Operators of equivalent sorting power and related Wilf-equivalences

Mathilde Bouvel
joint work with Michael Albert

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Previously, on groupe de travail CÉA...

We study permutations sortable by sorting operators which are compositions of stack sorting operators $S$ and reverse operators $R$.

From our previous work with O. Guibert, we have:

**Theorem**

There are as many permutations of $\mathfrak{S}_n$ sortable by $S \circ S$ as permutations of $\mathfrak{S}_n$ sortable by $S \circ R \circ S$, and many permutation statistics are equidistributed across these two sets.
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**Theorem**

*There are as many permutations of $\mathfrak{S}_n$ sortable by $S \circ S$ as permutations of $\mathfrak{S}_n$ sortable by $S \circ R \circ S$, and many permutation statistics are equidistributed across these two sets.*

Computer experiments then suggest that:

**Conjecture (The $(id, R)$ conjecture)**

*For any operator $A$ which is a composition of operators $S$ and $R$, there are as many permutations of $\mathfrak{S}_n$ sortable by $S \circ id \circ A$ as permutations of $\mathfrak{S}_n$ sortable by $S \circ R \circ A$. Moreover, many permutation statistics are equidistributed across these two sets.*
Our primary purpose is to prove the \((id, R)\) conjecture.

**Theorem**

*The \((id, R)\) conjecture holds.*

The proof uses:

- The characterization of preimages of permutations by \(S\)
- A new bijection (denoted \(P\)) between \(Av(231)\) and \(Av(132)\)
In this episode...

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**Theorem**

*The \((id, R)\) conjecture holds.*

The proof uses:

- The characterization of preimages of permutations by \(S\)
- A new bijection (denoted \(P\)) between \(Av(231)\) and \(Av(132)\)

The bijection \(P\) has nice properties, which allow us to derive unexpected enumerative results (Wilf-equivalences). For instance:

**Theorem**

\(Av(231, 31254)\) and \(Av(132, 42351)\) have the same enumerative sequence, and their common generating function is

\[
F_5(t) = \frac{t^3 - t^2 - 2t + 1}{2t^3 - 3t + 1}.
\]
Definitions
Permutations and patterns

**Permutation**: Bijection from \([1..n]\) to itself. Set \(\mathfrak{S}_n\).

We view permutations as **words**, \(\sigma = \sigma_1 \sigma_2 \ldots \sigma_n\)

**Example**: \(\sigma = 1 \ 8 \ 3 \ 6 \ 4 \ 2 \ 5 \ 7\).

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**Example**: $\sigma = 1\ 8\ 3\ 6\ 4\ 2\ 5\ 7$.

**Occurrence of a pattern**: $\pi \in \mathfrak{S}_k$ is a pattern of $\sigma \in \mathfrak{S}_n$ if $\exists\ i_1 < \ldots < i_k$ such that $\sigma_{i_1} \ldots \sigma_{i_k}$ is order isomorphic ($\equiv$) to $\pi$.

**Notation**: $\pi \preceq \sigma$.

**Equivalently**: The normalization of $\sigma_{i_1} \ldots \sigma_{i_k}$ on $[1..k]$ yields $\pi$.

**Example**: $2\ 1\ 3\ 4 \preceq 3\ 1\ 2\ 8\ 5\ 4\ 7\ 9\ 6$ since $3\ 1\ 5\ 7 \equiv 2\ 1\ 3\ 4$. 
Permutations and patterns

**Permutation:** Bijection from $[1..n]$ to itself. Set $\mathcal{S}_n$.

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**Notation:** $\pi \preceq \sigma$.

*Equivalently:* The normalization of $\sigma_{i_1} \ldots \sigma_{i_k}$ on $[1..k]$ yields $\pi$.

**Example:** $2 \ 1 \ 3 \ 4 \preceq 3 \ 1 \ 2 \ 8 \ 5 \ 4 \ 7 \ 9 \ 6$ since $3 \ 1 \ 5 \ 7 \equiv 2 \ 1 \ 3 \ 4$.

**Avoidance:** $Av(\pi, \tau, \ldots)$ = set of permutations that do not contain any occurrence of $\pi$ or $\tau$ or ...
The stack sorting operator $S$

Sort (or try to do so) using a stack satisfying the Hanoi condition.

