Studying permutation classes using the substitution decomposition

Mathilde Bouvel
(Institut für Mathematik, Universität Zürich)

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Permutation patterns and permutation classes
Permutation of size $n = \text{Bijection from } [1..n] \text{ to itself.}$
Set $\mathcal{S}_n$, and $\mathcal{S} = \bigcup_n \mathcal{S}_n$. 
**Permutation** of size $n = \text{Bijection from } [1..n] \text{ to itself.}

Set $\mathcal{S}_n$, and $\mathcal{S} = \bigcup \mathcal{S}_n$.

- **Two-line notation:**
  \[
  \sigma = \begin{pmatrix}
  1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
  1 & 8 & 3 & 6 & 4 & 2 & 5 & 7 
  \end{pmatrix}
  \]

- **One-line or word notation:**
  \[\sigma = 1\ 8\ 3\ 6\ 4\ 2\ 5\ 7\]

- **Description as a product of cycles:**
  \[\sigma = (1)\ (2\ 8\ 7\ 5\ 4\ 6)\ (3)\]
Permutations

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- **Two-line notation:**
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This talk is about **permutation patterns** and **permutation classes**.

### Graphical description, or diagram:

[Diagram showing the permutation $\sigma$ with dots indicating the mapping of each element to its image.]
The stack sorting operator $S$

Sort (or try to do so) using a stack satisfying the Hanoi condition.

$\text{6 1 3 2 7 5 4}$
The stack sorting operator $S$

Sort (or try to do so) using a stack satisfying the Hanoi condition.
The origin of permutation patterns: Stack sorting

**The stack sorting operator S**

Sort (or try to do so) using a *stack* satisfying the *Hanoi condition*.

```
1 6
```

```
3 2 7 5 4
```

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Permutation classes
The stack sorting operator $S$

Sort (or try to do so) using a stack satisfying the Hanoi condition.

1 \[\rightarrow\] 3 2 7 5 4

6
The stack sorting operator $S$

Sort (or try to do so) using a stack satisfying the Hanoi condition.

1  \[\rightarrow\]  2 7 5 4

3

6
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```
1 2
  
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\[ \begin{array}{cccccc}
1 & 2 & 3 & 6 & \leftarrow & 5 & 4 \\
& & & & 7 & \\
\end{array} \]
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6 1 3 2 7 5 4
The origin of permutation patterns: Stack sorting

The stack sorting operator $S$

Sort (or try to do so) using a stack satisfying the Hanoi condition.

$S(\sigma) = 1 \ 2 \ 3 \ 6 \ 4 \ 5 \ 7$ \hspace{1cm} $6 \ 1 \ 3 \ 2 \ 7 \ 5 \ 4 = \sigma$

Equivalently, $S(\varepsilon) = \varepsilon$ and $S(LnR) = S(L)S(R)n$, where $n = \max(LnR)$
The origin of permutation patterns: Stack sorting

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First result on permutation patterns [Knuth 68]:

A permutation $\sigma$ is stack-sortable iff $\sigma$ avoids the pattern 231
The origin of permutation patterns: Stack sorting

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meaning that there are no $i < j < k$ such that $\sigma_k < \sigma_i < \sigma_j$,

or equivalently no subsequences $\cdots \sigma_i \cdots \sigma_j \cdots \sigma_k \cdots$ of $\sigma$ whose elements

are in the same relative order as 231.
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Next: other sorting devices and patterns [Even & Itai 71, Tarjan 72, Pratt 73]
**Pattern relation** \( \preceq \):
\( \pi \in \mathfrak{S}_k \) is a pattern of \( \sigma \in \mathfrak{S}_n \) if \( \exists 1 \leq i_1 < \ldots < i_k \leq n \) such that \( \sigma_{i_1} \ldots \sigma_{i_k} \) is in the same relative order (\( \equiv \)) as \( \pi \).

Notation: \( \pi \preceq \sigma \).
Pattern relation \( \preceq \):  
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Notation: \( \pi \preceq \sigma \).

Equivalently: 
The normalization of \( \sigma_{i_1} \ldots \sigma_{i_k} \) on \( [1..k] \) yields \( \pi \).

Example: \( 2134 \preceq 312854796 \) since \( 3157 \equiv 2134 \).
**Pattern relation** $≼$:

$\pi \in \mathcal{S}_k$ is a pattern of $\sigma \in \mathcal{S}_n$ if $\exists 1 \leq i_1 < \ldots < i_k \leq n$ such that $\sigma_{i_1} \ldots \sigma_{i_k}$ is in the same relative order ($\equiv$) as $\pi$.

