Graphon limit of random cographs

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talk based on joint work with
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Mickaël Maazoun and Adeline Pierrot

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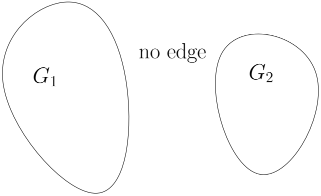
We have to start somewhere:
Setting the problem

Cographs

Definition: A cograph of size n is a graph $G = (V, E)$ with $|V| = n$ which can be constructed from graphs with one vertex by taking disjoint unions and joins.



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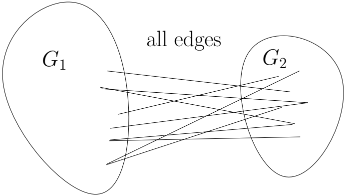

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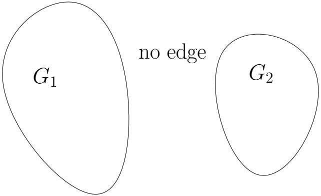

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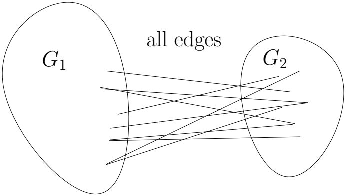

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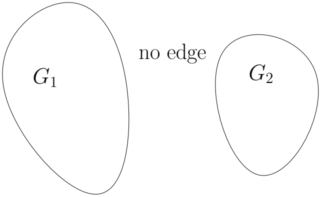

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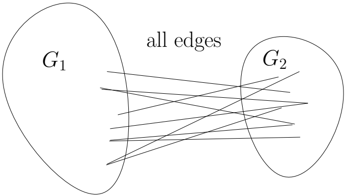

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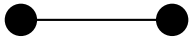

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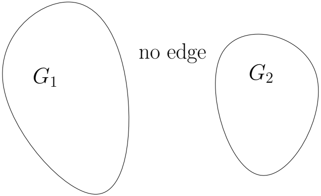

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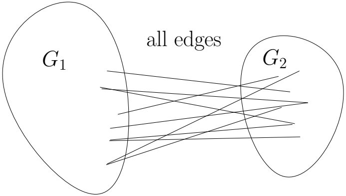

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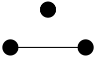

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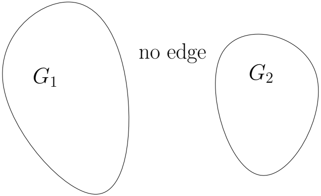

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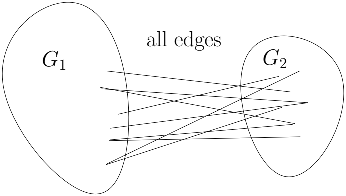

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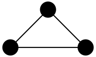

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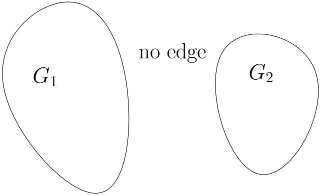

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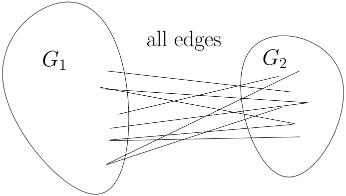

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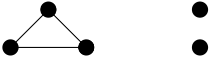

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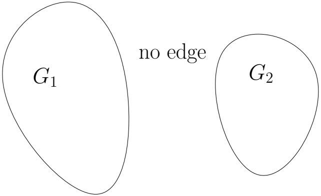

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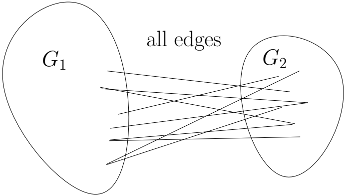

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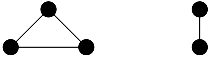