$\begin{array}{c}
\begin{array}{c}
6 \quad 1 \quad 3 \quad 2 \quad 7 \quad 5 \quad 4
\end{array}
\end{array}$
The stack sorting operator $S$

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Sort (or try to do so) using a stack satisfying the Hanoi condition.

\[
\begin{array}{cccccc}
1 & & & & & 3 2 7 5 4 \\
&&\downarrow&\downarrow&&\\
&\uparrow&6&
\end{array}
\]
The stack sorting operator $S$

Sort (or try to do so) using a stack satisfying the Hanoi condition.

1  

2 7 5 4

3 6
The stack sorting operator $S$

Sort (or try to do so) using a stack satisfying the Hanoi condition.
The stack sorting operator $S$

Sort (or try to do so) using a stack satisfying the Hanoi condition.

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1 2
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3 6

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7 5 4
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The stack sorting operator $S$

Sort (or try to do so) using a stack satisfying the **Hanoi condition**.

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1 2 3                  7 5 4
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6
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The stack sorting operator $S$

Sort (or try to do so) using a stack satisfying the Hanoi condition.

1 2 3 6

7 5 4
The stack sorting operator $S$

Sort (or try to do so) using a stack satisfying the Hanoi condition.

$$\begin{array}{ccccccc}
1 & 2 & 3 & 6 & \leftarrow & 7 & 5 & 4
\end{array}$$
The stack sorting operator $S$

Sort (or try to do so) using a stack satisfying the Hanoi condition.

1 2 3 6

5 7

4
The stack sorting operator $S$

Sort (or try to do so) using a stack satisfying the Hanoi condition.

1 2 3 6

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Operators of equivalent sorting power and related Wilf-equivalences
The stack sorting operator $S$:

Sort (or try to do so) using a **stack** satisfying the **Hanoi condition**.

$1 \ 2 \ 3 \ 6 \ 4$
The stack sorting operator $S$

Sort (or try to do so) using a stack satisfying the Hanoi condition.

1 2 3 6 4 5

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Operators of equivalent sorting power and related Wilf-equivalences
Definitions, context and main result

The stack sorting operator \( S \)

Sort (or try to do so) using a stack satisfying the Hanoi condition.

1 2 3 6 4 5 7
The stack sorting operator $S$

Sort (or try to do so) using a stack satisfying the Hanoi condition.

$$S(\sigma) = 1 \ 2 \ 3 \ 6 \ 4 \ 5 \ 7 \quad \quad 6 \ 1 \ 3 \ 2 \ 7 \ 5 \ 4 = \sigma$$

Equivalently, $S(\varepsilon) = \varepsilon$ and $S(LnR) = S(L)S(R)n$, $n = \max(LnR)$
The stack sorting operator $S$

Sort (or try to do so) using a stack satisfying the Hanoi condition.

$S(\sigma) = 1 \ 2 \ 3 \ 6 \ 4 \ 5 \ 7 \leftarrow \sigma = 6 \ 1 \ 3 \ 2 \ 7 \ 5 \ 4$

Equivalently, $S(\varepsilon) = \varepsilon$ and $S(LnR) = S(L)S(R)n$, $n = \max(LnR)$

- Permutations sortable by $S$: $Av(231)$, enumeration by Catalan numbers [Knuth 1975]
- Sortable by $S \circ S$: $Av(2341, 35241)$ [West 1993], enumeration by $\frac{2(3n)!}{(n+1)!(2n+1)!}$ [Zeilberger 1992]
- Sortable by $S \circ S \circ S$: characterization with (generalized) excluded patterns [Claesson, Úlfarsson 2012], no enumeration result
Main result

**Reverse operator** \( R \): \( R(\sigma_1\sigma_2\cdots\sigma_n) = \sigma_n\cdots\sigma_2\sigma_1 \)

**Theorem**

*For any operator \( A \) which is a composition of operators \( S \) and \( R \), there are as many permutations of \( \mathfrak{S}_n \) sortable by \( S \circ A \) as permutations of \( \mathfrak{S}_n \) sortable by \( S \circ R \circ A \). Moreover, many permutation statistics are equidistributed across these two sets.*

To prove it, we use:

- the characterization of preimages of permutations by \( S \) [Bousquet-Mélou, 2000]
- a new bijection (denoted \( P \)) between \( \text{Av}(231) \) and \( \text{Av}(132) \)
Definitions, context and main result

Main result, an equivalent statement

Recall that the set of permutations sortable by $S$ is $\text{Av}(231)$. Hence, the set of permutations sortable by $S \circ R$ is $\text{Av}(132)$.

