Introduction to Algebraic Topology

Mitschrift der Vorlesung von
Dr. M. Michalogiorgaki

Tobias Berner
Universität Zürich
Frühjahrsemester 2009
# Contents

1 Topology  
1.1 Topological spaces and continuous functions  
\hspace{0.5cm} 1.1.1 Topological spaces  
\hspace{0.5cm} 1.1.2 Continuous functions  
1.2 Connectedness & Compactness  
\hspace{0.5cm} 1.2.1 Connected spaces  
\hspace{0.5cm} 1.2.2 Compactness  

2 Algebraic topology  
2.1 Fundamental group  
\hspace{0.5cm} 2.1.1 Path homotopy  
2.2 The fundamental group  
2.3 Covering spaces  
2.4 Lifting properties  
2.5 The fundamental group of the circle and applications  
\hspace{0.5cm} 2.5.1 The Fundamental Theorem of Algebra  
\hspace{0.5cm} 2.5.2 Deformation retracts and homotopy type  
2.6 Seifert von Kampen Theorem  
\hspace{0.5cm} 2.6.1 Direct sums of abelian groups  
2.7 Free abelian groups  
2.8 Free products of groups  
\hspace{0.5cm} 2.8.1 Free groups  
\hspace{0.5cm} 2.8.2 The Seifert-van Kampen theorem  
2.9 CW complexes (cell complexes)  
2.10 Surfaces (two-dimensional manifolds)  
\hspace{0.5cm} 2.10.1 Fundamental group of surfaces  
\hspace{0.5cm} 2.10.2 Homology of surfaces  
\hspace{0.5cm} 2.10.3 Classification of surfaces  
2.11 Knot theory  
2.12 Classification of covering spaces  

Index  

4  
4  
4  
7  
9  
9  
12  
16  
16  
18  
21  
24  
27  
28  
29  
35  
35  
37  
38  
41  
43  
47  
49  
50  
51  
52  
58  
66
Introduction

In calculus, you have studied $\mathbb{R}^n$, $n \in \mathbb{N}$, as well as functions $f : \mathbb{R}^k \rightarrow \mathbb{R}^\ell$, $k, \ell \in \mathbb{N}$. You have studied notions such as neighbourhood of a point $x \in \mathbb{R}^n$, as well as convergence and continuity of a function $f$ at a point $x \in \mathbb{R}^k$. For this study, you have used the Euclidean metric.

For instance if $f : \mathbb{R} \rightarrow \mathbb{R}$, we say that $f$ is continuous at $x \in \mathbb{R}$ if $\forall \varepsilon > 0 \exists \delta > 0$ such that if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \varepsilon$.

We used the metric $d : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, with $(x, y) \mapsto |x - y|$. In general the Euclidean metric $d : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ with $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \mapsto \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$.

In topology, we study notions such as neighbourhood of a point $x \in X$, convergence, continuity for a general set $X$. For this study a metric is not necessary. What we will use is open sets in $X$.

In algebraic topology, we use abstract algebra to study topological properties.

Literature

- Muncres: Topology 2nd edition
- Jänich: Topologie
- Massey: Algebraic Topology: An Introduction
- Stöcker, Zieschang: Algebraische Topologie
- Lickorish: An introduction to knot theory
1 Topology

1.1. Topological spaces and continuous functions

1.1.1. Topological spaces

Consider a set $X$ and $\mathcal{P}(X) := \{ U \mid U \subseteq X \}$.

**Definition 1.1** A topology on $X$ is a collection $\mathcal{T} \subseteq \mathcal{P}(X)$ such that

1. $\emptyset, X \in \mathcal{T}$.
2. If $U_i \in \mathcal{T} \forall i \in I$, then $\bigcup_{i \in I} U_i \in \mathcal{T}$.
3. If $U_i \in \mathcal{T}, i \in \{1, \ldots, n\}$, then $\bigcap_{i \in \{1, \ldots, n\}} U_i \in \mathcal{T}$.

$(X, \mathcal{T})$ is called a topological space (sometimes we will only write $X$).

$U \subseteq X$ is called open, if $U \in \mathcal{T}$.

$U \subseteq X$ is called closed, if $X \setminus U \in \mathcal{T}$.

**Example 1.2**

1. $X$ some set.
   - $\mathcal{T}_d = \{ U \mid U \subseteq X \}$ discrete topology.
   - $\mathcal{T}_t = \{ \emptyset, X \}$ trivial topology.

2. $X = \{x_1, x_2, x_3\}$. Then the three collections
   - $\{\emptyset, X\}$
   - $\{\emptyset, \{x_1\}, \{x_1, x_2\}, X\}$
   - $\{\emptyset, \{x_2\}, \{x_1, x_2\}, \{x_2, x_3\}, X\}$

   are topologies on $X$.

3. $X$ is a set. Then the collections
   - $\mathcal{T}_f = \{ U \subseteq X \mid X \setminus U \text{ is finite or } X \setminus U = X \}$ finite topology.
   - $\mathcal{T}_c = \{ U \subseteq X \mid X \setminus U \text{ is countable or } X \setminus U = X \}$ countable topology.

are topologies on $X$. [Homework]
Suppose that \( B \) is a basis for a topological space \( X \). If \( T' \supseteq T \), then \( T' \) is called **finer** than \( T \). If \( T' \supsetneq T \), then \( T' \) is called **strictly finer** than \( T \). If \( T' \supseteq T \) or \( T \supseteq T' \), then \( T \) and \( T' \) are **comparable**.

**Definition 1.4** Let \( (X, T) \) be a topological space and \( A \subseteq X \).

The **interior** of \( A \) is

\[
\text{int} A := \bigcup_{U \in T, U \subseteq A} U.
\]

The **closure** of \( A \) is

\[
\overline{A} := \bigcap_{X \setminus U \in T, A \subseteq U} U.
\]

**Basis for a topology**

**Definition 1.5** A basis \( B \) for a topological space \( (X, T) \) is a collection of open subsets \( X \) (i.e. \( B \subseteq T \)), such that \( \forall U \in T \exists \{B_i\}_{i \in I}, B_i \in B, \forall i \in I, \text{ with } U = \bigcup_{i \in I} B_i \).

**Remark 1.6** A basis is not unique.

**Properties:** Basis for \( (X, T) \) then \( B \) has the following properties

1. \( \forall x \in X \exists B \in B \) with \( x \in B \).
   
   **Proof** \( X \in T \) and \( B \) is a basis \( \implies X = \bigcup_{i \in I} B_i, B_i \in B, \forall i \in I \). So \( x \in B_j \) for some \( j \in I \).

2. If \( x \in B_1 \cap B_2, B_1, B_2 \in B \) then \( \exists B_3 \in B \) such that \( x \in B_3 \subseteq B_1 \cap B_2 \).
   
   **Proof** \( B_1, B_2 \in B \subseteq T \implies B_1, B_2 \in T \implies B_1 \cap B_2 \in T \implies B_1 \cap B_2 = \bigcup_{i \in I} B_i, \ldots \)

Conversely: If a collection \( B \) of subsets of \( X \) satisfies properties 1. and 2. then there is a unique topology \( T \) for which \( B \) is a basis.

**Proof** We prove that \( T \) is a topology

1. \( X \in T \)?
   
   Property 1 \( \implies \) if \( x \in X \) then \( \exists B_x \in B \) with \( x \in B_x \subseteq X \). Therefore \( X = \bigcup_{x \in X} B_x \), i.e. \( X \in T \).

   \( \emptyset \in T \)?
   
   \( \emptyset \) is the empty union of elements in \( B \), so \( \emptyset \in T \).

2. If \( U_i \in T \ \forall i \in I \), does \( \bigcup_{i \in I} U_i \in T \)?
   
   \( U_i \in T \implies U_i = \bigcup_{j \in I} B_{i_j} \), So \( \bigcup_{i \in I} U_i = \bigcup_{i \in I} \bigcup_{j \in j} B_{i_j} \in T \).

3. If \( U_i \in T \) for \( i \in \{1, \ldots, n\} \), is \( \bigcap_{i \in \{1, \ldots, n\}} U_i \in T \)?
   
   We will show that if \( U_1, U_2 \in T \) then \( U_1 \cap U_2 \in T \).

   \( U_1 = \bigcup_{\lambda \in \Lambda} B_{\lambda}, U_2 = \bigcup_{k \in K} B_k \). Consider \( x \in U_1 \cap U_2 \). Then \( x \in B_{\lambda_x} \) and \( x \in B_{k_x} \), for some \( \lambda_x \in \Lambda \) and \( k_x \in K \). Thus \( x \in B_{\lambda_x} \cap B_{k_x} \implies \text{[by property 2.] } \exists B_x \in B \) with \( x \in B_x \subseteq B_{\lambda_x} \cap B_{k_x} \subseteq U_1 \cap U_2 \).

   Therefore \( U_1 \cap U_2 = \bigcup_{x \in \text{int} U_1 \cap \text{int} U_2} B_x \in T \).
Example 1.10

1. \( X = \mathbb{R} \), \( B = \{ (a, b) \mid a, b \in \mathbb{R}, a < b \} \). \( B \) satisfies properties 1. and 2. therefore it generates a topology on \( \mathbb{R} \). This is the standard topology on \( \mathbb{R} \).

2. \( X = \mathbb{R} \), \( B = \{ (a, b) \mid a, b \in \mathbb{Q}, a < b \} \)
   \( B \) satisfies properties 1. and 2.
   Claim [Homework]: The topology \( T \) generated by \( B \) is in fact the standard topology on \( \mathbb{R} \).

Lemma 1.11 Let \( B, B' \) be bases for the topologies \( T \) and \( T' \) on \( X \). Then the following statements are equivalent:

1. \( T' \supseteq T \).
2. \( \forall x \in X, \forall B \in B \) with \( x \in B \) there is \( B' \in B' \) such that \( x \in B' \subseteq B \).

Proof

1. \( \Rightarrow \) 2.
   Consider \( x \in X \) and \( B \in B \) with \( x \in B \). \( B \in T \subseteq T' \implies B \in T' \implies B = \bigcup_{\lambda \in \Lambda} B'_{\lambda}, B'_{\lambda} \in B' \).
   So \( x \in B \), therefore \( x \in B'_{\lambda} \), for some \( \lambda \in \Lambda \). i.e. we have \( B'_{\lambda} \in B' \), such that \( x \in B'_{\lambda} \subseteq B \).

2. \( \Rightarrow \) 1.
   Consider \( U \in T \) and \( x \in U \). Then \( x \in U = \bigcup_{\lambda \in \Lambda} B_{\lambda}, B_{\lambda} \in B \). That is \( x \in B_{\lambda} \), for some \( \lambda \in \Lambda \). 2. implies that there exists \( B'_{\lambda} \in B' \) such that \( x \in B'_{\lambda} \subseteq B_{\lambda} \).
   Then \( U \subseteq \bigcup_{x \in U} B'_{\lambda} \subseteq U \implies U = \bigcup_{x \in U} B'_{\lambda} \)
   \( \implies U \in T' \). Therefore \( T \subseteq T' \). \( \blacksquare \)

Product topology

Let \( (X, T_x) \), \( (Y, T_y) \) be two topological spaces. The product topology is the topology with basis the collection \( B = \{ U \times V \mid U \in T_x, \ V \in T_y \} \).

Metric topology

\( X \) is a set.

Definition 1.13 A metric on this set is a function \( d: X \times X \to \mathbb{R} \) with

1. \( d(x, y) \geq 0 \), and \( d(x, y) = 0 \) iff \( x = y \).
2. \( d(x, z) \leq d(x, y) + d(y, z) \) \( \forall x, y, z \in X \).
3. \( d(x, y) = d(y, x) \), \( \forall x, y \in X \).

We call the set \( B_d(x, \varepsilon) = \{ y \in X \mid d(y, x) < \varepsilon \} \) the \( \varepsilon \)-ball centered at \( x \).

\( \{ B_d(x, \varepsilon) \}_{x \in X, \varepsilon > 0} \) is a basis for a topology on \( X \), the metric topology.
Example 1.14 \(\mathbb{R}\) with the standard topology \((\mathcal{T}_{\text{stand}})\).
Consider \(\mathbb{R} \times \mathbb{R} = \mathbb{R}^2\).
Claim: The product topology on \(\mathbb{R}^2\) and the metric topology are the same.
For this, one has to show

\[ \mathcal{T}_{pr} \subseteq \mathcal{T}_m \]

By lemma 1.11 one has to show \(\forall x \in \mathbb{R}^2, \forall B_{pr} \in \mathcal{B}_{pr}, x \in B_{pr}, \exists B_m \in \mathcal{B}_m\) such that \(x \in B_m \subseteq B_{pr}\).

\[ \mathcal{T}_m \subseteq \mathcal{T}_{pr} \]

Subspace topology

Definition 1.15 \((X, \mathcal{T})\) is a topological space and \(Y \subseteq X\). The collection \(\mathcal{T}_Y := \{ U \cap Y \mid U \in \mathcal{T} \}\) is a topology on \(Y\), called the \textit{subspace topology}.

Example 1.16 \(Y = [0, 1] \cup \{2\} \subseteq X = \mathbb{R}\) with the standard topology.
Then the sets

\[ (a, b), \text{ with } a, b \in [0, 1], \]
\[ [0, b), \text{ with } b \in [0, 1], \]
\[ (a, 1], \text{ with } a \in [0, 1], \]
\[ (2), \]
\[ [0, 1] \]

are open sets in the subspace \(Y\).

1.1.2. Continuous functions

Definition 1.17 Let \((X, \mathcal{T}_X), (Y, \mathcal{T}_Y)\) be topological spaces and \(f : X \to Y\). \(f\) is called \textit{continuous} if \(f^{-1}(V) \in \mathcal{T}_X \forall V \in \mathcal{T}_Y\).

Claim \(f : \mathbb{R} \to \mathbb{R}\) (with standard topology). The \(\varepsilon-\delta\) definition of continuity is equivalent to the definition above.

Proof

\(\Leftarrow\) Suppose that \(f : \mathbb{R} \to \mathbb{R}\) is continuous with the definition above.
Consider \(x_0 \in \mathbb{R}\) and \(\varepsilon > 0\). Then \(V = (f(x_0) - \varepsilon, f(x_0) + \varepsilon) \in \mathcal{T}_{\text{stand}}\), so \(f^{-1}(V) \in \mathcal{T}_{\text{stand}}\) by our definition.
Now \(x_0 \in f^{-1}(V)\), so there exists \((a, b) \in \mathcal{B}_{\text{stand}}\) with \(x_0 \in (a, b) \subseteq f^{-1}(V)\).
Take \(\delta = \min\{x_0 - a, b - x_0\}\). Clearly \(\delta > 0\). Then \(|x - x_0| < \delta \implies x \in (a, b) \subseteq f^{-1}(V) \implies f(x) \in V \implies f(x) \in (f(x) - \varepsilon, f(x) + \varepsilon) \implies |f(x) - f(x_0)| < \varepsilon.\)
Theorem 1.19 Let \( X, Y \) be topological spaces and \( f : X \rightarrow Y \). The following are equivalent:

1. \( f \) is continuous.
2. for every closed subset of \( Y \) the inverse image of it is a closed subset of \( X \).

Proof

1.⇒2.
Let \( B \) be closed in \( Y \), then \( X \setminus f^{-1}(B) = f^{-1}(Y \setminus B) \) which is open in \( X \), i.e. \( f^{-1}(B) \) is closed in \( X \).

2.⇒1.
...

Lemma 1.21 (the pasting lemma)
Let \( X, Y \) be topological spaces and \( A, B \) closed subsets of \( X \) with \( X = A \cup B \). Let \( f : A \rightarrow Y, \ g : B \rightarrow Y \) be continuous with \( f(x) = g(x) \ \forall \ x \in A \cap B \). Then \( h : X \rightarrow Y \) with

\[
h(x) = \begin{cases} 
  f(x) & x \in A \\
  g(x) & x \in B
\end{cases}
\]

is continuous.

Proof Let \( C \) be a closed subset of \( Y \). Then \( h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C) \) where \( f^{-1}(C) \) is closed in \( A \) and \( g^{-1}(C) \) is closed in \( B \).

(a) \( f^{-1}(C) \) is closed in \( A \) \( \implies \) \( f^{-1}(C) = A \cap G \), where \( G \) closed in \( X \).

(b) \( g^{-1}(C) \) is closed in \( B \) \( \implies \) \( g^{-1}(C) = B \cap H \), where \( H \) closed in \( X \).

(a) \( \implies \) \( f^{-1}(C) \) is closed in \( X \),
(b) \( \implies \) \( g^{-1}(C) \) is closed in \( X \).

\( \implies \) \( h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C) \) is closed in \( X \).

Definition 1.23 Let \( X, Y \) be topological spaces. \( f : X \rightarrow Y \) is called a homeomorphism if \( f \) is bijective, and \( f, f^{-1} \) are continuous.

Bijective correspondence not only between \( X \) and \( Y \) but also between the collection of open sets in \( X \) and the collection of open sets \( Y \).
Thus any property of \( X \) that is expressed in terms of its open subsets yields via \( f \) the corresponding properties for \( Y \). Such a property is called a topological property.

Quotient topology

Example 1.24 Torus

Definition 1.25 Let \( X, Y \) be topological spaces, \( p : X \rightarrow Y \) a surjective map. The map \( p \) is called a quotient map if a subset \( U \) of \( Y \) is open in \( Y \), if and only if \( p^{-1}(U) \) is open in \( X \).
1.2 Conectedness & Compactness

**Definition 1.26** Let $X$ be a topological space, $Y$ be some set and $p : X \to Y$ a surjective map. The *quotient topology* on $Y$ induced by $p$ is defined as follows: A subset $U \subseteq Y$ is open if and only if $p^{-1}(U) \subseteq X$ is open.

The fact that this is a topology follows from $p^{-1}(\emptyset) = \emptyset$, $p^{-1}(Y) = X$, $p^{-1}\left(\bigcup_{a \in J} U_a\right) = \bigcup_{a \in J} p^{-1}(U_a)$, $p^{-1}\left(\bigcap_{i=1}^n U_i\right) = \bigcap_{i=1}^n p^{-1}(U_i)$.

**Remark 1.27** The quotient topology on $Y$ is the finest topology that makes $p$ continuous.

**Example 1.28**

1. $X = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2\pi, 0 \leq y \leq 1\} = [0, 2\pi] \times [0, 1] \subseteq \mathbb{R}^2$. $Y = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, 0 \leq z \leq 1\}$. 

   $f : X \to Y$, with $(x, y) \mapsto (\cos x, \sin x, y)$.

   $f$ is surjective. Using $f$ we can define the quotient topology in $Y$.

2. $p : \mathbb{R} \to \{x_1, x_2, c_3\}$

   $p(x) = \begin{cases} 
   x_1 & x > 0 \\
   x_2 & x < 0 \\
   x_3 & x = 0
   \end{cases}$

   The quotient topology on $\{x_1, x_2, x_3\}$ induced by $p$ is $\{\emptyset, \{x_1\}, \{x_2\}, \{x_1, x_2\}, \{x_1, x_2, x_3\}\}$.

1.2. Conectedness & Compactness

In calculus you have studied functions $f : [a, b] \to \mathbb{R}$ and you have proved three theorems for continuous functions $f : [a, b] \to \mathbb{R}$.

1. Intermediate Value Theorem (IVT),
2. Maximus Value Theorem (MVT),
3. Uniform Continuity Theorem (UCT).

These theorems rely on the continuity of $f$ and some topological properties of $[a, b] \subset \mathbb{R}$. In particular IVT relies on the connectedness of $[a, b]$. MVT and VCT rely on the compactness of $[a, b]$.

1.2.1. Connected spaces

**Definition 1.29** Consider $(X, T)$ topological space. A *separation* of $X$ is a pair $U, V$ where $U, V \in T$, $U \neq \emptyset$, $V \neq \emptyset$, $U \cap V = \emptyset$, and $X = U \cup V$. $X$ is called *connected* if there is no separation of $X$.

**Remark 1.30** Connectedness is a topological property.

**Example 1.31**

1. $\mathbb{R}, \mathcal{T}_{\text{stand}}, \mathbb{Q} = ((-\infty, a) \cap \mathbb{Q}) \cup ((a, \infty) \cap \mathbb{Q}), a \in \mathbb{R} \setminus \mathbb{Q}$.
2. $\mathbb{R}, [a, b], [a, b], (a, b], (a, b)$ are connected.
Claim: \( X \) is connected iff the only subsets of \( X \) that are both open and closed are \( \emptyset \) and \( X \).

**Proof**

\( \Rightarrow \) Suppose, that \( A \subseteq X, A \neq \emptyset, A \neq X, \) and \( A \in \mathcal{T}_X \), \( X \setminus A \in \mathcal{T}_X \). Then \( X = A \cup (X \setminus A) \implies A, X \setminus A \) is separation of \( X \), contradiction.

\( \Leftarrow \) If \( U, V \) is a separation of \( X \). Then \( U \in \mathcal{T}_X, X \setminus U = V \in \mathcal{T}_X, U \setminus X \neq \emptyset, U \cap (X \setminus U) = \emptyset \). \( V \in \mathcal{T}_X, X \setminus U \in \mathcal{T}_X, U \neq \emptyset, U \neq X \), contradiction. Therefore \( X \) is connected.

**Lemma 1.33** If \( C, D \) are a separation of \( X \) and \( Y \) is a connected subspace of \( X \), then \( Y \subseteq C \) or \( Y \subseteq D \).

**Proof** \( C \cap Y, D \cap Y \) are open subsets of \( Y \). In addition \( (C \cap Y) \cap (D \cap Y) = \emptyset \). If \( C \cap Y \) and \( D \cap Y \) were both nonempty, then they would form a separation of \( Y \). But \( Y \) is connected. So \( C \cap D = \emptyset \), or \( D \cap Y = \emptyset \).

\( \implies Y \subseteq C \) or \( Y \subseteq D \). 

**Theorem 1.35** Let \( \{A_i\}_{i \in I} \) be connected subspaces of \( X \) and \( p \in \cap_{i \in I} A_i \). Then \( \cup_{i \in I} A_i \) is connected subspace of \( X \).

**Proof** Assume that \( \cup_{i \in I} A_i \) is not connected, and \( \cup_{i \in I} A_i = C \cup D, C, D \) a separation of \( \cup_{i \in I} A_i \). Then \( p \in C \cup D \) and WLOG we can assume that \( p \in C \). \( A_i \) is a connected subspace.

\( \implies \) [Lemma] \( A_i \subseteq C \) or \( A_i \subseteq D \) for any \( i \in I, p \in \cap_{i \in I} A_i \).

\( \implies p \in A_i \forall i \in I \).

\( \implies A_i \subseteq C \forall i \in I \).

\( \implies \cup_{i \in I} A_i \subseteq C \).

\( \implies D = \emptyset \). Contradiction.

**Theorem 1.37** Let \( A \) be a connected subspace of \( X \) and \( A \subseteq B \subseteq \overline{A} \). Then \( B \) is also connected.

**Proof** Assume that \( C, D \) is a separation of \( B \). By Lemma \( A \subseteq C \) or \( A \subseteq D \). WLOG \( A \subseteq C \).

\( A \subseteq C \implies \overline{A} \subseteq \overline{C} \).

\( D \subseteq B \subseteq \overline{A} \subseteq \overline{C} \implies D \subseteq \overline{C} \).

Claim: \( D \cap \overline{C} = \emptyset \).

Proof: The closure of \( C \) in \( B \) is \( \overline{C} \cap B \), where \( \overline{C} \) is the closure of \( C \) in \( X \). \( C = \overline{C} \cap B = \overline{C} \cap (C \cup D) = (\overline{C} \cap C) \cup (\overline{C} \cap D) = C \cup (\overline{C} \cap D) \).

\( \implies \overline{C} \cap D = \emptyset \).

We have that \( D \subseteq \overline{C} \) and we showed that \( D \cap \overline{C} = \emptyset \). Therefore \( D = \emptyset \), contradiction. There is no separation of \( B \), i.e. \( B \) is connected.

**Theorem 1.39** Let \( f : X \to Y \) be a continuous function between topological spaces \( X \) and \( Y \). If \( X \) is connected, then \( f(X) \) is connected.
Consider \( f : X \rightarrow f(X), x \mapsto f(x) \). Then \( f \) is continuous. Indeed, if \( U \) open in \( f(X) \), then \( U = f(X) \cap V \) for some \( V \) open in \( Y \). \( \bar{f}(U) = \bar{f}^{-1}(f(X) \cap V) = \bar{f}^{-1}(f(X)) \cap \bar{f}^{-1}(V) = \bar{f}^{-1}(V) \cap f^{-1}(V) = f^{-1}(V) \) open in \( X \).

Suppose that \( f(X) = C \cup D, C, D \) separation of \( f(X) \). Then \( f^{-1}(C), f^{-1}(D) \) are open subsets of \( X \), disjoint and nonempty and \( X = f^{-1}(C) \cup f^{-1}(D) \). Contradiction (\( X \)

\textbf{Theorem 1.41} A finite (cartesian) product of connected spaces (in the product topology) is connected.

Proof: We start by proving that if \( X \) and \( Y \) are connected, then \( X \times Y \) is connected. Consider \( a \times b \) in \( X \times Y, X \times b \). \( X \times b \) is connected. Similarly \( X \times Y \) is connected for any \( x \in X \). As a result, \( T_x = (X \times b) \cup (x \times Y) \) is also connected (by previous theorem, since \( x \times b \in (X \times b) \cap (x \times Y) \)).

Therefore, \( \cup_{x \in X} T_x \), is connected, as the union of connected subspaces, \( T_x \) with common point \( a \times b \). Now \( X \times Y = \cup_{x \in X} T_x \), therefore \( X \times Y \) is connected.

