Studying permutation classes using the substitution decomposition

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Combinatorics and interactions,
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Permutation patterns
and permutation classes

Permutations

Permutation of size n = Bijection from $[1..n]$ to itself.
Set $\mathfrak{S}_n$, and $\mathfrak{S} = \bigcup_n \mathfrak{S}_n$.

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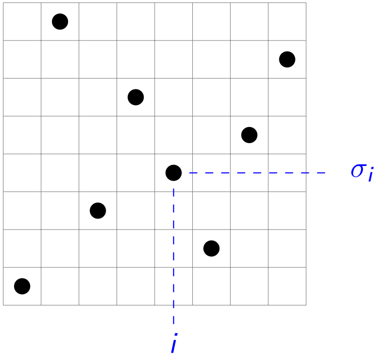
- One-line or word notation:
  $\sigma = 1\ 8\ 3\ 6\ 4\ 2\ 5\ 7$

- Description as a product of cycles:
  $\sigma = (1)\ (2\ 8\ 7\ 5\ 4\ 6)\ (3)$

- Graphical description, or diagram:



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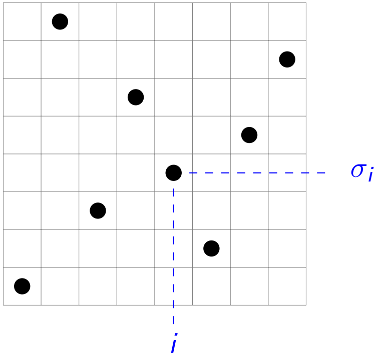

This talk is about **permutation patterns** and **permutation classes**.

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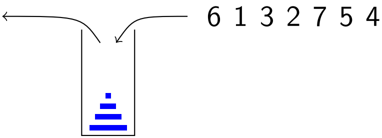
Sort (or try to do so) using a stack satisfying the Hanoi condition.



The origin of permutation patterns: Stack sorting

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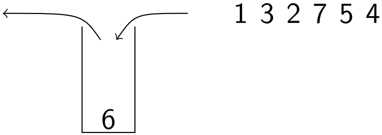

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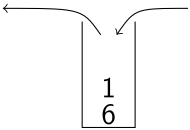

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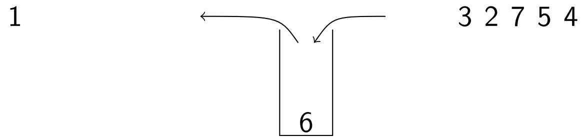

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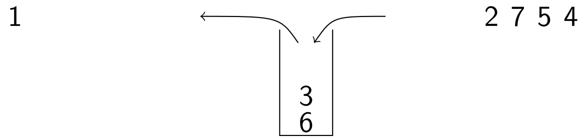

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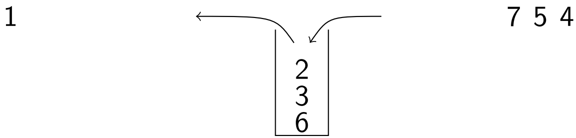

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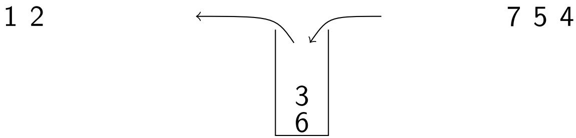

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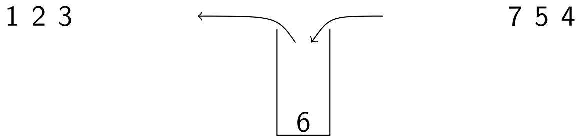

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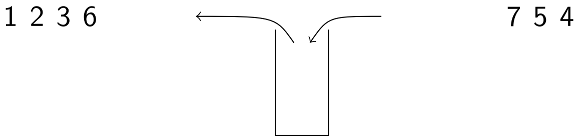

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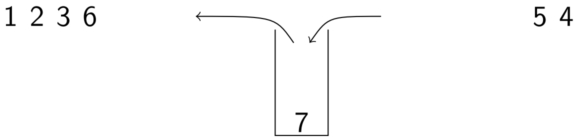

