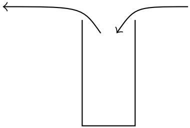

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- Permutations sortable by **S**: Av(231), enumeration by Catalan numbers [Knuth 1975]
- Sortable by $\mathbf{S} \circ \mathbf{S}$: Av$(2341, 3\bar{5}241)$[West 1993], enumeration by $\frac{2(3n)!}{(n+1)!(2n+1)!}$ [Zeilberger 1992]
- Sortable by $\mathbf{S} \circ \mathbf{S} \circ \mathbf{S}$: characterization with (generalized) excluded patterns [Claesson, Úlfarsson 2012], no enumeration result

Main result

Reverse operator **R**: $\mathbf{R}(\sigma_1\sigma_2\cdots\sigma_n) = \sigma_n\cdots\sigma_2\sigma_1$

Theorem

*For any operator **A** which is a composition of operators **S** and **R**, there are as many permutations of $\mathfrak{S}_n$ sortable by $\mathbf{S}\circ\mathbf{A}$ as permutations of $\mathfrak{S}_n$ sortable by $\mathbf{S}\circ\mathbf{R}\circ\mathbf{A}$. Moreover, many permutation statistics are equidistributed across these two sets.*

To prove it, we use:

- the characterization of preimages of permutations by **S** [Bousquet-Mélou, 2000]
- a new bijection (denoted P) between Av(231) and Av(132)

Main result, an equivalent statement

Recall that the set of permutations sortable by **S** is Av(231).
Hence, the set of permutations sortable by $\mathbf{S} \circ \mathbf{R}$ is Av(132).

Theorem

For any operator **A** *which is a composition of operators* **S** *and* **R**, *there is a size-preserving bijection between*

- *permutations of* Av(231) *that belong to the image of* **A**, *and*
- *permutations of* Av(132) *that belong to the image of* **A**,

that preserves the number of preimages under **A**.

We shall see later about the equidistributed statistics.

Preimages under S

from [Bousquet-Mélou, 2000]

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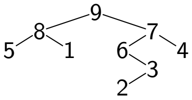
$\mathsf{T}_{\text{in}}(\theta) =$ 

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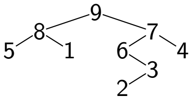
$\mathsf{T}_{\text{in}}(\theta) =$ 

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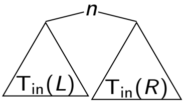
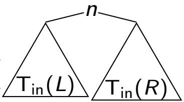
Proof: Since $\mathbf{S}(LnR) = \mathbf{S}(L)\mathbf{S}(R)n$, $\mathsf{T}_{\text{in}}(LnR) =$ 

and $\mathbf{Post}($ 

$) = \mathbf{Post}(\mathsf{T}_{\text{in}}(L))\,\mathbf{Post}(\mathsf{T}_{\text{in}}(R))n$. □

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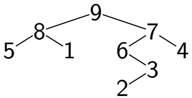
$\mathsf{T}_{\text{in}}(\theta) =$ 

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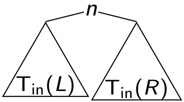
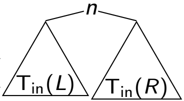
Proof: Since $\mathbf{S}(LnR) = \mathbf{S}(L)\mathbf{S}(R)n$, $\mathsf{T}_{\text{in}}(LnR) =$ 

and $\mathbf{Post}($ 

$) = \mathbf{Post}(\mathsf{T}_{\text{in}}(L))\,\mathbf{Post}(\mathsf{T}_{\text{in}}(R))n$. $\square$

Consequence: For $\pi \in \text{Im}(\mathbf{S})$, $\theta \in \mathbf{S}^{-1}(\pi)$ iff $\mathbf{Post}(\mathsf{T}_{\text{in}}(\theta)) = \pi$.

T_π, a canonical representative for $\mathbf{S}^{-1}(\pi)$

A decreasing binary tree T is canonical if $\forall x, z$ such that x is the left child of z, z has a right child, and the leftmost node in the right subtree of z is $y < x$.

Proposition: For $\pi \in \mathrm{Im}(\mathbf{S})$, there is a unique canonical tree T_π such that $\mathbf{Post}(T_\pi) = \pi$. In fact $T_\pi = \mathrm{T}_{\mathrm{in}}(\theta)$ where θ is the permutation having the greatest number of inversions in $\mathbf{S}^{-1}(\pi)$.

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Proposition: All $\theta \in \mathbf{S}^{-1}(\pi)$ may be described from T_π by *local re-rootings of subtrees*, or *wind blowing*.