By induction, the general proof follows.

\textbf{Theorem 1.43} (IVT)

Let \( f : X \rightarrow \mathbb{R} \) be a continuous function and \( X \) be a connected space. If \( a, b \in X \) and \( r \in (f(a), f(b)) \), then \( \exists c \in X \) with \( f(c) = r \).

Proof: \( A = f(X) \cap (-\infty, r), B = f(X) \cap (r, \infty) \). \( A, B \) are open in \( f(X) \), \( A \cap B = \emptyset \), \( A \neq \emptyset, B \neq \emptyset \) (as \( f(a) \in A, f(b) \in B \)).

If \( r \notin f(X) \), then \( f(X) = A \cup B \), i.e. \( A, B \) is a separation of \( f(X) \). However \( f(X) \) is connected, because \( X \) is connected and \( f \) is continuous. I.e. such a separation cannot exist. I.e. \( r \in f(X) \), that is, \( \exists c \in X \) such that \( f(c) = r \).

Remark 1.45 One can prove an even more generalised form of IVT, where instead of \( \mathbb{R} \) one considers any ordered set.

\textbf{Definition 1.46} Let \( X \) be a topological space and \( x, y \in X \). A \textit{path} in \( X \) from \( x \) to \( y \) is a continuous map \( f : [a, b] \rightarrow X \) with \( f(a) = x \) and \( f(b) = y \). \( X \) is called \textit{path connected}, if every pair of points in \( X \) can be joined by a path in \( x \).

Claim: If \( X \) is path connected, then \( X \) is connected.

Proof: Suppose \( X = C \cup D, C, D \) a separation of \( X \). Consider \( f : [a, b] \rightarrow X \) a path. \( f([a, b]) \) is connected, so \( f([a, b]) \subseteq C \) or \( f([a, b]) \subseteq D \). This implies that there is no path in \( X \) joining a point in \( C \) to a point in \( D \). \( \implies \) \( X \) not path connected, contradiction.

\textbf{Example 1.48} (Topologist's sine curve \( \overline{S} \))

Consider \( S = \left\{(x, \sin \frac{1}{x}) \mid 0 < x \leq 1 \right\} \subseteq \mathbb{R}^2 \).

\( f : (0, 1] \rightarrow \mathbb{R}, x \mapsto (x, \sin \frac{1}{x}) \). \( f \) is continuous, \( (0, 1] \) is connected, therefore \( S = f((0, 1]) \) is connected.

\( \overline{S} = S \cup \{0 \times [-1, 1] \} \). \( S \) is connected \( \implies \overline{S} \) is connected.

Claim: \( \overline{S} \) is not path connected.

Proof: Suppose \( f : [a, c] \rightarrow \overline{S} \) continuous with \( f(a) = (0, 0) \), and \( f(c) \in \overline{S} \). \( f^{-1}(0 \times [-1, 1]) \) is closed in \([a, c]\) therefore it has a largest element \( b \). Then \( f(b) \in 0 \times [-1, 1] \).
and \( f((b, c]) \in S \). Let \( f(t) = (x(t), y(t)), t \in [a, c] \). Then \( x(a) = 0, x(t) > 0 \) and 
\[ y(t) = \sin \frac{1}{t^2} \text{ for } t > b. \]
Given \( n \in \mathbb{N} \), choose \( u_n \) with \( 0 < u_n < x(b + \frac{1}{n^n}) \) such that 
\[ \sin \frac{1}{u_n} = (1 + 1)^n. \]
By the IVT there exists \( t_n \in (b, b + \frac{1}{n^n}) \) with \( x(t_n) = u_n \). Then \( t_n \to b \) but \( y(t_n) \) does not converge. Contradicting the continuity of \( f \).

Remarks on yesterday’s example

- If \( X = \mathbb{R}^2 \) and \( S \subseteq X \), then \( \mathcal{S} = \{ \text{limits of convergent sequences of points in } S \} \).
  
  More generally, this is true if \( X \) is first countable, e.g. \( X \) is a metric space.

- If \( f : [a, c] \to \mathcal{S} \) continuous. Of \( t \in [a, c] \) \( \forall n \in \mathbb{N} \), with \( t_n \to b, b \in [a, c] \), then 
  \[ f(t_n) \to f(b). \]
  
  In general, continuous implies sequentially continuous.

**Definition 1.49**

Given a topological space \( X \), define an equivalence relation \( \sim \) on \( X \), by setting \( x \sim y \) \((x, y \in X)\) if there exists a connected subspace of \( X \) with \( x, y \in Y \). The equivalence classes are called the **connected component** of \( X \).

Define another equivalence relation \( \sim \) by setting \( x \sim y \) \((x, y \in X)\) if there is a path in \( X \) from \( x \) to \( y \). The equivalence classes are called **path components** of \( X \).

### 1.2.2. Compactness

**Definition 1.50**

A topological space \( X \) is called **compact** if for any open covering \( \{U_\lambda \}_{\lambda \in \Lambda} \) of \( X \), i.e. \( U_\lambda \text{ open } \forall \lambda \in \Lambda \) and \( X = \bigcup_{\lambda \in \Lambda} U_\lambda \), there exist finitely many \( \lambda_1, \ldots, \lambda_n \in \Lambda \) such that \( X = \bigcup_{i=1}^n U_{\lambda_i} \).

Compactness is a topological property.

**Example 1.51**

1. \( \mathbb{R} \) is not compact, \( \mathbb{R} = \bigcup_{r \in \mathbb{R}}(r - \frac{1}{2}, r + \frac{1}{2}) \), or \( \mathbb{R} = \bigcup_{n \in \mathbb{N}}(n, n + 2) \).
2. \((a, b], a, b \in \mathbb{R} \) is not compact. \((a, b] = \bigcup_{n \in \mathbb{N}}(a + \frac{1}{n}, b] \).
3. Subspaces of \( X \) with finitely many points are obviously compact.
4. \( X = \{0\} \cup \{ \frac{1}{n} \mid n \in \mathbb{N} \} \) is compact. Consider \( \{U_\lambda \}_{\lambda \in \Lambda} \) an open covering of \( X \) and consider \( \lambda_0 \in \Lambda \) with \( 0 \in U_{\lambda_0} \). Then there exists \( N_{\lambda_0} \) such that \( \frac{1}{n} \in U_{\lambda_0} \) \( \forall n \geq N_{\lambda_0} \).

  For each \( n \in \{1, \ldots, N_{\lambda_0} - 1\} \) choose \( U_{\lambda_n} \) containing it. Then \( \bigcup_{\lambda=0}^{N_{\lambda_0}} U_{\lambda} = X \).

**Order topology**

**Definition 1.52**

A relation \( C \) on a set \( X \) is an **order relation** if it has the following properties:

1. If \( x, y \in X, x \neq y \) then \( xCy \) or \( yCx \).
2. For no \( x \in X \) does \( xCx \) hold.
3. If \( xCy, yCz \) then \( xCz, x, y, z \in X \).
Example 1.53 $X = \mathbb{R}$, $C = \leq$. If $X$ is a set with an order relation $<$ and $a, b \in X$, $a < b$.

\[
\begin{align*}
(a, b) &= \{x \in X \mid a < x < b\}, \\
(a, b] &= \{x \in X \mid a < x \leq b\}, \\
[a, b) &= \{x \in X \mid a \leq x < b\}, \\
[a, b] &= \{x \in X \mid a \leq x \leq b\}.
\end{align*}
\]

**Definition 1.54** $X$ is a set with an order relation and more than two elements. Let $\mathcal{B}$ be the collection of all sets of the following types:

- $(a, b)$, $a, b \in X$,
- $(a, b_0]$, $a \in X$, $b_0$ the largest element (if any) in $X$,
- $[a_0, b)$, $b \in X$, $a_0$ the smallest element (if any) in $X$.

$\mathcal{B}$ is a basis for a topology, and we call the topology generated by $\mathcal{B}$ the *order topology*.

**Example 1.55** $\mathbb{R}^2$ with the dictionary order, i.e. $a \times b < c \times d$ if $a < c$ or $a = c$ and $b < d$. $\mathbb{R} \times \mathbb{R}$ has no largest or smallest element. (Graph ... open sets ... ]

An ordered set $X$ has the *least upper bound property* if every subset of $X$ that is bounded above has a least upper bound.

**Theorem 1.56** Let $X$ be an ordered set with the least upper bound property. Then any closed interval $[a, b]$ in $X$ is compact.

[Without proof]

**Corollary 1.57** In $\mathbb{R}$ ($T_{\text{stand}} = T_{\text{ordered}}$), $[a, b]$ is compact for any $a, b \in \mathbb{R}$.

**Theorem 1.58** A closed subspace $Y$ of a compact space $X$ is compact.

**Proof** Consider $\{U_\lambda\}_{\lambda \in \Lambda}$ an open covering of $Y$. Then $U_\lambda = Y \cap U'_\lambda$ for some $U'_\lambda$ open in $X$, $\forall \lambda \in \Lambda$. Denote $U' = \{U'_\lambda\}_{\lambda \in \Lambda}$. Now $U'' = U' \cup \{X \setminus Y\}$ is an open covering of $X$. $X$ is compact, so a finite subcollection of $U''$ covers $X$. If this contains $X \setminus Y$, discard it. If not I leave the subcollection unchanged. What we obtain is a finite subcollection of $U'$, say $\{U'_i\}_{i \in \{1, \ldots, n\}}$ with $Y \subseteq \bigcup_{i=1}^n U'_i$. This implies, that $\{U_{\lambda_i}\}_{i \in \{1, \ldots, n\}}$ is a finite subcovering of $Y$.

**Theorem 1.60** $X$ compact topological spaces. $Y$ topological space and $f : X \to Y$ continuous. Then $f(X)$ is compact.

**Proof** Let $U = \{U_\lambda\}_{\lambda \in \Lambda}$ be an open covering of $f(X)$. Then $\forall \lambda \in \Lambda U_\lambda = U'_\lambda \cap f(X)$ for some $U'_\lambda$ open in $Y$. Denote $U' = \{U'_\lambda\}_{\lambda \in \Lambda}$, $\{f^{-1}(U'_\lambda)\}_{\lambda \in \Lambda}$ is an open covering of $X$. $X$ is compact, hence it has a finite subcovering $\{U_{\lambda_i}\}_{i \in \{1, \ldots, n\}}$. Then $\{U_{\lambda_i}\}_{i \in \{1, \ldots, n\}}$ is a finite covering of $X$.
Theorem 1.62  The product of finitely many compact spaces is compact.

[Proof omitted]

Theorem 1.63  (Extreme value theorem)

\[ X \text{ is a compact topological space and } Y \text{ is an ordered set in the order topology. If } f: X \to Y \text{ is continuous then } \exists c, d \in X \text{ with } f(c) \leq f(x) \leq f(d) \forall x \in X. \]

Proof: \( X \) is compact and \( f \) is continuous, so \( f(X) \) is compact. Define \( (-\infty, y) := \{ a \in Y \mid a < y \} \) and note that if \( Y \) has no smallest element, then \( (-\infty, y) = \bigcup_{a \in Y} (a, y) \) and if \( Y \) has a smallest element \( a_0 \), then \( (-\infty, y) = [a_0, y) \).

\( (-\infty, y) \) is an open set in \( Y \).

If \( f(X) \) has no largest element, then \( \{(-\infty, y) \cap f(X)\}_{y \in f(X)} \) is an open covering of \( f(X) \). Since \( f(x) \) is compact, there exist \( y_i, i \in \{1, \ldots, n\} \), such that \( \{(-\infty, y_i) \cap f(X)\}_{i \in \{1, \ldots, n\}} \) covers \( f(X) \). If \( y_j = \max\{y_i\}_{i \in \{1, \ldots, n\}} \), then \( y_j \in f(X) \), but \( y_j \notin \{(-\infty, y_i) \cap f(X)\}_{i \in \{1, \ldots, n\}} \), contradiction, since \( \{(-\infty, y_i) \cap f(X)\}_{i \in \{1, \ldots, n\}} \) is an open covering of \( f(X) \).

Similarly we can prove, that \( f(X) \) has a smallest element \( f \).

Hausdorff spaces

Terminology: \( X \) topological space, \( x \in X, U \) open set in \( X \) with \( x \in U \), then \( U \) is called an open neighbourhood of \( x \).

Definition 1.65  A topological space \( X \) is called Hausdorff if for every pair \( x_1, x_2 \in X \) with \( x_1 \neq x_2 \) there exist open neighbourhoods \( U_1, U_2 \) of \( x_1, x_2 \) respectively with \( U_1 \cap U_2 = \emptyset \).

Facts:

- Every ordered set with the order topology is a Hausdorff space.
- If \( X, Y \) are Hausdorff spaces, then \( X \times Y \) is Hausdorff.
- A subspace of a Hausdorff space is Hausdorff.
- Every finite point set in a Hausdorff space is closed.

Theorem 1.66  Every compact subspace of a Hausdorff space is closed.

Proof: \( X \) Hausdorff space, \( Y \subseteq X \) compact \( \implies Y \) is closed.

We will prove, that \( X \setminus Y \) is open.

Consider \( x \in X \setminus Y \). \( \forall y \in Y \) chose \( U_y, V_y \) open in \( X \) with \( x \in U_y, y \in V_y \) and \( U_y \cap V_y = \emptyset \) (this can be done, because \( X \) is Hausdorff). Then \( \{Y \cap V_y\}_{y \in Y} \) is an open covering of \( Y \). \( Y \) is compact, so \( \exists \{Y \cap V_{y_i}\}_{i \in \{1, \ldots, n\}} \) a finite subcovery of \( Y \). Note that \( Y \subseteq V_{y_1} \cup \cdots \cup V_{y_n} =: V \). \( U_x := U_{y_1} \cap \cdots \cap U_{y_n} \). Then \( U_x \cap V = \emptyset \). Indeed, if \( z \in U_x \cap V \), then \( z \in U_{y_i} \forall i \in \{1, \ldots, n\} \) and \( z \in V_{y_j} \) for some \( j \in \{1, \ldots, n\} \) but \( V_{y_i} \cap U_{y_j} = \emptyset \). So \( U_x \cap V = \emptyset \). We have constructed \( U_x \) open in \( X \) with \( x \in U_x \) and \( U_x \cap Y \subseteq U_x \cap V = \emptyset \), i.e. \( U_x \subseteq X \setminus Y \).

So \( X \setminus Y = \bigcup_{x \in X \setminus Y} U_x \) and \( X \setminus Y \) is open.
Remark 1.68 \((X, \mathcal{T})\) topological space, \(x \in X\). Some authors (e.g. Munkres) define: A neighbourhood of \(X\) is a set \(U \in \mathcal{T}\) such that \(x \in U\). Other authors (e.g. Janich) define: A neighbourhood of \(X\) is a set \(U \subseteq X\) such that \(\exists V \in \mathcal{T}\) with \(x \in V \subseteq U\).

**Theorem 1.69** Let \(f : X \to Y\) be a continuous bijective function. If \(X\) is compact, and \(Y\) is Hausdorff, then \(f\) is a homeomorphism.

**Proof** We have to prove that \(f^{-1}\) is continuous. Equivalently, tat if \(U\) is open in \(X\), then \((f^{-1})^{-1}(U)\) is open in \(Y\). Equivalently, that if \(U\) is closed in \(X\), then \(f(X)\) is closed in \(Y\).

Indeed, \(U\) is closed in \(X\) and \(X\) compact, so \(f(U) \subseteq Y\) is compact. \(Y\) is Hausdorff, therefore \(f(U)\) is closed in \(Y\). 

Local connectedness and local path connectedness

**Definition 1.71** \(X\) is called locally (path) connected at \(x \in X\) if for every open neighbourhood \(U\) of \(x\) there is a (path) connected open neighbourhood \(V\) of \(x\) with \(V \subseteq U\). If \(Y\) is locally (path) connected at every \(x \in X\), then \(X\) is called locally (path) connected.
2 Algebraic topology

Determining whether two spaces are homeomorphic and studying continuous functions between topological spaces are two of the central problems in topology.

To show that $X, Y$ are homeomorphic, we need to construct $f : X \rightarrow Y$ bijective, continuous, with $f^{-1}$ continuous.

To show that $X, Y$ are not homeomorphic, we can for instance show that $X$ has a topological property (e.g. connectedness or compactness) but $Y$ does not have this property.

Any continuous map between topological spaces induces a homeomorphism between their fundamental groups.

$x_0 \in X \xrightarrow{f} 1_{(X, x_0)}$ fundamental group. We will show that if $X \simeq Y$ then their fundamental group is isomorphic.

2.1 Fundamental group

2.1.1 Path homotopy

Notation: $I := [0, 1]$.

**Definition 2.1** Let $f : X \rightarrow Y, f' : X \rightarrow Y$ be continuous maps. $f, f'$ are called homotopic if there exists a continuous map $F : X \times I \rightarrow Y$ with $F(x, 0) = f(x)$ $\forall x \in X$ and $F(x, 1) = f'(x)$ $\forall x \in X$.

$F$ is called a homotopy between $f$ and $f'$.

If $f, f'$ are homotopic, we write $f \simeq f'$.

If $f$ is homotopic to a constant map then $f$ is called nullhomotopic.

We focus on the special case $X = [a, b]$.

Then if $f : [a, b] \rightarrow Y$ is a continuous function, $f$ is called a path. $f(a)$ is called the initial point, and $f(b)$ the final point.

WLOG we can assume, that the domain of $f$ is $[0, 1]$.

**Definition 2.2** Let $f : I \rightarrow X, f' : I \rightarrow X$ two paths in $X$. $f, f'$ are called path homotopic if $f(0) = f'(0), f(1) = f'(1)$ and there exists continuous map $F : I \times I \rightarrow X$, with $F(x, 0) = f(x) \forall x \in I$ and $F(x, 1) = f'(x) \forall x \in I$ and $F(0, t) = f(0) \forall t \in I, F(1, t) = f'(1) \forall t \in I$.

If $f, f'$ are path homotopic, we write $f \simeq_p f'$.

Claim: $\simeq_p$ and $\simeq_p$ are equivalence relations.

**Proof** We prove this for $\simeq_p$.

1. $f \simeq_p f'$:
   \[ F(x, t) := f(x). \]
2. \( f \simeq_p f' \implies f' \simeq_p f \):

If \( F \) is a path homotopy from \( f \) to \( f' \), then \( G(x, t) = F(x, 1 - t) \) is a path homotopy from \( f' \) to \( f \).

3. \( f \simeq_p f', f' \simeq_p f'' \implies f \simeq_p f'' \):

If \( F \) is a path homotopy from \( f \) to \( f' \) and \( F' \) is a path homotopy from \( f' \) to \( f'' \), then

\[
G(x, t) = \begin{cases} 
F(x, 2t) & t \in [0, 1/2] \\
F'(x, 2t - 1) & t \in [1/2, 1]
\end{cases}
\]

is a path homotopy from \( f \) to \( f'' \). \( G(x, 1/2) = F'(x, 1) = F'(x, 0) = f'(x) \). i.e. \( G \) is well defined. Is \( G \) continuous? \( G \) is continuous in \( I \times [0, 1/2] \) and \( G \) is continuous in \( [1/2, 1] \) \[\implies\] pasting lemma (lemma 1.21) \( G \) is continuous.

\[
G(0, t) = \begin{cases} 
F(0, 2t) & t \in [0, 1/2] \\
F'(0, 2t - 1) & t \in [1/2, 1]
\end{cases} = \begin{cases} 
f(0) = f'(0) \\
f'(0) = f''(0)
\end{cases}
\]

\( G(1, t) = f(1) \),
\( G(x, 0) = f(x) \forall x \in I \) (since \( G(x, 0) = F(x, 0) = f(x) \forall x \in I \).
\( G(x, 1) = f''(x) \forall x \in I \) (since \( G(x, 1) = F'(x, 1) = f''(x) \forall x \in I \).

If \( f \) is a path, we denote its homotopy class by \([f]\).

**Example 2.4**

1. Let \( f : I \to \mathbb{R}^2, g : I \to \mathbb{R}^2 \), paths with \( f(0) = g(0) \) and \( f(1) = g(1) \). Then \( F : I \times I \to \mathbb{R}^2 \) with \((x, t) \mapsto (1 - t)f(x) + tg(x)\) is a path homotopy between \( f \) and \( g \).

   \( F \) is called a linear homotopy.

2. \( f : I \to \mathbb{R}^2, s \mapsto (\cos(\pi s), \sin(\pi s)), g : I \to \mathbb{R}^2, t \mapsto (\cos(\pi t), 3\sin(\pi t)) \).

   \( f(0) = (1, 0) = g(0) \), and \( f(1) = (-1, 0) = g(1) \).

   \( f, g \) are path homotopic. Indeed, consider the linear homotopy \( F \) described in 1.

3. \( f : I \to \mathbb{R}^2 \setminus \{(0, 0)\} \) with \( s \mapsto (\cos(\pi s), \sin(\pi s)) \), and \( g : I \to \mathbb{R}^2 \setminus \{(0, 0)\} \) with \( t \mapsto (\cos(\pi t), \sin(\pi t)) \).

   Are \( f, g \) path homotopic? Yes. Prove using the linear homotopy.

4. \( f : I \to \mathbb{R}^2 \setminus \{(0, 0)\} \), with \( s \mapsto (\cos(\pi s), \sin(\pi s)) \), and \( g : I \to \mathbb{R}^2 \setminus \{(0, 0)\} \) with \( t \mapsto (\cos(\pi t), -3\sin(\pi t)) \).

   The linear homotopy is not a path homotopy between \( f \) and \( g \).

   Infact, there exists no path homotopy between \( f \) and \( g \).

We define the following operation:

**Definition 2.5** Let \( f : I \to X \) path in \( X \) with \( f(0) = x_0, f(1) = x_1, x_0, x_1 \in X \).

\( g : I \to X \) path in \( X \) with \( g(0) = x_1, g(1) = x_2, x_2 \in X \).

We define the product \( f \cdot g \) of \( f, g \) as the path \( h \) with

\[
h(s) = \begin{cases} 
f(2s) & s \in [0, 1/2] \\
g(2s - 1) & s \in [1/2, 1]
\end{cases}
\]
Remark 2.6 \( h \) is well defined, since \( f(1) = g(0) = h(1/2) \).

\( h \) is continuous by the pasting lemma (lemma 1.21).

The product \( f \cdot g \) induces a well defined product on path homotopy equivalence classes defined by \([f] \cdot [g] = \{f \cdot g\} \).

Indeed, if \( f, f' \) is a path in X with \([f] = \{f'\} \), and \( g, g' \) is a path in X with \([g] = \{g'\} \). Consider path homotopies \( F, G \) between \( f \) and \( f' \), \( g \) and \( g' \) respectively. Then

\[
H(s, t) := \begin{cases} 
F(2s, t) & t \in [0, 1/2] \\
G(2s - 1, t) & s \in [1/2, 1] 
\end{cases}
\]

is a path homotopy between \( f \cdot g \) and \( f' \cdot g' \) (homework).

Therefore \([f \cdot g] = \{f' \cdot g'\}\), i.e. \([f] \cdot [g] = \{f' \cdot g'\}\) and the induced product is well-defined.

Reparametrisation

Define a reparametrisation of a path \( f \) to be a composition \( f \circ \varphi \), where \( \varphi : I \to I \) continuous map, such that \( \varphi(0) = 0 \), and \( \varphi(1) = 1 \).

Reparametrisation a path preserves its homotopy class (i.e. \([f \circ \varphi] = \{f\}\)). Indeed, consider the path homotopy \( f \circ \varphi_t \) where \( \varphi_t(s) := (1-t)\varphi(s) + ts \).

\[
\varphi_0(s) = \varphi(s) \Rightarrow f \circ \varphi_0 = f \circ \varphi, \varphi_1 = s \Rightarrow f \circ \varphi_1 = f
\]

Note that \((1-t)\varphi(s) + ts\) lies between \( \varphi(s) \) and \( s \), hence \((1-t)\varphi(s) + ts\) lies in \( I \), so \( f \circ \varphi_t \) is defined.

2.2. The fundamental group

We restrict our attention to paths \( f : I \to X \) with \( f(0) = f(1) \). Call \( x_0 := f(0) = f(1) \).

Such paths are called loops in \( X \) at the basepoint \( x_0 \). The set \( \{[f] | f : I \to X \} \) is denoted by \( \pi_1(X, x_0) \).

If \([f], [g] \in \pi_1(X, x_0)\), then \( f, g \) are loops in \( X \) at the basepoint \( x_0 \), so \( f \cdot g \) is defined.

The induced multiplication \([f] \cdot [g] = \{f \cdot g\}\) is well-defined.

Proposition 2.7 \( \pi_1(X, x_0) \) is a group.

**Proof**

1. associativity: \([f] \cdot ([g] \cdot [h]) = ([f] \cdot [g]) \cdot [h] \), \([f], [g], [h] \in \pi_1(X, x_0)\).

\[
\iff [f] \cdot [g \cdot h] = [f \cdot g] \cdot [h] \\
\iff [f \cdot (g \cdot h)] = ([f \cdot g]) \cdot [h]
\]

Claim: \( f \cdot (g \cdot h) \) is a reparametrisation of \( (f \cdot g) \cdot h \) by the piecewise linear function \( \varphi \): where

\[
\varphi(t) = \begin{cases} 
\frac{1}{2}t & t \in [0, 1/2] \\
\frac{1}{4}t - \frac{1}{4} & t \in [1/2, 3/4] \\
2t - 1 & t \in [3/4, 1]
\end{cases}
\]

Proof:

\[
(f \cdot (g \cdot h))(t) = \begin{cases} 
f(2t) & t \in [0, 1/2] \\
g(4t - 2) & t \in [1/2, 3/4] \\
h(4t - 3) & t \in [3/4, 1]
\end{cases}
\]

Analogously

\[
((f \cdot g) \cdot h)(s) = \begin{cases} 
(f \cdot g)(2s) & s \in [0, 1/2] \\
h(2s - 1) & s \in [1/2, 1]
\end{cases}
\]

\[
\begin{cases} 
f(4s) & s \in [0, 1/4] \\
g(4s - 1) & s \in [1/4, 1/2] \\
h(2s - 1) & s \in [1/2, 1]
\end{cases}
\]
2.2 The fundamental group

So if one easily checks, that \(((f \circ g) \cdot h)(s) = (f \cdot (g \circ h))(s)\).