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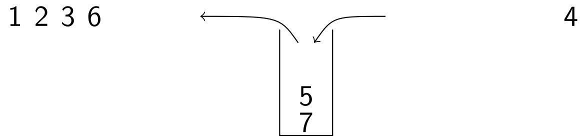

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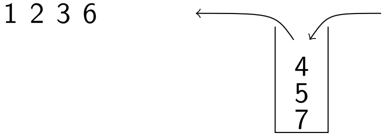

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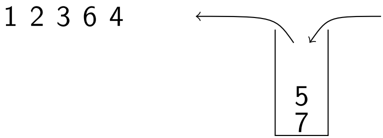

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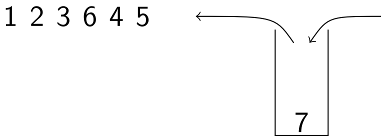

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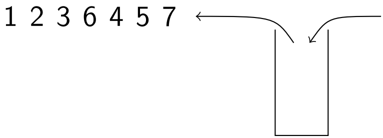

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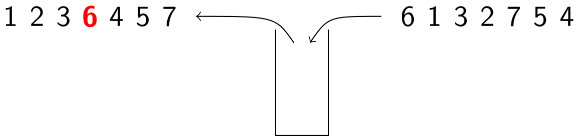

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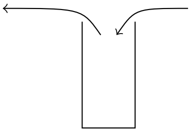
$\mathbf{S}(\sigma) = 1\ 2\ 3\ 6\ 4\ 5\ 7$  $6\ 1\ 3\ 2\ 7\ 5\ 4 = \sigma$

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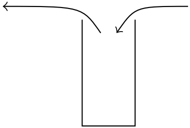

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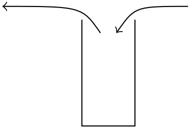

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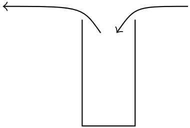
$\mathbf{S}(\sigma) = 1\ 2\ 3\ \mathbf{6}\ 4\ 5\ 7$  $\mathbf{6}\ 1\ 3\ 2\ \mathbf{7}\ 5\ \mathbf{4} = \sigma$

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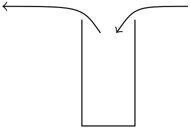
$\mathbf{S}(\sigma) = 1\ 2\ 3\ \mathbf{6}\ 4\ 5\ 7$  $\mathbf{6}\ 1\ 3\ 2\ \mathbf{7}\ 5\ \mathbf{4} = \sigma$

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Next: other sorting devices and patterns [Even & Itai 71, Tarjan 72, Pratt 73]

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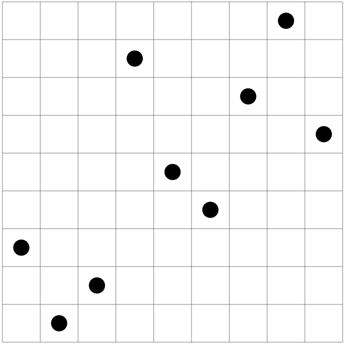

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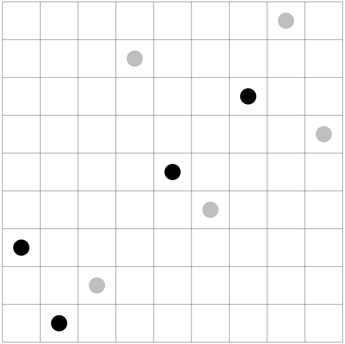

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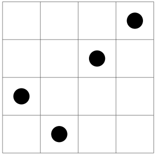

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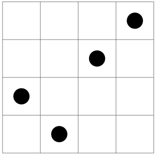

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Crucial remark: $\preccurlyeq$ is a partial order on $\mathfrak{S}$ and "[$\preccurlyeq$] *is even more interesting than the [sorting] networks we were characterizing*" [Pratt 73].

This is the key to defining permutation classes.

Permutation classes

• A **permutation class** is a set $\mathcal{C}$ of permutations that is downward closed for $\preccurlyeq$, i.e. whenever $\pi \preccurlyeq \sigma$ and $\sigma \in \mathcal{C}$, then $\pi \in \mathcal{C}$.