**Theorem**

*For any operator $A$ which is a composition of operators $S$ and $R$, there is a size-preserving bijection between*

- permutations of $\text{Av}(231)$ that belong to the image of $A$, and
- permutations of $\text{Av}(132)$ that belong to the image of $A$, that preserves the number of preimages under $A$.

We shall see later about the equidistributed statistics.
Preimages under $S$

from [Bousquet-Mélou, 2000]
Stack sorting on trees

The stack sorting of $\theta$ is equivalent to the post-order reading of the in-order tree $T_{\text{in}}(\theta)$ of $\theta$: $S(\theta) = \text{Post}(T_{\text{in}}(\theta))$
Stack sorting on trees

The stack sorting of \( \theta \) is equivalent to the post-order reading of the in-order tree \( T_{\text{in}}(\theta) \) of \( \theta \): \( S(\theta) = \text{Post}(T_{\text{in}}(\theta)) \)

Example: \( \theta = 5 \ 8 \ 1 \ 9 \ 6 \ 2 \ 3 \ 7 \ 4 \), giving \( S(\theta) = 5 \ 1 \ 8 \ 2 \ 3 \ 6 \ 4 \ 7 \ 9 \).
Stack sorting on trees

The stack sorting of $\theta$ is equivalent to the post-order reading of the in-order tree $T_{\text{in}}(\theta)$ of $\theta$: $S(\theta) = \text{Post}(T_{\text{in}}(\theta))$

Example: $\theta = 5 \ 8 \ 1 \ 9 \ 6 \ 2 \ 3 \ 7 \ 4$, giving $S(\theta) = 5 \ 1 \ 8 \ 2 \ 3 \ 6 \ 4 \ 7 \ 9$.

$T_{\text{in}}(\theta) = 5 \ 8 \ 1 \ 6 \ 7 \ 4 \ 3 \ 2 \ 9$ and $\text{Post}(T_{\text{in}}(\theta)) = 5 \ 1 \ 8 \ 2 \ 3 \ 6 \ 4 \ 7 \ 9$. 
Stack sorting on trees

The stack sorting of $\theta$ is equivalent to the post-order reading of the in-order tree $T_{in}(\theta)$ of $\theta$: $S(\theta) = \text{Post}(T_{in}(\theta))$

Example: $\theta = 5\ 8\ 1\ 9\ 6\ 2\ 3\ 7\ 4$, giving $S(\theta) = 5\ 1\ 8\ 2\ 3\ 6\ 4\ 7\ 9$.

$$T_{in}(\theta) = \begin{array}{c} 9 \\ 8 \ 1 \\ 6 \ 7 \ 4 \\ 2 \ 3 \ 4 \end{array}$$ and $\text{Post}(T_{in}(\theta)) = 5\ 1\ 8\ 2\ 3\ 6\ 4\ 7\ 9$.

Proof: Since $S(LnR) = S(L)S(R)n$, $T_{in}(LnR) = \begin{array}{c} n \\ T_{in}(L) \\ T_{in}(R) \end{array}$ and $\text{Post}(T_{in}(LnR)) = \text{Post}(T_{in}(L)) \text{Post}(T_{in}(R))n$. 

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Stack sorting on trees

The stack sorting of $\theta$ is equivalent to the post-order reading of the in-order tree $T_{\text{in}}(\theta)$ of $\theta$: $S(\theta) = \text{Post}(T_{\text{in}}(\theta))$

Example: $\theta = 5\ 8\ 1\ 9\ 6\ 2\ 3\ 7\ 4$, giving $S(\theta) = 5\ 1\ 8\ 2\ 3\ 6\ 4\ 7\ 9$.

$T_{\text{in}}(\theta) = \begin{array}{c} 5 \\ 8 \\ 9 \\ 1 \\ 6 \\ 7 \\ 4 \\ 2 \\ 3 \\ 7 \\ 4 \end{array}$ and $\text{Post}(T_{\text{in}}(\theta)) = 5\ 1\ 8\ 2\ 3\ 6\ 4\ 7\ 9$.