Consequence: $|\mathbf{S}^{-1}(\pi)|$ depends only on the shape of T_π (and in particular, not on its labeling).

Example of canonical tree

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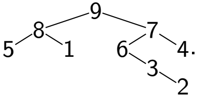

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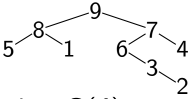
The canonical tree T_π is: 

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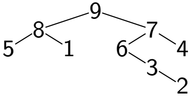
The canonical tree T_π is: 

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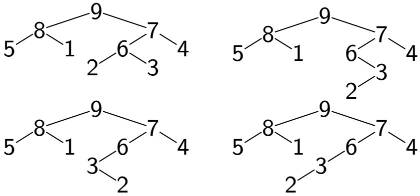

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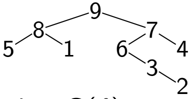
The canonical tree T_π is: 

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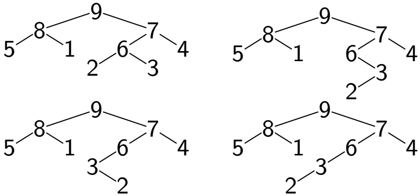

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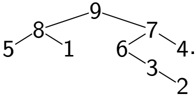
The canonical tree T_π is: 

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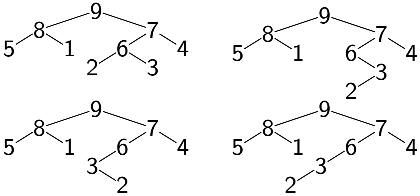

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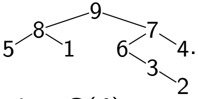
The canonical tree T_π is: 

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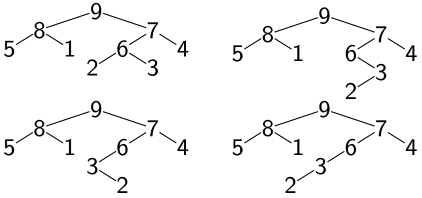

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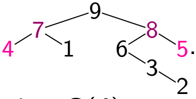
The canonical tree T_π is: 

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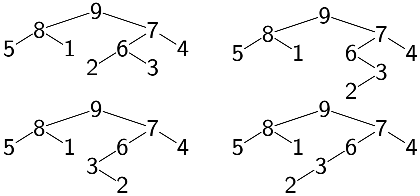
$\theta = 5\,8\,1\,9\,6\,3\,2\,7\,4$ is such that $\mathbf{S}(\theta) = \pi$ and $\mathrm{T_{in}}(\theta) = T_\pi$.

There are 4 other permutations in $\mathbf{S}^{-1}(\pi)$: those whose in-order trees are:



In particular, $|\mathbf{S}^{-1}(\pi)| = 5$.

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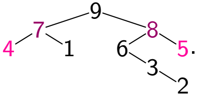
The canonical tree T_π is: .

$\theta = 4\,7\,1\,9\,6\,3\,2\,8\,5$ is such that $\mathbf{S}(\theta) = \pi$ and $\mathsf{T}_{\mathsf{in}}(\theta) = T_\pi$.

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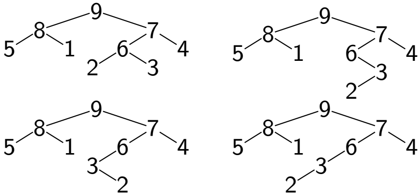

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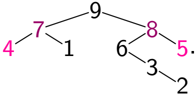

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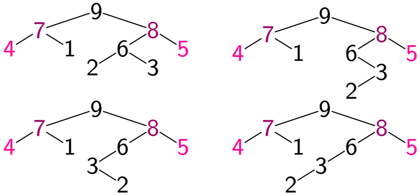

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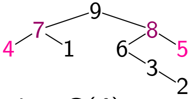

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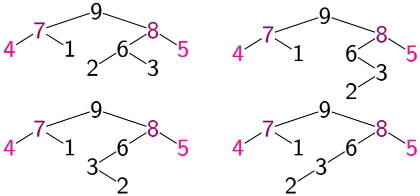

In particular, $|\mathbf{S}^{-1}(\pi)| = 5$ is **unchanged**.