2. Identity: Consider \(c : I \to X\) with \(c(s) = x_0 \forall s \in I\). Then \(f \cdot c\) is a reparametrisation of \(f\) via \(\varphi\):

\[
\varphi(t) = \begin{cases} 
    2t & t \in [0, 1/2] \\
    1 & t \in [1/2, 1]
\end{cases}
\]

i.e. \(f \cdot c = f \circ \varphi\).

\([f] = [f \circ \varphi] = [f \cdot c]\).

\([f] = [\varphi] \cdot [c]\).

3. inverse: Consider \([f] \in \pi_1(X, x_0)\) and \(\overline{f}(s) = f(1 - s)\).

Claim: \(f \cdot \overline{f} \simeq c \simeq f \cdot f\).

Proof: Consider the homotopy

\[
H(s, t) = \begin{cases} 
    f(2s) & s \in [0, (1-t)/2] \\
    f(1 - t) & [(1-t)/2, (1+t)/2] \\
    f(2 - 2s) & [(1 + t)/2, 1]
\end{cases}
\]

\(\varphi(0, t) = f(0) = x_0\),

\(\varphi(1, t) = f(0) = x_0\).

\(\varphi(s, 0) = \begin{cases} 
    f(2s) & s \in [0, 1/2] \\
    f(2 - 2s) & s \in [1/2, 1]
\end{cases} = f \cdot \overline{f}\)

\(\varphi(s, 1) = x_0\).

\(\varphi \) is a homotopy between \(f \cdot \overline{f}\) and \(c\) i.e. \([f] \cdot [\overline{f}] = [c]\).

Analogously \([\overline{f}] \cdot [f] = [c]\).

\[\text{ imagining } [\overline{f}] \cdot [f] = [c].\]

\[\Rightarrow [\overline{f}] \cdot [f] = [c].\]

\[\Rightarrow \text{ is the inverse of } [f].\]

\[\text{ Definition 2.9 } \pi_1(X, x_0) \text{ is called the fundamental group of } X \text{ at } x_0.\]

\[\text{ Example 2.10 Consider } X \subseteq \mathbb{R}^n \text{ convex. } x_0 \in X.\]

\[\pi_1(X, x_0) = \{[f] \mid f : I \to X \text{ path, } f(0) = f(1) = x_0\}.\]

If \(f_0, f_1\) are two paths in \(X\) at the basepoint \(x_0\), then \(f_t(s) = (1 - t)f_0(s) + tf_1(s)\) is a linear homotopy between \(f_0\) and \(f_1\).

Let \(x_0, x_1 \in X, \pi_1(X, x_0), \pi_1(X, x_1)\).

\[\text{ Proposition 2.11 If } x_0, x_1 \in X \text{ are in the same path component of } X, \text{ then } \pi_1(X, x_0), \pi_1(X, x_1) \text{ are isomorphic.}\]

\[\text{ Proof } x_0, x_1 \in X. \text{ Consider } h : I \to X \text{ a path with } h(0) = x_0, h(1) = x_1 \text{ and } \overline{h} : I \to X, \overline{h}(s) = h(1 - s).\]

Define \(\hat{h} : M_1(X, x_1) \to \pi_1(X, x_0) \text{ where } [f] \mapsto [h \cdot f \cdot \overline{h}].\)

- \(\hat{h}\) is well-defined: If \([f] = [g] \in \pi_1(X, x_1)\) and \(f_t\) is a homotopy between \(f\) and \(g\) then \(h \cdot f_t \cdot \overline{h}\) is a homotopy between \(h \cdot f \cdot h\) and \(h \cdot g \cdot \overline{h}\) i.e. \([h \cdot f \cdot h] = [h \cdot g \cdot h]\).

- \(\hat{h}\) is a homomorphism: \(\hat{h}([f] \cdot [g]) = \hat{h}([f]) \cdot \hat{h}([g]) = [h \cdot f \cdot h] \cdot [h \cdot g \cdot h] = [h \cdot f \cdot h][h \cdot g \cdot h] = \hat{h}([f]) \cdot \hat{h}([g]).\)
• $\hat{h}$ is bijective: $\hat{h} = (\hat{h})^{-1}$.
  $$(\hat{h}\hat{h})[f] = \hat{h}(\hat{h}(f)) = \hat{h}([\hat{h}fh]) = \cdots = [f].$$

**Definition 2.13** $X$ is simply connected if it is path connected and $\pi_1(X)$ is trivial.

**Proposition 2.14** $X$ is simply connected iff there is a unique homotopy class of paths connecting any two points in $X$.

**Proof** Path connectedness is the existence of paths connecting every pair of points in $X$. So we only need to check uniqueness.

“$\Rightarrow$” Suppose $\pi_1(X) = 0$. Let $x_0, x_1 \in X$. If $f, g$ are two paths from $x_0$ to $x_1$, then $f \simeq f \circ g \simeq g \Rightarrow [f] = [g]$.

“$\Leftarrow$” If there is a unique homotopy class of paths connecting any two points, then there is a unique homotopy class of paths connecting a point to itself. $\Rightarrow \pi_1(X) = 0$.

**Induced homomorphisms**

$X, Y$ topological spaces, $x_0 \in X, y_0 \in Y$ and $h : X \to Y$ a continuous map with $h(x_0) = y_0$. We will denote this by $h : (X, x_0) \to (Y, y_0)$.

If $f : I \to X$ is a loop in $X$, based at $x_0$, then $h \circ f$ is a loop in $Y$ based at $y_0$.

**Definition 2.16** Let $h : (X, x_0) \to (Y, y_0)$ continuous. Define $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ where $[f] \mapsto [h \circ f]$. $h_*$ is called the homomorphism induced by $h$ relative to $x_0$.

**Remark 2.17** $h_*$ is well-defined.

If $f, g$ loops in $X$ based at $x_0$ with $[f] = [g]$. Consider $H$ a homotopy from $f$ to $g$. Then $h \circ H$ is a homotopy from $h \circ f$ to $h \circ g$.

I.e. $[h \cdot f] = [h \cdot g] \iff h_*([f]) = h_*([g])$.

**Remark 2.18** $h_*$ is a group homomorphism.

Just calculate and compare the following two expressions:

$$h_*([f] \cdot [g]) = h_*([f \cdot g]) = [h \circ (f \cdot g)]$$
$$h_*([f]) \cdot h_*([g]) = [h \circ f] \cdot [h \circ g] = [(h \circ f) \cdot (h \circ g)].$$

**Remark 2.19** $h_*$ depends on the basepoint $x_0$. So strictly speaking we should have written $(h_{x_0})_*$.

**Properties of induces homomorphisms**

1. If $h : (X, x_0) \to (Y, y_0), g : (Y, y_0) \to (Z, z_0)$ continuous, then $(g \circ h)_* = g_* \circ h_*$.

2. If $1_X : (X, x_0) \to (X, x_0)$, then $(1_x)_* = 1_{\pi_1(X, x_0)}$.

**Proposition 2.20** If $H : (X, x_0) \to (Y, y_0)$ a homoeomorphism, then $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is a group isomorphism.
2.3 Covering spaces

Proof Consider $h^{-1}(Y, y_0) \to (X, x_0)$ and $(h^{-1})_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$

$(h^{-1})_* \circ h_* = (h^{-1} \circ h)_* = 1, = 1.$
$h_* \circ (h^{-1})_* = (h \circ h^{-1})_* = 1, = 1.$

$\pi_1$ gives a covariant functor from the category with objects topological spaces with basepoint and morphism basepoint preserving continuous maps to the category with objects groups and morphisms group homomorphisms.

A category $\mathcal{D}$ consists of

1. a collection of objects $\text{Ob}(\mathcal{D})$,
2. sets of morphisms $\text{Mor}(X, Y)$ for each $X, Y \in \text{Ob}(\mathcal{D})$ with a distinguished identity morphism $1_X \in \text{Mor}(X, X)$,
3. a composition of morphisms $\circ : \text{Mor}(X, Y) \times \text{Mor}(Y, Z) \to \text{Mor}(X, Z)$ for each triple $X, Y, Z \in \text{Ob}(\mathcal{D})$ with the properties $f \circ 1 = 1 \circ f = f$ and $(f \circ g) \circ h = f \circ (g \circ h)$.

Example 2.22

1. $\text{Ob}(\mathcal{D}) = \{G \mid G \text{ group}\}$.
   \[
   \text{Mor}(G, H) = \{f : G \to H \mid f \text{ group homomorphism}\}.
   \]
2. $\text{Ob}(\mathcal{D}) = \{(X, x_0) \mid X \text{ topological space, } x_0 \text{ basepoint}\}$.
   \[
   \text{Mor}((X, x_0), (Y, y_0)) = \{f : (X, x_0) \to (Y, y_0) \mid f \text{ continuous}\}.
   \]

A covariant functor $F$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ assigns to each $X \in \text{Ob}(\mathcal{C})$ an $F(X) \in \text{Ob}(\mathcal{D})$ to each $f \in \text{Mor}(X, Y)$ a $F(f)$ such that $F(1_X) = 1_{F(X)}$ and $F(f \circ g) = F(f) \circ F(g)$.

Example 2.23 $\pi_1$ is a covariant functor from category $\mathcal{C}_2$ in example 2 to the category $\mathcal{C}_1$ in example 1.

$(X, x_0) \in \text{Ob}(\mathcal{C}_2) \xrightarrow{\pi_1} \pi_1(X, x_0) \in \text{Ob}(\mathcal{C}_1)$.

$f \in \text{Mor}((X, x_0), (Y, y_0)) \xrightarrow{\pi_1} f_* \in \text{Mor}(\pi_1(X, x_0), (Y, y_0))$

$(1_{(X, x_0)})_* = 1_{\pi_1(X, x_0)}$.

$(f \circ g)_* = f_* \circ g_*$.}

2.3. Covering spaces

- useful for computing $\pi_1$,
- algebraic features of $\pi_1$ can be translated into geometric features of the spaces.

Definition 2.24 Consider $p : \tilde{X} \to X$ a continuous and surjective map. If there exists an open cover $\{U_a\}_{a \in A}$ of $X$ such that $\forall a \in A, p^{-1}(U_a) = \bigcup_{b \in B_a} V_b^a$, where $V_b^a \cap V_{b'}^a = \emptyset$ for any $b, b' \in B_a$, $b \neq b'$, $p : V_b^a \to U_a$ is a homeomorphism $\forall b \in B_a$ and $V_b^a$ open in $\tilde{X}$, $\forall b \in B_a$, then $p$ is called a covering map. $\tilde{X}$ is called a covering space of $X$.

Example 2.25
1. \( S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \} \). \( p : \mathbb{R} \to S^1 \) where \( s \mapsto (\cos 2\pi s, \sin 2\pi s) \).

   \( p \) is a covering map, \( p \) is continuous (calculus), and \( p \) is surjective.

   \( U_1 := \{(x, y) \in S^1 \mid x > -\sqrt{2}/2\} \)

   \( U_2 := \{(x, y) \in S^1 \mid x < \sqrt{2}/2\} \).

\( \{U_1, U_2\} \) is an open cover of \( S^1 \)

\( p^{-1}(U_1) = \cup_{n \in \mathbb{Z}} V_n \), where \( V_n := (n - 3/8, n + 3/8) \).

\( V_n \) is open in \( \mathbb{R} \) \( \forall n \in \mathbb{Z} \).

\( V_n \cap V_n = \emptyset \), for any \( n, m \in \mathbb{Z}, n \neq m \).

\( p : V_n \to U_1 \) is a homeomorphism \( \forall n \in \mathbb{Z} \).

We can make the analogous construction for \( U_2 \). So \( p \) is a covering map.

**Remark 2.26.** You can prove that \( p \) is a covering by using any open cover of \( S^1 \) by two open subsets of \( S^1 \neq S^1 \).

2. \( S^1 = \{z \in \mathbb{C} \mid |z| = 1\}, n \in \mathbb{N}, n \geq 1 \).

   \( p_n : S^1 \to S^1, z \mapsto z^n \). \( p_n \) is a covering map (problem sheet 5).

   \( f : X \to Y \) is called an **embedding** if \( f : X \to f(X) \) is a homeomorphism.

Consider the solid torus \( S^1 \times D^2 \) where \( D^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\} \), and its boundary \( \partial(S^1 \times D^2) = S^1 \times S^1 \).

Consider \( f : S^1 \to \partial(S^1 \times D^2) \) an embedding such that \( f(S^1) \) wraps around the first \( S^1 \) three times.

Lastly consider the projection \( \pi : S^1 \times D^2 \to S^1 \times \{(0, 0)\} \) and restrict it to \( f(S^1) \).

3. \( f : \mathbb{R}^+ \to S^1 \), where \( s \mapsto (\cos 2\pi s, \sin 2\pi s) \) is not a covering map.

Consider an open cover \( \{U_a\}_{a \in A} \) of \( S^1 \). Then \( \exists U \in \{U_a\}_{a \in A} \) with \( x \in U \). \( p^{-1}(U) = V_0 \cup (\cup_{a \in B} V_a) \).

\( p : V_0 \to U \) is not a homeomorphism.

**Theorem 2.27** If \( p : \tilde{X} \to X \) is a covering map, \( X_0 \subseteq X \), and \( \tilde{X}_0 = p^{-1}(X_0) \), then \( p_0 : \tilde{X}_0 \to X_0 \) obtained by restricting \( p \) to \( \tilde{X}_0 \) is a covering map.

**Proof** \( p_0 \) is continuous (restricting the domain or the range of a continuous function gives a continuous function). \( p_0 \) is surjective.

Consider an open cover \( \{U_a\}_{a \in A} \) of \( X \) with the properties in the definition of covering map.

22
Then \( \forall a \in A \ p^{-1}(U_a) = \cup_{b \in B_a} V_b \).

If \( x_0 \in X_0 \), then there exists \( a \in A \) with \( x_0 \in U_a \). Now \( U_a \cap X_0 \) is an open set in \( X_0 \).

\[
p^{-1}(U_a \cap X_0) = p^{-1}(U_a) \cap p^{-1}(X_0) = (\cup_{b \in B_a} V_b) \cap \overline{X_0} = (\cup_{b \in B_a} V_b) \cap \overline{X_0} = (\cup_{b \in B_a} V_b) \cap \overline{X_0} = \cup_{b \in B_a} (V_b \cap \overline{X_0})
\]

We have

- \( V_b \cap \overline{X_0} \) open in \( \overline{X_0} \),
- \( (V_b \cap \overline{X_0}) \cap (V_{b'} \cap \overline{X_0}) = \emptyset \) for any \( b, b' \in B_a, b \neq b' \).
- \( p_0 : V_b \cap \overline{X_0} \to U_a \cap X_0 \) is a homeomorphism.

\( \{U_a \cap X_0\}_{a \in A} \) is an open cover of \( X_0 \) with the desired properties \( \implies p_0 : \overline{X_0} \to X_0 \) is a covering map.

**Theorem 2.29** If \( p : X \to X' \), \( p' : \overline{X}' \to X' \) are covering maps, then \( p \times p' : \overline{X} \times \overline{X}' \to X \times X' \) is a covering map.

**Example 2.30**

1. Consider \( p : \mathbb{R} \to S^1 \), where \( s \mapsto (\cos 2\pi s, \sin 2\pi s) \).

   The map \( p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1 \) is a covering map.

2. \( X_0 = (S^1 \times p(0)) \cup (p(0) \times S^1) \).
   \( \overline{X_0} = (p \times p)^{-1}(X_0) = (\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R}) \).
   \( (p \times p)_0 : \overline{X_0} \to X_0 \) is a covering map (by theorem).
   Two circles \( S^1 \) with a one common point is called wedge of two circles (notation \( S^1 \vee S^1 \)).

3. Other covering spaces of \( S^1 \vee S^1 \).

Consider \( \overline{X} \) as in the following picture

Then \( p : \overline{X} \to S^1 \vee S^1 \), where \( x_1 \mapsto x \) and \( x_2 \mapsto x \). \( p \) maps each edge of \( \overline{X} \) to the edge of \( X \) with the same label by a map that is a homeomorphism and preserves the orientation.
4. Consider \( p \times 1_{\mathbb{R}} : \mathbb{R} \times \mathbb{R} \to S^1 \times \mathbb{R} \) (where \( p : \mathbb{R} \to S^1 ; s \mapsto (\cos 2\pi s, \sin 2\pi s) \)) and \( f : S^1 \times \mathbb{R} \to \mathbb{R}^2 \setminus \{(0, 0)\} \), where \((x, t) \mapsto t \cdot x\).

\( f \) is a homeomorphism.

\[ f \circ (p \times 1_{\mathbb{R}}) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2 \setminus \{(0, 0)\} \]

\( f \) is a covering map.

**Proof.** \( p \times p' \) is continuous (problem sheet 3). \( p \times p' \) is surjective (if \((x, x') \in X \times X'\), then \( x \in X \) and \( x' \in X' \implies (p, p') \) is surjective) \( \exists \overline{x}, \overline{x}' \in \overline{X}, \overline{x}' \in \overline{X}' \), such that \( p(\overline{x}) = x \)

and \( p'(\overline{x}') = x' \implies \exists (\overline{x}, \overline{x}') \in \overline{X} \times \overline{X}' \) such that \( f(p \times p'(\overline{x}, \overline{x}')) = (x, x') \).

Consider \( \{U_a\}_{a \in A}, \{U'_a\}_{a \in C} \) open covers of \( X \) and \( X' \) as in the definition of covering maps.

If \((x, x') \in X \times X'\) then \( x \in X \) and \( x' \in X' \) i.e. \( \exists U \in \{U_a\}_{a \in A} \) with \( x \in U \) and \( \exists U' \in \{U'_a\}_{a \in C} \) with \( x' \in U' \).

\( p^{-1}(U) = \bigcup_{b \in B} V_b, V_b \) open \( V_b \cap V_{b'} = \emptyset \) for \( b \neq b' \) and \( p : V : b \to U \) homeomorphism \( \forall b \in B, (p')^{-1}(U') = \bigcup_{d \in D} V'_d, V'_d \) open, \( V'_d \cap V'_{d'} = \emptyset \) for \( d \neq d' \) and \( p' : V'_d \to U' \) homeomorphism \( \forall d \in D \).

\((x, x') \in U \times U', U \times U' \) is open in \( X \times X' \),

\((p \times p')^{-1}(U \times U') = \bigcup_{i \in I} (V_b \times V')_i \), where \( V_b \times V'_d \) open in \( X \times X' \), \( V_b \times V'_d \) disjoint, \( p \times p' : V_b \times V'_d \to U \times U' \) is a homeomorphism. I.e. \( \{U_a \times U'_a\}_{a \in A} \) is the open cover of \( X \times X' \). That shows that \( p \times p' \) is a covering map. ■

### 2.4. Lifting properties

We will discuss two properties of covering spaces.

**Definition 2.32** Let \( p : \overline{X} \to X \) be a map. If \( f : Y \to X \) is continuous a **lifting of \( f \)** is a map \( \overline{f} : Y \to \overline{X} \) such that \( p \circ \overline{f} = f \).

**Example 2.33** \( p : \mathbb{R} \to S^1, s \mapsto (\cos 2\pi s, \sin 2\pi s) \), \( f : I \to S^1 \) the path \( f(s) = (\cos \pi s, \sin \pi s) \).

Then \( \overline{f} : I \to \mathbb{R}, \overline{f}(s) = \frac{s}{2} \) is a lifting of \( f \).

**Proposition 2.34** (path lifting property)

Let \( p : \overline{X} \to X \) be a covering map with \( p(\overline{x}_0) = x_0 \). If \( f : I \to X \) is a path with \( f(0) = x_0 \), then \( \exists ! \) lifting of \( f \) to a path \( \overline{f} : I \to \overline{X} \) with \( \overline{f}(0) = \overline{x}_0 \).

**Proof**

1. Consider the open cover \( \{U_a\}_{a \in A} \) of \( X \) (as in the definition of covering map). Let \( t \in (0, 1) \), and \( f(t) \in X \implies f(t) \in U_a \) for some \( a \in A \). \( f : I \to X \) is continuous \( \implies \exists (a_t, b_t) \subseteq (0, 1) \) with \( t \in (a_t, b_t) \) and \( f([a_t, b_t]) \subseteq U_a \).

Analogously for \( t = 0 \exists [0, b_0] \subseteq [0, 1] \) with \( f([0, b_0]) \subseteq U_a \), and for \( t = 1 \exists (a_0, 1] \subseteq [0, 1] \) with \( f([a_0, 1]) \subseteq U_a \).

\( I \) is compact, and \( \{(a_i, b_i)\}_{i \in \{0, 1\}} \) together with \( [0, b_0] \) and \( (b_1, 1] \) is an open cover of \( I \). We can choose a finite subcover, say \( [0, b_0], (a_0, 1], (a_1, b_1), \ldots, (a_m, b_m) \).

Consider \( \{(a_i, b_i)\}_{i \in \{0, \ldots, m\}} \) and order its elements. (Possibly) rename the elements as follows: \( 0 < a_1 < \cdots < a_{2m+2} < 1 \). this gives a subdivision \( 0 = s_0 < s_1 < \cdots < s_n = 1 \) of \( I \) with the property, that \( \forall i \in \{0, \ldots, n - 1\} f([s_i, s_{i+1}]) \subseteq U_a \) for some \( a \in A \).
2. Define $\tilde{f}(0) = \tilde{x}_0$. Suppose that $\tilde{f}$ is defined in $[0, s_1]$. Define $\tilde{f}$ in $[s_1, s_{i+1}]$ as follows: $\tilde{f}([s_1, s_{i+1}]) \subseteq U$ for some $U$ in $\{U_a\}_{a \in A}$.

Let $p^{-1}(U) = \cup b \in B V_b$. Now $(p \circ \tilde{f})(s_i) = f(s_i) \in U$.

$\implies \tilde{f}(s_i) \in p^{-1}(U) = \cup b \in B V_b$. $\implies \tilde{f}(s_i) \in V_0$ for some $V_0 \in \{V_b\}_{b \in B}$.

Define $\tilde{f}(s) = (p_1 V_b)^{-1}(f(s)), s \in [s_i, s_{i+1}]$.

$\tilde{f}$ is continuous in $[s_i, s_{i+1}]$, since $p_1 V_b$ is a homeomorphism. Thus we have defined $\tilde{f} : [0, 1] \to \tilde{X}$ continuous with $\tilde{f}(0) = \tilde{x}_0$.

3. Suppose $\tilde{f}$ is another lifting of $f$ with $\tilde{f}(0) = x_0$. Then $\tilde{f}(0) = f(0) = x_0$.

Suppose that $\tilde{f}(s) = \tilde{f}(s)$ in $[0, s_i]$. $p \circ \tilde{f}([s_i, s_{i+1}]) = f([s_i, s_{i+1}]) \subseteq U_a$ for some $a \in A$.

$\implies \tilde{f}([s_i, s_{i+1}]) \subseteq p^{-1}(U_a) = \cup b \in B V_b$. $\tilde{f}([s_i, s_{i+1}])$ is connected and $\tilde{f}(s_i) = \tilde{f}(s_i) \in V_0$.

For $s \in S, p \circ \tilde{f}(s) = f(s)$

$\implies \tilde{f}(s) \in p^{-1}(f(s))$.

Also $\tilde{f} \in V_0$. So $\tilde{f}(s) = (p_1 V_b)^{-1}(f(s)) = \tilde{f}(s)$.

I.e. $\tilde{f}(s) = \tilde{f}(s)$.

**Proposition 2.36** Let $p : \tilde{X} \to X$ be a covering map with $p(\tilde{x}_0) = x_0$. Let $F : I \times I \to X$ be continuous with $F(0, 0) = x_0$ Then there exists a lifting of $F$ to a continuous map $\tilde{F} : I \times I \to \tilde{X}$ with $\tilde{F}(0, 0) = \tilde{x}_0$. If $F$ is a path homotopy then $\tilde{F}$ is a path homotopy.

**Proof** Consider $\{U_a\}_{a \in A}$ an open cover of $X$ with the properties as in the definition of covering map.

1. Define $\tilde{F}((0, 0)) = \tilde{x}_0$.

   Extend $\tilde{F}$ to $I \times \{0\}$ and $\{0\} \times I$ using the proposition from last time. Extend $\tilde{F}$ to $I \times I$, as follows: Choose subdivision $s_0 < s_1 < \cdots < s_m, t_0 < t_1 < \cdots < t_n$ of $I$ with the property that for each such rectangle $I_i \times J_j = [s_i, s_{i+1}] \times [t_j, t_{j+1}]$, $F(I_i \times J_j) \subseteq U_a$ for some $a \in A$. Define $\tilde{F}$ first in $I_i \times J_j$, then $I_2 \times J_1, \ldots, I_m \times J_1$, then $I_1 \times J_2, I_2 \times J_2$ and so on.

   Consider $I_{i_0} \times J_{j_0}$ and suppose that $\tilde{F}$ is defined on the union $A$ of the rectangles $I_i \times J_j$ with $j < j_0$ or $j = j_0$ and $i < i_0$.