Proof: Since $S(LnR) = S(L)S(R)n$, $T_{\text{in}}(LnR) = \begin{array}{c} n \\ T_{\text{in}}(L) \\ T_{\text{in}}(R) \end{array}$

and $\text{Post}(\begin{array}{c} n \\ T_{\text{in}}(L) \\ T_{\text{in}}(R) \end{array}) = \text{Post}(T_{\text{in}}(L))\ \text{Post}(T_{\text{in}}(R))n.$

Consequence: For $\pi \in \text{Im}(S)$, $\theta \in S^{-1}(\pi)$ iff $\text{Post}(T_{\text{in}}(\theta)) = \pi$. 
A decreasing binary tree $T$ is **canonical** if $\forall x, z$ such that $x$ is the left child of $z$, $z$ has a right child, and the leftmost node in the right subtree of $z$ is $y < x$.

**Proposition:** For $\pi \in \text{Im}(S)$, there is a unique canonical tree $T_\pi$ such that $\text{Post}(T_\pi) = \pi$. In fact $T_\pi = T_{\text{in}}(\theta)$ where $\theta$ is the permutation having the greatest number of inversions in $S^{-1}(\pi)$. 
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**Proposition:** For $\pi \in \text{Im}(S)$, there is a unique canonical tree $T_\pi$ such that $\text{Post}(T_\pi) = \pi$. In fact $T_\pi = T_{\text{in}}(\theta)$ where $\theta$ is the permutation having the greatest number of inversions in $S^{-1}(\pi)$.

**Proposition:** All $\theta \in S^{-1}(\pi)$ may be described from $T_\pi$ by **local re-rootings of subtrees**, or **wind blowing**.

**Consequence:** $|S^{-1}(\pi)|$ depends only on the **shape** of $T_\pi$ (and in particular, not on its labeling).
Example of canonical tree

\[ \pi = 5\ 1\ 8\ 2\ 3\ 6\ 4\ 7\ 9 \in \text{Im}(S) \]
The canonical tree $T_\pi$ is:

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 /   \\
5 - 8 - 1
  \ /  \
 6 - 9 - 7
  /   \
3 - 2 - 4
  /\  
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$\pi = 518236479 \in \text{Im}(S)$
Example of canonical tree

\[ \pi = 518236479 \in \text{Im}(S) \]

The canonical tree \( T_\pi \) is:

\[ \begin{array}{c}
5 & 8 & 1 & 9 & 6 & 3 & 2 & 7 & 4 \\
\end{array} \]

\( \theta = 581963274 \) is such that \( S(\theta) = \pi \) and \( T_{\text{in}}(\theta) = T_\pi \).
Example of canonical tree

\[ \pi = 5 1 8 2 3 6 4 7 9 \in \text{Im}(S) \]

The canonical tree \( T_\pi \) is:

\[ \begin{align*}
9 & \quad 8 \quad 1 \\
6 & \quad 7 \\
2 & \quad 3
\end{align*} \]

\[ \theta = 5 8 1 9 6 3 2 7 4 \] is such that \( S(\theta) = \pi \) and \( T_{\text{in}}(\theta) = T_\pi \).

There are 4 other permutations in \( S^{-1}(\pi) \): those whose in-order trees are:

\[ \begin{align*}
\begin{array}{c}
5 \quad 8 \quad 1 \\
2 \quad 6 \\
3 \quad 7 \\
\end{array} & \quad \begin{array}{c}
5 \quad 8 \quad 1 \\
2 \quad 3 \\
6 \quad 7 \\
4
\end{array} \\
\begin{array}{c}
5 \quad 8 \quad 1 \\
2 \quad 6 \\
3 \quad 7 \\
\end{array} & \quad \begin{array}{c}
5 \quad 8 \quad 1 \\
2 \quad 3 \\
6 \quad 7 \\
4
\end{array}
\end{align*} \]
Example of canonical tree

\[ \pi = 5\,1\,8\,2\,3\,6\,4\,7\,9 \in \text{Im}(S) \]

The canonical tree \( T_\pi \) is:

\[ \begin{array}{c}
9 \\
8 \quad 1 \\
6 \quad 7 \quad 4
\end{array} \]

\( \theta = 5\,8\,1\,9\,6\,3\,2\,7\,4 \) is such that \( S(\theta) = \pi \) and \( T_{\text{in}}(\theta) = T_\pi \).