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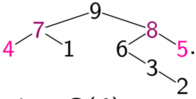
The canonical tree T_π is: 

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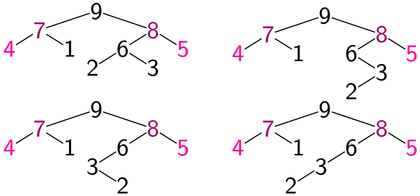

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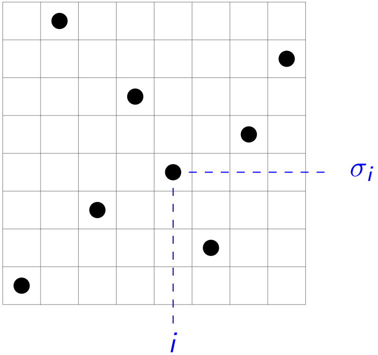
Conclusion: $|\mathbf{S}^{-1}(\pi)|$ is determined by the shape of T_π.

Bijection $\mathrm{Av}(231) \stackrel{P}{\longleftrightarrow} \mathrm{Av}(132)$

Diagrams of permutations; Sum and skew sum

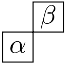
Diagram of $\sigma = 1\ 8\ 3\ 6\ 4\ 2\ 5\ 7$:



α a permutation of $\mathfrak{S}_a$,
β a permutation of $\mathfrak{S}_b$

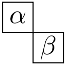
- Sum:

$$\alpha \oplus \beta = \alpha\,(\beta + a) =$$

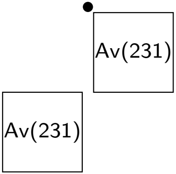


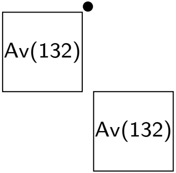
- Skew sum:

$$\alpha \ominus \beta = (\alpha + b)\,\beta =$$



Describing permutations in Av(231) and Av(132)

- $\text{Av}(231) = \varepsilon +$ 

- $\text{Av}(132) = \varepsilon +$ 

- Any $\pi \neq \varepsilon \in \text{Av}(231)$ is decomposed as

$$\pi = \alpha \oplus (1 \ominus \beta)$$

with $\alpha, \beta \in \text{Av}(231)$.

- Any $\pi \neq \varepsilon \in \text{Av}(132)$ is decomposed as

$$\pi = (\alpha \oplus 1) \ominus \beta$$

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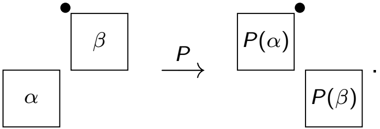
Bijection P from Av(231) to Av(132)

P is recursively defined as:

- If $\pi = \alpha \oplus (1 \ominus \beta)$ then $P(\pi) = (P(\alpha) \oplus 1) \ominus P(\beta)$.
- or equivalently,



with $\alpha, \beta \in$ Av(231).

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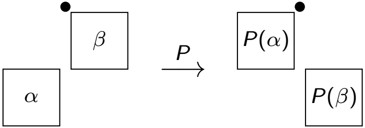
- or equivalently, 

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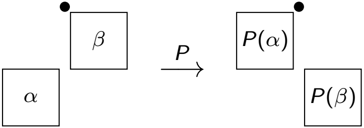
- or equivalently, 

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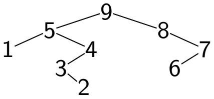
Remark: P is the identity map on Av(231, 132).

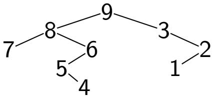
Some properties of P

Proposition: P preserves the shape of in-order trees.

Proof: From the recursive definition of P.

Example: For $\pi = 1\,5\,3\,2\,4\,9\,8\,6\,7$ (and $P(\pi) = 7\,8\,5\,4\,6\,9\,3\,1\,2$):

$T_{in}(\pi) =$ 

$T_{in}(P(\pi)) =$ 

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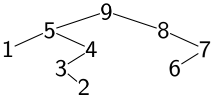
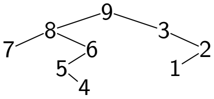
$T_{in}(\pi) =$ 

$T_{in}(P(\pi)) =$ 

Consequence: P preserves the following statistics:

- number and positions of the right-to-left maxima,
- number and positions of the left-to-right maxima,
- up-down word.

Proof: These are determined by the shape of in-order trees.