   Consider $C = A \cap (I_{i_0} \times J_{j_0})$. Choose $U \in \{U_a\}_{a \in A}$ with $F(I_{i_0} \times J_{j_0}) \subseteq U$.

   $\tilde{F}$ is defined in $C$, $C$ is connected. So, $\tilde{F}(C)$ is connected, i.e. $\exists V_0$ such that $\tilde{F}(C) \subseteq V_0$.

   If $p_0 = p_{i_0} : V_0 \to U$, then $p_0 \circ \tilde{F}(x) = p \circ \tilde{F}(x) = F(x) \implies F(x) = (p_0)^{-1}(F(x))$.

   Define $\tilde{F}(x) = p_0^{-1}(F(x)) \forall x \in I_{i_0} \times J_{j_0}$.

   $\tilde{F}$ is continuous, according to the pasting lemma.

2. Check that at every step, there is a unique way to define $\tilde{F}$. 

25
3. If $F$ is a path homotopy, then $F((0) \times I) = x_0$. Also, $\overline{F}((0) \times I) \subseteq p^{-1}((x_0))$. $p^{-1}((x_0))$ has the discrete topology as a subspace of $\overline{X}$. Anlose $\overline{F}((0) \times I)$ is connected. Thus $\overline{F}((0) \times I) = \{x_0\}$.

Similarly $\overline{F}((1) \times I)$ is a set with one point.

Thus, $\overline{F}$ is a path homotopy.

\begin{proposition}[Homotopy lifting property]
Let $p : \overline{X} \to X$ covering map with $p(\overline{x}_0) = x_0$. Consider $f, g$ the paths in $X$ from $x_0$ to $x_1$ and $\overline{f}, \overline{g}$ their liftings to paths in $\overline{X}$ with $\overline{f}(0) = \overline{g}(0) = \overline{x}_0$. If $f \equiv g$, then $\overline{f} \equiv \overline{g}$.

\begin{proof}
Consider $F$ the path homotopy between $f$ and $g$. Then $F((0), 0)) = x_0$. Let $\overline{F} : I \times I \to \overline{X}$ the lifting of $f$ to $\overline{X}$ with $\overline{F}((0), 0)) = \overline{x}_0$. Then $F((0) \times I) = \{x_0\}$ and $\overline{F}((1) \times I) = \{\overline{x}_1\}$. $\overline{F}|_{I \times \{0\}}$ is a path in $\overline{X}$ starting at $\overline{x}_0$ that is a lifting of $F|_{I \times \{0\}}$. I.e. $\overline{F}|_{I \times \{0\}} = \overline{f}$ (uniqueness of path liftings).

$\overline{F}|_{I \times \{1\}} = \overline{g}$.

So $\overline{f}(1) = \overline{g}(1) = \overline{x}_1$ and $\overline{F} : I \times I \to \overline{X}$ is a path homotopy from $\overline{f}$ to $\overline{g}$.
\end{proof}

\begin{definition}
Let $p : \overline{X} \to X$ be a covering map with $p(\overline{x}_0) = x_0$. Given a loop $f$ in $X$ based $x_0$, let $\overline{f}$ be the lifting of $f$ in $\overline{X}$ with $\overline{f}(0) = \overline{x}_0$. Define

$$
\phi : \pi_1(X, x_0) \to p^{-1}((x_0)),
$$

$$
[f] \mapsto \overline{f}(1).
$$

$\phi$ is called the lifting correspondence derived from the map $p$.
\end{definition}

\begin{claim}
$\phi$ is well-defined.
\end{claim}

\begin{proof}
If $[f] = [f']$, then $f, f'$ are path homotopic. We can lift this homotopy to a path homotopy between $\overline{f}$ and $\overline{f}' \implies \overline{f}(1) = \overline{f}'(1)$.
\end{proof}

\begin{theorem}
1. If $p : \overline{X} \to X$ be covering map with $p(\overline{x}_0) = x_0$. If $\overline{X}$ is path connected, then $\phi$ is surjective.

2. If $\overline{X}$ is simply connected, then $\phi$ is bijective.
\end{theorem}

\begin{proof}
1. Let $\overline{x}_1 \in p^{-1}((x_0))$. Then since $\overline{X}$ is path connected, there is a path $\overline{f} : I \to \overline{X}$ with $\overline{f}(0) = \overline{x}_0$ and $\overline{f}(1) = \overline{x}_1$. Then $f = p \circ \overline{f}$ is a loop in $X$ based at $x_0$, since $f(0) = p(\overline{x}_0) = x_0, f(1) = p(\overline{x}_1) = x_0$ and $\phi([f]) = \overline{f}(1) = \overline{x}_1$.

2. We only need to check that $\phi$ is injective. Indeed consider $[f], [g] \in \pi_1(X, x_0)$ with $\phi([f]) = \phi([g])$. Then $\overline{f}(1) = \overline{g}(1)$, where $\overline{f}, \overline{g}$ are the liftings of $f, g$ with $\overline{f}(0) = \overline{g}(0) = \overline{x}_0$. Since $\overline{X}$ is simply connected, we have that $[f] = [g]$, i.e. there is a path homotopy $\overline{F}$ between $\overline{f}$ and $\overline{g}$. Then $F = p \circ \overline{F}$ is a path homotopy between $f$ and $g$, i.e $[f] = [g]$.
\end{proof}
2.5 The fundamental group of the circle and applications

Remark 2.44 $S^1$ is path connected, so we will write $\pi_1(S^1)$.

Theorem 2.45

$$(\pi_1(S^1), \cdot) \cong (\mathbb{Z}, +).$$

Proof Let $p : \mathbb{R} \to S^1$, $p(t) = (\cos 2\pi t, \sin 2\pi t)$. Denote $p(0) = (1, 0) = x_0$. Then $p^{-1}(\{x_0\}) = \mathbb{Z}$. Consider $\phi : \pi_1(S^1) \to \mathbb{Z}$, where $[f] \mapsto \tilde{f}(1)$.

By exercise 1 of sheet 3 $\mathbb{R}$ is simply connected. By the second point of the theorem $\phi$ is bijective.

It remains to show that $\phi$ is a group homomorphism, i.e. $\phi([f] \cdot [g]) = \phi([f]) + \phi([g])$.

Furthermore, $(f \cdot g)(s) = f(s) \cdot g(s)$.

Definition 2.47 A retraction of a $X$ onto $A \subset X$ is a continuous map $r : X \to A$ with $r(a) = a \ \forall a \in A$.

If such a map exists $A$ is called a retract of $X$.

Claim: If $A$ is a retract of $X$ and $i : A \hookrightarrow X$ is the inclusion map, then $i_*$ is injective.

Proof $r \circ i : A \to A. r \circ i = 1_A$

$$(r \circ i)_* = (1_A)_*.$$ 

$r_* \circ i_* = 1_{\pi_1(X, x_0)}$

$i_*$ is injective.

Proposition 2.49 There is no retraction of $D^2$ onto $S^1$.

Proof If $S^1$ was a retract of $D^2$ and $i : S^1 \to D^2$ the inclusion. Then $i_* : \pi_1(S^1) \to \pi_1(D^2)$ would be injective. But $\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_1(D^2)$ is trivial as a convex subset of $\mathbb{R}^2$.

Lemma 2.51 Let $h : S^1 \to X$ be a continuous map. The following are equivalent:

1. $h$ is nullhomotopic.
2. There exists a continuous map $\tilde{h} : D^2 \to X$ with $\tilde{h}|_{S^1} = h$.
3. $h_*$ is trivial.

Proof
1. “1.⇒2.” Consider the homotopy \( H : S^1 \times I \to X \) between \( h \) and the constant map \( c \). Let \( q : S^1 \times I \to D^2 \) with \( (x,t) \mapsto t \cdot x \).

\( q \) is continuous, \( q \) is open and surjective.

2. 

3. 

### Vorlesung von Di 17. März fehlt noch

**Remark 2.53** Ha the corollary: If \( S^2 = A_1 \cup A_2 \cup A_3 \), \( A_1 \) closed \( \forall i \in \{1, 2, 3\} \), then \( \exists x \in S^2, \ i \in \{1, 2, 3\} \) such that \( \{x, -x\} \subseteq A_i \).

Inscribe a sphere in the tetrahedron. Project each face of the tetrahedral radially onto the sphere. Then one obtains four closed sets \( A_i, i \in \{1, \ldots, 4\} \) with \( S^2 = A_1 \cup A_2 \cup A_3 \cup A_4 \), but none of the \( A_i \) contains a pair of antipodal points.

#### 2.5.1. The Fundamental Theorem of Algebra

(Any non-constant polynomial in \( \mathbb{C}[x] \) has a root in \( \mathbb{C} \).

- proof in algebra,
- proof in complex analysis (corollary of Liouville’s theorem),
- proof in topology.

**Theorem 2.54 (Fundamental Theorem of Algebra)**

If \( a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{C}[x] \), \( n > 0 \), then \( \exists x_0 \in \mathbb{C} \) with \( a_n x_0^n + \cdots + a_1 x_0 + a_0 = 0 \).

**Proof** Let \( a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{C}[x] \) and \( n > 0 \).

1. We can assume \( a_n = 1 \) (\( a_n(x^n + \frac{a_{n-1}}{a_n} x^{n-1} + \cdots + \frac{a_0}{a_n}) \)).

We can assume \( \sum_{i=0}^{n-1} |a_i| < 1 \). Indeed choose \( c \in \mathbb{R}_{>0} \) and set \( x = cy \). This gives \( c^n y^n + \cdots + a_1 cy + a_0 = c^n(y^n + \frac{a_{n-1}}{c} y^{n-1} + \cdots + \frac{a_0}{c^n}) \). Choose \( c \) large enough such that \( \frac{|a_{n-1}|}{c} + \cdots + \frac{|a_0|}{c^n} < 1 \). Now if \( y_0 \) is a root of \( y^n + \frac{a_{n-1}}{c} y^{n-1} + \cdots + \frac{a_0}{c^n} \), then \( cy_0 \) is a root of \( x^n + \cdots + a_0 \).

2. Consider \( f : S^1 \to S^1, z \mapsto z^n \) and \( p : I \to S^1, s \mapsto (\cos 2\pi s, \sin 2\pi s) = e^{2\pi is} \). Then \( f_* : \pi_1(S^1, (1,0)) \to \pi_1(S^1, (1,0)) \). \( f_*([p]) = [f \circ p] \), where \( f \circ p(s) = (\cos 2\pi ns, \sin 2\pi ns) \).

This shows that \( f_* \) is injective. Also if \( \iota : S^1 \to \mathbb{R}^2 \setminus \{(0,0)\} \) (inclusion), then \( \iota_* \) is injective, since \( S^1 \) is a retract of \( \mathbb{R}^2 \setminus \{(0,0)\} \).

I.e. \( \iota_* \circ f_* = (\iota \circ f)_* \) is injective, which implies (Lemma ??) that \( \iota \circ f \) is not nullhomotopic (a).

3. Suppose, that our polynomial has no root in \( D^2 \). Consider \( \tilde{f} : D^2 \to \mathbb{R}^2 \setminus \{(0,0)\}, z \mapsto z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \). By lemma ?? \( \tilde{f} \) is nullhomotopic (b). Furthermore: \( \tilde{F} : S^1 \times I \to \mathbb{R}^2 \setminus \{(0,0)\} \) where \( (z,t) \mapsto z^n + t(a_{n-1} z^{n-1} + \cdots + a_1 z + a_0) \). This is a homotopy between \( \tilde{f} \) and \( \iota \circ f \).
\[ F(z, t) = |z^n + t(a_{n-1}z^{n-1} + \cdots + a_0)| \geq |z^n| - |t||a_{n-1}z^{n-1} + \cdots + a_0| \geq 1 - |t| \sum_{i=0}^{n-1} |a_i| > 0. \]

(a), (b), (c) contradiction, i.e. our polynomial has a root in \( D^2 \).

Exercise: Any root of such a polynomial \( z^n + \cdots + a_0 \) with \( \sum |a_i| < 1 \) is in \( D^2 \).

### 2.5.2. Deformation retracts and homotopy type

**Definition 2.56** Let \( A \subseteq X \), \( A \) is called a deformation retract of \( X \), if there is a continuous map \( H : X \times I \to rA \) with \( H(x, 0) = x \) \( \forall x \in X \), \( H(x, 1) \in A \) \( \forall x \in X \), \( H(a, t) = a \) \( \forall a \in A \), \( \forall i \in I \). The homotopy \( H \) is called a deformation retraction of \( X \) onto \( A \).

**Remark 2.57**

1. \( r : X \to A \), \( r(x) = H(x, 1) \) is a retraction of \( X \) onto \( A \).
2. \( H \) is a homotopy between \( 1_X \) and \( \iota \circ r \), where \( \iota : A \to X \) inclusion map.

**Example 2.58**

1. \( \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \neq (0, 0), z = 0\} =: A \) is a deformation retract of \( \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \neq (0, 0)\} =: X. \)

   \[ H : X \times I \to X, H(x, y, z, t) := (x, y, (1-t)z). \]

   - \( H \) is continuous,
   - \( H(x, y, z, 0) = (x, y, z) \) \( \forall (x, y, z) \in X \),
   - \( H(x, y, z, 1) = (x, y, 0) \in A \) \( \forall (x, y, z) \in X \),
   - if \( (x, y, z) \in A \), then \( z = 0 \) and thus \( H(x, y, z, t) = (x, y, 0) \in A. \)

2. Let \( p, q \in \mathbb{R}^2 \), \( p \neq q \). Then \( S^1 \cup S^1 \) is a deformation retract of \( \mathbb{R}^2 \setminus \{p, q\} \).

\[ \bullet p \quad \bullet q \]

3. \( p, q \in \mathbb{R}^2 \), \( p \neq q \), \( \mathbb{R}^2 \setminus \{p, q\} \) deform retracts onto the theta space:

   \[ \text{(Homeomorph to } S^1 \cup (0 \times [-1, 1])\text{).} \]

4. Möbius band

\[ \text{\begin{tikzpicture}
\filldraw[fill=white](0,0) rectangle (1,1);
\end{tikzpicture}} \]
Let $r: (X, x_0) \to (A, x_0)$, $r(x) = H(x, 1)$ Then $r \circ \iota: (A, x_0) \to (A, x_0)$ is the id map $1_A$ and $(r \circ \iota)_* = (1_A)_*$. Furthermore $\iota_* \circ r_* = (1 \circ \iota)_* = 1_{\pi_1(A, x_0)}$. If $f: I \to X$ loop, $f(0) = x_0$, then $H \circ (f \times 1_I): I \times I \to X$ is a path homotopy between $1_X \circ f$ and $(\iota \circ r) \circ f$, (path homotopy: $H \circ (f \times 1_I)(0, t) = H(x_0, t) = x_0 \forall t \in I$, $H \circ (f \times 1_I)(1, t) = H(x_0, t) = x_0 \forall t \in I$).

So $[1_X \circ f] = [(\iota \circ r) \circ f] \Rightarrow (1 \circ f) = (\iota \circ r) \ast (f)$. $\iota_* = 1_{\pi_1(X, x_0)} = (1 \circ r)_* \Rightarrow \iota_* r_* = 1_{\pi_1(A, x_0)}$. 

**Corollary 2.61** If $\iota: S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ the inclusion map, $x_0 \in S^n$, then $\iota_*: \pi_1(S^n, x_0) \to \pi_1(\mathbb{R}^{n+1} \setminus \{0\})$ is isomorphism.

**Proof** Define $H: (\mathbb{R}^{n+1} \setminus \{0\}) \times I \to \mathbb{R}^{n+1} \setminus \{0\}$, $H(x, t) := (1-t)x + t \frac{x}{|x|}$. $H$ is a deformation retraction of $\mathbb{R}^{n+1} \setminus \{0\}$ onto $S^n$. Now use theorem.

**Example 2.63** We can look at example 2.58 again:

1. $\pi_1(X) \cong \pi_1(A) \cong \pi_1(S^1) \cong \mathbb{Z}$.

2. “figure eight” space is deformation retraction of $\mathbb{R}^2 \setminus \{p, q\}$, $\pi_1(A_1) \cong \pi_1(\mathbb{R}^2 \setminus \{p, q\})$.

3. “theta” space is deformation retraction of $\mathbb{R}^2 \setminus \{p, q\}$, $\pi_1(A_2) \cong \pi_1(\mathbb{R}^2 \setminus \{p, q\})$.

4. $\pi_1(A_1) \cong \pi_1(A_2)$.

Comment: “figure eight” and “theta” space have isomorphic fundamental group, but none is a deformation retract of the other (exercise).

**Definition 2.64** Let $f: X \to Y$ be a continuous map. If there exists a continuous map $g: Y \to X$ such that $g \circ f: X \to X$ is homotopic to $1_X: X \to X$ and $f \circ g: Y \to Y$ is homotopic to $1_Y: Y \to Y$, then $f$ is called a homotopy equivalence.

$g$ is called the homotopy inverse of $f$. $X, Y$ are called homotopy equivalent. If $X, Y$ are homotopy equivalent, we say that $X, Y$ have the same homotopy type.

Check: Homotopy equivalence is an equivalence relation.

**Remark 2.65** Homotopy equivalence is more general than deformation retraction.

1. Suppose that $H: X \times I \to X$ is a deformation retraction of $X$ onto $A$. Let $\iota: A \to X$ the inclusion map and $r: X \to A$, $r(x) = H(x, 1)$. Then $r \circ \iota: A \to A$ is the id map $1_A$, and $H$ is a homotopy between $r \circ \iota$ and $1_X$.
 Lemma 2.66 Let $h, k : X \to Y$ homotopic continuous maps with $h(x_0) = y_0, k(x_0) = y_1$. If $H : X \times Y \to Y$ is the homotopy between $h$ and $k$ and if $a(t) = H(x_0, t)$, then $k_* = \alpha \circ h_*$. 

\[
\begin{array}{c}
\pi_1(X, x_0) \xrightarrow{h_*} \pi_1(Y, y_0) \\
\xrightarrow{k_*} \pi_1(Y, y_1)
\end{array}
\]

 Proof \textit{...der Beweis ist noch sehr fehlerhaft, noch fehlerhafter als gewohnt :P...}

Let $f : I \to X$ a loop based at $x_0$.

We want to show that $k_*([f]) = \alpha \circ h_*([f])$

\[
\begin{align*}
\iff [k \circ f] &= \alpha(h \circ f) \\
\iff [k \circ f] &= [\alpha h \circ f][\alpha] \\
\iff [a(k \circ f)] &= [h \circ f][\alpha] \\
\iff [a(k \circ f)] &= (h \circ f)[\alpha].
\end{align*}
\]

$h(x) = H(x, 0) \forall x \in X$, so $h(f(s)) = H(f(s), 0) = (H \circ f_0)(s)$ where $f_0 : I \to X \times I$, $f_0(s) = (f(s), 0)$, i.e. $h \circ f = H \circ f_0$.

$k(x) = H(x, 1) \forall x \in X$, so $k(f(s)) = H(f(s), 1) = (H \circ f_1)(s)$ where $f_1 : I \to X \times I$, $f_1(s) = (f(s), 1)$, i.e. $k \circ f = \alpha f_1$.

Consider $c : I \to X \times I, c(t) = (x_0, t)$. Then $H(c(t)) = a(t)$.

$F : I \times I \to X \times I, F(s, t) := (f(s), t)$.

$G_{0\gamma_1}$ is homotopic to $\gamma_0 B_1$, call $G$ the path homotopy between them.

Then $F \circ G : I \times I \to X \times I$, continuous, $(F \circ G)(s, 0) = F(G(s), 0) = \begin{cases} F(\beta_0(s), 0), & s \in [0, \frac{1}{2}] \\ F(1, \gamma_1(s)) & s \in [\frac{1}{2}, 1] \end{cases}$

Where $F \circ G$ is a path homotopy from $c \cdot f_1$ to $c \cdot f_1$.

Lastly, $H \circ (F \circ G)$ is a path homotopy from $H(f_0, c) = H(f_0) \cdot H(c) = (h \circ f) \cdot a$ to $H(c \cdot f_1) = H(c) \cdot H(f_1) = a \cdot (k \circ f)$

Corollary 2.68 Let $h, k : (X, x_0) \to (Y, y_0)$ homotopic continuous maps. If $H : X \times I \to Y$ is the homotopy between $h$ and $k$ and $H(x_0, t) = y_0 \forall t \in [0, 1]$, then $k_* = h_*$. 

Theorem 2.69 Let $f : X \to Y$ homotopy equivalence with $f(x_0) = y_0$. Then $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.

 Proof Let $g : Y \to X$ be a homotopy inverse of $f$, and let $g(y_0) = x_1$ and $f(x_1) = y_1$.

\[
\begin{array}{c}
(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (X, x_1) \xrightarrow{f} (Y, y_1)
\end{array}
\]

$g \circ f_{x_0}$ is homotopy to $1_X \implies$ [Lemma] there is a path $a : I \to X$, such that $(g \circ f_{x_0})_* = \alpha \circ (1_X)_* = \alpha$.

31
\(\tilde{a}\) is a group isomorphism, so \(g_* \circ f_*\) is a group isomorphism.
\[\implies g_* \text{ is surjective (1).}\]
Similarly \((f_* \circ g)_*\) is a group isomorphism.
\[\iff (f_* \circ g)_* \text{ is a group isomorphism.}\]
\[\implies g_* \text{ is injective (2).}\]
\[(1), (2) \implies g_* \text{ is a group isomorphism,} \]
\[\implies (f_* \circ g)_* = (g_*)^{-1} \circ \tilde{a}\] is a group isomorphism.

**Theorem 2.71** \(X, Y\) have the same homotopy type iff \(X, Y\) are homeomorphic to to deformation retracts of a space \(Z\).

[Proof: hard, we omit it]

**Theorem 2.72** \(\pi_1(S^1 \vee S^1)\) is not abelian.

**Proof** \(X := S^1 \vee S^1\).

Consider the following covering space

\[p : \tilde{X} \to X,\ p \text{ maps each } A_i \text{ homeomorphically onto } A \forall i \in \mathbb{Z} \setminus \{0\}, \text{ and } B_i \text{ homeomorphically to } B \forall i \in \mathbb{Z} \setminus \{0\}.\ p \text{ maps all tangency points to } X_0.\ p \text{ wraps } A_0 \text{ around } A, B_0 \text{ around } B.\]

\(p\) is a covering map (check). Consider \(\tilde{f} : I \to \tilde{X}, f(s) = (s, 0). \tilde{g} : I \to \tilde{X}, g(s) = (0, s)\) and \(f = p \circ \tilde{f}, g = p \circ \tilde{g}.\)

Also consider \(f \cdot g\) and \(g \cdot f\) and lift these two paths to paths \(\tilde{f} \cdot \tilde{g}, \tilde{g} \cdot \tilde{f}\) in \(\tilde{X}\) both starting at \((0, 0)\). \(\tilde{f} \cdot \tilde{g}\) is a path in \(\tilde{X}\) that goes from \((0, 0)\) to \((1, 0)\) along \(x\)-axis and then once around \(B_1, g \cdot f\) is a path in \(\tilde{X}\) that goes from \((0, 0)\) to \((0, 1)\) along the \(y\)-axis and then once around \(A_1,\)

\(\tilde{f} \cdot \tilde{g}(1) \not\equiv \tilde{g} \cdot \tilde{f}(1).\) So \(f \cdot g\) and \(g \cdot f\) are not path-homotopic. \([f \cdot g] \not\equiv [g \cdot f]\) i.e. \([f] \cdot [g] \not\equiv [g] \cdot [f]\)
\[\implies \pi_1(S^1 \vee S^1)\] is not abelian.

\[\Box\]
2.5 The fundamental group of the circle and applications

**Theorem 2.74** \( \pi_1(\mathbb{X} \times \mathbb{Y}, x_0 \times y_0) \cong \pi_1(\mathbb{X}, x_0) \times \pi_1(\mathbb{Y}, y_0) \).

**Proof** Consider the projections \( p : \mathbb{X} \times \mathbb{Y} \to \mathbb{X} \), \( x \times y \mapsto x \) and \( q : \mathbb{X} \times \mathbb{Y} \to \mathbb{Y} \), \( x \times y \mapsto y \), and \( p_* : \pi_1(\mathbb{X} \times \mathbb{Y}, x_0 \times y_0) \to \pi_1(\mathbb{X}, x_0) \) and \( q_* : \pi_1(\mathbb{X} \times \mathbb{Y}, x_0 \times y_0) \to \pi_1(\mathbb{Y}, y_0) \).

Define \( \Phi : \pi_1(\mathbb{X} \times \mathbb{Y}, x_0 \times y_0) \to \pi_1(\mathbb{X}, x_0) \times \pi_1(\mathbb{Y}, y_0) \), \( [f] \mapsto (p_*(f), q_*(f)) \).

Claim: \( \Phi \) is a group homomorphism. 

Proof: \( \Phi([f][g]) = \Phi([fg]) = (p_*(fg), q_*(fg)) = (p \circ (fg), g \circ (fg)) = ((p \circ f)(p \circ g), [g \circ g]) \).

Claim: \( \Phi \) is surjective. 

Proof: Consider \( g : I \to \mathbb{X} \) Loop based at \( x_0 \), and \( h : I \to \mathbb{Y} \) loop based at \( y_0 \). Define \( f : I \to \mathbb{X} \times \mathbb{Y}, f(s) = (g(s), h(s)) \). Then \( f \) is a loop based at \( (x_0, y_0) \) and \( \Phi([f]) = (p_*(f)), q_*(f)) = ([g],[h]) \).

Claim: \( \Phi \) is injective. 