Notation: $\pi \not\preceq \sigma$.

*Equivalently*:
The normalization of $\sigma_{i_1} \ldots \sigma_{i_k}$ on $[1..k]$ yields $\pi$.

**Example**: $2\ 1\ 3\ 4 \not\preceq 3\ 1\ 2\ 8\ 5\ 4\ 7\ 9\ 6$
since $3\ 1\ 5\ 7 \equiv 2\ 1\ 3\ 4$. 
Pattern relation \( \preceq \):
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**Crucial remark:** \( \preceq \) is a partial order on \( \mathcal{S} \) and “[\( \preceq \)] is even more interesting than the [sorting] networks we were characterizing” [Pratt 73]. This is the key to defining permutation classes.
A **permutation class** is a set $\mathcal{C}$ of permutations that is downward closed for $\preceq$, i.e. whenever $\pi \preceq \sigma$ and $\sigma \in \mathcal{C}$, then $\pi \in \mathcal{C}$. 
• A permutation class is a set $C$ of permutations that is downward closed for $\preccurlyeq$, i.e. whenever $\pi \preccurlyeq \sigma$ and $\sigma \in C$, then $\pi \in C$. 

$\sigma \preccurlyeq \pi$ means $\pi \preccurlyeq \sigma$. 

\[
\begin{array}{cccccccc}
1234 & \ldots & 1423 & \ldots & 3142 & \ldots & 4321 \\
123 & 132 & 213 & 231 & 312 & 321 \\
12 & 21 \\
1 \\
\end{array}
\]
Permutation classes

- A **permutation class** is a set \( C \) of permutations that is downward closed for \( \preceq \), i.e. whenever \( \pi \preceq \sigma \) and \( \sigma \in C \), then \( \pi \in C \).

\[ \sigma \text{ means } \pi \preceq \sigma \]

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• **Notations:** $Av(\pi) = \text{the set of permutations that avoid the pattern } \pi$

\[ Av(B) = \bigcap_{\pi \in B} Av(\pi) \]

• **Fact:** For every permutation class $\mathcal{C}$, $\mathcal{C} = Av(B)$ for

\[ B = \{ \sigma \notin \mathcal{C} : \forall \pi \preceq \sigma \text{ such that } \pi \neq \sigma, \pi \in \mathcal{C} \}. \]

$B$ is an antichain (set of elements incomparable for $\preceq$), called the **basis** of $\mathcal{C}$.
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• **Remarks:**

  * Conversely, every set $Av(B)$ is a permutation class.

  * There exist **infinite antichains** in the permutation pattern poset, hence some permutation classes have **infinite basis**.
A biased overview of important results
Specific enumeration results

For $C$ a permutation class, $C_n$ is the set of permutations of size $n$ in $C$ and $C(z) = \sum_n |C_n|z^n$ is its generating function.
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- **One excluded pattern:**
  - of size 3: By symmetry, focus on $Av(321)$ and $Av(231)$ only.
    - Description of $Av(321)$ [MacMahon 1915] and $Av(231)$ [Knuth 68].
    - Enumeration by the Catalan numbers in both cases.
    - Bijections: [Simion, Schmidt 85] [Claesson, Kitaev 08].
    - But these two classes have a very different structure.
Specific enumeration results

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- **One excluded pattern:**
  - **of size 3:** By symmetry, focus on $Av(321)$ and $Av(231)$ only.
  - **of size 4:** Only three different enumerations. Representatives are:
    - $Av(1342)$ [Bóna 97], algebraic generating function
    - $Av(1234)$ [Gessel 90], holonomic (or $D$-finite) generating function
    - $Av(1324)$ . . . remains an open problem
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- **Systematic enumeration** of $Av(B)$ when $B$ contains small excluded patterns (size 3 or 4).

Often combining general methods briefly discussed later.

[Simion & Schmidt, Gessel, Bóna, Gire, Guibert, Stankova, West... in the nineties]
[Albert, Atkinson, Brignall, Callan, Kremer, Pantone, Shiu, Vatter, ... nowadays]
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- Enumeration of classes (with more excluded patterns) appearing in a different context (e.g. indices of Schubert varieties [Albert, Brignall 13])
Growth rates of permutation classes

- Upper growth rate: \( \overline{Gr}(C) = \limsup_n \sqrt[n]{|C_n|} \)
- Lower growth rate: \( \underline{Gr}(C) = \liminf_n \sqrt[n]{|C_n|} \)

**Marcus-Tardos theorem** (2004, former Stanley-Wilf conjecture):

\( \overline{Gr}(C) < \infty \) for any class \( C \neq \mathcal{S} \).

