First-order logic for permutations

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talk based on joint work with M. Albert and V. Féray

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80th Séminaire Lotharingien de Combinatoire, Lyon, March 2018

What is a permutation (of size n)?

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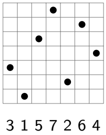

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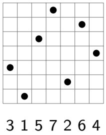
Ex.: $\sigma = (1,3,5,2)(4,7)(6) = 3\ 1\ 5\ 7\ 2\ 6\ 4 =$ 

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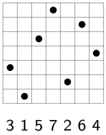
Ex.: $\sigma = (1, 3, 5, 2)(4, 7)(6) = 3\ 1\ 5\ 7\ 2\ 6\ 4 =$ 

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Goal: Give a "proof" that the two points of view are hardly reconciled.

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To prove that the two points of view are essentially different, we study the expressivity of the theories:

- describe properties expressible in each theory,
- show that the properties expressible in both theories are trivial.

Two logics for permutations

TOOB: models

TOOB: the Theory Of One Bijection *(already appeared in the literature)*

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Permutations are models, and every model is a permutation.
(Possibly, up to a conjugating by a bijection between X and $\{1, 2, \ldots, n\}$.)

The relation R_σ associated to σ of size n is given by:

$$i \; R_\sigma \; \sigma(i) \text{ for all } i \leq n$$

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A model of a sentence ψ is a model which in addition satisfies ψ.

Ex.: The models of $\exists x\, xRx$ are the permutations having a fixed point.

TOOB: expressivity

A property of permutations is expressible in a theory (here, TOOB) if it can be described by a sentence, *i.e.*, there is a sentence whose models are exactly the permutations for which this property holds.

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In TOOB, properties about the cycle decomposition of a permutation are expressible.

Thm.: If $\sigma \models \psi$, then for any τ in the conjugacy class of σ, $\tau \models \psi$.

In other words, TOOB does not distinguish between conjugate permutations.

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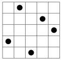
  - Ex.: $\sigma =$  25143 is represented for instance by $(\{a, b, c, d, e\}, \lhd, \blacktriangleleft)$

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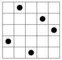
  - Ex.: $\sigma =$  2 5 1 4 3 is represented for instance by $(\{a, b, c, d, e\}, \lhd, \blacktriangleleft)$

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Summary of differences:

- TOOB speaks about the cycle structure but the total order on $\{1, 2, \ldots, n\}$ is lost.
- TOTO speaks about the relative order of the elements, but the cycle structure is lost.

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Some concepts expressible in TOTO:

- Containment/avoidance of a classical pattern;

  Ex.: Containment of 231 is expressed by the sentence

$$\exists x \, \exists y \, \exists z \quad (x <_P y <_P z) \quad \wedge \quad (z <_V x <_V y)$$

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- Being simple;
- Being West-k-stack sortable, for any k (+ construction of the corresponding sentences)

Inexpressibility results in TOTO

Inexpressibility of fixed points

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Proof strategy:

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- Exhibit two permutations σ and σ' such that
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To show that two permutations satisfy the same sentences, use the Ehrenfeucht-Fraïssé Theorem:

> Two permutations σ and σ' satisfy the same sentences of quantifier depth at most k if and only if Duplicator wins the EF-game with k rounds on σ and σ'.

EF-games (a.k.a. Duplicator-Spoiler games)

The setting:

- Two players: Duplicator (D) and Spoiler (S).
- They play on a pair of permutations σ and σ'.
- Goal of D: show that σ and σ' cannot be distinguish in k rounds.
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Winner of the EF-game with k rounds:

- D if $\mathbf{s} = (s_1, \dots, s_k)$ and $\mathbf{s}' = (s'_1, \dots, s'_k)$ are isomorphic, *i.e.*, if the position- and value-orders on $\mathbf{s}$ and $\mathbf{s}'$ are identical;
- S otherwise.

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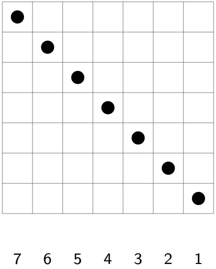

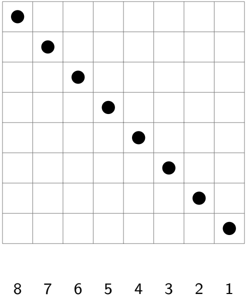

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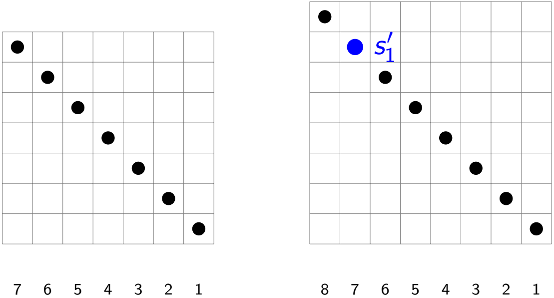

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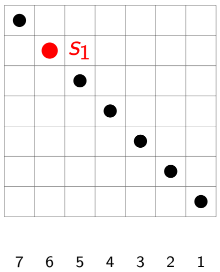