Proof: Suppose that \( f : I \to \mathbb{X} \times \mathbb{Y} \) is a loop based at \( (x_0, y_0) \) with \( \Phi([f]) \) is the identity element in \( \pi_1(\mathbb{X}, x_0) \times \pi_1(\mathbb{Y}, y_0) \). Then \( p \circ f \) is path homotopic to \( c_{x_0} \) via a path homotopy \( G \) and \( q \circ f \) is path homotopic to \( c_{y_0} \) via a path homotopy \( H \). Define \( F : I \times I \to \mathbb{X} \times \mathbb{Y} \) by \( F(s,t) = (G(s,t), H(s,t)) \). \( F \) is a path homotopy from \( f \) to \( c_{(x_0,y_0)} \). 

**Corollary 2.76** Consider the torus \( T \cong S^1 \times S^1 \). \( \Longrightarrow \pi_1(T) \cong \mathbb{Z} \times \mathbb{Z} \).

**Definition 2.77** 1. Let \( \{X_i\}_{i \in I} \) an indexed collection of sets. The **disjoint union** of these sets is \( \bigcup_{i \in I} X_i \equiv \bigcup_{i \in I} X_i \times \{i\} \).

2. Suppose that \( X_i \) is a topological space \( \forall i \in I \). The **disjoint union topology** on \( \bigcup_{i \in I} X_i \) is the finest topology, such that all of the following maps are continuous: \( \varphi_i : X_i \to \bigcup_{i \in I} X_i, x \mapsto x \times \{i\} \).

   Explicitly, \( U \) is open in \( \bigcup_{i \in I} X_i \) iff \( \varphi_i^{-1}(U) \) is open \( \forall i \in I \).

3. Let \( X, Y \) be topological spaces, \( A \subseteq X \), and \( f : A \to Y \) continuous. The **adjunction space** \( X \cup_f Y \) (adjunction of \( X \) to \( Y \) along \( f \) (along \( A \))) is the quotient space 

   \[ X \cup_f Y / \sim, \]

   where \( a \sim f(a) \forall a \in A \). \( f \) is called the **attaching map**.

**Remark 2.78** If \( X_i \cap X_j = \emptyset \) for any \( i, j \in I, i \neq j \), then \( \bigcup_{i \in I} X_i \cong \bigcup_{i \in I} X_i \).

**Example 2.79** \( X_1, X_2 \subseteq (R,T,S^2), X_1 \cup X_2 \).

**Definition 2.80** The **wedge sum** of \( X \) and \( Y \) is 

\[ X \vee Y := X \cup Y / x_0 \sim y_0, \]

for \( x_0 \in X, y_0 \in Y \).
If \( X_i \) are topological spaces, \( x_i \in X_i, i \in I \), then

\[
\vee_{i \in I} X_i = \bigcup_{i \in I} X_i \setminus \{x_i \sim x_j, i, j \in I\}.
\]

**Definition 2.81** Let \( X, Y \) \( n \)-manifolds. Consider \( U, V \), open discs \( D^2 \) in \( X, Y \) respectively, and \( f : \partial U \to \partial V \) a homeomorphism. The **connected sum** of \( X \) and \( Y \) is

\[
X \# Y := (X \setminus U) \cup (Y \setminus V) \setminus \sim,
\]

where \( x \sim y \) if \( x \in \partial U, y \in \partial V \) and \( f(x) = y \).

**Remark 2.82** Although this construction involves the choice of the discs \( U, V \) the resulting space is unique up to homeomorphism.

Claim: \( Y := S^1 \vee S^1 \) is a retract of \( X := T \# T \).

**Proof** (hier bräuchte es ne Graphik...) Consider \( f : X \to Y \), where \( f \) maps the dotted circle to \( y \), and \( f \) restricted to \( S^1 \vee S^1 \subseteq X \) is a homeomorphism \( h \), and \( f : 1:1 \) everywhere else. Further consider \( r \), where \( r \) retracts \( Y \) onto \( S^1 \vee S^1 \) by mapping each cross-sectional circle to the point where it intersects \( S^1 \vee S^1 \). Use \( h^{-1} \) to map \( S^1 \vee S^1 \) in \( Y \) to \( S^1 \vee S^1 \) in \( X \).

\[ \sim \rightarrow \text{retraction of } T \# T \text{ onto } S^1 \vee S^1. \]

**Corollary 2.84** \( \pi_1(T \# T) \) is not abelian.

**Proof** If \( i : S^1 \vee S^1 \to T \# T \) the inclusion map, then \( i_* \) is injective (follows from claim).

\[ i_* : \pi_1(S^1 \vee S^1) \to \pi_1(T \# T). \]

So \( \pi_1(T \# T) \) is not abelian.

**Corollary 2.86** \( T, T \# T \) are not homotopy equivalent.

**Proof** obvious.

**Lemma 2.88** If \( P = (0, \ldots, 0, 1) \in S^n \subseteq \mathbb{R}^{n+1} \), then \( S^n \setminus \{P\} \) is homeomorphic to \( \mathbb{R}^n \).

**Proof** Define \( f : S^n \setminus \{P\} \to \mathbb{R}^n \) (stereographic projection), where \( (x_1, \ldots, x_{n+1}) \mapsto \frac{1}{1-x_{n+1}}(x_1, \ldots, x_n) \).

\( f \) is continuously and furthermore \( g : \mathbb{R}^n \to S^n \setminus \{P\} \) where \( (y_1, \ldots, y_n) \mapsto \left( \frac{2}{1+\|y\|^2} y_1, \ldots, \frac{2}{1+\|y\|^2} y_n, 1-\frac{2}{1+\|y\|^2} \right) \) is continuous and \( f \circ g = 1_{\mathbb{R}^n} \) and \( g \circ f = \text{id}_{S^n \setminus \{P\}} \).

**Proposition 2.90** \( \pi_1(S^n) = 0 \) \( \forall n \in \mathbb{N}, n \geq 2 \).

**Proof** Consider \( f : I \to S^n \) a loop based at \( x_0 \in S^n \). Let \( x \in S^n \setminus \{x_0\} \) and \( B \) a small open ball in \( S^n \setminus \{x_0\} \) with \( x \in X \).

Then \( f^{-1}(B) \subseteq (0, 1) \) and \( f^{-1}(B) \) is open. So \( f^{-1}(B) = \bigcup_i (a_i, b_i) \).

\( f^{-1}((x)) \) is a closed subset of the compact space \( I \), i.e. \( f^{-1}((x)) \) is compact.

\( f^{-1}((x)) \subseteq f^{-1}(B) = \bigcup_i (a_i, b_i) \implies \exists (a_1, b_1), \ldots, (a_n, b_n) \) such that \( f^{-1}((x)) \subseteq (a_1, b_1) \cup \cdots \cup (a_n, b_n) \).

Consider \( f_i = f|_{[a_i, b_i]} \). Then \( f((a_i, b_i)) \subseteq B \) and \( f((a_i, b_i)) = f((a_i, b_i)) \subseteq \overline{f((a_i, b_i))} \subseteq B \).

\( f(a_i), f(b_i) \in \partial B \).

Since \( n \geq 2 \), choose \( g_i : I \to \partial B \) path, such that \( g_i(0) = f(a_i) \) and \( g_i(1) = f(b_i) \).

Furthermore, \( B \) is convex, so \( \pi_1(B) = 0 \). \( \implies f_i \simeq g_i \).
Repeating this process for all intervals \([a_i, b_i], i \in \{1, \ldots, n\}\), we obtain a loop \(g : I \to S^n\), homotopic to \(f\), and with the property that \(g(I) \subseteq S^n \setminus \{x\}\), \([f] = [g]\).

The lemma implies that \([g] = [c_{\infty}]\), so \([f] = [c_{\infty}]\) and \(\pi_1(S^n, x_0) = 0\).

**Corollary 2.92** \(\mathbb{R}^2\) and \(\mathbb{R}^n\) are not homeomorphic for any \(n \neq 2\).

**Proof** Suppose that \(f : \mathbb{R}^2 \to \mathbb{R}^n\) a homeomorphism \((n \neq 2)\). Then

- \(n = 1: \mathbb{R}^2 \setminus \{0\} \cong \mathbb{R} \setminus \{f(0)\}\), but \(\mathbb{R}^2 \setminus \{0\}\) is path connected, but \(\mathbb{R} \setminus \{f(0)\}\) is not.
- Recall \(\pi_1(\mathbb{R}^2 \setminus \{P\}) \cong \pi_1(S^1) = \mathbb{Z}, \pi_1(\mathbb{R}^n \setminus \{f(P)\}) \cong \pi_1(S^{n-1}) = 0\) for \(n \geq 3\), \(f_{\mathbb{R}^2 \setminus \{P\}} : \mathbb{R}^2 \setminus \{P\} \to \mathbb{R}^n \setminus \{f(P)\}\) homeomorphism, contradiction.

Recall:

<table>
<thead>
<tr>
<th>space (\subseteq \mathbb{R}^n) convex, (x_0 \in X)</th>
<th>(\pi_1)</th>
<th>using</th>
</tr>
</thead>
<tbody>
<tr>
<td>(X)</td>
<td>0</td>
<td>definition</td>
</tr>
<tr>
<td>(S^1)</td>
<td>(\mathbb{Z})</td>
<td>covering spaces</td>
</tr>
<tr>
<td>(\mathbb{R}^2 \setminus {(0,0)})</td>
<td>(\mathbb{Z})</td>
<td>deformation retracts</td>
</tr>
<tr>
<td>(\mathbb{R}^3 \setminus {(z\text{-axis)}} \cong \mathbb{R}^3)</td>
<td>(\mathbb{Z})</td>
<td>deformation retracts</td>
</tr>
<tr>
<td>(S^1 \vee S^1)</td>
<td>(\pi_1(S^1 \vee S^1) \cong \pi_1(\mathbb{R}^2 \setminus {p, q}))</td>
<td>deformation retracts</td>
</tr>
<tr>
<td>(\theta)-space</td>
<td>(\pi_1(\theta) \cong \pi_1(\mathbb{R}^2 \setminus {p, q}))</td>
<td>deformation retracts</td>
</tr>
<tr>
<td>Möbius-band (M)</td>
<td>(\mathbb{Z})</td>
<td>deformation retracts</td>
</tr>
<tr>
<td>(S^1 \vee S^1)</td>
<td>not abelian</td>
<td>covering spaces</td>
</tr>
<tr>
<td>(T # T)</td>
<td>not abelian</td>
<td>retraction</td>
</tr>
<tr>
<td>(S^n, n \in \mathbb{N}, n \geq 2)</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>(T)</td>
<td>(\mathbb{Z} \times \mathbb{Z})</td>
<td></td>
</tr>
</tbody>
</table>

### 2.6. Seifert von Kampen Theorem

#### 2.6.1. Direct sums of abelian groups

**Definition 2.94** Let \(G\) be an abelian group and \(\{G_a\}_{a \in A}\) a family of subgroups of \(G\).

We say that the groups \(G_a\) *generate* \(G\) if every element \(x\) of \(G\) can be written as a finite sum of elements of the groups \(G_a\).

Since \(G\) is abelian, we can always write this sum in the form \(X = X_{a_1} + \cdots + X_{a_k}\), \(X_a \in G_a, a_i \neq a_j, i \neq j\).

In this case we often write \(x = \sum_{a \in A} X_a\).

If the groups \(G_a\) generate \(G\), then \(G\) is called the *sum* of \(G_a\) and we write \(G = \sum_{a \in A} G_a\) or \(G = G_1 + \cdots + G_n\).

If \(G_a\) generate \(G\) and \(\forall x \in G \exists ! A\text{-tuple} (x_a)_{a \in A}\) with \(x_a = 0\), for all but finitely many \(a\) in \(A\), and \(G = \sum_{a \in A} X_a\) then \(G\) is called the *direct sum* of \(G_a\), and we write \(G = \oplus_{a \in A} G_a\), or \(G = G_1 \oplus \cdots \oplus G_n\).

**Example 2.95**

1. \(\mathbb{R}^2, G_1 = \{(x, 0) \mid x \in \mathbb{R}\}, G_2 = \{(0, y) \mid y \in \mathbb{R}\}, \mathbb{R}^2 = G_1 \oplus G_2\).

2. \(\mathbb{R}^\infty\): the set of all sequences of real numbers that are eventually zero. \(\mathbb{R}^\infty\) is a group under coordinate addition. \(G_n = \{(0, \ldots, 0, x_n, 0, \ldots) \mid x_n \in \mathbb{R}\}\) are subgroups of \(\mathbb{R}^\infty\) \(\forall n \in \mathbb{N}\) and \(\mathbb{R}^\infty = \bigoplus_{n \in \mathbb{N}} G_n\).
Lemma 2.96 (Extension condition) Let $G$ be an abelian group and $\{G_a\}_{a \in A}$ be a family of subgroups. If $G = \oplus_{a \in A} G_a$, then $G$ satisfies the following condition:

\[(\ast) \text{ If } H \text{ is an abelian group and } h_a : G_a \to H \text{ is a family of homomorphisms, then there exists a homomorphism } h : G \to H \text{ such that } h_{|G_a} = h_a \forall a \in A.\]

Furthermore, this is unique. Conversely if $G_a$ generate $G$ and $(\ast)$ holds, then $G = \oplus_{a \in A} G_a$.

**Proof** Suppose that $G = \oplus_{a \in A} G_a$. Consider $H$ abelian group, and $h_a : G_a \to H$ family of homomorphisms. Define $h : G \to H$ in the following way: If $x \in G$, then $x = \sum_{i=1}^n x_i, h(x) = \sum_{i=1}^n h_{a_i}(x_i)$. $h$ is well-defined, because there is a unique way of writing $x$ as $\sum_{i=1}^n x_i$. $h$ is a homomorphism and $h_{|G_a} = h_a$.

Suppose that $h' : G \to H$ with $h'_a = h_a, \forall a \in A$. Then $h'(x) = h'(\sum_{i=1}^n x_i) = \sum_{i=1}^n h'(x_i) = \sum_{i=1}^n h_{a_i}(x_i) = h(x)$.

Converse: Suppose that $x = \sum_{a \in A} x_a = \sum_{a \in A} y_a$ for some $x \in G$. Choose $b \in B$ and let $H = G_b$. Define $h_a : G_a \to H$ by $h_a = 1_{G_b}$ if $a = b$, and $h_a$ trivial if $a \neq b$. Let $h : G \to H$ be the extension of the homomorphisms $h_a$. Then $h(x) = \sum_{a \in A} h(x_a) = \sum_{a \in A} h(y_a) = \sum_{a \in A} h_a(y_a) = Y_b$.

$$\Rightarrow X_b = Y_b, \forall b \in A,$$

$$\Rightarrow X_a = Y_a \forall a \in A.$$  

Corollary 2.98 Let $G = G_1 \oplus G_2$ and $G_1 = \oplus_{a \in A} H_a$, and $G_2 = \oplus_{b \in B} H_b$, where the index sets $A, B$ are disjoint. Then $G = \oplus_{a \in A \cup B} H_a$.

[Proof: Problem Set 7, Exercise 4.a]]

Corollary 2.99 If $G = G_1 \oplus G_2$, then $G \cong G_1$. 

[Proof: Problem Set 7, Exercise 4.b]]

Given a family of abelian groups $\{G_a\}_{a \in A}$, find a group $G$ that contains subgroups $G'_a \cong G_a, \forall a \in A$ and $G = \oplus_{a \in A} G'_a$.

**Definition 2.100** Let $\{G_a\}_{a \in A}$ be a family of abelian groups, $G$ be an abelian group, and $i_a : G_a \to G$ a family of monomorphisms such that $G = \oplus_{a \in A} i_a(G_a)$. Then $G$ is called the external direct sum of the groups $G_a$, relative to monomorphisms $i_a$.

**Notation:** $\leq$: Subgroup.

**Theorem 2.101** If $\{G_a\}_{a \in A}$ is family of abelian groups, then there exists an abelian group $G$, and a family of monomorphisms $i_a : G_a \to G$, such that $G = \oplus_{a \in A} i_a(G_a)$.

**Proof** Consider the product $\prod_{a \in A} G_a$ (with coordinate wise addition), which is an abelian group. Let $G = \{ (x_a)_{a \in A} \}$ for all but finitely many indices in $A$ \leq $\prod_{a \in A} G_a$.

Given $\beta \in A$, define $i_{\beta} : G_{\beta} \to G$, by letting $i_{\beta}(x)$ the tuple with $x$ as its $\beta$-th coordinate and 0 in every other coordinate. $i_{\beta}$ is a monomorphism (follows from the definition), and if $x \in G$, then $x$ has finitely many non zero coordinates, i.e. $x = (x_1, \ldots, x_n, 0, \ldots)$ and $x = i_1(x_1) + \cdots + i_n(x_n)$. Furthermore, this expression is unique.
Lemma 2.103 (extension condition)
Let \( \{ G_a \}_{a \in A} \) be a family of abelian groups, \( G \) be an abelian group, \( i_a : G_a \to G \) family of homomorphisms. If \( i_a \) is a monomorphism \( \forall a \in A \), and \( G = \oplus_{a \in A} i_a(G_a) \), then the following condition holds:

\[ \text{(*) Given any abelian group } H \text{ and any family of homomorphisms, } h_A : G_a \to H, \]
then there exists a homomorphism \( h : G \to H \), with \( h \circ i_a = h_a \forall a \in A \).

Furthermore \( h \) is unique.
Conversely if \( i_a(G_a) \) generate \( G \) and \( (*) \) holds, then \( i_a \) is a monomorphism \( \forall a \in A \), and \( G = \oplus_{a \in A} i_a(G_a) \).

[Proof: Problem Set 7, Exercise 4.c]

Theorem 2.104 (Uniqueness of direct sums)
Let \( \{ G_a \}_{a \in A} \) be a family of abelian groups, \( G, G' \) be abelian groups, \( i_a : G_a \to G \), \( i'_a : G_a \to G' \) families of monomorphisms, such that \( G = \oplus_{a \in A} i_a(G_a) \) and \( G' = \oplus_{a \in A} i'_a(G_a) \). Then \( \exists! \varphi : G \to G' \) isomorphism, such that \( \varphi \circ i_a = i'_a \forall a \in A \).

Proof
- Lemma 2.103 \((H = G') \Rightarrow \exists! \varphi : G \to G' \), such that \( \varphi \circ i_a = i'_a \forall a \in A \).
- Lemma 2.103 \((H = G) \Rightarrow \exists! \psi : G' \to G \), such that \( \psi \circ i'_a = i_a \forall a \in A \).

Now (Lemma 2.103) \( \psi \circ \varphi \circ i_a = i'_a \forall a \in A \).

\[ \Rightarrow \psi \circ \varphi = 1_G, \] analogously \( \varphi \circ \psi = 1_{G'} \).

2.7. Free abelian groups

Definition 2.106 Let \( G \) be an abelian group and \( \{ x_a \}_{a \in A} \) a family of elements in \( G \). Let \( G_a = \langle x_a \rangle \), i.e. the subgroup generated by \( x_a \). If \( G_a \) generate \( G \), we also say that \( x_a \) generate \( G \).
If \( G_a = \langle x_a \rangle \) is infinite cyclic \( \forall a \in A \), and \( G = \oplus_{a \in A} (x_a) \), then \( G \) is called a free abelian group with basis \( \{ x_a \}_{a \in A} \).

Lemma 2.107 (extension condition)
Let \( G \) be an abelian group and \( \{ x_a \}_{a \in A} \) a family of elements of \( G \) that generates \( G \). Then \( G \) is a free abelian group with basis \( \{ x_a \}_{a \in A} \) iff for any abelian group \( H \) and any family \( \{ y_a \}_{a \in A} \) of elements in \( H \), there is a homomorphism \( h : G \to H \), such that \( h(x_a) = y_a \forall a \in A \). In such a case \( h \) is unique.

Proof
\("\Rightarrow" \) Given \( H, \{ y_a \}_{a \in A} \) define homomorphisms \( h_a : G_a \to H \), with \( h_a(x_a) = y_a \).
\((G_a \text{ is cyclic, so the condition } h(x_a) = y_a \text{ determines uniquely a homomorphism } h_a : G_a \to H)\). By lemma 2.96 \( \exists! h : G \to H \) homomorphism, with \( h|_{G_a} = h_a \forall a \in A \). Furthermore \( h \) is unique.

\("\Leftarrow" \) Suppose that for some \( \beta \in A, \{ x_\beta \} \) is finite. Then for \( H = \mathbb{Z} \) and \( y_a = 1 \forall a \in A \), there is no homomorphism \( h : G \to H \) with \( h(x_a) = 1 \forall a \in A \).
By lemma 2.96, we get that \( G = \oplus_{a \in A} G_a \), i.e. \( G \) is a free abelian group with basis \( \{ x_a \}_{a \in A} \).
Remark 2.109 Let \( G \) be free abelian with basis \( \{x_1, \ldots, x_n\} \). \( G = \langle x_1 \rangle \oplus \cdots \oplus \langle x_n \rangle \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \).

Proposition 2.110 If \( G \) is free abelian with basis \( \{x_1, \ldots, x_n\} \), then \( n \) is uniquely determined by \( G \).

\[
\text{Proof.} \quad G \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \cong 2\mathbb{Z} \oplus \cdots \oplus 2\mathbb{Z},
\]

\[
\mathbb{Z} \oplus \cdots \oplus 2\mathbb{Z} = \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2.
\]

\[|\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2| = 2^n.\]

Definition 2.112 If \( G \) is a free abelian group with basis \( \{x_1, \ldots, x_n\} \), then \( n \) is called the rank of \( G \).

2.8. Free products of groups

Definition 2.113 Let \( G \) be a group and \( \{G_a\}_{a \in A} \) be a family of subgroups of \( G \). We say that \( G_a \) generate \( G \), if \( \forall x \in G \exists (x_1, \ldots, x_n) \) with \( x_i \) element of some \( G_a \forall i \in \{1, \ldots, n\} \), and \( x = x_1 \ldots x_n \).

Such a sequence is called a word of length \( n \) in the groups \( G_a \) that represents \( x \).

If \( x_i, x_{i+1} \in G_a \) for some \( a \in A \), we group them together to obtain the word

\[
(x_1, x_2, \ldots, x_{i-1}, x_i x_{i+1}, x_{i+2}, \ldots, x_n)
\]

of length \( n - 1 \). If \( x_i = 1 \), we delete \( x_i \) from our sequence.

Repeating these reduction operations, we obtain \( (y_1, \ldots, y_m) \) representing \( x \), such that no group \( G_a \) contains \( y_i \) and \( y_i \neq 1 \forall i \in \{1, \ldots, m\} \). We call \( (y_1, \ldots, y_m) \) a reduced word.

Convention: \( \varnothing \) is a reduced word representing \( 1 \in G \).

Definition 2.114 Let \( G \) be a group and \( \{G_a\}_{a \in A} \) a family of subgroup that generates \( G \) and \( G_a \cap G_b = \{1\} \forall a, b \in A, a \neq b \). If \( \forall x \in G \) there is only one reduced word in the group \( \{G_a\}_{a \in A} \) that represents that element, then \( G \) is called the free product of the groups \( \{G_a\}_{a \in A} \).

\( G = \ast_{a \in A} G_a \), or \( G = G_1 \ast \cdots \ast G_n \).

Example 2.115 \( G_1 = \{1, x_1\}, G_2 = \{1, x_2\}. x \in G = G_1 \ast G_2 \). Elements of \( G \) are for example \( x_1, x_2, x_1 x_2, x_2 x_1, x_1 x_2 x_1, x_2 x_1, x_2 x_1 x_2, \ldots \).

Example 2.116 \( G = \{\varphi : \{0, 1, 2\} \rightarrow \{0, 1, 2\} \mid \varphi \text{ bijection}\}. \varphi_1 : \{0, 1, 2\} \rightarrow \{0, 1, 2\}, \varphi_1(2) = 2, \varphi_1(1) = 0, \varphi_1(0) = 1 \)

\( \varphi_2 : \{0, 1, 2\} \rightarrow \{0, 1, 2\}, \varphi_2(0) = 0, \varphi_2(1) = 2, \varphi_2(2) = 1 \).

\(|\{\varphi_1\}| = 2, |\{\varphi_2\}| = 2, \langle \varphi_1 \rangle, \langle \varphi_2 \rangle \text{ generate } G, \text{ but } B \neq \langle \varphi_1 \rangle \ast \langle \varphi_2 \rangle \).

Indeed the reduced words \( \langle \varphi_1, \varphi_2, \varphi_1 \rangle \) and \( \langle \varphi_2, \varphi_1, \varphi_2 \rangle \) represent the same element in \( G \).
Claim: Suppose that \( G_a \) generate \( G \). \( G_a \cap G_b = \{1\} \ \forall a,b \in A \) with \( a \neq b \). If the representation of 1 by the empty word is unique then \( G = *_{a \in A} G_a \).

**Proof** Let \( x \in G \) and suppose that \( (x_1, \ldots, x_n), (y_1, \ldots, y_m) \) are two reduced words representing \( x \), where \( x_i \in G_{a_i}, y_i \in G_{b_i}, a_i \in A, i \in \{1, \ldots, n\}, b_j \in A, j \in \{1, \ldots, m\} \).

\[
x_1 \ldots x_n = x = y_1 \ldots y_m \implies x_1 \ldots x_n y_m^{-1} \ldots y_1^{-1}.
\]

We obtain \( (x_1, \ldots, x_{n-1}, x_n y_m^{-1}, y_m^{-1}, \ldots, y_1^{-1}) \).

Now we must have \( x_n y_m^{-1} = 1 \implies x_n = y_m \).

Continue this process, to conclude that \( n = m \) and \( x_i = y_i, \forall i \in \{1, \ldots, n\} \).