That is to say, permutation classes grow at most exponentially.
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That is to say, permutation classes grow at most exponentially.

Conjecture: For any class $C$, $\overline{Gr}(C) = \underline{Gr}(C)$. Growth rate, denoted $Gr(C)$. This holds for all principal classes, i.e., $C = Av(\pi)$, and more generally for all sum-closed or skew-closed classes.
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**Arratia’s (false) conjecture**:
\( Gr(Av(\pi)) \leq (k - 1)^2 = Gr(Av(k \ldots 21)) \) for \( |\pi| = k \)

- \( Gr(Av(1324)) > 9.47 \) [Albert, Elder, Rechnitzer, Westcott, Zabrocki 06]
- Remark: \( Gr(Av(1324)) \) is \( > 9.81 \) [Bevan 15], \( < 13.74 \) [Bóna 15] and conjectured to be \( \approx 11.60 \) [Conway, Guttmann 15]
- \( Gr(Av(\pi)) \) is typically exponential in \( |\pi| \) [Fox, 2017+]
Growth rates of permutation classes

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Classification of growth rates:
Exactly which numbers can occur as (upper) growth rates is known, except between \( \xi \approx 2.305 \) and \( \lambda < 2.36 \) [Vatter and collaborators].

- Before \( \xi \): countably many growth rates, all characterized
- After \( \lambda \): all real numbers
A variety of behaviors can occur: rational, algebraic, D-finite, non D-finite.

- For $Av(231)$ and $Av(321)$: Catalan numbers, algebraic GF. But:
  - All proper subclasses of $Av(231)$ are rational [Albert, Atkinson 05].
  - $Av(321)$ contains non D-finite subclasses.
  - However, every proper subclass of $Av(321)$ which has finite basis or is wqo is rational [Albert, Brignall, Ruškuc, Vatter 2017+].
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- (Tight?) connection between wqo and nice GF:
  - A class is wqo (well quasi-ordered) if it contains no infinite antichains.
  - If a class $C$ and all its subclasses are algebraic, then $C$ is wqo.
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- Sufficient algebricity condition [Albert, Atkinson 05]:
  When a class contains finitely many simple permutations.
A probabilistic look at permutation classes

Typical diagrams of large permutations in classes: what do they look like?
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- One excluded pattern of size 3:

\[ \text{Av}(231) \]

[Miner, Pak 14] [Hoffman, Rizzolo, Slivken 16]
Typical diagrams of large permutations in classes: what do they look like?

- One excluded pattern of size 3:
  - Precise local description of the asymptotic shape [Miner, Pak 14] [Madras and collaborators].
  - Scaling limits and link with the Brownian excursion (for the fluctuations around the main diagonal) [Hoffman, Rizzolo, Slivken 16].
  - For any pattern $\pi$, the following quantity converges in distribution to a strictly positive random variable [Janson 16]:

$$\frac{\text{number of occurrences of } \pi \text{ in uniform } \sigma \in \text{Av}_n(132)}{n(|\pi| + \text{number of descents of } \pi+1))/2}.$$
Typical diagrams of large permutations in classes: what do they look like?

- One excluded pattern of size $3$:
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- Other known cases:
  - Connected monotone grid classes (deterministic limit) [Bevan 15]
  - Separable permutations (non-deterministic limit) [Bassino, B., Féray, Gerin, Pierrot 2017+]
Some general methods

To prove general results on families of permutation classes (e.g. growth rates, nature of GF), some general methods are often used, which each capture a notion of nice structure of permutations in these classes:
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- Generating trees
- Substitution decomposition
- Merging and splitting
- (Geometric) grid classes
- Encodings by words over a finite alphabet
- ...
Some general methods

To prove general results on families of permutation classes (e.g. growth rates, nature of GF), some **general methods** are often used, which each capture a notion of **nice structure** of permutations in these classes:

- Generating trees
- **Substitution decomposition**
- Merging and splitting
- (Geometric) grid classes
- Encodings by words over a finite alphabet
- ... 