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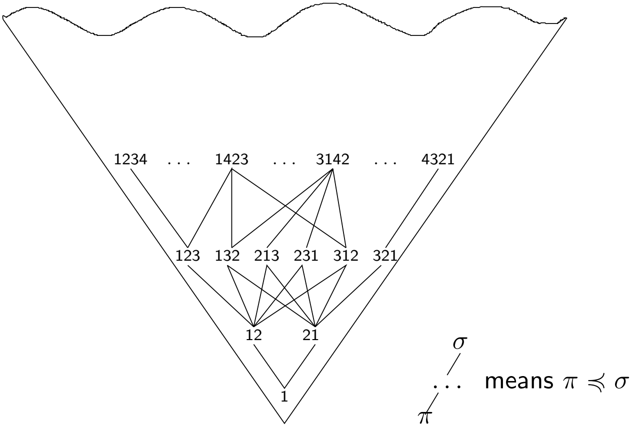

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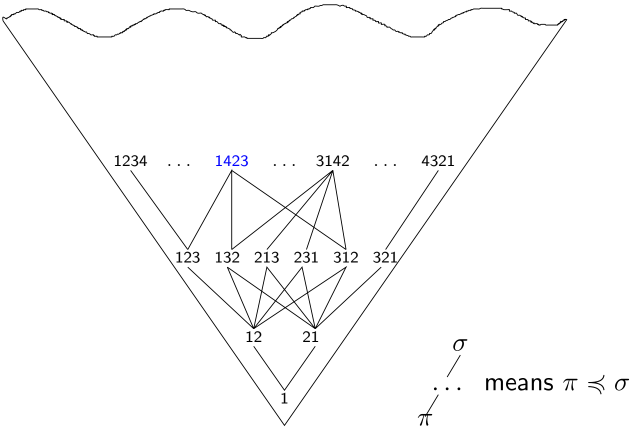

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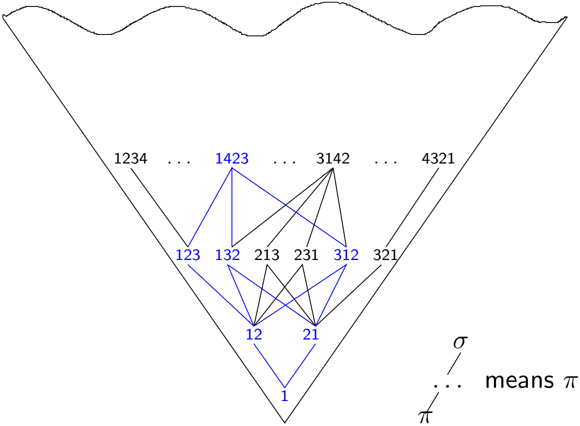

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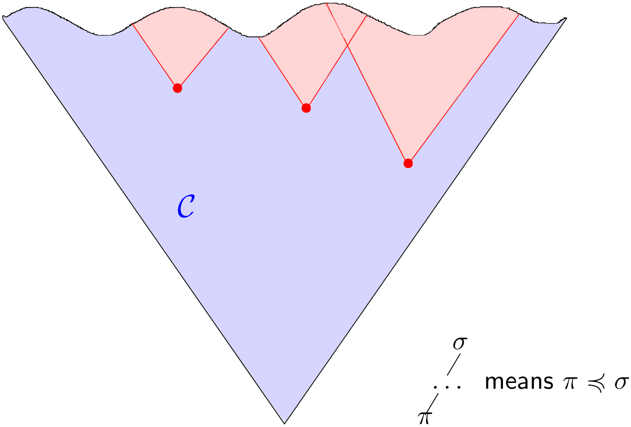

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• Notations: $Av(\pi)$ = the set of permutations that avoid the pattern π

$$Av(B) = \bigcap_{\pi \in B} Av(\pi)$$

• Fact: For every permutation class $\mathcal{C}$, $\mathcal{C} = Av(B)$ for

$B = \{\sigma \notin \mathcal{C} : \forall \pi \preccurlyeq \sigma$ such that $\pi \neq \sigma, \pi \in \mathcal{C}\}$.