**Lemma 2.118** (extension condition / universal mapping property) Let \( G \) be a group, \( \{G_a\}_{a \in A} \) be a family of subgroups. If \( G = *_{a \in A} G_a \), then it satisfies the following extension condition: Given any group \( H \) and any family of homomorphisms \( h_a : G_a \to H \), there exists a homomorphism \( h : G \to H \), such that \( h|_{G_a} = h_a \). In addition \( h \) is unique.

**Proof** Define \( h : G \to H \) as follows:

\[
h(1) = 1,
\]

\[
\text{If } x \in G, x \neq 1, \text{ let } (x_1, \ldots, x_n) \text{ be the (unique) reduced word representing } x. \text{ Set } h(x) = h_{a_1}(x_1) \ldots h_{a_n}(x_n), \text{ where } a_i \text{ is the index for which } x_i \in G_{a_i}.
\]

Then

\[
\begin{align*}
h & \text{ is well-defined,} \\
h|_{G_a} = h_a \ \forall a \in A, \\
h & \text{ is a group homomorphism (problem sheet 8).}
\end{align*}
\]

In fact, uniqueness of \( h \) follows from the fact that \( h \) must satisfy \( h(x) = h(x_1, \ldots, x_n) = h(x_1) \ldots h(x_n) = h_{a_1}(x_1) \ldots h_{a_n}(x_n) \).

**External free product**

**Definition 2.120** Let \( \{G_a\}_{a \in A} \) be a family of groups. Suppose that \( G \) s a group and \( \{i_a : G_a \to G\}_{a \in A} \) a family of monomorphisms with \( G = *_{a \in A} i_a(G_a) \), then \( G \) is called the **external free product** of the groups \( G_a \) relative to the monomorphisms \( i_a \).

Does such a group \( G \) exist?

**Theorem 2.121** Let \( \{G_a\}_{a \in A} \) be a family of groups. There exists a group \( G \) and a family of monomorphisms \( i_a : G_a \to G \), such that \( G = *_{a \in A} i_a(G_a) \).

**Lemma 2.122** (extension condition)

\( \{G_a\}_{a \in A} \) family of groups, \( G \) group. \( i_a : G_a \to G \) family of homomorphisms. If \( i_a \) is a monomorphism \( \forall a \in A \) and \( G = *_{a \in A} i_a(G_a) \), then the following extension condition holds: Given any group \( H \) and a family of homomorphisms \( h_a : G_a \to H \), \( \exists : G \to H \) homomorphism such that \( h \circ i_a = h_a \ \forall a \in A \). In addition \( h \) is unique.

[Proof ommitted]
Theorem 2.123 (uniqueness of free product)
Let \( \{G_a\}_{a \in A} \) be a family of groups, \( G, G' \) be groups, \( \{i_a : G_a \to G\}_{a \in A}, \{i'_a : G_a \to G'\}_{a \in A} \) families of monomorphism such that \( \{i_a(G_a)\}_{a \in A}, \{i'_a(G_a)\} \) generate \( G, G' \) respectively.
If \( G, G' \) satisfy the extension condition, then \( \exists! \) isomorphism \( \varphi : G \to G' \) such that \( \varphi \circ i_a = i'_a \), \( \forall a \in A \).

[Proof: analogous to the proof of uniqueness of direct sums (problem sheet 8)]

Lemma 2.124 Let \( \{G_a\}_{a \in A} \) be a family of groups, \( G \) be a group and \( i_a : G_a \to G \) be a family of homomorphisms. If \( i_a(G_a) \) generate \( G \), and the extension condition holds, then \( i_a \) is a monomorphism \( \forall a \in A \), and \( G = \ast_{a \in A} G_a \).

Proof: Consider \( b \in A \). Set \( H = G_b, h_b : G_a \to H \) the identity homomorphism \( 1_{G_b} \) if \( a = b \), and \( h_a : G_a \to H \) the trivial homomorphism, if \( a \neq b \).
Let \( h : G \to H \) be the homomorphism given by the extension condition.
Then \( h \circ i_b = h_b \implies h \circ i_b = 1_{G_b} \implies i_b \) is 1:1.
I.e., \( i_b \) is a homomorphism \( \forall a \in A \).
Existence theorem \( \implies \) there exists a group \( G' \) and \( \{i'_a : G_a \to G'\}_{a \in A} \) a family of monomorphisms such that \( G' = \ast_{a \in A} i'_a(G_a) \). \( G,G' \) both satisfy the extension condition and are generated by \( \{i_a(G_a)\}_{a \in A}, \) respectively \( \{i'_a(G_a)\}_{a \in A} \).
Uniqueness theorem \( \implies \) there is an isomorphism \( \varphi : G \to G' \) with \( \varphi \circ i_a = i'_a \). \( G' = \ast_{a \in A} i'_a(G_a) \).
\( \implies G = \ast_{a \in A} i_a(G_a) \).

Corollary 2.126 If \( G = G_1 \ast G_2, G_1 = \ast_{a \in A} H_a, G_2 = \ast_{b \in B} H_b, A \cap B = \emptyset \), then \( G = \ast_{\gamma \in A \cup B} H_{\gamma} \).
\( N \) normal subgroup of \( G \): \( N \not\subseteq G \).

Theorem 2.127 Let \( G = G_1 \ast G_2, N_i \subseteq G_i, i \in \{1,2\} \). If \( N \) is the smallest normal subgroup of \( G \) that contains \( N_i \), and \( N : \) then
\[ G/_{N} = (G_1/_{N_1}) \ast (G_2/_{N_2}). \]

Proof: Consider the inclusion homomorphism \( i : G_1 \to G_1 \ast G_2, i' : G_2 \to G_1 \ast G_2, \) and the projection homomorphism \( p : G_1 \ast G_2 \to G_1 \ast G_2/N, g \mapsto gN \).
Let \( n_i \in N_i \). Then \( (p \circ i')(n_i) = n_iN = N \). This implies that \( N_i \subseteq \ker(p \circ i) \) and \( p \circ i \)
induces \( \langle p \circ i' \rangle : G_1/_{N_1} \to G_1 \ast G_2/_{N} \).
Analogously, \( \langle p \circ i' \rangle : G_2/_{N_2} \to G_1 \ast G_2/_{N} \).
We will apply lemma 2.124 for the homomorphisms \( i_1, i_2 \).

- Check that the extension condition holds.
  Let \( h_1 : G_1/_{N_1} \to H_1, h_2 : G_2/_{N_2} \to H \) arbitrary homomorphisms and \( p_1 : G_1 \to G_1/_{N_1}, p_2 : G_2 \to G_2/_{N_2} \) the projection homomorphism. Then \( h_1 \circ p_1 : G \to H, h_2 \circ p_2 : G_2/_{N_2} \to H \) homomorphisms. The extension condition for \( G_1 \ast G_2 \)
implies that \( \exists! h : G_1 \ast G_2 \to H, \) with \( h_{G_1} = h_1 \circ p_1, i \in \{1,2\} \).
  If \( n_i \in N_i, i \in \{1,2\} \), then \( h(n_i) = 1_H \). I.e., \( N_i \subseteq \ker h \forall i \in \{1,2\} \).
\( \implies N \subseteq \ker h \).
This implies that \( h : G_1 \ast G_2 \to H \) induces a homomorphism \( h' : G_1 \ast G_2/_{N} \to H \),
with $h' \circ i_1 = h_1$, and $h' \circ i_2 = h_2$.

$(h' \circ i_1(N_1)) = h' \circ (p \circ i)(g_1(N_1)) = h'(p(i(g_1))) = h'(p(g_1)) = h'(g_1) = h_1(g_1(N_1))$.

- Check that $i_1(G_1/N_1)$, $i_2(G_2/N_2)$ generate $G_1 \ast G_2/N$.

$L_{1}(G_1/N_1) = (p \circ i)(G_1/N_1) = p(G_1) = G_1/N$, and similarly $i_2(G_2/N_2) = G_2/N_2$.

(Lemma 2.124) $\Rightarrow i_1, i_2$ are monomorphisms and $(G_1 \ast G_2)/N = i_1(G_1/N_1) \ast i_2(G_2/N_2)$.

**Corollary 2.129** If $N$ is the smallest normal subgroup of $G_1 \ast G_2$ that contains $G_1$, then $G_1 \ast G_2/N \simeq G_2$.

**Proof** $N_1 \simeq G_1$, $N_2 = \{1_{G_2}\}$. $\blacksquare$

$G$ group, $\{G_a\}_{a \in A}$ family of groups, $\{i_a : G_a \rightarrow G\}_{a \in A}$ family of homomorphisms.

Then the following statements are equivalent:

1. $(i_a)$ is a monomorphism $\forall a \in A$ and $G = \ast_{a \in A} i_a(G_a)$.

2. Given any group $H$ and any family of homomorphisms $h_a : G_a \rightarrow H$, $\exists!$ homomorphism $h : G \rightarrow H$ with $h \circ i_a = h_{i_a}$, $\forall a \in A$.

### 2.8 Free products of groups

**Definition 2.131** Let $G$ be a group and $\{x_a\}_{a \in A}$ a family of $G$ with $\langle x_a \rangle$ infinite cyclic $\forall a \in A$. If $G = \ast_{a \in A} \langle x_a \rangle$, then $G$ is called a **free group** and $\{x_a\}_{a \in A}$ is called a system of free generators.

In this case, if $x \in G \setminus \{1\}$, then $x$ can be written uniquely as $x = (x_{a_i})^{n_i} \cdot (x_{a_k})^{n_k}$ where $a_i \neq a_k \neq a_{i+1}, \forall i \in \{1, \ldots, k-1\}$ and $n_i, n_k \in \mathbb{Z} \setminus \{0\}$ $\forall i \in \{1, \ldots, k\}$.

Free groups are characterized by the following:

**Lemma 2.132 (extension condition)** Let $G$ be a group and $\{x_a\}_{a \in A}$ be a family of elements in $G$. If $G$ is a free group with $\{x_a\}_{a \in A}$ system of free generators, then $G$ satisfies

\[(*) \quad \text{Given any group } H \text{ and any family } \{y_a\}_{a \in A} \text{ of elements of } H \text{ there exists a homomorphism } h : G \rightarrow H \text{ with } h(x_a) = y_a \forall a \in A.\]

In addition $h$ is unique. Conversely, if $\{x_a\}_{a \in A}$ generates $G$, and $(*)$ holds, then $G$ is a free group with system of free generators $\{x_a\}_{a \in A}$.

**Proof** Lemma 2.122.

For the converse, consider $b \in A$. Then there exists a homomorphism $h_b : G \rightarrow \mathbb{Z}$ with $h_b(x_a) = 1$, and $h_b(x_a) = 0 \forall a \in A, a \neq b$. $(H = \mathbb{Z}, \{y_a\}_{a \in A} = \{0, 1\})$. This implies, that $\langle x_b \rangle$ is infinite cyclic. Therefore $\langle x_a \rangle$ infinite cyclic $\forall a \in A$.

Then lemma 2.124.

**Theorem 2.134** Let $G = G_1 \ast G_2$, $G_1, G_2$ are free groups with $\{x_a\}_{a \in A}$, $\{x_a\}_{a \in B}$ respective free systems of generators with $A \cap B = \emptyset$, then $G$ is a free group with $\{x_a\}_{a \in A \cup B}$ free system of generators.
**Definition 2.135** Let \( \{x_a\}_{a \in A} \) be an arbitrary indexed family and \( G_a \) be the set of all symbols \( x_a^n, n \in \mathbb{Z} \). Define \( x_a^n \cdot x_a^m = x_a^{n+m} \). This makes our set \( G_a \) a group. The external free product of the groups \( G_a \) is called the free group on the elements \( x_a \).

**Notation:** We will identify the elements \( x_a^n \) of \( G_a \) with their images \( i_a(x_a^n) \) in \( G \).

**Relation between free groups and free abelian groups.**

**Theorem 2.136** If \( G \) is a free group and \( \{x_a\}_{a \in A} \) is a system of free generators, then \( G \) is the free abelian group with basis the \( \{[x_a]\}_{a \in A} \) (where \( [x_a] \) is the coset \( x_a[G, G] \)).

**Proof** We will show, that 1. \( \{[x_a]\}_{a \in A} \) generate \( G \sim [G, G] \) and 2. if \( H \) is an abelian group and \( \{y_a\}_{a \in A} \) a family of elements in \( H \), then there is a homomorphism \( h' : G \to H \) with \( h'([x_a]) = y_a \forall a \in A \). Lemma 2.107 will then directly imply the proof.

1. \( \{x_a\}_{a \in A} \) generate \( G \sim [G, G] \).

2. Consider an abelian group \( H \) and \( \{y_a\}_{a \in A} \) a family of elements in \( H \). \( G \) free \( \implies \exists h : G \to H \) homomorphism with \( h(x_a) = y_a \forall a \in A \).

\[ H \text{ abelian} \implies [G, G] \leq \ker h. \implies \text{there exists an induced morphism } h' : G \to H \text{ with } h'([x_a]) = y_a \forall a \in A. \]

**Corollary 2.138** If \( G \) is a free group with \( n \) free generators, then an system of free generators of \( G \) has \( n \) elements.

**Proof** \( G \sim [G, G] \) free abelian group with basis \( \{[x_i]\}_{i=1, \ldots, n} \).

**Definition 2.140** If \( G \) is a free group, then

\[ \text{rank}(G) := \text{rank}(G \sim [G, G]). \]

**Properties:**

<table>
<thead>
<tr>
<th>( G ) free abelian group</th>
<th>( G ) free group</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H \leq G \to H ), is free abelian</td>
<td>( H \leq G \sim H ) is free.</td>
</tr>
<tr>
<td>( \text{rank}(G) = n ), and ( H \leq G \implies \text{Consider the free group } G \text{ and two elements } {x_1, x_2}, [G, G] \leq G \text{ is freely generated by } {x_1^n, x_2^m}, m, n \in \mathbb{Z} \setminus {0} ). In fact a free group with rank greater than 1 has subgroups of all countable orders.</td>
<td></td>
</tr>
<tr>
<td>( \text{rank}(H) \leq n ).</td>
<td></td>
</tr>
</tbody>
</table>

Groups up to isomorphism:

\( G \) free abelian, then the isomorphism type is determined by the cardinality of the basis.

\( G \) free, then the isomorphism type is determined by the cardinality of systems of generators.

\( G \) finitely generated abelian group, then \( G \simeq H \oplus T \), where \( H \) is free-abelian and \( T \) is torsion subgroup, so the isomorphism type is defined by the rank of \( H \) and elementary divisors.
**Group representation** Let \( G \) be a group and \( \{ x_a \}_{a \in A} \) be a family that generates \( G \). Consider the free group \( F \) on \( \{ x_a \}_{a \in A} \). Define an epimorphism \( h : F \to G, x_a \mapsto x_a \). \( G \simeq F \big/ \ker h \).

\( N := \ker h \). \( N \) is called the **relation subgroup**. If \( n \in N \), then \( n \) is called a relation on \( F \). \( N \not\subseteq F \) and we can specify \( N \) by specifying a family \( \{ r_b \}_{b \in B} \) of elements in \( F \) such that \( N \) is the smallest normal subgroup containing \( \{ r_b \}_{b \in B} \). (Exercise 4, problem sheet 8). Such a family \( \{ r_b \}_{b \in B} \) is called a complete set of relations.

**Definition 2.141** If \( G \) is a group, a **presentation** of \( G \) is family of generators \( \{ x_a \}_{a \in A} \) of \( G \) together with a complete set of relations \( \{ r_b \}_{b \in B} \) for \( G \). If both \( \{ x_a \}_{a \in A} \) and \( \{ r_b \}_{b \in B} \) are finite, then \( G \) is called **finitely presented**. If \( \{ x_a \}_{a \in A} \) is finite, then \( G \) is called **finitely generated**.

A presentation determines \( G \) uniquely up to isomorphism. Given two different presentations, it is hard to decide whether they determine isomorphic groups (Unsolvability of the isomorphism problem for groups).

**Example 2.142**

1. \( G \) cyclic group of order \( n \). Let \( x \in G \) be generator. \( \langle x \mid x^n \rangle \).

2. \( D_{2n} \) dihedral group. \( \langle r, s \mid r^n, s^2, (rs)^2 \rangle \).

3. \( G = \langle a, b \mid a^3b^{-2} \rangle, G' := \langle x, y \mid xyx^{-1}y^{-1} \rangle \). Is \( G \simeq G' \)?

\[ F = \langle a, b \rangle, F' = \langle x, y \rangle. \]

Define \( h : F \to G' \) group homomorphism with \( h(a) = xy \)

and \( h(b) = xyx \).

\[ h(a^3b^{-2}) = (h(a))^3(h(b))^{-2} = xyxxyx^{-1}y^{-1}x^{-1}y^{-1}x^{-1} = xyxxyxxy^{-1}y^{-1}x^{-1}y^{-1} = 1_G. \]

If \( N \) is the smallest normal subgroup containing \( a^3b^{-2} \) then (exercise 4, problem sheet 8) \( N = \langle ga^3b^{-2}g^{-1}, g \in F \rangle \).

Then \( N \subseteq \ker h \).

\( h' : F / \ker h \to G', \) i.e. \( h' : G \to G' \).

Define \( g : F' \to G \) group homomorphism, with \( g(x) := a^{-1}b \) and \( g(y) := b^{-1}a^2 \).

Do the same as above, then, \( g \) induces a homomorphism \( g' : G' \to G \).

Check that \( h' \circ g = 1_{G'} : G' \to G' \) and \( g' \circ h' = 1_G : G \to G \).

### 2.8.2. The Seifert-van Kampen theorem

Let \( X \) be a topological space, \( x_0 \in X \), and \( X = \cup_{\alpha \in A} A_\alpha \), where \( A_\alpha \) open path-connected subset of \( X \) with \( x_0 \in A_\alpha \) \( \forall \alpha \in A \). Then we can consider the homomorphism \( j_\alpha : \pi_1(A_\alpha) \to \pi_1(X) \) induced by the inclusion \( A_\alpha \to X \) (for brevity we write \( \pi_1(A_\alpha) \) instead of \( \pi_1(A_\alpha, x_0) \) and we write \( \pi_1(X) \) instead of \( \pi_1(X, x_0) \)). These homomorphism extend to a homomorphism \( \Phi : *_{\alpha \in A} \pi_1(U_\alpha) \to \pi_1(X) \) (extension condition for free product).

The Seifert-van Kampen theorem tells us, that \( \Phi \) is “very often” surjective, but in general not injective.

Consider \( i_{\alpha \beta} : \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha) \) the homomorphism induced by \( A_\alpha \cap A_\beta \to A_\alpha \).

Then \( j_\alpha \circ i_{\alpha \beta} = j_\beta \circ i_{\beta \alpha} \). Let \( \omega \in \pi_1(A_\alpha \cap A_\beta) \). Then \( \Phi(i_{\alpha \beta}(\omega) \cdot (i_{\beta \alpha}(\omega))^{-1}) = \)
\[ \Phi(i_{\alpha\beta}(\omega))(\Phi(i_{\beta\alpha}(\omega)))^{-1} = (j_{\alpha} \circ i_{\alpha\beta}(w)) \cdot (j_{\beta} \circ i_{\beta\alpha}(w))^{-1} = 1_{\pi_1(X)}. \] I.e. \( i_{\alpha\beta}(\omega) \cdot i_{\beta\alpha} \in \ker \Phi. \)

**Theorem 2.143 (Seifert-van Kampen)**

If \( X = \bigcup_{\alpha \in A} A_{\alpha} \) and \( A_{\alpha} \) is open and path connected for any \( \alpha \in A \), then \( \Phi \) is surjective. If furthermore \( \bigcap \) is path-connected, for any triple \( (\alpha, \beta, \gamma) \) in \( A \), then \( \ker \Phi = N \), where \( N \) is the smallest normal subgroup containing all the elements of the form \( i_{\alpha\beta}(\omega) \cdot i_{\beta\alpha}^{-1} \), so \( \Phi \) induces an isomorphism \( \pi_1(A_\alpha) / N \simeq \pi_1(X) \).

**Proof**

- **surjectivity of \( \Phi : \ast_\alpha \pi_1(A_\alpha) \to \pi_1(X) \).**

  Let \( f \) be a loop in \( \pi_1(X) \) based at \( x_0 \). If \( f \) is continuous and \( A_{\alpha} \) is open for all \( \alpha \in A \), then \( f \) is a path in \( A_{\alpha} \) for any \( \alpha \in A \). We have \( f = \bigcup_{\alpha \in A} f_{\alpha} \) and \( f \) is compact, so it is covered by finitely many of the \( V_{\alpha} \). Then endpoints of these intervals define a partition \( 0 = s_0 < s_1 < s_2 < \cdots < s_m = 1 \) such that for each \( i \) there is some subset \( A_{\alpha_i} \) with \( f([s_{i-1}, s_i]) \subseteq A_{\alpha_i} \).

  Denote the \( A_{\alpha} \) containing \( f([s_{i-1}, s_i]) \) by \( A_i \), and the path \( f_{[s_{i-1}, s_i]} = f_i \). Then

  \[
  f = f_1 \cdots f_m, f_i \text{ path in } A_i.
  \]

  Consider the loop \( (f_1 \cdot g_1) \cdot (g_2 \cdot f_2) \cdots (g_{m-1} \cdot f_m) \), which is homotopic to \( f \).

  This is a composition of loops based at \( x_0 \) each lying in a single \( A_i \). Hence \( [f] = [f_1 \cdot g_1] \cdot [g_2 \cdot f_2] \cdots [g_{m-1} \cdot f_m] \in \ker \Phi \).

**Notation:** A factorization of \( [f] \in \pi_1(X) \) is a formal product \( [f_1] \cdots [f_k] \), where each \( f_i \) is a loop in some \( A_{\alpha_i} \) based at \( x_0 \), and \( f \) is homotopic to \( f_1 \cdots f_k \).

A factorization of \( [f] \) is a word in \( \ast_{\alpha \in A} \pi_1(A_{\alpha}) \), possibly unreduced, with \( \Phi([f_1] \cdots [f_k]) = [f] \).

Two factorizations of \( [f] \) are equivalent, if they are related by a sequence of the following two kinds of moves or their inverses: 1. Combine adjacent terms \( [f_i] \cdot [f_{i+1}] \) into \( [f_i \cdot f_{i+1}] \) if \( [f_i], [f_{i+1}] \) lie in the same \( \pi_1(A_{\alpha}) \). 2. regard the term \( [f_i] \) as lying in \( \pi_1(A_{\beta}) \), if \( f_i \) is a loop in \( A_{\alpha} \cap A_{\beta} \).

Let \( Q = \ast_{\alpha \in A} \pi_1(A_{\alpha}) / N \). The first move doesn't change the element of \( \ast_{\alpha \in A} \pi_1(A_{\alpha}) \) defined by the factorization. The second move doesn't change the image of this element in \( Q \).

- **\( \ker \Phi \).**

Recall that \( N \leq \ker \Phi \). If I show that any two factorizations of \( f \) are equivalent, this will imply that \( \Phi' = Q \to \pi_1(X) \) is injective, hence \( \ker \Phi = N \).

Let \( [f_1] \cdots [f_k] \) and \( [f'_1] \cdots [f'_k] \) be two factorizations of \( [f] \) and let \( F : I \times I \to X \) a homotopy from \( f_1' \cdots f_k' \) to \( f_1' \cdots f_k' \). Then there exists partitions \( 0 = s_0 < \cdots < s_m = 1 \) and \( 0 = t_0 < \cdots < t_n = 1 \), such that each rectangle \( R_{ij} = [s_{i-1}, s_i] \times [t_{j-1}, t_j] \) is mapped by \( f \) into a single \( A_{\alpha_i} \), which we label by \( A_{ij} \). These partitions are obtained by
– covering \( I \times I \) with finitely many rectangles \([a, b] \times [c, d]\) each mapped into a single \( A_\alpha \),

– partitioning \( I \times I \) by the union of all horizontal and vertical lines containing edge of the rectangle above.

We can also assume that the \( s \)-partition subdivides the partitions given by the products \( f_1 \cdots f_s, f_1' \cdots f_s' \). We may also perturb the vertical sides of the rectangles \( R_{ij} \), so that each point in \( I \times I \) lies in at most three \( R_{ij} \)'s. Relabel the new rectangles \( R_{mn} \) ordering them as in the picture.

Consider \( \gamma_r, r \in \{0, 1, \ldots, mn\} \), the path in \( I \times I \) from the left edge \( \{0\} \times I \) to the right edge \( \{1\} \times I \) that separates \( R_1, \ldots, R_r \) from \( R_{r+1}, \ldots, R_{mn} \). We will call the corners of the rectangles vertices. For each vertex \( v \) with \( F(v) \neq x_0 \) let \( g_v \) be a path from \( x_0 \) to \( F(v) \) in the intersection of the two or three \( A_{ij} \)'s corresponding to the rectangles containing \( v \). Insert into \( F_{|\gamma_r} \) the appropriate paths \( f_v \) at successive vertices (same idea as in the first part of the proof). This gives us a factorization of \( [F_{|\gamma_r}] \) by regarding the loop corresponding to a horizontal or vertical segment between two adjacent vertices as lying in the \( A_{ij} \) for either of the rectangles containing the segment.

Different choices of \( A_{ij} \) give equivalent factorizations. Also the factorizations associated to successive paths \( \gamma_r, \gamma_{r+1} \) are equivalent, because pushing \( \gamma_r \) to \( \gamma_{r+1} \) across \( R_{r+1} \) changes \( F_{|\gamma_r} \) to \( F_{|\gamma_{r+1}} \) by the homotopy within \( A_{ij} \) corresponding to \( R_{r+1} \).