There are 4 other permutations in \( S^{-1}(\pi) \): those whose in-order trees are:

In particular, \( |S^{-1}(\pi)| = 5 \).
Example of canonical tree

\[ \pi = 4 \, 1 \, 8 \, 2 \, 3 \, 6 \, 5 \, 7 \, 9 \in \text{Im}(S) \]

The canonical tree \( T_{\pi} \) is:

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Example of canonical tree

\( \pi = 4 \ 1 \ 7 \ 2 \ 3 \ 6 \ 5 \ 8 \ 9 \in \text{Im}(S) \)

The canonical tree \( T_\pi \) is: 

\[
\begin{array}{c}
5 & 8 & 9 \\
1 & 6 & 7 \\
3 & 2 & .
\end{array}
\]

\( \theta = 5 \ 8 \ 1 \ 9 \ 6 \ 3 \ 2 \ 7 \ 4 \) is such that \( S(\theta) = \pi \) and \( T_{\text{in}}(\theta) = T_\pi \).

There are 4 other permutations in \( S^{-1}(\pi) \): those whose in-order trees are:

In particular, \( |S^{-1}(\pi)| = 5 \).
Example of canonical tree

\( \pi = 4 1 7 2 3 6 5 8 9 \in \text{Im}(S) \)

The canonical tree \( T_\pi \) is: 4\,\overleftrightarrow{7}\,1\,\overleftrightarrow{6}\,3\,2\,5.

\( \theta = 5 8 1 9 6 3 2 7 4 \) is such that \( S(\theta) = \pi \) and \( T_{\text{in}}(\theta) = T_\pi \).

There are 4 other permutations in \( S^{-1}(\pi) \): those whose in-order trees are:

In particular, \( |S^{-1}(\pi)| = 5 \).
Example of canonical tree

$\pi = 4\,1\,7\,2\,3\,6\,5\,8\,9 \in \text{Im}(S)$

The canonical tree $T_{\pi}$ is:

\[
\begin{array}{c}
\text{4} \\
\text{7} \\
\text{1} \\
\text{6} \\
\text{3} \\
\text{2} \\
\text{5} \\
\end{array}
\]

$\theta = 4\,7\,1\,9\,6\,3\,2\,8\,5$ is such that $S(\theta) = \pi$ and $T_{\text{in}}(\theta) = T_{\pi}$.

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Example of canonical tree

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\pi = 4 \ 1 \ 7 \ 2 \ 3 \ 6 \ 5 \ 8 \ 9 \in \text{Im}(S)
\]

The canonical tree \( T_\pi \) is:

\[
\begin{align*}
4 & \quad 7 & \quad 9 \\
1 & \quad 6 & \quad 8 & \quad 5 \\
\end{align*}
\]

\[
\theta = 4 \ 7 \ 1 \ 9 \ 6 \ 3 \ 2 \ 8 \ 5 \text{ is such that } S(\theta) = \pi \text{ and } T_{\text{in}}(\theta) = T_\pi.
\]

There are 4 other permutations in \( S^{-1}(\pi) \): those whose in-order trees are:

\[
\begin{align*}
4 & \quad 7 & \quad 9 & \quad 6 & \quad 8 & \quad 5 \\
1 & \quad 2 & \quad 3 & \quad 5 \\
\end{align*}
\]

In particular, \( |S^{-1}(\pi)| = 5 \).
Example of canonical tree

\[ \pi = 4 \ 1 \ 7 \ 2 \ 3 \ 6 \ 5 \ 8 \ 9 \in \text{Im}(S) \]

The canonical tree \( T_\pi \) is: \( \begin{tikzpicture} \end{tikzpicture} \)

\[ \theta = 4 \ 7 \ 1 \ 9 \ 6 \ 3 \ 2 \ 8 \ 5 \] is such that \( S(\theta) = \pi \) and \( T_{\text{in}}(\theta) = T_\pi \).

There are 4 other permutations in \( S^{-1}(\pi) \): those whose in-order trees are:

In particular, \( |S^{-1}(\pi)| = 5 \) is unchanged.

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Operators of equivalent sorting power and related Wilf-equivalences
Example of canonical tree

\[ \pi = 4 \; 1 \; 7 \; 2 \; 3 \; 6 \; 5 \; 8 \; 9 \in \text{Im}(S) \]

The canonical tree \( T_\pi \) is: \( 4 \begin{array}{c} 7 \\ 1 \end{array} \ 
\begin{array}{c} 9 \\ 6 \end{array} \ 
\begin{array}{c} 8 \\ 3 \end{array} \ 
\begin{array}{c} 5 \end{array} \).

\[ \theta = 4 \; 7 \; 1 \; 9 \; 6 \; 3 \; 2 \; 8 \; 5 \] is such that \( S(\theta) = \pi \) and \( T_{\text{in}}(\theta) = T_\pi \).