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- Remarks:
  - Conversely, every set $Av(B)$ is a permutation class.
  - There exist infinite antichains in the permutation pattern poset, hence some permutation classes have infinite basis.

A biased overview of important results

Specific enumeration results

For $\mathcal{C}$ a permutation class, $\mathcal{C}_n$ is the set of permutations of size n in $\mathcal{C}$ and $C(z) = \sum_n |\mathcal{C}_n| z^n$ is its generating function.

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- One excluded pattern:
  - of size 3: By symmetry, focus on $Av(321)$ and $Av(231)$ only.
    - Description of $Av(321)$ [MacMahon 1915] and $Av(231)$ [Knuth 68].
    - Enumeration by the Catalan numbers in both cases.
    - Bijections: [Simion, Schmidt 85] [Claesson, Kitaev 08].
    - But these two classes have a very different structure.

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  - of size 4: Only three different enumerations. Representatives are:
    - $Av(1342)$ [Bóna 97], algebraic generating function
    - $Av(1234)$ [Gessel 90], holonomic (or D-finite) generating function
    - $Av(1324)$ . . . remains an open problem

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Often combining general methods briefly discussed later.

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- Enumeration of classes (with more excluded patterns) appearing in a different context (e.g. indices of Schubert varieties [Albert, Brignall 13])

Growth rates of permutation classes

- Upper growth rate: $\overline{Gr}(\mathcal{C}) = \limsup_n \sqrt[n]{|\mathcal{C}_n|}$
- Lower growth rate: $\underline{Gr}(\mathcal{C}) = \liminf_n \sqrt[n]{|\mathcal{C}_n|}$

Marcus-Tardos theorem (2004, former Stanley-Wilf conjecture):

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Conjecture: For any class $\mathcal{C}$, $\overline{Gr}(\mathcal{C}) = \underline{Gr}(\mathcal{C})$. Growth rate, denoted $Gr(\mathcal{C})$.
This holds for all principal classes, *i.e.*, $\mathcal{C} = Av(\pi)$,
and more generally for all sum-closed or skew-closed classes.

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Arratia's (false) conjecture:
$Gr(Av(\pi)) \leq (k-1)^2 = Gr(Av(k \dots 21))$ for $|\pi| = k$

- $Gr(Av(1324)) > 9.47$ [Albert, Elder, Rechnitzer, Westcott, Zabrocki 06]
- Remark: $Gr(Av(1324))$ is > 9.81 [Bevan 15], < 13.74 [Bóna 15] and conjectured to be ≈ 11.60 [Conway, Guttmann 15]
- $Gr(Av(\pi))$ is typically exponential in $|\pi|$ [Fox, 2017+]

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Classification of growth rates:
Exactly which numbers can occur as (upper) growth rates is known, except between $\xi \approx 2.305$ and $\lambda < 2.36$ [Vatter and collaborators].

- Before ξ: countably many growth rates, all characterized
- After λ: all real numbers

Nature of the generating functions of permutation classes

A variety of behaviors can occur: rational, algebraic, D-finite, non D-finite.

- For $Av(231)$ and $Av(321)$: Catalan numbers, algebraic GF. But:
  - All proper subclasses of $Av(231)$ are rational [Albert, Atkinson 05].
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- Sufficient algebricity condition [Albert, Atkinson 05]:

When a class contains finitely many simple permutations.

A probabilistic look at permutation classes

Typical diagrams of large permutations in classes: what do they look like?

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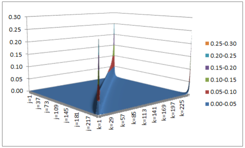
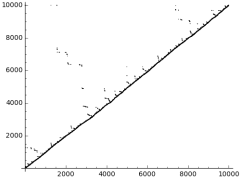
- One excluded pattern of size 3:

$Av(231)$





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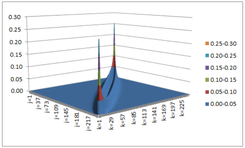

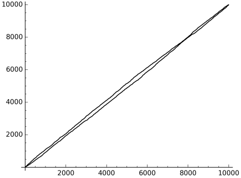

[Miner, Pak 14] [Hoffman, Rizzolo, Slivken 16]

A probabilistic look at permutation classes

Typical diagrams of large permutations in classes: what do they look like?