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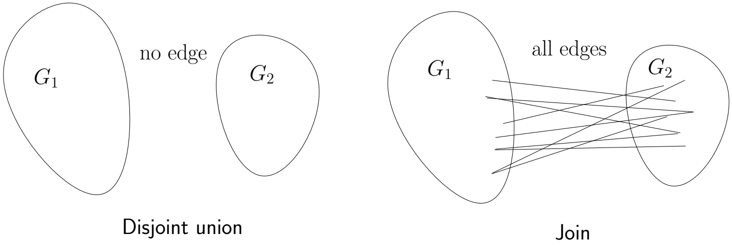

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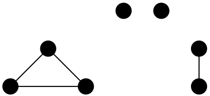

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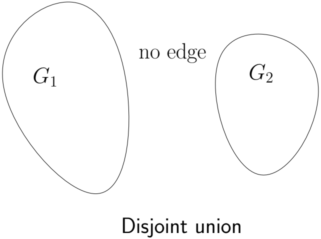

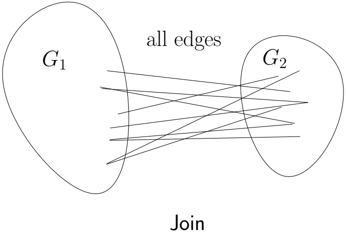

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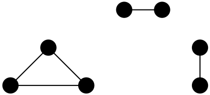

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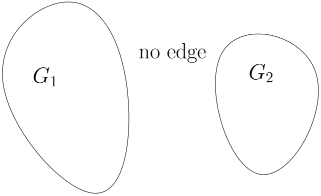

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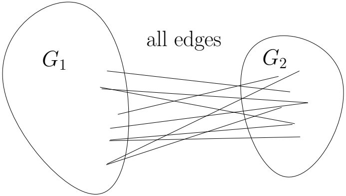

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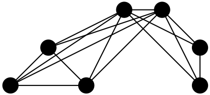

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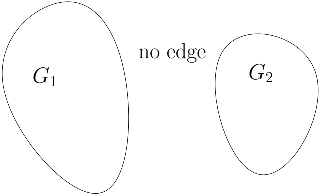

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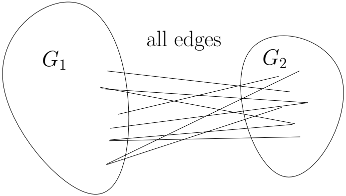

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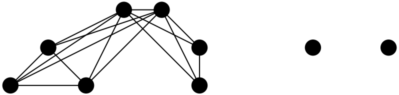

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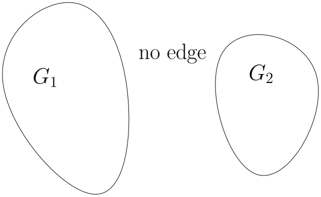

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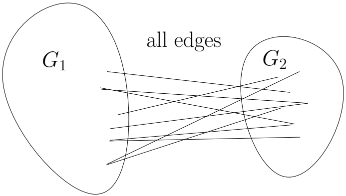

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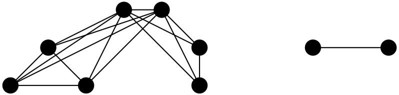

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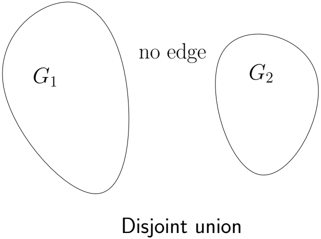

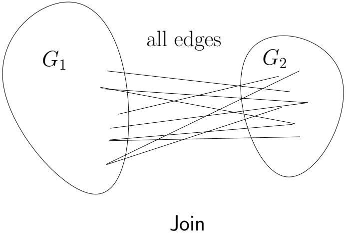

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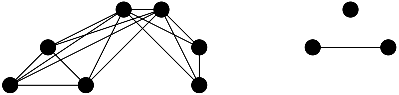

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Central question for this talk:

What does a uniform random cograph of size n look like, when n goes to infinity?

Cographs: one of many cases

One question studied in the (huge!) random graph literature is the following:

> For $\mathcal{F}$ a family of graphs,
> what is the typical behavior of a large graph in $\mathcal{F}$?