These methods are also sometimes used to prove results about (or enumerate) specific classes.
Substitution decomposition
Substitution for permutations

Substitution is an operation building a permutation from smaller ones. Notation for **substitution** (or inflation): $\sigma = \pi[\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)}]$ with $k = \text{size of } \pi$. 
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**Example:** Here, \( \pi = 132 \), and

\[
\begin{align*}
\alpha^{(1)} &= 21 = \\
\alpha^{(2)} &= 132 = \\
\alpha^{(3)} &= 1 =
\end{align*}
\]

Hence \( \sigma = 132[21, 132, 1] = 214653 \).
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Substitution is an operation building a permutation from smaller ones. Notation for **substitution** (or inflation): \( \sigma = \pi[\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)}] \) with \( k = \text{size of } \pi \).

Example: Here, \( \pi = 1 \, 3 \, 2 \), and

\[
\begin{align*}
\alpha^{(1)} &= 2 \, 1 = \begin{array}{|c|}
\hline
2 & 1 \\
\hline
\end{array} \\
\alpha^{(2)} &= 1 \, 3 \, 2 = \begin{array}{|c|c|}
\hline
1 & 3 \\
\hline
3 & 2 \\
\hline
\end{array} \\
\alpha^{(3)} &= 1 = \begin{array}{|c|}
\hline
1 \\
\hline
\end{array}
\end{align*}
\]

Hence \( \sigma = 1 \, 3 \, 2[2 \, 1, 1 \, 3 \, 2, 1] = 2 \, 1 \, 4 \, 6 \, 5 \, 3 \).

In general, **many** substitutions give \( \sigma \), but we will see a canonical one.
Simple permutations

Interval (or block) = set of elements of $\sigma$ whose positions and values form intervals of integers
Example: $5 7 4 6$ is an interval of $2 5 7 4 6 1 3$

Simple permutation = permutation with no interval, except the trivial ones: $1, 2, \ldots, n$ and $\sigma$
Example: $3 1 7 4 6 2 5$ is simple
**Simple permutations**

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The smallest simple permutations:

12, 21, 2413, 3142, 6 of size 5, ...

But: For us, it is convenient to consider that 12 and 21 are not simple permutations.
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The smallest simple permutations:
$12, 21, \ 2413, 3142, \ 6$ of size $5, \ldots$

But: For us, it is convenient to consider that $12$ and $21$ are not simple permutations.

Remark: Enumeration of simple permutations:
- Generating function is not D-finite
- Asymptotically $\frac{n!}{e^2}$ of size $n$ [Albert, Atkinson, Klazar 03]
Substitution decomposition theorem for permutations

Notation:

- $\oplus$ represents any permutation $12 \ldots k$ for $k \geq 2$
- $\ominus$ represents any permutation $k \ldots 21$ for $k \geq 2$
- $\oplus$-indecomposable: that cannot be written as $\oplus[\beta^{(1)}, \beta^{(2)}]$
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Theorem: [Albert, Atkinson, Klazar 03]
Every $\sigma$ ($\neq 1$) is uniquely decomposed as
- $\oplus[\alpha^{(1)}, \ldots, \alpha^{(k)}]$, where the $\alpha^{(i)}$ are $\oplus$-indecomposable
- $\ominus[\alpha^{(1)}, \ldots, \alpha^{(k)}]$, where the $\alpha^{(i)}$ are $\ominus$-indecomposable
- $\pi[\alpha^{(1)}, \ldots, \alpha^{(k)}]$, where $\pi$ is simple of size $k \geq 4$

Proof idea: The $\alpha^{(i)}$ represent the maximal proper intervals of $\sigma$. 
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**Proof idea:** The $\alpha^{(i)}$ represent the maximal proper intervals of $\sigma$.
Decomposing recursively inside the $\alpha^{(i)} \Rightarrow$ decomposition tree
Example: Decomposition tree of
\[ \sigma = 10 \ 13 \ 12 \ 11 \ 14 \ 1 \ 18 \ 19 \ 20 \ 21 \ 17 \ 16 \ 15 \ 4 \ 8 \ 3 \ 2 \ 9 \ 5 \ 6 \ 7 \]

Notation and properties:

- Nodes labeled by \( \oplus \), \( \ominus \) or \( \pi \) simple of size \( \geq 4 \).
- No edge \( \oplus - \oplus \) nor \( \ominus - \ominus \).
- Rooted ordered trees.
- These conditions characterize decomposition trees.

\[ \sigma = 3 \ 1 \ 4 \ 2 [\oplus[1, \ominus[1, 1, 1], 1], 1, \ominus[\ominus[1, 1, 1], 1, 1, 1], 2 \ 4 \ 1 \ 5 \ 3[1, 1, \ominus[1, 1], 1, 1, 1]] \]
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The substitution decomposition theorem provides a **bijection** between permutations of size \( n \) and decomposition trees with \( n \) leaves.
Decomposition tree

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The substitution decomposition theorem provides a \textit{bijection} between permutations of size \( n \) and decomposition trees with \( n \) leaves.