Proof of the main result: Some key ideas

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Theorem

For any operator **A** *which is a composition of operators* **S** *and* **R**, *there are as many permutations of* $\mathfrak{S}_n$ *sortable by* $\mathbf{S} \circ \mathbf{A}$ *as permutations of* $\mathfrak{S}_n$ *sortable by* $\mathbf{S} \circ \mathbf{R} \circ \mathbf{A}$. *Moreover, many permutation statistics are equidistributed across these two sets.*

Definition of $\Phi_{\mathbf{A}}$

For $\pi \in \text{Av}(231)$, we may see $P(\pi) \in \text{Av}(132)$ as obtained from π by some relabeling of $\{1, 2, \ldots, n\}$, denoted λ_π, *i.e.* $P(\pi) = \lambda_\pi \circ \pi$.

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Definition:

- Take θ a permutation sortable by $\mathbf{S} \circ \mathbf{A}$.
- Set $\pi = \mathbf{A}(\theta)$. $\pi \in$ Av(231).
- Consider λ_π such that $P(\pi) = \lambda_\pi \circ \pi$.
- Define $\Phi_{\mathbf{A}}(\theta) = \lambda_\pi \circ \theta$.

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- Define $\Phi_{\mathbf{A}}(\theta) = \lambda_\pi \circ \theta$.

Theorem: $\Phi_{\mathbf{A}}$ is a bijection between the set of permutation sortable by $\mathbf{S} \circ \mathbf{A}$ and the set of those sortable by $\mathbf{S} \circ \mathbf{R} \circ \mathbf{A}$.

Proving that $\Phi_{\mathbf{A}}$ is a bijection

Definition: $\mathbf{A}$ respects P if, for all $\pi \in \mathrm{Av}(231) \cap \mathrm{Im}(\mathbf{A})$:

- For each θ such that $\mathbf{A}(\theta) = \pi$, we have $\mathbf{A}(\Phi_{\mathbf{A}}(\theta)) = P(\pi) = \lambda_\pi \circ \pi$
- some condition (??) on canonical trees...

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- the correspondence $\Phi_{\mathbf{A}} : \theta \mapsto \Phi_{\mathbf{A}}(\theta)$ is a bijection between $\mathbf{A}^{-1}(\pi)$ and $\mathbf{A}^{-1}(P(\pi))$.

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Proposition: The identity operator respects P.

Proposition: If $\mathbf{A}$ respects P then so does $\mathbf{A} \circ \mathbf{R}$.

Proposition: If $\mathbf{A}$ respects P then so does $\mathbf{A} \circ \mathbf{S}$.

Proving that $\Phi_{\mathbf{A}}$ is a bijection

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Proposition: If **A** respects P then so does $\mathbf{A} \circ \mathbf{R}$.

Proposition: If **A** respects P then so does $\mathbf{A} \circ \mathbf{S}$.

Theorem: Every operator **A** respects P.

Consequence: $\Phi_{\mathbf{A}}$ is a bijection between the set of permutations sortable by $\mathbf{S} \circ \mathbf{A}$ and those sortable by $\mathbf{S} \circ \mathbf{R} \circ \mathbf{A}$.

Statistics preserved by $\Phi_{\mathbf{A}}$

Theorem: $\Phi_{\mathbf{A}}$ preserves the shape of in-order trees.

Consequence: $\Phi_{\mathbf{A}}$ preserves the following statistics:

- number and positions of the right-to-left maxima,
- number and positions of the left-to-right maxima,
- up-down word.

Other statistics preserved:

- Zeilberger's statistics when $\mathbf{A} = \mathbf{A}_0 \circ \mathbf{S}$:

$\text{zeil}(\theta) = \max\{k \mid n(n-1)\dots(n-k+1) \text{ is a subword of } \theta\}$

- the reversed Zeilberger's statistics when $\mathbf{A} = \mathbf{A}_0 \circ \mathbf{S}$ and $\mathbf{A}_0$ contains at least a composition $\mathbf{S} \circ \mathbf{R}$:

$\text{Rzeil}(\theta) = \max\{k \mid (n-k+1)\dots(n-1)n \text{ is a subword of } \theta\}$

Who is $\Phi_{\mathbf{S}}$?

- $\Phi_{\mathbf{S}}$ provides a bijection between the set of permutations sortable by $\mathbf{S} \circ \mathbf{S}$ and those sortable by $\mathbf{S} \circ \mathbf{R} \circ \mathbf{S}$.
- With O. Guibert, we gave a common generating tree for those two sets, providing a bijection between them.

Problem

Are these two bijections the same one?

It is not as easy as it seems...

More about the bijection

$$\mathrm{Av}(231) \stackrel{P}{\longleftrightarrow} \mathrm{Av}(132)$$

Related Wilf-equivalences

. . . Next talk!