Already studied for:

- perfect graphs [McDiarmid-Yolov, 2019]
- planar graphs [Noy, 2014]
- graphs embeddable in a surface of given genus [Dowden-Kang-Sprüssel, 2017]
- graphs in subcritical classes [Panagiotou-Stufler-Weller, 2016]
- large hereditary classes [Hatami-Janson-Szegedy, 2018]
- addable classes [McDiarmid-Steger-Welsh, 2006 ; Chapuy-Perarnau, 2019]

Which model?

Which discrete objects? Graphs may be

- labeled: in this case, vertices are numbered from 1 to n;
- or unlabeled: vertices are indistinguishable.

Unlabeled graphs are equivalence classes of labeled graphs under the action of relabeling the vertices.

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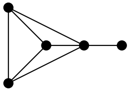
Which continuous limit? We describe the limit in the space of graphons.
Graphons were introduced by Lovász and co-authors in 2008, and attracted a lot of interest.
Graphons are appropriate to describe limits of dense graphs.

Some basics on graphons

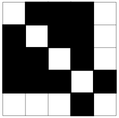
What is (informally) a graphon?

In the discrete setting:

(Unlabeled) graph G $\longrightarrow$ Adjacency matrix M_G (symmetric) $\longrightarrow$ Function $w_G : [0,1]^2 \to [0,1]$



$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$



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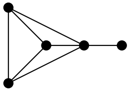

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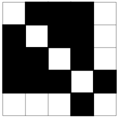

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Remarks:

- W_G does not depend on the order of the vertices chosen to write M_G.
- If G is labeled, W_G is the graphon of the unlabeled version of G.

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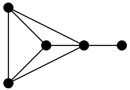
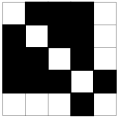
		$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$		

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Continuous extension:

In general, a graphon is obtained as above, from a symmetric matrix M, possibly with a continuum of rows and columns, and with values in $[0,1]$.

It is an equivalence class of symmetric functions from $[0,1]^2 \to [0,1]$ under the action of permuting rows and columns of M.

Characterization of (deterministic) graphon convergence

(Non-)definition:
The space of graphons is (up to technicalities) metric, for the cut-distance (and in addition is compact).

So, it makes sense to study convergence of a sequence of graphons $(W_n)_{n\geq 0}$ to a graphon W (for this cut-distance). We write $W_n \to W$.

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Typically, $W_n = W_{G_n}$, the graphon associated to a graph G_n, with the sequence of graphs (G_n) such that the size of G_n grows to infinity with n. In this case, we also write $G_n \to W$.

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Combinatorial characterization of convergence:
For (W_n) a sequence of graphons and W a graphon, $W_n \to W$ iff for any (finite) graph g, $\mathrm{Dens}(g, W_n) \to \mathrm{Dens}(g, W)$.

Let us now define the density of a graph g in a graphon.

Subgraph densities in graphs and graphons

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Densities: Fix g a graph with k vertices, unlabeled.

- For a graph G, $\mathrm{Dens}(g, G) = \mathbb{P}(\mathrm{SubGraph}_k(G) = g)$, where $\mathrm{SubGraph}_k(G)$ is the (random) subgraph of G induced by a k-tuple of i.i.d. uniform random vertices of G.

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Remark: For any graph G, $\text{Dens}(g, W_G) = \text{Dens}(g, G)$.

Characterization of graphon convergence: the random case

Reminder: $G_n \to W$ iff $\mathrm{Dens}(g, G_n) \to \mathrm{Dens}(g, W)$ for all g, for (G_n) a sequence of (deterministic) graphs and W a (deterministic) graphon.

Characterization of graphon convergence: the random case

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What if we take $(\boldsymbol{G_n})$ random? ($\text{Dens}(g, \boldsymbol{G_n})$ being then a real r.v.)

Theorem [Diaconis-Janson, 2008]:
The distribution of a random graphon $\boldsymbol{W}$ is characterized by all expected subgraph densities $\mathbb{E}[\text{Dens}(g, \boldsymbol{W})]$ (for all g).

Theorem [Diaconis-Janson, 2008]:
Let $(\boldsymbol{G_n})$ be a sequence of random graphs. TFAE:

- $\boldsymbol{G_n}$ tends in distribution to some random graphon, $\boldsymbol{W}$.
- For all g, $\mathbb{E}[\text{Dens}(g, \boldsymbol{G_n})]$ converges to some value $\Delta_g \in [0, 1]$.