There are 4 other permutations in \( S^{-1}(\pi) \): those whose in-order trees are:

In particular, \( |S^{-1}(\pi)| = 5 \) is unchanged.

Conclusion: \( |S^{-1}(\pi)| \) is determined by the shape of \( T_\pi \).
Bijection $\text{Av}(231) \leftrightarrow^{P} \text{Av}(132)$
A new bijection between $\text{Av}(231)$ and $\text{Av}(132)$

Diagrams of permutations; Sum and skew sum

**Diagram** of $\sigma = 1 \ 8 \ 3 \ 6 \ 4 \ 2 \ 5 \ 7$:

- $\alpha$ a permutation of $\mathcal{S}_a$,
- $\beta$ a permutation of $\mathcal{S}_b$

- **Sum**:
  \[ \alpha \oplus \beta = \alpha (\beta + a) = \begin{array} \alpha \end{array} \begin{array} \beta \end{array} \]

- **Skew sum**:
  \[ \alpha \ominus \beta = (\alpha + b) \beta = \begin{array} \alpha \end{array} \begin{array} \beta \end{array} \]
A new bijection between $\text{Av}(231)$ and $\text{Av}(132)$

Describing permutations in $\text{Av}(231)$ and $\text{Av}(132)$

- **$\text{Av}(231) = \varepsilon + \text{Av}(231)$**
- **$\text{Av}(132) = \varepsilon + \text{Av}(132)$**

Any $\pi \neq \varepsilon \in \text{Av}(231)$ is decomposed as

$$\pi = \alpha \oplus (1 \ominus \beta)$$

with $\alpha, \beta \in \text{Av}(231)$.

Any $\pi \neq \varepsilon \in \text{Av}(132)$ is decomposed as

$$\pi = (\alpha \oplus 1) \ominus \beta$$

with $\alpha, \beta \in \text{Av}(132)$.
A new bijection between $Av(231)$ and $Av(132)$

**Bijection $P$ from $Av(231)$ to $Av(132)$**

$P$ is recursively defined as:

- If $\pi = \alpha \oplus (1 \ominus \beta)$ then $P(\pi) = (P(\alpha) \oplus 1) \ominus P(\beta)$.

  or equivalently, $\alpha \beta \rightarrow P(\alpha) P(\beta)$.

with $\alpha, \beta \in Av(231)$.

**Example:** For $\pi = 1\,5\,3\,2\,4\,9\,8\,6\,7 \in Av(231)$,

$$P(\pi) =$$

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A new bijection between $Av(231)$ and $Av(132)$

**Bijection $P$ from $Av(231)$ to $Av(132)$**

$P$ is recursively defined as:

- If $\pi = \alpha \oplus (1 \ominus \beta)$ then $P(\pi) = (P(\alpha) \oplus 1) \ominus P(\beta)$.

- or equivalently, $\begin{array}{c} \alpha \\ \beta \end{array} \xrightarrow{P} \begin{array}{c} P(\alpha) \\ P(\beta) \end{array}$.

with $\alpha, \beta \in Av(231)$.

**Example:** For $\pi = 153249867 \in Av(231)$,

$P(\pi) = 785469312$. 

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Operators of equivalent sorting power and related Wilf-equivalences
A new bijection between $Av(231)$ and $Av(132)$

**Bijection $P$ from $Av(231)$ to $Av(132)$**

$P$ is recursively defined as:

- If $\pi = \alpha \oplus (1 \ominus \beta)$ then $P(\pi) = (P(\alpha) \oplus 1) \ominus P(\beta)$.

- or equivalently, $\begin{array}{c}
\bullet \\
\beta \\
\end{array}$ \quad \xrightarrow{P} \quad \begin{array}{c}
\bullet \\
P(\alpha) \\
\end{array}$.

with $\alpha, \beta \in Av(231)$.