Very convenient, since “\textit{trees are the prototypical recursive structure}” [Flajolet, Sedgewick 09]
A tree grammar for permutations

With $S$ the set of simple permutations, the substitution decomposition theorem says:

$$G = \bullet + \oplus G^+ \ldots G^- + \ominus G^+ \ldots G^- + \sum_{\pi \in S} \pi$$

$$G^+ = \bullet + \oplus G^- \ldots G^- + \sum_{\pi \in S} \pi$$

$$G^- = \bullet + \oplus G^+ \ldots G^+ + \sum_{\pi \in S} \pi$$
A tree grammar for permutations

With $S$ the set of simple permutations, the substitution decomposition theorem says:

$$
S = \mathcal{G} - \mathcal{G} + \mathcal{G} \sum_{\pi \in S} \pi
$$

$$
\mathcal{G} = \bullet + \mathcal{G}^{+} \oplus \mathcal{G}^{-} + \mathcal{G} \sum_{\pi \in S} \pi
$$

$$
\mathcal{G}^{+} = \bullet + \mathcal{G}^{-} \mathcal{G}^{+} + \mathcal{G}^{-} \sum_{\pi \in S} \pi
$$

$$
\mathcal{G}^{-} = \bullet + \mathcal{G}^{+} \mathcal{G}^{-} + \mathcal{G}^{+} \sum_{\pi \in S} \pi
$$

Can we specialize this tree grammar to subsets of $S$, and in particular to permutation classes $C = Av(B)$?

Can we do it automatically? even algorithmically?

What kind of results can be obtained from such a tree grammar describing a permutation class $C$?
Some (general) results obtained using substitution decomposition
How it all started

- **Theorem** [Albert, Atkinson 05]: For any permutation class $C$, if $C$ contains finitely many simple permutations, then $C$ has a finite basis and an algebraic generating function $C(z)$. 
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- **Theorem** [Albert, Atkinson 05]: For any permutation class $C$, if $C$ contains finitely many simple permutations, then $C$ has a finite basis and an algebraic generating function $C(z)$.

- **Constructive proof** (of the $GF$ part of the theorem):
  - Propagate avoidance constraints in

\[
\begin{align*}
\mathcal{G} &= \bullet + \mathcal{G}^+ \oplus \mathcal{G}^+ + \mathcal{G}^- \oplus \mathcal{G}^- + \sum_{\pi \in S} \mathcal{G}^\pi \\
\mathcal{G}^+ &= \bullet + \sum_{\pi \in S} \mathcal{G}^\pi \\
\mathcal{G}^- &= \bullet + \sum_{\pi \in S} \mathcal{G}^\pi
\end{align*}
\]

  - Obtain a (possibly ambiguous) context-free tree grammar for $C$.
  - Inclusion-exclusion gives a polynomial system for $C(z)$.
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\[
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G &= \bullet + G_+ \oplus G_+ \oplus G_- \oplus G_- + \sum_{\pi \in S} \pi \\
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\end{align*}
\]

Obtain a (possibly ambiguous) context-free tree grammar for $C$.

Inclusion-exclusion gives a polynomial system for $C(z)$.

Next steps: Automatic computation of a tree grammar for $C$, possibly unambiguous (=combinatorial specification).
• **Input**: a finite basis $B$ defining $C = Av(B)$
Algorithmization

- **Input**: a finite basis $B$ defining $C = Av(B)$

- Decide whether $C$ contains finitely many simples:
  - Naive semi-decision procedure [Schmerl, Trotter 93]
  - Decision procedure [Brignall, Ruškuc, Vatter 08]
  - “Much more practical” algorithm [Bassino, B., Pierrot, Rossin 15]
**Algorithmization**

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- Compute the **set of simples** in $C$:
  - In a naive way [Albert, Atkinson 05] using [Schmerl, Trotter 93]
  - Using the structure of the poset of simples [Pierrot, Rossin 2017+]
Algorithmization