If this holds, in addition we have:
for all g, $\mathbb{E}[\text{Dens}(g, \boldsymbol{W})] = \Delta_g$, so that $(\Delta_g)_g$ characterizes $\boldsymbol{W}$.

Our work in a nutshell

Main result and proof strategy

Theorem:
For all n, let $\boldsymbol{G_n}$ (resp. $\boldsymbol{G_n^u}$) be a uniform random labeled (resp. unlabeled) cograph with n vertices.
We have that $\boldsymbol{G_n}$ (resp. $\boldsymbol{G_n^u}$) converges in distribution to a random graphon $\boldsymbol{W}^{1/2}$ called the Brownian cographon of parameter $1/2$.

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Proof strategy (labeled case):

- Compute $\Delta_g = \mathbb{E}[\mathrm{Dens}(g, \boldsymbol{W}^{1/2})]$ for all cographs g
- Express $\mathbb{E}[\mathrm{Dens}(g, \boldsymbol{G}_n)]$ as a quotient of coefficients of generating functions, starting from
$$\mathbb{E}[\mathrm{Dens}(g, \boldsymbol{G}_n)] = \frac{\left|\left\{(G, I) : \begin{array}{c} G=(V,E) \text{ labeled cograph of size } n, \\ I \in V^k \text{ which induces } g \end{array}\right\}\right|}{|\{G \text{ labeled cograph of size } n\}| \cdot n^k}$$
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Essential tool: encoding of cographs by cotrees.

Outline of (the rest of) the talk

About the main theorem:

- Cographs and cotrees
- Combinatorial proof of convergence in the labeled case
- Description of the Brownian cographon
- Corollary: average degree distribution in cographs
- How to deal with the unlabeled case

Additional results, questions, comments:

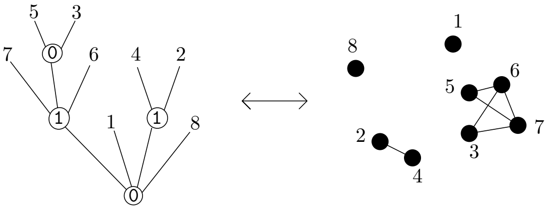
- Vertex connectivity distinguishes between the labeled and the unlabeled settings
- A parallel with permutations, yielding new problems to work on
- Independence number of cographs

Cotrees and how to use them to compute $\lim_{n\to\infty} \mathbb{E}[\mathrm{Dens}(g, G_n)]$

Cographs and cotrees

A labeled cotree of size n is a rooted tree t with leaves $\{1, \ldots, n\}$ s.t.

- t is not plane (*i.e.* the children of every internal node are not ordered);
- every internal node has at least two children;
- every internal node carries a decoration 0 or 1.



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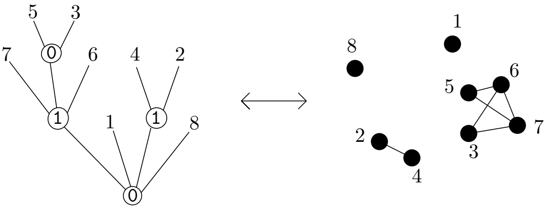

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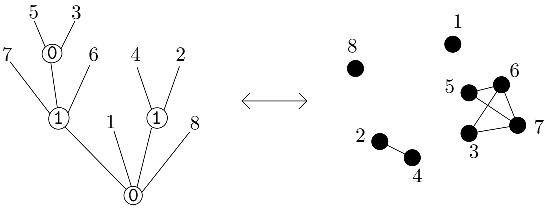

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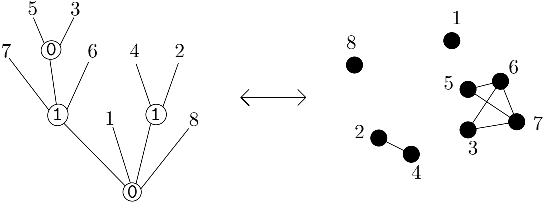
t is canonical if 0 and 1 alternate on every branch from the root to a leaf.