**Example:** For $\pi = 1\, 5\, 3\, 2\, 4\, 9\, 8\, 6\, 7 \in Av(231)$, $P(\pi) = 7\, 8\, 5\, 4\, 6\, 9\, 3\, 1\, 2$.

**Remark:** $P$ is the identity map on $Av(231, 132)$.
A new bijection between $\text{Av}(231)$ and $\text{Av}(132)$

Some properties of $P$

**Proposition:** $P$ preserves the shape of in-order trees.

**Proof:** From the recursive definition of $P$.

**Example:** For $\pi = 1\ 5\ 3\ 2\ 4\ 9\ 8\ 6\ 7$ (and $P(\pi) = 7\ 8\ 5\ 4\ 6\ 9\ 3\ 1\ 2$):

$T_{\text{in}}(\pi) = 1 \overset{5}{\quad} \overset{9}{\quad} \overset{8}{\quad} \overset{7}{\quad} \overset{3}{\quad} \overset{2}{\quad} \overset{4}{\quad} \overset{6}{\quad} \overset{1}{\quad}$

$T_{\text{in}}(P(\pi)) = 7 \overset{8}{\quad} \overset{9}{\quad} \overset{3}{\quad} \overset{6}{\quad} \overset{5}{\quad} \overset{4}{\quad} \overset{1}{\quad}$
Some properties of \( P \)

**Proposition:** \( P \) preserves the shape of in-order trees.

**Proof:** From the recursive definition of \( P \).

**Example:** For \( \pi = 1\ 5\ 3\ 2\ 4\ 9\ 8\ 6\ 7 \) (and \( P(\pi) = 7\ 8\ 5\ 4\ 6\ 9\ 3\ 1\ 2 \)):

\[
T_{in}(\pi) = 1\ 5\ 3\ 2\ 4\ 9\ 8\ 7
\]

\[
T_{in}(P(\pi)) = 7\ 8\ 5\ 4\ 6\ 9\ 3\ 1\ 2
\]

**Consequence:** \( P \) preserves the following statistics:

- number and positions of the right-to-left maxima,
- number and positions of the left-to-right maxima,
- up-down word.

**Proof:** These are determined by the shape of in-order trees.
Proof of the main result:
Some key ideas
Proof of the main result:
Some key ideas

**Theorem**

For any operator $A$ which is a composition of operators $S$ and $R$, there are as many permutations of $\mathfrak{S}_n$ sortable by $S \circ A$ as permutations of $\mathfrak{S}_n$ sortable by $S \circ R \circ A$. Moreover, many permutation statistics are equidistributed across these two sets.
Definition of $\Phi_A$

For $\pi \in \text{Av}(231)$, we may see $P(\pi) \in \text{Av}(132)$ as obtained from $\pi$ by some relabeling of $\{1, 2, \ldots, n\}$, denoted $\lambda_\pi$, i.e. $P(\pi) = \lambda_\pi \circ \pi$. 
Idea of the proof of the main result

Definition of $\Phi_A$

For $\pi \in \operatorname{Av}(231)$, we may see $P(\pi) \in \operatorname{Av}(132)$ as obtained from $\pi$ by some relabeling of $\{1, 2, \ldots, n\}$, denoted $\lambda_\pi$, i.e. $P(\pi) = \lambda_\pi \circ \pi$.

Definition:

- Take $\theta$ a permutation sortable by $S \circ A$.
- Set $\pi = A(\theta)$. $\pi \in \operatorname{Av}(231)$.
- Consider $\lambda_\pi$ such that $P(\pi) = \lambda_\pi \circ \pi$.
- Define $\Phi_A(\theta) = \lambda_\pi \circ \theta$. 

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Operators of equivalent sorting power and related Wilf-equivalences
### Definition of $\Phi_A$

For $\pi \in \text{Av}(231)$, we may see $P(\pi) \in \text{Av}(132)$ as obtained from $\pi$ by some relabeling of $\{1, 2, \ldots, n\}$, denoted $\lambda_\pi$, i.e. $P(\pi) = \lambda_\pi \circ \pi$.

**Definition:**

- Take $\theta$ a permutation sortable by $S \circ A$.
- Set $\pi = A(\theta)$. $\pi \in \text{Av}(231)$.
- Consider $\lambda_\pi$ such that $P(\pi) = \lambda_\pi \circ \pi$.
- Define $\Phi_A(\theta) = \lambda_\pi \circ \theta$.

**Theorem:** $\Phi_A$ is a bijection between the set of permutation sortable by $S \circ A$ and the set of those sortable by $S \circ R \circ A$. 