- One excluded pattern of size 3:
  - Precise local description of the asymptotic shape [Miner, Pak 14] [Madras and collaborators].
  - Scaling limits and link with the Brownian excursion (for the fluctuations around the main diagonal) [Hoffman, Rizzolo, Slivken 16].
  - For any pattern π, the following quantity converges in distribution to a strictly positive random variable [Janson 16]:

$$\frac{\text{number of occurrences of } \pi \text{ in uniform } \sigma \in Av_n(132)}{n^{(|\pi| + \text{ number of descents of } \pi + 1))/2}}.$$

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- Other known cases:
  - Connected monotone grid classes (deterministic limit) [Bevan 15]
  - Separable permutations (non-deterministic limit) [Bassino, B., Féray, Gerin, Pierrot 2017+]

Some general methods

To prove general results on families of permutation classes (*e.g.* growth rates, nature of GF), some general methods are often used, which each capture a notion of nice structure of permutations in these classes:

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These methods are also sometimes used to prove results about (or enumerate) specific classes.

Substitution decomposition

Substitution for permutations

Substitution is an operation building a permutation from smaller ones.

Notation for **substitution** (or inflation): $\sigma = \pi[\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)}]$ with k = size of π.

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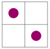
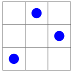
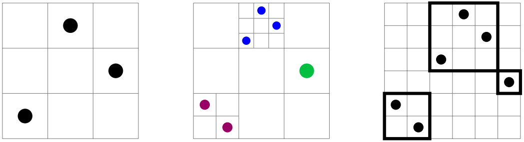
Example: Here, $\pi = 1\,3\,2$, and

$\alpha^{(1)} = 2\,1 =$ 

$\alpha^{(2)} = 1\,3\,2 =$ 

$\alpha^{(3)} = 1 =$  .



Hence $\sigma = 1\,3\,2[2\,1, 1\,3\,2, 1] = 2\,1\,4\,6\,5\,3$.

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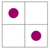
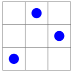
$\alpha^{(1)} = 2\,1 =$ 

$\alpha^{(2)} = 1\,3\,2 =$ 

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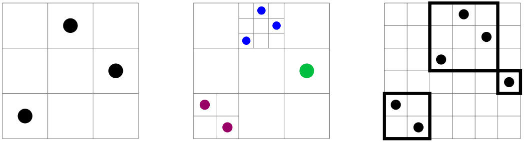

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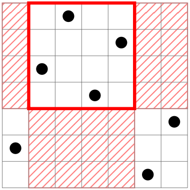
In general, many substitutions give σ, but we will see a canonical one.

Simple permutations

Interval (or block) = set of elements of σ whose positions **and** values form intervals of integers
Example: 5 7 4 6 is an interval of 2 5 7 4 6 1 3

Simple permutation = permutation with no interval, except the trivial ones: $1, 2, \ldots, n$ and σ
Example: 3 1 7 4 6 2 5 is simple

Not simple:



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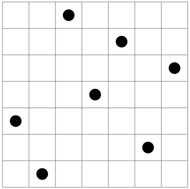

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The smallest simple permutations:
12, 21,   2413, 3142,   6 of size 5, ...

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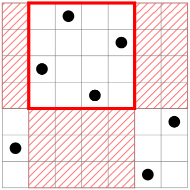

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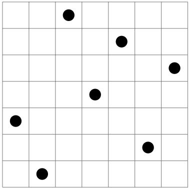

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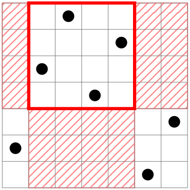

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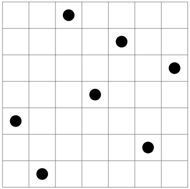

Remark: Enumeration of simple permutations:

- Generating function is not D-finite
- Asymptotically $\frac{n!}{e^2}$ of size n [Albert, Atkinson, Klazar 03]

Substitution decomposition theorem for permutations

Notation:

- $\oplus$ represents any permutation $12\dots k$ for $k \geq 2$
- $\ominus$ represents any permutation $k \dots 21$ for $k \geq 2$
- $\oplus$-indecomposable: that cannot be written as $\oplus[\beta^{(1)}, \beta^{(2)}]$
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Every σ ($\neq 1$) is uniquely decomposed as

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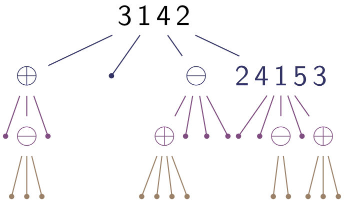
Decomposing recursively inside the $\alpha^{(i)} \Rightarrow$ **decomposition tree**

Decomposition tree

Example: Decomposition tree of

$\sigma =$ 10 13 12 11 14 1 18 19 20 21 17 16 15 4 8 3 2 9 5 6 7



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Notation and properties:

- Nodes labeled by $\oplus$, $\ominus$ or π simple of size ≥ 4.
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- These conditions characterize decomposition trees.

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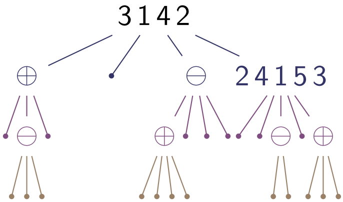

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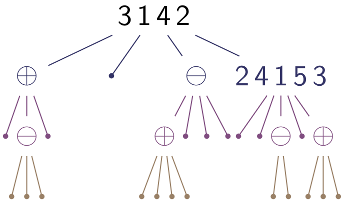

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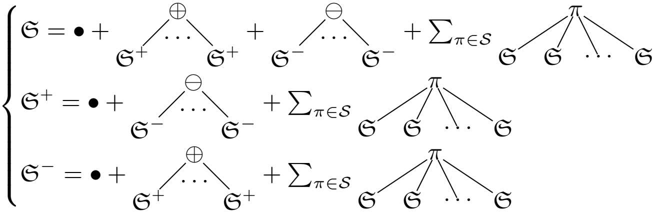
Very convenient, since "*trees are the prototypical recursive structure*" [Flajolet, Sedgewick 09]

A tree grammar for permutations

With $\mathcal{S}$ the set of simple permutations, the substitution decomposition theorem says:



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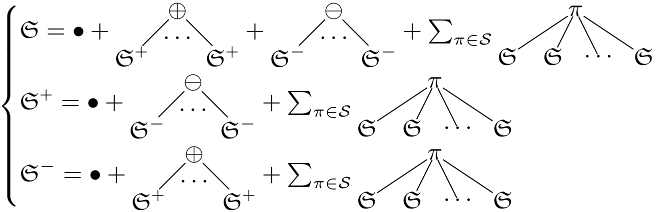

Can we specialize this tree grammar to subsets of $\mathfrak{S}$, and in particular to permutation classes $\mathcal{C} = Av(B)$?

Can we do it automatically? even algorithmically?

What kind of results can be obtained from such a tree grammar describing a permutation class $\mathcal{C}$?

Some (general) results obtained using substitution decomposition

How it all started

• Theorem [Albert, Atkinson 05]: For any permutation class $\mathcal{C}$,
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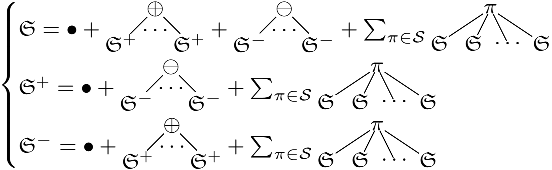

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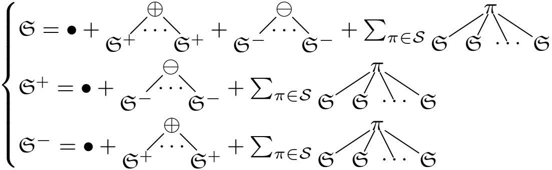

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• Next steps: Automatic computation of a tree grammar for $\mathcal{C}$,
possibly unambiguous (=combinatorial specification).