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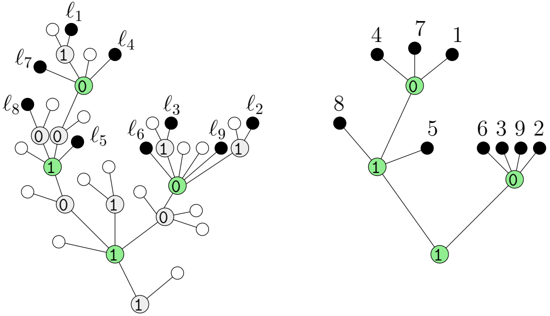
Prop.: This mapping restricted to canonical cotrees is a bijection.

t a canonical cotree	$\leftrightarrow$	G the corresponding cograph
a k-tuple $\boldsymbol{\ell} = (\ell_1, \dots, \ell_k)$ of leaves	$\leftrightarrow$	a k-tuple I of vertices

Subtree of t induced by $(\ell_1, \dots, \ell_k)$ = the cotree **labeled from** $\boldsymbol{\ell}$ whose leaves are $(\ell_1, \dots, \ell_k)$ and whose internal structure is inherited from t.



Induced subgraphs in cographs on their cotrees

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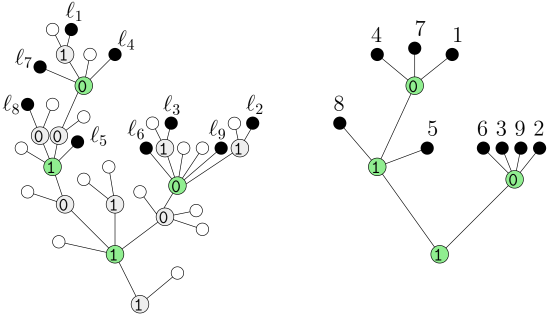

Prop.: Forgetting the labelings, the subgraph of G induced by I is the cograph corresponding to the subtree of t induced by $(\ell_1, \dots, \ell_k)$

Variations on $\mathbb{E}[\mathrm{Dens}(g, \boldsymbol{G_n})]$

- **Reminder**: $\mathrm{Dens}(g, G) = \mathbb{P}(\mathrm{SubGraph}_k(G) = g)$, where $\mathrm{SubGraph}_k(G)$ is the (random) subgraph of G induced by a k-tuple of i.i.d. uniform random vertices of G.

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Notation: for all n, and all $k \leq n$,
$\boldsymbol{t}^{(n)}$ is a uniform random labeled canonical cotree of size n, and
$\boldsymbol{t}_k^{(n)}$ is the subtree of $\boldsymbol{t}^{(n)}$ induced by a uniform k-tuple of distinct leaves.

For any cograph g, we have:

$$\mathbb{E}[\mathrm{Dens}^{inj}(g, \boldsymbol{G}_n)] = \mathbb{P}(\mathrm{SubGraph}_k^{inj}(\boldsymbol{G}_n) = g) = \sum \mathbb{P}(\boldsymbol{t}_k^{(n)} = t_0),$$

where the sum runs over all cotrees t_0 corresponding to g.

Combinatorics of the labeled case: Finding $\lim_{n\to\infty} \mathbb{P}(t_k^{(n)} = t_0)$

Expressing $\mathbb{P}(\boldsymbol{t}_k^{(n)} = t_0)$

Notation:

- $\mathcal{M}$: the set of labeled canonical cotrees
- for any cotree t_0 with k leaves, $\mathcal{M}_{t_0}$: the set of labeled canonical cotrees with a marked k-tuple of distinct leaves, which induce t_0.

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Estimate the limit as $n \to \infty$ using analytic combinatorics,

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- $\mathcal{L}$: the set of non-plane rooted trees, labeled on their leaves, where internal nodes have ≥ 2 children.
  Trees of $\mathcal{L}$ are just like cotrees without the decorations on internal nodes.