We can arrange that the factorization associated to \( \gamma_0 \) is equivalent to \([f_1] \cdots [f_s]\) and by choosing \( g_v \) for each vertex \( v \) along \( I \times \{0\} \subseteq I \times I \), to lie not just in the the two \( A_{ij} \)'s corresponding to the rectangles containing \( v \), but also to lie in the \( A_\alpha \) for the \( f_i \) containing \( v \) in its domain.

Similarly, the factorization associated to \( \gamma_{mn} \) is equivalent to \([f_1'] \cdots [f_s']\). This gives us that \([f_1] \cdots [f_s], [f_1'] \cdots [f_s']\) are equivalent.

**Corollary 2.145** Let \( X = U \cup V, U, V \) open, \( U, V, U \cap V \) path connected. Let \( x_0 \in U \cap V \). Then \( \Phi \) induces an isomorphism \( \pi_1(X) \cong \pi_1(U) * \pi_1(V) / N \), where \( N \) is the normal subgroup generated by all elements \( i_1(\omega) \cdot i_2(\omega)^{-1} \), where \( i_1 : \pi_1(U \cap V) \to \pi_1(U) \), \( i_2 : \pi_1(U \cap V) \to \pi_1(V) \) morphisms induced by the inclusion and \( \omega \in \pi_1(U \cap V) \).

[Proof: Immediate from Seifert-van Kampen theorem].

**Corollary 2.146** Consider the assumption of corollary 2.145. If in addition \( U \cap V \) is simply connected, then \( \Phi \) induces an isomorphism \( \pi_1(X) \cong \pi_1(U) * \pi_1(V) \).

[Proof: Immediate from corollary 2.145].

**Example 2.147**

1. Let \( X \) be a theta space. Recall \( \pi_1(X) \) is not abelian. \((S^1 \vee S^1) \) and theta space are deformation retracts of \( \mathbb{R}^2 \setminus \{p, q\} \) therefore \( \pi((S^1 \vee S^1)) = \pi_1(X) \).
Let $X$ be the wedge of two circles, i.e. $X = S^1 \vee S^1$. $U := X \setminus \{a\}, V := X \setminus \{c\}$. $U \cap V = X \setminus \{a,c\}$ is path connected, in addition $U \cap V$ is contractible.

By corollary 2.146 $\pi_1(X) \simeq \pi_1(U) \ast \pi_1(V)$. $U, V$ have the homotopy type of $S^1$ implies $\pi_1(U) \simeq \mathbb{Z}, \pi_1(V) \simeq \mathbb{Z}$.

Remark 2.148

1. Necessity of path connectedness of $A_a \cap A_b$.
   Consider

   \[ A_a := X \setminus \{a\}, A_b := X \setminus \{b\}, A_c := X \setminus \{c\}, \text{ then } A_a \cap A_b \cap A_c = X \setminus \{a,b,c\} \]

   not path connected, $\pi_1(X) \simeq \mathbb{Z} \ast \mathbb{Z}$.

2. Necessity of path-connectedness of $A_a \cap A_b \cap A_c$.
   Consider the theta space

   \[ A_a := X \setminus \{a\}, A_b := X \setminus \{b\}, A_c := X \setminus \{c\}, \text{ then } A_a \cap A_b \cap A_c = X \setminus \{a,b,c\} \]

   not path connected, $\pi_1(X) \simeq \mathbb{Z} \ast \mathbb{Z}$.

Example 2.149 (The shrinking wedge of circles)

Consider $C_n$ the circle with center $(1/n, 0)$ and radius $1/n$ in $\mathbb{R}^2$. Let $X = \cup_{n \in \mathbb{N}} C_n$ with the subspace topology.
2.9 CW complexes (cell complexes)

Consider the retraction \( r_n : X \to C_n \) collapsing all \( C_i \)’s except \( C_n \) to \((0, 0)\). Each \( r_n \) induces a surjection \((r_n)_* : \pi_1(X) \to \pi_1(C_n)\) (basepoint \((0, 0)\)). The product of these surjections \((\text{i.e. } \rho([f]) = (r_1([f]), r_2([f]), \ldots))\) gives a homomorphism \( \rho : \pi_1(X) \to \prod_{\infty} \mathbb{Z} (= \text{direct product of countably infinite many copies of } \mathbb{Z})\).

**Claim:** \( \rho \) is surjective \((\Rightarrow \mathbb{1}(X) \text{ is uncountable})\).

**Proof:** For any sequence of integers \((K_n)_{n \in \mathbb{N}}\) we can construct a loop \( f : I \to X \) based at \((0, 0)\) and going \( K_n \) times around the circle \( C_n \) in time \([1 - 1/n, 1 - 1/(n + 1)]\).

\( f \) is indeed continuous:

- \( \forall t \in [0, 1), f \) is continuous at \( t \).
- \( f \) is continuous at \( 1 \in I \iff (\forall \) neighbourhoods \( V \) of \( f(1) = (0, 0) \exists \) neighbourhood \( U \) of \( 1 \) such that \( f(U) \subseteq V \).

\( V \) contains all but finitely many of the circles \( C_n \), and this proves continuity of \( f \) at \( 1 \).

2.9. CW complexes (cell complexes)

We have seen, that the torus \( T \) can be constructed from

\[ \begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{b}
\end{array} \\
\begin{array}{c}
\text{a} \\
\text{b}
\end{array}
\end{array} \]

The interior of the rectangle can be though of as an open disc or a 2-cell attached to the union of two circles \( a, b \).

The union of the two circles can be thought of as obtained from their point of intersection \( P \) by attaching to it two open arcs, 1-cells.

In general construct a space \( X \) by the following procedure:

1. Start with a discrete set \( X^0 \), whose points are regarded as 0-cells.

2. Inductively construct the \( n \)-skeleton \( X^n \) from the \((n - 1)\)-skeleton \( X^{n-1} \) via continuous maps \( \varphi_\alpha : S^{n-1} \to X^{n-1} \).

\[ X^n = X^{n-1} \bigcup_{\alpha} \varphi_\alpha(S^{n-1}) / ~, \]

where \( ~ \) the equivalence relation generated by \( x \sim \varphi_\alpha(x), x \in \partial S^n \).

3. Either stop this inductive process after finitely many steps, setting \( X = X^n \), or continue indefinitely, setting \( X = \bigcup_{n \in \mathbb{N}} X^n \). In latter case, \( X \) is given the weak topology, i.e. \( A \subset X \) is open, iff \( A \cap X^n \) is open in \( X^n \forall n \).

A space constructed in this way is called a CW-complex, where C stands for closure-finiteness: the closure of each cell intersects only finitely many other cells, W stands for weak-topology.

**Example 2.150** \( S^2 \).
Suppose we attach a collection $e^2_a$ of 2-cells to a path connected space $X$

$$Y := X \coprod \bigoplus_{\alpha} e^2_{\alpha} / \sim,$$

where $x \sim \varphi_\alpha(x)$ for $x \in \partial e^2_{\alpha}$.

If $s_0$ is a base-point of $S^1$, then $\varphi_\alpha$ determines a loop in $X$ based at $\varphi_\alpha(s_0)$.

**Proposition 2.151** If $i : X \to Y$ is the inclusion map, then $i_* : \pi_1(X, x_0) \to \pi_1(Y, x_0)$ is surjective and $\ker i_* = N$, i.e. $\pi_1(Y, x_0) \cong \pi_1(X, x_0) / N$.

$X$ path-connected space, $Y$ space constructed from $X$ by attaching a family of 2-cells $e^2_{\alpha}$ via maps $\varphi_\alpha : S^1 \to X$. If $s_0$ is a basepoint of $S^1$, then $\varphi_\alpha$ determines a loop in $X$ based at $\varphi_\alpha(s_0)$. Let $x_0 \in X$ a basepoint and let $\gamma_\alpha : I \to X$ a path with $\gamma(0) = x_0$ and $\gamma(1) = \varphi_\alpha(s_0)$.

Then $\gamma_\alpha : \varphi_\alpha \tilde{\gamma_\alpha}$ is a loop in $X$ based at $x_0$, $\gamma_\alpha \varphi_\alpha \tilde{\gamma_\alpha}$ is nullhomotopic in $Y \forall \alpha$.

Let $N$ be the normal subgroup of $\pi_1(X, x_0)$ generated by all elements $\gamma_\alpha \varphi_\alpha \tilde{\gamma_\alpha}$.

If $i_* : \pi_1(X, x_0) \to \pi_1(Y, x_0)$ is the homomorphism induced by $i : X \to Y$, then $N \leq \ker i_*$.

**Proposition 2.152** If $i : X \to Y$ is the inclusion map, then $i_* : \pi_1(X, x_0) \to \pi_1(Y, x_0)$ is surjective, and $\ker i_* = N$.

**Proof** Consider the space $Z$ obtained from $Y$ by attaching rectangular strips $S_\alpha = I \times I$ with $I \times \{0\}$ is attached along $\gamma_\alpha$, $I \times \{1\}$ is not attached to anything, $\{0\} \times I$ are identified for all $\alpha$. $I \times \{1\}$ is attached along an arc in $e^2_{\alpha}$.

$Z$ deformation retracts onto $Y$.

$A := Z \setminus \{y_\alpha\}_\alpha$, where $y_\alpha$ is a point in $e^2_{\alpha}$ not in the arc along which $S_\alpha$ is attached.

$B := Z \setminus X$.

We apply the Seifert-van Kampen theorem for $Z = A \cup B$. $A = Z \setminus \{y_\alpha\}_\alpha$ deformation retracts onto $X$. $B$ is contractible. $A, B$ satisfy the conditions of the Seifert-van Kampen theorem.

$$\Longrightarrow \pi_1(Y) \cong \pi_1(Z) \cong \pi_1(A) / N,$$ where $N$ is the normal subgroup generated by the image of $\pi_1(A \cap B) \to \pi_1(A)$.

Consider the cover $A_\alpha = A \cap B \cup \beta_{\beta \neq \alpha} e^2_{\beta}$.

$A_\alpha$ deformation retracts onto a circle in $e^2_{\alpha} \setminus \{y_\alpha\}$, so $\pi_1(A_\alpha) \cong \mathbb{Z}$ and $\pi_1(A_\alpha)$ is generated by a loop homotopic to $\gamma_\alpha \varphi_\alpha \tilde{\gamma_\alpha}$. So $\pi_1(A \cap B)$ is generated by loops homotopic to $\gamma_\alpha \varphi_\alpha \tilde{\gamma_\alpha}$ for all $\alpha$.

**Example 2.154** $T$ has a cell-structure with one 0-cell, two 1-cells and one 2-cell. Let $X = X^1$ be the 1-skeleton of $T$, $X^1 = S^1 \vee S^1$ and attach to $X^1$ one 2-cell $e^2_\alpha$ via $\varphi_1 : S^1 \to X$ with $\varphi_1(S^1) = \alpha \cdot [a] \cdot [b] \cdot \beta$, to produce $Y = T$.

![Diagram](image)

It follows from the proof, that $\pi_1(T) \cong \pi_1(X^1) / N$, where $N$ is the normal subgroup of $\pi_1(X^1)$ generated by $\varphi_1$.

Now $\pi_1(X^1) \cong \mathbb{Z} \ast \mathbb{Z}$, with generators $[a], [b]$, so $\pi_1(T) = \langle [a], [b] \mid [a][b][a]^{-1}[b]^{-1} \rangle$. $\alpha = [a], \beta = [b]$. Let $F = \langle \alpha, \beta \rangle$ be the free group on two generators $\alpha, \beta$ and $N$ be the smallest normal subgroup containing $[\alpha, \beta]$. Then $N \leq \langle F, F \rangle$. Furthermore $F / N$ is abelian, then $\langle F, F \rangle \leq N$. $\Longrightarrow N = \langle F, F \rangle$, and $\langle \alpha, \beta \mid [\alpha, \beta] \rangle = \pi_1(T)$.
2.10. Surfaces (two-dimensional manifolds)

2.10.1. Fundamental group of surfaces

\[ T_n := T^n \cdot \ldots \cdot T \text{ } n \text{-fold Torus.} \]

Example 2.155 \(T^n \cdot T\) has CW-cell structure one 0-cell, four 1-cells, one 2-cell.

Example 2.156 \(T^n \cdot T^n \cdot T\)

Example 2.157 \(n\)-fold torus: \(4n\)-gon with labelling \((a_1b_1a_1b_1) \cdot (a_2b_2a_2b_2) \cdot \ldots \cdot (a_nb_nb_na_nb_n)\)

Proposition 2.158

\[ \pi_1(T_n) \simeq \langle \alpha_1, \beta_1, \ldots, \alpha_n, \beta_n \mid [\alpha_1, \beta_1][\alpha_2, \beta_2] \ldots [\alpha_n, \beta_n] \rangle. \]

Proof: Completely analogous to the case \(n = 1\), i.e. \(T\). \(T_n\) has a cell structure with one 0-cell, 2n 1-cells and one 2-cell. The attaching map \(\varphi_1 : S^1 \to (T_n)^1\) of the 2-cell \(e_2^1\) determines a loop in \(T_n\) with \([\varphi_1] = [a_1][b_1][a_1]^{-1}[b_1]^{-1} \ldots [a_n][b_n][a_n]^{-1}[b_n]^{-1} \cdot \)

The real projective plane \(\mathbb{R}P^2\)

Notation: \(P := \mathbb{R}P^2\).

\[ \mathbb{R}P^2 := \mathbb{R}^3 \setminus \{0\} / \sim, \]

where \(x \sim y :\iff \exists \lambda \neq 0 \text{ with } x = \lambda y.\)

We can also look at it as \(S^2 / \sim\)

where \(p, q \in S^2, p \sim q :\iff p = -q.\)

\(\mathbb{R}P^2\) can also be described as the quotient of \(D^2\) with antipodal points of \(\partial D^2\) identified.

As \(U = \{(x, y, z) \in S^2 \mid z \geq 0\} \simeq D^2.\)

Proposition 2.160 \(\pi_1(\mathbb{R}P^2) \simeq \langle \alpha \mid \alpha^2 \rangle \simeq \mathbb{Z}_2.\)

Proof: Method 1: Using the cell structure of \(\mathbb{R}P^2\) given by the picture

\[ \text{Method 2: } p : S^2 \to \mathbb{R}P^2 \text{ is a covering map, since } \pi_1(S^2) = 0, \phi : \pi_1(\mathbb{R}P^2, x) \to p^{-1}(x) \]

\(x \in \mathbb{R}P^2\) is bijective.

Consider \(P_n := P^n \cdot \ldots \cdot P\).

\(P_n := \mathbb{R}P^2\). has cell structure

Proposition 2.162 \(\pi_1(P_n) \simeq \langle \alpha_1, \ldots, \alpha_n \mid \alpha_1^2 \ldots \alpha_n^2 \rangle.\)

Proof: Using cell-structure of \(P_n.\)

Remark 2.164
1. For every group $G$, there exists a CW-complex $X_G$ with $\pi_1(X_G) \simeq G$: Consider $(g_\alpha, r_\beta)$ a representation of $G$ and take $V_\alpha S^1_\alpha$. Then attach 2-cells $e^2_\beta$ along the loops specified by the relations $r_\beta$.

2. $\mathbb{R}P^2 = P$

- $P$ cannot be embedded in $\mathbb{R}^3$ (i.e. there does not exist a map $f : P \to \mathbb{R}^3$, such that $f : P \to f(P)$ is a homeomorphism).
- $P$ can be immersed in $\mathbb{R}^3$ (i.e. there exists a differentiable map $f : P \to \mathbb{R}^3$ with $D_pf : T_pP \to T_{f(p)}\mathbb{R}^3$ is injective $\forall p \in P$.

**Question:** Can we deduce, that $T_n, T_m$ are not homotopy equivalent, if $n \neq m$?

**Answer:** Yes, for $T$ and $T_2$. In general not yet. (It is not a trivial matter to compare two group presentations). We turn to study $\pi_1/\langle \pi_1, \pi_1 \rangle$.

### 2.10.2. Homology of surfaces

Let $X$ be a path connected space, $x_0, x_1 \in X$, $a : I \to X$ path from $a(0) = x_0$ to $a(1) = x_1$. Then we can construct an isomorphism

\[
\tilde{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1),
\]

where $[f] \to [\alpha][f][a]$ and

\[
\tilde{\alpha}_{ab} : \pi_1(X, x_0) \to [\pi_1(X, x_0), \pi_1(X, x_0)] \to \pi_1(X, x_1) / [\pi_1(X, x_1), \pi_1(X, x_1)].
\]

If $b : I \to X$ is a path in $X$ with $b(0) = x_0$ and $b(1) = x_1$ and $g = a\tilde{\alpha}$, then

\[
\tilde{g}_{ab} : \pi_1(X, x_0) / [\pi_1(X, x_0), \pi_1(X, x_0)] \to \pi_1(X, x_1) / [\pi_1(X, x_1), \pi_1(X, x_1)],
\]

where

\[
[f][\pi_1(X, x_0), \pi_1(X, x_0)] \to [\alpha][f][g][\pi_1(X, x_0), \pi_1(X, x_0)] = [f][\pi_1(X, x_0), \pi_1(X, x_0)].
\]

So

\[
\tilde{g}_{ab} = 1_{\pi_1(X, x_0)} / [\pi_1(X, x_0), \pi_1(X, x_0)].
\]

That is, the isomorphism $\tilde{\alpha}_{ab}$ is independent of the choice of path $a$.

**Definition 2.165** If $X$ is a path connected space and $x_0 \in X$, let

\[
H_1(X) := \pi_1(X, x_0) / [\pi_1(X, x_0), \pi_1(X, x_0)].
\]

$H_1(X)$ is called the **first homology group** of $X$.

[We omit the basepoint, since there is in fact a unique isomorphism between the corresponding groups of different basepoints].

One can define homology groups $H_n(x) \forall n \in \mathbb{N}$. 50
Lemma 2.166 Let $F$ be a group and $N \leq F$. Then

$$F / N \cong F / [F, F] / \{n[F, F] \mid n \in N\}.$$ 

[Proof omitted].

Let $F$ be a free group with free generators $\alpha_1, \ldots, \alpha_n$. Let $x \in F$ and $N$ be the smallest normal subgroup containing $x$. Finally, let $G = F / N$. $F / [F, F]$ is free abelian group with basis $\alpha_1[F, F], \ldots, \alpha_n[F, F]$. $N / [F, F]$ is the subgroup generated by $x[F, F]$.

The lemma implies that $G \cong [G, G]$ is isomorphic to the quotient of a free abelian group with basis $\{\alpha_1[F, F], \ldots, \alpha_n[F, F]\}$ by the subgroup $\langle x[F, F]\rangle$.

**Theorem 2.167** $H_0(T_n)$ is a free abelian group of rank $2n \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$. 

**Proof** $\pi_1(T_n) = \langle \alpha_1, \beta_1, \ldots, \alpha_n, \beta_n \mid [\alpha_1, \beta_1] \cdots [\alpha_n, \beta_n] \rangle$.

$H_1(T_n) = \pi_1(T_n) / \langle \pi_1(T_n), \pi_1(T_n) \rangle$ is the quotient of a free abelian group of rank $2n$ by the group generated by $\langle \alpha_1, \beta_1, \ldots, \alpha_n, \beta_n \rangle[\pi_1(T_n), \pi_1(T_n)] = 1_{\pi_1(T_n)}[\pi_1(T_n), \pi_1(T_n)]$. 

i.e. $H_1(T_n) \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$. 

**Theorem 2.169** $H_1(P_n)$ has a torsion subgroup $T(P_n)$ of order $2$ and $H_1(P_n) \cong T(P(n))$ is a free abelian group of order $n - 1$.

**Proof** $\pi_1(P_n) = \langle \alpha_1, \ldots, \alpha_n \mid \alpha_1^2 \cdots \alpha_n^2 \rangle$. $H_1(P_n) = \pi_1(P_n) / \langle \pi_1(P_n), \pi_1(P_n) \rangle$ is the quotient of a free abelian group of rank $n$ (basis $\{\pi_1, \ldots, \pi_n\}$) by the subgroup generated by $\alpha_1^2 \cdots \alpha_n^2 \langle \pi_1(P_n), \pi_1(P_n) \rangle$. Since we compute in an abelian group we can use additive notation and write $2 \alpha_1 + \cdots + 2 \alpha_n + [\ldots]$. Change the basis $\{\pi_1, \ldots, \pi_n\}$ to the basis $\langle \pi_1, \ldots, \pi_{n-1}, \pi_1 + \cdots + \alpha_n \rangle$. This shows that $H_1(P_n)$ is isomorphic to the quotient of the free abelian group with basis $\langle \pi_1, \ldots, \pi_{n-1}, \pi_1 + \cdots + \alpha_n \rangle$ by the subgroup $2(\pi_1 \cdots + \pi_{n-1})$. So $H_1(P_n) = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \mathbb{Z}_2$. 

**Theorem 2.171** The surfaces $S^2$, $T$, $T_2$, $\ldots$, $P$, $P_2$, $\ldots$ are topologically distinct (i.e. any two of these have different homotopy type).

[Proof: Follows immediately from the previous theorems].

2.10.3. Classification of surfaces

**Theorem 2.172** (Classification of closed connected surfaces)

Let $S$ be a closed surface (i.e. compact without boundary). Then $S$ is either homeomorphic to $S^2$, or to $T_n$ for some $n \in \mathbb{N}$, or to $P_m$ for some $m \in \mathbb{N}$. 

Ideas of the proof:
Algebraic topology

Triangulation of a closed surface

Definition 2.173  A triangulation of a closed surface $S$ is a finite family of closed subsets $(T_1, \ldots, T_n)$ that cover $S$ together with homeomorphisms $\varphi_i : T'_i \to T_i$ where each $T'_i$ is a triangle in $\mathbb{R}^2$. The subsets $T_i$ are called triangles. The subsets of $T_i$ that are images of vertices and edges are called vertices, respectively edges. We require that if $T_i, T_j$ are distinct, $i, j \in \{1, \ldots, n\}$, then $T_i \cap T_j = \emptyset$, or $T_i \cap T_j$ is vertex or $T_i \cap T_j$ is a common edge.

Example 2.174  Torus:

\[
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & 1 & 3 \\
\hline
2 & 3 & 2 & 4 \\
\hline
3 & 4 & 3 & 1 \\
\hline
4 & 1 & 4 & 2 \\
\hline
\end{array}
\]

Theorem 2.175  (T. Radó, 1925)
Any closed surface admits a triangulation.

[Proof: Using the Jordan curve theorem].

Cutting and pasting polygonal regions in $\mathbb{R}^2$

Theorem 2.176  If $S$ is a closed surface, then $S$ is homeomorphic to a space obtained from a polygonal region in $\mathbb{R}^2$ by gluing its edges together in pairs.

Theorem 2.177  If $X$ is the quotient space obtained from a polygonal region in $\mathbb{R}^2$ by gluing its edges together, then $X$ is homeomorphic either to $S^2$ or to $T_n$, for some $n \in \mathbb{N}$ or to $P_m$ for some $m \in \mathbb{N}$.

2.11. Knot theory

First attempt to define the notion of a knot  A knot is a simple closed curve in $\mathbb{R}^3$ (i.e. $\exists \gamma : S^1 \to \mathbb{R}^3$ continuous and injective, further, as $S^1$ is compact and $\mathbb{R}^3$ Hausdorff, we have $f$ is closed) $\iff \gamma : S^1 \to \gamma(S^1)$ is a homeomorphism $\iff \gamma$ is an embedding of $S^1$ in $\mathbb{R}^3$.

Problem: “wild” knots are allowed in this definition.
**Definition 2.178** A link $L$ of $m$ components is a subset of $\mathbb{R}^3$ consisting of $m$ disjoint, piecewise linear, simple closed curves. A link of one component is a knot $K$.

Each of the curves is a union of finitely many line segments attached end to end.

**Example 2.179**

1. The unknot $U$:

   ![Unknot](image)

   ($U$ is the only knot bounding a disc embedded in $\mathbb{R}^3$).

2. The trivial knot of $m$ components:

   ![Trivial Knot](image)

3. Hopf Link

   ![Hopf Link](image)

4. Trefoil knot

5. Borromean rings

**Definition 2.180** Two links $L_1, L_2$ in $\mathbb{R}^3$ (respectively in $S^3 = \mathbb{R}^3 \cup \{\infty\}$) are called equivalent if there is an orientation preserving piecewise linear homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ (respectively $h : S^3 \to S^3$) with $h(L_1) = L_2$.

**Question:** Given two links $L_1, L_2$ how can we decide if they are equivalent.

- If $L_1, L_2$ are equivalent, then try to deform one into the other.

- If $L_1$ and $L_2$ are not equivalent, use invariants to distinguish them.

If $L_1, L_2$ are equivalent, ten $\mathbb{R}^3 \setminus L_1$ and $\mathbb{R}^3 \setminus L_2$ are homeomorphic. The homeomorphism type of $\mathbb{R}^3 \setminus L$ is such an invariant.

How strong an invariant is it?

**Theorem 2.181** (Gordon-Luecue, 1989)

The knots $K_1, K_2$ are equivalent $\iff S^3 \setminus K_1$ and $S^3 \setminus K_2$ are homeomorphic.

[Proof: Very difficult; omitted].

**Note:** The theorem says, that the homeomorphism type of $S^3 \setminus K$ is a complete invariant.
Remark 2.182 The theorem does not hold for links (with more than one component).

Determining the homeomorphism type of $S^3 \setminus L$ is not so simple. We turn to study $\pi_1(S^3 \setminus L)$. That is $\pi_1(S^3 \setminus L) \neq \pi_1(S^3 \setminus L_2) \implies S^3 \setminus L_1 \neq S^3 \setminus L_2 \implies L_1$ and $L_2$ are not equivalent.