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- Compute an unambiguous tree grammar for $\mathcal{C}$:
  - With query-complete sets (not effective) [Brignall, Huczynska, Vatter 08]
  - Algorithm propagating pattern avoidance and containment constraints in the tree grammar [Bassino, B., Pierrot, Pivoteau, Rossin 2017+]

Experimenting with the results of this algorithm

The algorithm produces a combinatorial specification for $\mathcal{C}$.
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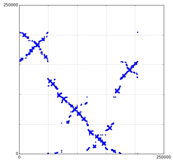
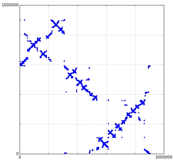
Example: $\mathcal{C} = Av(2413, 3142)$ the class of separable permutations:
Two separable permutations of size 204523 and 903073, drawn uniformly at random among those of the same size:





The algorithm produces a combinatorial specification for $\mathcal{C}$.
From it, we automatically derive a Boltzmann sampler of permutations in $\mathcal{C}$ [Flajolet, Fusy, Pivoteau 07].

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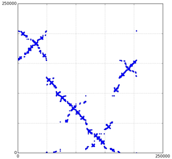

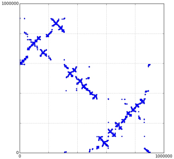

Goal: Explain these diagrams, by describing the “limit shape” of random separable permutations of size $n \to +\infty$.

Proportion of patterns in separable permutations

- Notation:
  - $\widetilde{\text{occ}}(\pi, \sigma) = \frac{\text{number of occurrences of } \pi \text{ in } \sigma}{\binom{n}{k}}$ for $n = |\sigma|$ and $k = |\pi|$
  - $\boldsymbol{\sigma}_n$ = a uniform random separable permutation of size n

- Theorem [Bassino, B., Féray, Gerin, Pierrot 2017+]:

There exist random variables (Λ_π), π ranging over all permutations, such that for all π, $0 \leq \Lambda_\pi \leq 1$ and when $n \to +\infty$, $\widetilde{\text{occ}}(\pi, \boldsymbol{\sigma}_n)$ converges in distribution to Λ_π.

Substitution decomposition is essential to the proof.

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Substitution decomposition is essential to the proof.

Moreover,

- We describe a construction of Λ_π.
- This holds jointly for patterns $\pi_1, \ldots, \pi_r$.
- If π is separable of size at least 2, Λ_π is non-deterministic.
- Combinatorial formula for all moments of Λ_π.

What does this say about limit shapes of diagrams?

- Permutons and permuton convergence:
  - Permuton = measure on $[0,1]^2$ with uniform marginals
    $\approx$ diagram of a finite or infinite permutation.
  - The convergence of $\widetilde{\mathrm{occ}}(\pi, \sigma)$ for all π characterizes the convergence of permutons [Hoppen, Kohayakawa, Moreira, Rath, Sampaio 13; brought to a probabilistic setting].
  - Hence, denoting μ_σ the permuton associated with σ, there exists a random permuton $\boldsymbol{\mu}$ such that $\mu_{\boldsymbol{\sigma}_n}$ tends to $\boldsymbol{\mu}$ in distribution (in the weak convergence topology).

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- Properties of $\boldsymbol{\mu}$:
  - $\boldsymbol{\mu}$ is not deterministic [Bassino, B., Féray, Gerin, Pierrot 2017+].
  - Construction of $\boldsymbol{\mu}$ directly in the continuum [Maazoun 2017+].
  - $\boldsymbol{\mu}$ has Hausdorff dimension 1 [Maazoun 2017+].

Extension to substitution-closed classes

A permutation class $\mathcal{C}$ is substitution-closed when:

- $\pi[\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)}]$ belongs to $\mathcal{C}$ as soon as π and all $\alpha^{(i)}$ do;
- equivalently, the decomposition trees of permutations in $\mathcal{C}$ are all decomposition trees built using simple permutations in $\mathcal{C}$.

Remark: The class of separable permutations is the smallest (non-trivial) substitution-closed class (it contains no simples).

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Let $\mathcal{C}$ be a substitution-closed class, whose set S of simple permutations satisfies (mild?) enumeration conditions.
(e.g. S finite, or $|S_n|$ uniformly bounded, or GF of S rational or of radius of convergence $> \sqrt{2} - 1$, ... are sufficient conditions)

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Thank you for listening!