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Estimate the limit as $n \to \infty$ using analytic combinatorics,
on $L(z)$ and variants, relating $M(z)$ and $M_{t_0}(z)$ to $L(z)$

Behavior of $L(z)$ and $M(z)$

Study of $L(z)$:
From [Flajolet-Sedgewick] (rather a variant on trees counted by leaves):

- $L(z)$ satisfies $L(z) = z + \exp(L(z)) - 1 - L(z)$.

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- Near $z = \rho$, $M(z) = 1 - 2\sqrt{\rho}\sqrt{1 - z/\rho} + \mathcal{O}(1 - z/\rho)$.
- From the transfer theorem,

$$n(n-1)\dots(n-k+1)[z^n]M(z) \underset{n\to+\infty}{\sim} \frac{n^{k-3/2}}{\rho^{n-1/2}\sqrt{\pi}}.$$

Relating $M_{t_0}(z)$ to variations on $L(z)$ (1/2)

Recall: Trees of $\mathcal{M}_{t_0}$ are trees of $\mathcal{L}$ with k marked leaves inducing t_0, and in addition a decoration on the root

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Thus, the singular behavior of $L(z)$ determines the one of these four series.

Prop.: If t_0 with k leaves has n_v internal vertices, $n_=$ edges of the form $0-0$ or $1-1$, and $n_{\neq}$ edges of the form $0-1$ or $1-0$, then

$$M_{t_0} = (L')(\exp(L))^{n_v}(L^{\bullet})^k(L^{\text{odd}})^{n_=}(L^{\text{even}})^{n_{\neq}}.$$

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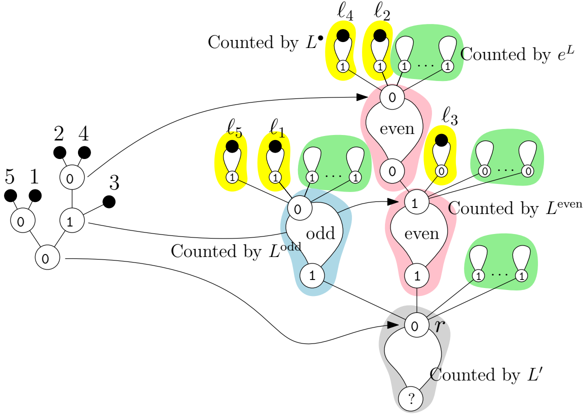

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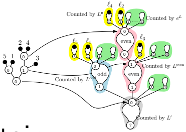

Corollary: Like before, we obtain

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Relating $M_{t_0}(z)$ to variations on $L(z)$ (2/2)

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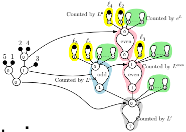

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More precisely, we have

$$[z^n]M_{t_0}(z) \underset{n\to+\infty}{\sim} \frac{(k-1)!}{(2k-2)!}\frac{n^{k-3/2}}{\rho^{n-1/2}\sqrt{\pi}},$$

if t_0 is binary (which implies $n_v = k-1$ and $n_= + n_{\neq} = k-2$).

Conclusion of the combinatorial study (labeled case)

Notation (reminder):

- $\boldsymbol{t}^{(n)}$: uniform random labeled canonical cotree of size n
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What we proved: If t_0 is binary, then $\lim_{n\to\infty} \mathbb{P}(\boldsymbol{t}_k^{(n)} = t_0) = \frac{(k-1)!}{(2k-2)!}$.

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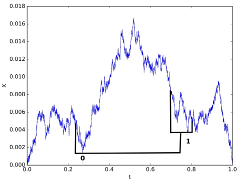
Consequence: If t_0 is not binary, then $\lim_{n\to\infty} \mathbb{P}(\boldsymbol{t}_k^{(n)} = t_0) = 0$.

Remark/reminder:
Summing over all t_0 encoding a cograph g, this gives $\lim_{n\to\infty} \mathbb{E}[\mathrm{Dens}(g, \boldsymbol{G}_n)]$.