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Idea of the proof of the main result

Proving that $\Phi_A$ is a bijection

**Definition:** $A$ respects $P$ if, for all $\pi \in \text{Av}(231) \cap \text{Im}(A)$:

- For each $\theta$ such that $A(\theta) = \pi$, we have $A(\Phi_A(\theta)) = P(\pi) = \lambda_{\pi} \circ \pi$
- some condition (??) on canonical trees...
Idea of the proof of the main result

Proving that $\Phi_A$ is a bijection

**Definition:** $A$ respects $P$ if, for all $\pi \in \text{Av}(231) \cap \text{Im}(A)$:

- For each $\theta$ such that $A(\theta) = \pi$, we have $A(\Phi_A(\theta)) = P(\pi) = \lambda_\pi \circ \pi$ and $T_{\text{in}}(\Phi_A(\theta)) = \lambda_\pi(T_{\text{in}}(\theta))$,
- the correspondence $\Phi_A : \theta \mapsto \Phi_A(\theta)$ is a bijection between $A^{-1}(\pi)$ and $A^{-1}(P(\pi))$. 
Proving that $\Phi_A$ is a bijection

**Definition**: $A$ respects $P$ if, for all $\pi \in \text{Av}(231) \cap \text{Im}(A)$:

- For each $\theta$ such that $A(\theta) = \pi$, we have $A(\Phi_A(\theta)) = P(\pi) = \lambda_\pi \circ \pi$ and $T_{\text{in}}(\Phi_A(\theta)) = \lambda_\pi(T_{\text{in}}(\theta))$,
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**Proposition**: The identity operator respects $P$.

**Proposition**: If $A$ respects $P$ then so does $A \circ R$.

**Proposition**: If $A$ respects $P$ then so does $A \circ S$. 

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Operators of equivalent sorting power and related Wilf-equivalences
Definitions

Preimages under $S$

$P : \text{Av}(231) \leftrightarrow \text{Av}(132)$

Proof of main result

Idea of the proof of the main result

Proving that $\Phi_A$ is a bijection

**Definition:** $A$ respects $P$ if, for all $\pi \in \text{Av}(231) \cap \text{Im}(A)$:

- For each $\theta$ such that $A(\theta) = \pi$, we have $A(\Phi_A(\theta)) = P(\pi) = \lambda_\pi \circ \pi$ and $T_{\text{in}}(\Phi_A(\theta)) = \lambda_\pi(T_{\text{in}}(\theta))$,
- the correspondence $\Phi_A : \theta \mapsto \Phi_A(\theta)$ is a bijection between $A^{-1}(\pi)$ and $A^{-1}(P(\pi))$.

**Proposition:** The identity operator respects $P$.

**Proposition:** If $A$ respects $P$ then so does $A \circ R$.

**Proposition:** If $A$ respects $P$ then so does $A \circ S$.

**Theorem:** Every operator $A$ respects $P$.

**Consequence:** $\Phi_A$ is a bijection between the set of permutations sortable by $S \circ A$ and those sortable by $S \circ R \circ A$. 

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Operators of equivalent sorting power and related Wilf-equivalences
Statistics preserved by $\Phi_A$

Theorem: $\Phi_A$ preserves the shape of in-order trees.

Consequence: $\Phi_A$ preserves the following statistics:
- number and positions of the right-to-left maxima,
- number and positions of the left-to-right maxima,
- up-down word.

Other statistics preserved:
- Zeilberger’s statistics when $A = A_0 \circ S$:
  $\text{zeil}(\theta) = \max\{k \mid n(n-1) \ldots (n-k+1) \text{ is a subword of } \theta\}$
- the reversed Zeilberger’s statistics when $A = A_0 \circ S$ and $A_0$ contains at least a composition $S \circ R$:
  $R\text{zeil}(\theta) = \max\{k \mid (n-k+1) \ldots (n-1)n \text{ is a subword of } \theta\}$
Who is $\Phi_S$?

- $\Phi_S$ provides a bijection between the set of permutations sortable by $S \circ S$ and those sortable by $S \circ R \circ S$.
- With O. Guibert, we gave a common generating tree for those two sets, providing a bijection between them.

Problem

*Are these two bijections the same one?*

It is not as easy as it seems...
More about the bijection $\text{Av}(231) \overset{P}{\leftrightarrow} \text{Av}(132)$

Related Wilf-equivalences

... Next talk!