- **Input**: a finite basis $B$ defining $\mathcal{C} = Av(B)$

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- Compute the **set of simples** in $\mathcal{C}$:
  - In a naive way [Albert, Atkinson 05] using [Schmerl, Trotter 93]
  - Using the structure of the poset of simples [Pierrot, Rossin 2017+]

- Compute an **unambiguous tree grammar** for $\mathcal{C}$:
  - With query-complete sets (not effective) [Brignall, Huczynska, Vatter 08]
  - Algorithm propagating pattern avoidance and containment constraints in the tree grammar [Bassino, B., Pierrot, Pivoteau, Rossin 2017+]
Experimenting with the results of this algorithm

The algorithm produces a combinatorial specification for $C$. From it, we automatically derive a **Boltzmann sampler** of permutations in $C$ [Flajolet, Fusy, Pivoteau 07].
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Example: \( C = Av(2413, 3142) \) the class of separable permutations:
Two separable permutations of size 204523 and 903073, drawn uniformly at random among those of the same size:
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Example: $C = Av(2413, 3142)$ the class of separable permutations:
Two separable permutations of size 204523 and 903073, drawn uniformly at random among those of the same size:

Goal: Explain these diagrams, by describing the “limit shape” of random separable permutations of size $n \to +\infty$. 
Proportion of patterns in separable permutations

- **Notation:**
  - $\tilde{\text{occ}}(\pi, \sigma) = \frac{\text{number of occurrences of } \pi \text{ in } \sigma}{\binom{n}{k}}$ for $n = |\sigma|$ and $k = |\pi|$
  - $\sigma_n$ = a uniform random separable permutation of size $n$

- **Theorem** [Bassino, B., Féray, Gerin, Pierrot 2017+]:
  There exist random variables $(\Lambda_\pi)$, $\pi$ ranging over all permutations, such that for all $\pi$, $0 \leq \Lambda_\pi \leq 1$ and when $n \to +\infty$, $\tilde{\text{occ}}(\pi, \sigma_n)$ converges in distribution to $\Lambda_\pi$.

  Substitution decomposition is essential to the proof.
Proportion of patterns in separable permutations

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- **Theorem** [Bassino, B., Féray, Gerin, Pierrot 2017+]:
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  Substitution decomposition is essential to the proof.

Moreover,

- We describe a construction of \( \Lambda_\pi \).
- This holds jointly for patterns \( \pi_1, \ldots, \pi_r \).
- If \( \pi \) is separable of size at least 2, \( \Lambda_\pi \) is non-deterministic.
- Combinatorial formula for all moments of \( \Lambda_\pi \).
What does this say about limit shapes of diagrams?

- **Permutons and permuton convergence:**
  - **Permuton** = measure on $[0, 1]^2$ with uniform marginals
    $\approx$ diagram of a finite or infinite permutation.
  - The convergence of $\widetilde{\text{occ}}(\pi, \sigma)$ for all $\pi$ characterizes the convergence of permutons [Hoppen, Kohayakawa, Moreira, Rath, Sampaio 13; brought to a probabilistic setting].
  - Hence, denoting $\mu_\sigma$ the permuton associated with $\sigma$, there exists a random permuton $\mu$ such that $\mu_{\sigma_n}$ tends to $\mu$ in distribution (in the weak convergence topology).
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- **Properties of $\mu$:**
  - $\mu$ is not deterministic [Bassino, B., Féray, Gerin, Pierrot 2017+].
  - Construction of $\mu$ directly in the continuum [Maazoun 2017+].
  - $\mu$ has Hausdorff dimension 1 [Maazoun 2017+].
A permutation class $C$ is **substitution-closed** when:

- $\pi[\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)}]$ belongs to $C$ as soon as $\pi$ and all $\alpha^{(i)}$ do;
- equivalently, the decomposition trees of permutations in $C$ are all decomposition trees built using simple permutations in $C$.

**Remark:** The class of separable permutations is the smallest (non-trivial) substitution-closed class (it contains no simples).
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Theorem [Bassino, B., Féray, Gerin, Maazoun, Pierrot 2017+]:
Let $C$ be a substitution-closed class, whose set $S$ of simple permutations satisfies (mild?) enumeration conditions.
(e.g. $S$ finite, or $|S_n|$ uniformly bounded, or GF of $S$ rational or of radius of convergence $> \sqrt{2} - 1$, . . . are sufficient conditions)

There exists a random permuton $\mu^C$ (a one-parameter deformation of $\mu$) which is the limit of permutons of uniform random permutations in $C$. 
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Thank you for listening!