The Brownian cographon and its expected subgraph densities (or something close to it)

Defining the Brownian cographon

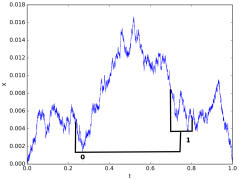
Decorated Brownian excursion:



- $\mathbf{e}$: Brownian excursion of length 1.
- $(\boldsymbol{b}_i)_{i\geq 1}$: enumeration of the locations of the local minima of $\mathbf{e}$ (which exists).
- $\boldsymbol{S}^p = (\boldsymbol{s}_1, \dots)$: sequence of i.i.d. r.v. in $\{0, 1\}$, independent from $\mathbf{e}$, with $\mathbb{P}(\boldsymbol{s}_1 = 0) = p$.

In the decorated Brownian excursion $(\mathbf{e}, \boldsymbol{S}^p)$, we think of the decorations $\boldsymbol{s}_i$ as attached to the local minimum at $\boldsymbol{b}_i$.

Defining the Brownian cographon



Decorated Brownian excursion:

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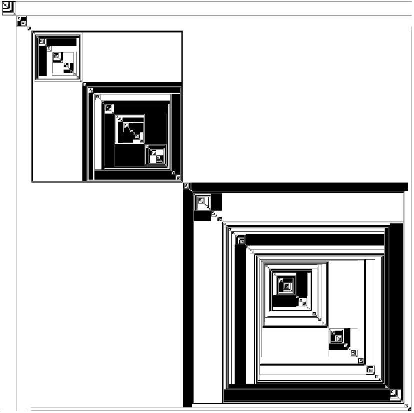
Brownian cographon of parameter $p \in [0, 1]$, $\boldsymbol{W}^p$:

- for any $x, y \in [0, 1]$, $\mathrm{Dec}(x, y; \mathbf{e}, \boldsymbol{S}^p) \in \{0, 1\}$ = decoration of the local minimum of $\mathbf{e}$ on $[x, y]$ (or $[y, x]$) (a.s. unique and $\neq x, y$)
- $\boldsymbol{W}^p$ = graphon associated with the function

$$\begin{array}{rccc} \boldsymbol{w}^p : & [0,1]^2 & \to & \{0,1\}; \\ & (x,y) & \mapsto & \mathrm{Dec}(x, y; \mathbf{e}, \boldsymbol{S}^p). \end{array}$$

“Here is” $W^{1/2}$



This is actually the adjacency matrix of a uniform random labeled cograph of size 4482, where the order of the vertices to plot the matrix is the depth-first search on the associated cotree.

Distribution of induced subgraphs of $\boldsymbol{W}^p$

Notation:

- $\boldsymbol{W}^p$: Brownian cographon of parameter p
- $\mathrm{Sample}_k(W)$: subgraph of W induced by k i.i.d. uniform "vertices" $x_1, \ldots, x_k \in [0,1]$
- $\boldsymbol{b}_k^p$: uniform labeled binary tree with k leaves, where internal vertices carry $\{0,1\}$ decorations with $\mathbb{P}(0) = p$.

Prop.: $\mathrm{Sample}_k(\boldsymbol{W}^p) \stackrel{(d)}{=}$ the unlabeled version of $\mathrm{Cograph}(\boldsymbol{b}_k^p)$.

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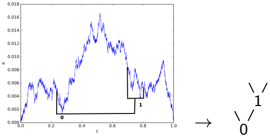
Proof idea:

- $\boldsymbol{b}_k^p$ is the cotree extracted from $(\mathbf{e}, \boldsymbol{S}^p)$ and $x_1, \dots, x_k$.



- $\text{Sample}_k(\boldsymbol{W}^p)$ is the associated cograph since decorations indicate edges similarly in $\boldsymbol{W}^p$ and in $\text{Cograph}(\boldsymbol{b}_k^p)$.