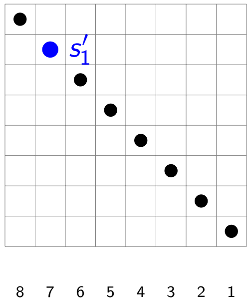

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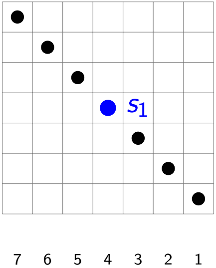

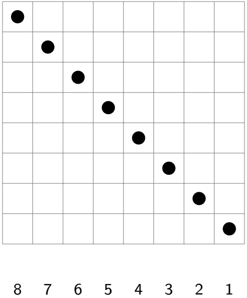

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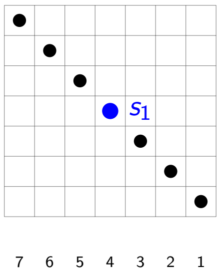

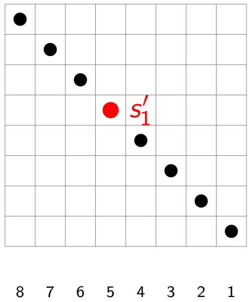

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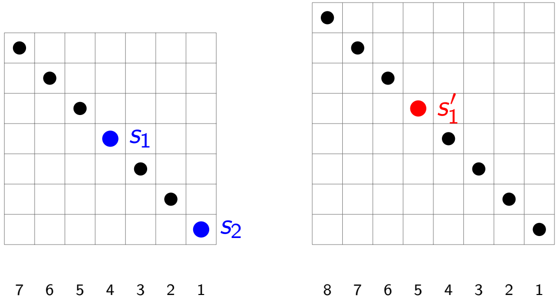

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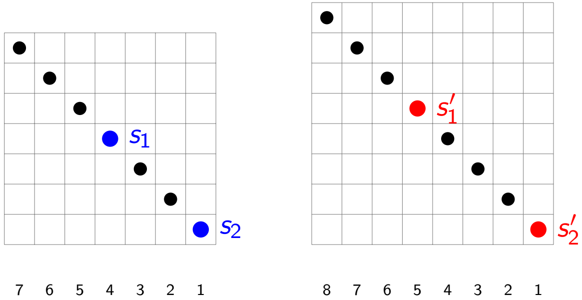

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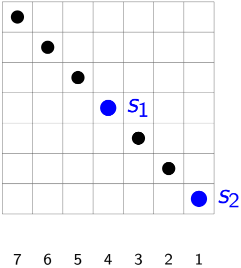

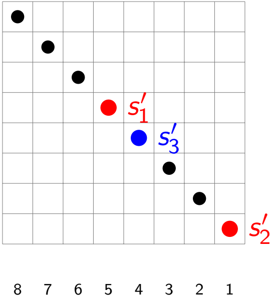

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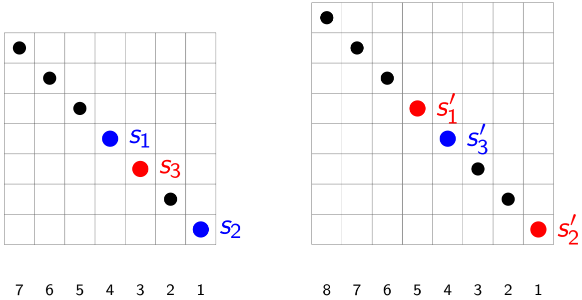

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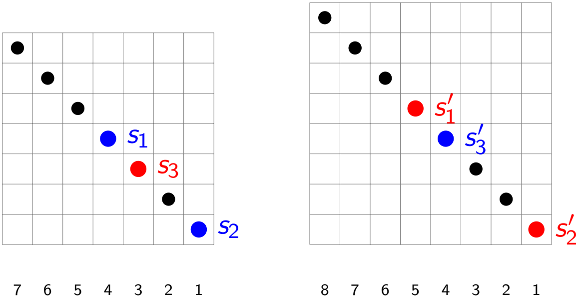

S and D alternate turns. After 3 rounds, D wins!

Intersection of TOTO **and** TOOB

Properties expressible in one/both theories

Examples of properties expressible in one of TOOB and TOTO only:

- Having a fixed point: expressible in TOOB but not in TOTO;
- Containing a 231-pattern: expressible in TOTO but not in TOOB.
  (TOOB does not distinguish between $231 = (1,2,3)$ and $312 = (1,3,2)$)

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Which properties are expressible in both TOOB and TOTO?

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Which properties are expressible in both TOOB and TOTO?

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A possible proof uses EF-games.

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Rk.: In addition, we have a complete characterization of the properties expressible in both theories.

Some other things we know (or not)

- Characterization of the permutation classes $\mathcal{C}$ such that "having a fixed point" is expressible in the restriction of TOTO to $\mathcal{C}$.

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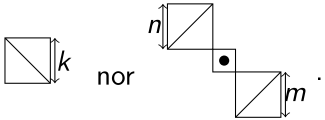

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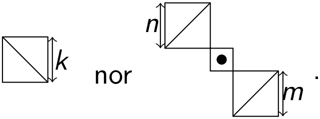

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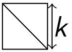

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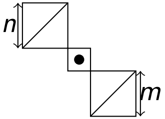

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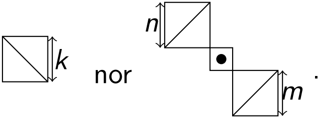

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- Further project with M. Noy: Prove convergence laws in permutation classes (for properties expressible in TOTO).