Characterization of convergence to $\boldsymbol{W}^{1/2}$

Prop.: For $(\boldsymbol{t}^{(n)})_n$ a sequence of random cotrees s.t. $\text{size}(\boldsymbol{t}^{(n)}) = n$, let $\boldsymbol{t}_k^{(n)}$ be the subtree of $\boldsymbol{t}^{(n)}$ induced by a unif. k-tuple of distinct leaves.
If for any binary cotree t_0 we have $\mathbb{P}(\boldsymbol{t}_k^{(n)} = t_0) \xrightarrow[n\to\infty]{} \frac{(k-1)!}{(2k-2)!}$, $(\star)$
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- $(\star)$ says $\boldsymbol{t}_k^{(n)}$ is asymptotically uniform on labeled binary cotrees with k leaves, which is distributed like $\boldsymbol{b}_k^{1/2}$
- Take cographs and forget labels
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Corollary: $(\boldsymbol{G}_n)$ converges to $\boldsymbol{W}^{1/2}$.
Uniform random labeled cographs converge to the Brownian cographon.
(apply the prop. to $\boldsymbol{t}^{(n)}$ = unif. random canonical labeled cotree of size n)

Additional results

Average degree distribution

Degree distribution of graphs and graphons:

- (Rescaled) degree distribution of G: $D_G = \frac{1}{n} \sum_{v \text{ vertex}} \delta_{\deg(v)/n}$
- It generalizes to graphons: for w representing W, D_W is defined by $\int_{[0,1]} f(x) D_W(dx) = \int_{[0,1]} f\left(\int_{[0,1]} w(u,v)dv\right) du, \forall f$ cont. bounded
- D_G and D_W are probability measures on $[0,1]$

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Prop.: For the Brownian cographon, $I[D_{\boldsymbol{W}^{1/2}}]$ is uniform on $[0,1]$.

Corollary: The rescaled degree of a uniform random vertex $\boldsymbol{v}_n$ in $\boldsymbol{G}_n$ is asymptotically uniform in $[0,1]$.

The unlabeled case

Same results:

- *Definition*: $\boldsymbol{G}_n^u =$ uniform random unlabeled cograph with n vertices
- *Theorem*: $(\boldsymbol{G}_n^u)_n$ converges to the Brownian cographon $\boldsymbol{W}^{1/2}$
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How to modify the proof:

- Same strategy of analytic combinatorics, using unlabeled cotrees.
- With Pólya operators, it is difficult to count objects with marked leaves (inducing a given subtree t_0).
- Instead of $\mathcal{L}$ as before, we study

$$\mathcal{U} = \{(t,a) : t \in \mathcal{L}, a \text{ a root-preserving automorphism of } t\}.$$

  Using $\mathcal{U}$, we can interpret Pólya operators combinatorially, in a way that allows to keep track of marked leaves.

Vertex connectivity

Remark: For their graphon limit (and average degree distribution), labeled and unlabeled cographs display the same behavior.

Question: Are there some statistics which behave differently in the labeled and unlabeled case?

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Example of the vertex connectivity:

- $\kappa(G)$ = minimal number of vertices whose removal disconnects G
- For a connected cograph G with canonical cotree T (with root 1), $\kappa(G) = |G| - |T_{max}|$, where T_{max} is the largest component of T
- Using again analytic combinatorics, we express, for all $j \geq 1$,
  $\lim_{n\to\infty} \mathbb{P}(\kappa(\boldsymbol{G_n}) = j)$ using $L(z)$ as $1/2 \cdot \rho_L^j \, [z^j] \, (e^{L(z)} - 1)$
  $\lim_{n\to\infty} \mathbb{P}(\kappa(\boldsymbol{G_n^u}) = j)$ using $U(z)$ as $1/2 \cdot \rho_U^j \, [z^j] \, (2U(z) - z)$
  (the limiting probability of having $\kappa(\boldsymbol{G_n})$ or $\kappa(\boldsymbol{G_n^u}) = 0$ being $1/2$).
- These limit distributions are different.

A parallel with permutations *via* inversion graphs

Separable permutations

- encoding by decomposition trees
- convergence to the Brownian separable permuton (BSP)
- [BBFGP18, Maazoun16, BBFS20]

Cographs

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- [this talk, Stufler21]

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Classes of graphs closed for the substitution operation of the modular decomposition

- encoding by modular decomposition trees
- Expected universality of the BCG for "small" classes

Beyond these families

- Other graph decompositions?

Independence number and longest increasing subsequences

see Arxiv:2104.07444, by the same group + Michael Drmota

Results:

- The size of the largest independent set of a uniform random cograph is sublinear.
  (hence P_4 does not have the asymptotic linear Erdős-Hajnal property)
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Thank you for being there!