**Definition 2.183** If $L$ is a link in $S^3$, then $\pi_1(S^3 \setminus L)$ is called the group of the link $L$. It is sometimes denoted by $\pi_1(L)$.

Remark 2.184 The inclusion $\mathbb{R}^3 \to S^3$ induces an inclusion $i : \mathbb{R}^3 \setminus L \to S^3 \setminus L$ with $i_* : \pi_1(\mathbb{R}^3 \setminus L) \to \pi_1(S^3 \setminus L)$ an isomorphism. So it is therefore equivalent to consider $L \subset \mathbb{R}^3$ or $L \subset S^3$.

**Torus knots** Consider $T \subset \mathbb{R}^3$ (rotate $C_1 = \{(x, y, z) \in \mathbb{R}^3 \mid y = 0, (x - 1)^2 + z^2 = \frac{1}{3}\}$ about the $z$-axis). We want to study knots on $T$.

$T$ can be thought of as the quotient space of $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$ with opposite edges identified or as the quotient space obtained from $\mathbb{R}^2$ by identifying two points $(x, y), (x', y')$ if $x - x' \in \mathbb{Z}$ and $y - y' \in \mathbb{Z}$.

Let $q : \mathbb{R}^2 \to T$ be the identification map described and $L = \{(x, y) \in \mathbb{R}^2 \mid |y = m/n x, m, n \in \mathbb{Z}, 1 < m < n, (m, n) = 1\}$. Take $T_{(m,n)} := q(L) \subset T$.

This is a simple closed curve on $T$ that wraps $m$ times around a line through the hole in the torus and $n$-times around a circle inside $T$. $T_{(m,n)}$ is called an $(m, n)$-torus knot. $T_{(m,n)}$ can be given by the following parametrisation:

$$
\begin{align*}
x &= (2 + \cos(n\varphi/m)) \cos \varphi, \\
y &= (2 + \cos(n\varphi/m)) \sin \varphi, \\
z &= \sin(n\varphi/m),
\end{align*}
$$

with $\varphi \in [0, 2mn]$. This lies on the torus given by $(r - 2)^2 + z^2 = 1$ in cylindrical coordinates.

**Remark 2.185**

1. $T_{(1,n)}$ is equivalent to $U$ the unknot.
2. $T_{(m,n)}, T_{(n,m)}$ are equivalent.
3. $(m, n) \neq 1$: $T_{(m,n)}$ torus link.

We want to compute $\pi_1(S^3 \setminus K)$, where $K$ is the unknot $U$, or $K$ is a torus knot.

**Decomposition of $S^3$** $A := \{(x_1, x_2, x_3, x_4) \in S^3 \mid x_1^2 + x_2^2 \leq x_3^2 + x_4^2\}$ and $B := \{(x_1, x_2, x_3, x_4) \in S^3 \mid x_1^2 + x_2^2 \geq x_3^2 + x_4^2\}$. Then $A, B$ are closed subsets of $S^3$ and $S^3 = A \cup B$. Further $A \cap B = \{(x_1, x_2, x_3, x_4) \in S^3 \mid x_1^2 + x_2^2 = x_3^2 + x_4^2\}$. So $A \cap B$ is the cartesian products of the circle $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1/2\}$ with the circle $\{(x_3, x_4) \in \mathbb{R}^2 \mid x_3^2 + x_4^2 = 1/2\}$, that is $A \cap B$ is a torus.
Applying the Seifert-van Kampen theorem gives
\[ (x_1, x_2, x_3, x_4) \rightarrow (x_1, x_2, \sqrt{2}x_3\sqrt{1 - (x_1^2 + x_2^2)}, \sqrt{2}x_4\sqrt{1 - (x_1^2 + x_2^2)}) \]

\( f \) is obviously continuous and injective. One can easily verify, that it is also surjective. Further (as \( D^2 \times S^1 \) compact, A Hausdorff) \( f \) homeomorphism.

Analogously, we can prove that \( B \) is homeomorphic to \( D^2 \times S^1 \).

\[ S^3 = A \cup B. \]

We can now compute:

1. \( \pi_1(S^3 \setminus U) \):
   Take \( U \) to be the core circle of the \( A \) torus, i.e. \( U = \{ (x_1, x_2, x_3, x_4) \in A \mid x_1 = 1, x_2 = 0 \} \). The boundary \( A \cap B \) of \( A \) is a deformation retract of \( A \setminus U \). So \( B \) is a deformation retract of \( S^3 \setminus U = (A \cup B) \setminus U \).
   \[ \Rightarrow \pi_1(S^3 \setminus U) \simeq \pi_1(U) \simeq \mathbb{Z}. \]

2. \( \pi_1(S^3 \setminus K) \) where \( K \) is an \( (m, n) \) torus knot \( T_{(m, n)} \):
   Consider \( K \) as a subset of \( A \cap B \) in \( S^3 \), \( S^3 \setminus K = (A \setminus K) \cup (B \setminus K) \), \( A \setminus K \), \( B \setminus K \), \( A \setminus K \cap (B \setminus K) \) path connected. However \( (A \setminus K), (B \setminus K), (A \setminus K) \cap (B \setminus K) \) are not open subsets so we need to modify them in order to apply the Seifert-van Kamepen theorem.

   Choose \( \varepsilon > 0 \) such that there exists a tubular neighbourhood \( N \) of \( K \) in \( S^3 \) with radius \( \varepsilon \). \( S^3 \setminus N \) is a deformation retract of \( S^3 \setminus K \). Then consider \( \frac{1}{2}\varepsilon \) open neighbourhoods \( A', B' \) of \( A \) and \( B \) respectively.

   \( A', B' \) are homeomorphic to the product of \( S^1 \) with an open disk. And \( A' \cap B' \) is homomorphic to \( (A \cap B) \times (\frac{1}{2}\varepsilon, \frac{1}{2}\varepsilon) \). Lastly \( A, B \) are deformation retracts of \( A' \) and \( B' \) respectively.

We can apply the Seifert-van Kamepen theorem for \( S^3 \setminus N = (A' \setminus N) \cup (B' \setminus N) \) to compute \( \pi_1(S^3 \setminus N) \simeq \pi_1(S^3 \setminus K) \).

- \( A' \setminus N \) deformation retracts onto the core circle of the torus \( A \) i.e. \( \pi_1(A' \setminus N) \simeq \mathbb{Z} \), \( B' \setminus N \) deformation retracts onto the wire circle of \( B \), i.e. \( \pi_1(B' \setminus N) \simeq \mathbb{Z} \).
- \( (A' \setminus N) \cap (B' \setminus N) = (A' \cap B') \setminus N \) has the homotopy type of \( (A \cap B) \setminus K \). So \( \pi_1((A' \setminus N) \cap (B' \setminus N)) \simeq \pi_1((A \cap B) \setminus K) \simeq \mathbb{Z} \).
- \( \pi_1(A' \cap B' \setminus N) = \langle \gamma \rangle \rightarrow \pi_1(A' \setminus N) = \langle \alpha \rangle, \pi_2(A' \cap B' \setminus N) = \langle \gamma \rangle \rightarrow \pi_1(B' \setminus N) = \langle \beta \rangle \).
- \( i_1(\gamma) = a^m, i_2(\gamma) = \beta^n \).
- Applying the Seifert-van Kamepen theorem gives

**Proposition 2.186** \( \pi_1(S^3 \setminus T_{(m, n)}) = \{ \alpha, \beta \mid a^m \beta^{-n} \} \).

[Proof: Follows from the above analysis].

**Proposition 2.187** If the knots \( T_{(m, n)}, T_{(m', n')} \) are equivalent, then \( m = m' \) and \( n = n' \). Furthermore \( T_{(m, n)} \) and \( U \) are not equivalent.
Consider the element \( m = b^n = z \in \pi_1(S^3 \setminus T_{(m,n)}) =: G \), and the subgroup \( N \) of \( G \) generated by \( z \). So \( N \leq \pi_1(G) \).

Then \( N \leq G \) and we can consider the quotient \( G / N \). \( G \) has the following presentation: \( G / N \simeq \mathbb{Z}_m \ast \mathbb{Z}_n \). Thus \( \pi_1(G) \simeq \mathbb{Z}_m \ast \mathbb{Z}_n \).

Sine \( \pi_1(G) \simeq \mathbb{Z}_m \ast \mathbb{Z}_n \), it follows that \( \pi_1(G) \simeq \mathbb{Z}_m \ast \mathbb{Z}_n \).

Consider the Wirtinger presentation. Procedure for writing a presentation of the group of a knot \( K \) starting from a diagram for \( K \subset \mathbb{R}^3 \).

1. Label the arcs by \( a_1, \ldots, a_n \), so that each \( a_i \) is connected to \( a_{i-1} \) and \( a_{i+1} \) (mod \( n \)).
2. Assume that the arcs are oriented compatibly with their labelling.
3. Draw an arrow \( x_i \) passing under each \( a_i \) in a right-left direction. Each \( x_i \) represents a loop in \( \mathbb{R}^3 \setminus K \) as follows. Suppose that the black board is the \( xy \)-plane \( P \). Consider as basepoint \( * \) the point \( (0,0,1) \). The loop consists of a segment from \( * \) to the tail of \( x_i \), and then the arrow, then a segment from the head of \( x_i \) to \( * \).
4. At each crossing, there is a certain relation:

\[
\text{In total there are } n \text{ relations.}
\]

**Theorem 2.189** \( \pi_1(\mathbb{R}^3 \setminus K) = \langle x_1, \ldots, x_m \mid r_1, \ldots, r_n \rangle \).

**Example 2.190** Trefoil

\[
\pi_1(\text{trefoil}) = \langle x_1, x_2, x_3 \mid x_3 x_1 = x_2 x_3, x_2 x_3 = x_1 x_2, x_1 x_2 = x_3 x_1 \rangle = \langle x_1, x_2 \mid x_1 x_2 x_1 = x_2 x_1 x_2 \rangle.
\]
Remark 2.191 In example 3 of group presentations we proved that \( \langle x_1, x_2 \mid x_1 x_2 x_1 = x_2 x_1 x_2, x_3 \rangle = \langle \alpha, \beta \mid \alpha^3 = \beta^3 \rangle \).

Proof (of theorem 2.189)

We will apply the Seifert-van Kampen theorem for \( \mathbb{R}^3 \setminus K \). Assume that \( K \) lies on the \( xy \)-plane except where it goes down by distance \( \varepsilon > 0 \), at each crossing. Set the basepoint at \((0,0,1)\).

Let \( A = \{(x, y, z) \in \mathbb{R}^3 \mid z > -2\varepsilon/3\} - K \). Then \( A \simeq_{\mathrm{homeo}} \) open 3-dim Ball \( \setminus \{ n \text{ unknotted arcs with endpoints on the boundary of the ball} \} \).

So \( \pi_1(A) = \langle x_1, \ldots, x_n \rangle \).

For each crossing \( a_k \)

\[
\begin{array}{c|c}
\rightarrow & \leftarrow \\
\downarrow & \downarrow \\
a_i & a_{i+1}
\end{array}
\]

let \( Y_\ell \) be an open rectangular box around it at level \(-2\varepsilon < z < -\varepsilon/3\), and set \( B_\ell = (Y_\ell \setminus K) \cup \{ \text{open neighbourhood of an arc from a point on } \partial Y_\ell \text{ to } * \} \).

Then \( B_\ell \simeq_{\mathrm{homeo}} \) open 3-dim Ball \( \setminus \{ \text{one unknotted arc with endpoints on the boundary} \} \).

So \( \pi_1(B_\ell) = \langle y_\ell \rangle \).

Finally \( C = \text{open neighbourhood of } (\mathbb{R}^3 \setminus A \cup n_{i=1} B_\ell) \cup \{ \text{open neighbourhood of an arc to } * \} \).

\( \mathbb{R}^3 \setminus K = A \cup B_1 \cup \cdots \cup B_\ell \cup C \).

\( \pi_1(A \cup B_1) = \) ?.

\( A \cap B_1 = ?. \)

\[
i_1 : \pi_1(A \cap B_1) \to \pi_1(A), \quad c_1 \mapsto x_k x_k x_k^{-1}, \quad b_1 \mapsto x_{i+1}.
\]

\[
i_2 : \pi_1(A \cap B_1) \to \pi_1(B_1), \quad c_1 \mapsto y_1, \quad c_1 \mapsto y_1.
\]

So we have \( i_1(c_1)(i_2(c_1))^{-1} = x_k x_k x_k^{-1} y_1^{-1} \)

So \( \pi_1(A \cup B_1) = \langle x_1, \ldots, x_n, y_1 \mid x_{i+1} y_1^{-1}, x_k x_k x_k^{-1} y_1^{-1} \rangle = \langle x_1, \ldots, x_n \mid x_k x_k x_k^{-1} x_{i+1}^{-1} \rangle \).

Repeat this process for each \( B_i \) to get \( \pi_1(A \cup B_1 \cup \cdots \cup B_n) = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle \).

Since \( C \) and \( C \cap (A \cup B_1 \cup \cdots \cup B_n) \) are simply connected, applying the Seifert-van Kampen theorem we get that \( \pi_1(K) = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle \).

Study \( \pi_1(S^3 \setminus K) (\simeq \pi_1(\mathbb{R}^3 \setminus K)) \) to show that any one of the \( r_i \) can be ommitted.

Let \( A' = A \cup \{ \infty \}, \quad C' = C \cup B_n \cup \{ \infty \}. \) Then \( \pi_1(A') = \pi_1(A) \) and \( \pi_1(A' \cup B_1 \cup \cdots \cup B_{n-1}) = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_{n-1} \rangle \).

\( S^3 \setminus K = A' \cup B_1 \cup \cdots \cup B_{n-1} \cup C'. \)

Now we can compute that \( \pi_1(C') = \langle y_n \rangle \) and also that \( \pi_1(C' \cap (A' \cup B_1 \cup \cdots \cup B_{n-1})) = \langle b_n \rangle \).
Algebraic topology

\[ \pi_1(C' \cap (A' \cup B_1 \cup \cdots \cup B_{n-1})) \to \pi_1(C'), \quad b_n \mapsto g_n. \]
\[ \pi_1(C' \cap (A' \cup B_1 \cup \cdots \cup B_{n-1})) \to \pi_1(A' \cup B_1 \cup \cdots \cup B_{n-1}), \quad b_n \mapsto x_{i+1}. \]

Applying the Seifert-van Kampen theorem we obtain that \( \langle x_1, \ldots, x_n \mid r_1, \ldots, r_{n-1} \rangle \) \( \pi_1(S^3 \setminus K) \equiv \pi_1(\mathbb{R}^3 \setminus K). \)

**Remark 2.193** The above reasoning applies also to compute \( \pi_1(S^3 \setminus L) \) where \( L \) is a link and \( \pi_1(S^3 \setminus L) = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle. \)

**Example 2.194**

1. 
2. 

**Theorem 2.195** The group of a knot is not a complete knot invariant.

### 2.12. Classification of covering spaces

\( p : \tilde{X} \to X \) covering map with \( p(\tilde{x}_0) = x_0, p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0) \).

We will study \( \tilde{X} = X \cup_{p_*} \tilde{X} \)

**Lemma 2.196** Let \( p : \tilde{X} \to X \) be a covering map with \( p(\tilde{x}_0) = x_0 \). Then

1. \( p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0) \) is a monomorphism.
2. The lifting correspondence \( \phi : \pi_1(X, x_0) \to p^{-1}(x_0) \) where \( [f] \mapsto \tilde{f}(1) \) induces an injective map \( \Phi : \pi_1(X, x_0) \to H \to p^{-1}(x_0) \), where by \( \pi_1(X, x_0) \to H \) we denote all the right cosets of \( H \) in \( \pi_1(X, x_0) \). If \( \tilde{X} \) is path connected, then \( \Phi \) is bijective.
3. If \( f : I \to X \) is a loop based at \( x_0 \), then \( [f] \in H \) iff \( f \) lifts to a loop in \( \tilde{X} \) based at \( \tilde{x}_0 \).

**Proof**

1. Immediate, it follows from homotopy lifting property.
2. If \( f : I \to X, g : I \to X \) are loops based at \( x_0 \), then let \( \tilde{f} : I \to \tilde{X}, \tilde{g} : I \to \tilde{X} \) be their lifts with \( \tilde{f}(0) = \tilde{g}(0) = \tilde{x}_0 \). If \( [f] \in H[g] \), then \( [f] = [g] \), where \( h \in H := p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \) i.e. \( h = p \circ h \) for some loop \( h : I \to \tilde{X} \) based at \( \tilde{x}_0 \). Now \( \tilde{h} \cdot \tilde{g} \) is defined and it is a lift of \( h \cdot g \). Since \( [f] = [h \cdot g] \), \( \tilde{f}(1) = \tilde{h} \cdot \tilde{g}(1) \).

\[ \implies \tilde{f}(1) = \tilde{g}(1). \]
\[ \implies \phi([f]) = \phi([g]). \] That shows \( \Phi \) is well-defined.

3. Check that \( \Phi \) is injective: If \( \phi([f]) = \phi([g]) \), then \( \tilde{f}(1) = \tilde{g}(1) \) and we can define \( \tilde{f} : \tilde{g} \) loop based at \( \tilde{x}_0 \). Therefore \( \tilde{f}(1) = \tilde{g}(1) \).

If \( \tilde{F} \) is a path homotopy between \( (\tilde{f}, \tilde{g}) \) and \( \tilde{F} \), then \( F := p \circ \tilde{F} \) is a path homotopy between \( p(\tilde{f}) \cdot ^p \tilde{g} \) and \( p(f) \).

4. If \( \tilde{X} \) is path connected, then \( \phi \) is surjective. So \( \Phi \) is also surjective.
3. $\Phi$ is injective, hence $\phi([f]) = \phi([g]) \iff [f] \in H[g]$. For $g$ the constant loop based at $x_0$ this gives $[f] \in H \iff \phi([f]) = \overline{x_0} \iff \overline{f}(1) = \overline{x_0}$.

**Lemma 2.198** Let $X$ be path connected and locally path connected. If $\overline{x}_0$ is path component of $\overline{X}$, then $p|\overline{x}_0 : \overline{x}_0 \to X$ is a covering map.

**Proof:** Let $x \in p(\overline{x}_0)$ and $U$ be a path connected open neighbourhood of $x$ such that $p^{-1}(U) = \cup_{x \in V_\alpha} V_\alpha$, where $V_\alpha$ pairwise disjoint open, $p|V_\alpha : V_\alpha \to U$ homomorphism $\forall \alpha$.

Since $U$ contains a point of $p(\overline{x}_0)$, $V_\alpha \cap \overline{x}_0 = \emptyset$ for some $\alpha$. Now $V_\alpha$ is path connected. Then $p(V_\alpha) = U \subseteq p(\overline{x}_0)$, i.e. $x \in p(\overline{x}_0)$. So $p(\overline{x}_0) = p(\overline{X})$.

$p(\overline{X}) \subseteq X$ is both open and closed
\[ \Rightarrow p(\overline{x}_0) = X \]
$p(\overline{x}_0)$ is surjective.

Now let $x \in X$ and chose an open neighbourhood $U$ of $x$ as before. If $V_\alpha \cap \overline{x}_0 = \emptyset$, then $V_\alpha \subseteq X$. Therefore $(p|\overline{x}_0)^{-1}(U)$ is the union of those $V_\alpha$’s that intersect $\overline{x}_0$. Each of these is open with $p|V_\alpha : V_\alpha \to U$ is a homeomorphism.

$p : \overline{X} \to X$ covering map. We restrict to spaces $X$ that are locally path connected. Then $\overline{X}$ is also locally path connected. We can also assume that $X$ is path connected. Lastly we assume, that $\overline{X}$ is also path connected. So, we can determine all covering spaces of a locally path connected space $X$ by determining all path connected covering spaces of $X$ components of $X$.

From now on, whenever we write $p : \overline{X} \to X$ is a covering map, we imply that $X, \overline{X}$ are locally path connected, and path connected.

**Lemma 2.200** (lifting lemma) Let $p : \overline{X} \to X$ covering map with $p(\overline{x}_0) = x_0$, and $f : Y \to X$ continuous. With $Y$ path connected and locally path connected. $f(y_0) = x_0$.

Then $f$ can be lifted to a continuous map $\overline{f} : Y \to \overline{X}$ with $f(\overline{y}_0) = \overline{x}_0$ iff $f_*((\pi_1(Y), y_0)) \subseteq p_*((\pi_1(\overline{X}, \overline{x}_0)))$. Also if such a lift exists, it is unique.

**Proof**

$\Rightarrow$ If $\overline{f}$ exists, then $f_*((\pi_1(Y, y_0)) = (p \circ \overline{f})_*((\pi_1(Y, y_0))) = p_*((\overline{f}(\pi_1(Y, y_0)))) \subseteq p_*((\pi_1(\overline{X}, \overline{x}_0)))$.

$\Leftarrow$ Given $y_1 \in Y$, choose a path $\alpha : I \to Y$ with $\alpha(0) = y_0, \alpha(1) = y_1$. Lift $f \circ \alpha : I \to X$ to $f \circ \alpha : I \to \overline{X}$ with $f \circ \alpha(0) = \overline{x}_0$. Define $\overline{f} : Y \to \overline{X}$ by $\overline{f}(y_1) = f \circ \alpha(1)$.

- $\overline{f}$ is well-defined:
  Let $\beta : I \to Y$ be a path with $\beta(0) = y_0$ and $\beta(1) = y_1$. Lift $f \circ \beta : I \to X$ to a path $f \circ \beta : I \to \overline{X}$ with $f \circ \beta(0) = f \circ \alpha(1)$. Then $f \circ \alpha \cdot f \circ \beta$ is a lift of the loop $f \cdot (\alpha \beta)$. By assumption $f_*((\pi_1(Y, y_0)) \subseteq p_*((\pi_1(\overline{X}, \overline{x}_0)))$. So $[f \circ (\alpha \beta)] \in p_*((\pi_1(\overline{X}, \overline{x}_0)))$. Lemma 2.196 implies that its lift $f \circ \alpha \cdot f \circ \beta$ is a loop based at $\overline{x}_0$. So $f \circ \alpha(1) = f \circ \beta(1)$.

- $\overline{f}$ is continuous:
  Let $y_1 \in Y$ and $N$ an open neighbourhood of $f(y_1)$. We have to show that there is an open neighbourhood $W$ of $y_1$ with $\overline{f}(W) \subseteq N$. Start by choosing a path connected open neighbourhood $U$ of $f(y_1)$ such that $p^{-1}(U) = \cup_{x \alpha} V_\alpha$, $V_\alpha$ disjoint open sets. $p|_{V_\alpha} : V_\alpha \to U$ homeomorphism $\forall \alpha$. $f$ is
continuous at \( y_1 \) and \( Y \) is locally path connected, so there is a path connected open neighbourhood \( W \) of \( y_1 \) with \( f(W) \subseteq U \). We will show that \( \overline{f(W)} \subseteq V_0 \), where \( V_0 \) is the subset containing \( f(y_0) \). Let \( y \in W \), and choose a path \( b: I \to W \) with \( b(0) = y_1 \) and \( b(1) = y \). Since \( f \) is well-defined \( \overline{f(y)} \) can be obtained by taking the path \( \alpha \beta \) from \( y_0 \) to \( y \), lifting \( f \circ (\alpha \beta) \) to a path \( f \circ (\alpha \beta): I \to \tilde{X} \) with \( f \circ (\alpha \beta)(0) = \tilde{x}_0 \) and setting \( \overline{f(y)} = f \circ (\alpha \beta)(1) \).

Now \( f \circ \alpha \) is a lift of \( f \circ \beta \) with \( f \circ \alpha(0) = \tilde{x}_0 \). Since \( f \circ \beta(I) \subseteq U \), the path \( (\pi_{V_0})^{-1} \circ f \circ \beta \) is a lift of \( f \circ \beta \) with \( ((\pi_{V_0})^{-1} \circ f \circ \beta)(0) = \overline{f}(y_1) \). Then \( (f \circ \alpha)((\pi_{V_0})^{-1} \circ f \circ \beta)(0) = \tilde{x}_0 \). But \( (f \circ \alpha)((\pi_{V_0})^{-1} \circ f \circ \beta)(1) = ((\pi_{V_0})^{-1} \circ f \circ \beta)(1) \in V_0 \), hence \( \overline{f(W)} \subseteq V_0 \).

Let \( y_1 \in Y \) and \( \alpha: U \to Y \), with \( \alpha(0) = y_0 \), and \( \alpha(1) = y_1 \). Consider the path \( f \circ \alpha: I \to X \) and lift it to a path \( \overline{f} \circ \alpha: I \to \tilde{X} \) with \( \overline{f} \circ \alpha(v) = \tilde{x}_0 \). Then \( \overline{y}_1 = \overline{f}(\alpha(1)) \) and \( \overline{f}(\alpha(1)) = \overline{f} \circ \alpha(1) \) since \( \overline{f} \circ \alpha \) is a lift of \( f \circ \alpha \) with \( \overline{f}(\alpha(0)) = \overline{f}(y_0) = x_0 \). So we see that if such an \( \overline{f} \) exists, it is unique.

**Example 2.202** Space that is path connected, but not locally path connected.

```
0  1  2
```

And consider a point \( (0, y) \), where \( y > 0 \).

**Example 2.203** Space that is locally path connected, but not path connected.

Equivalent covering spaces

**Definition 2.204** Let \( p: \tilde{X} \to X, p': \tilde{X}' \to X \) be covering maps, \( p, p' \) are called **equivalent** if there exists a homeomorphism \( h: \tilde{X} \to \tilde{X}' \) with \( p = p' \circ h \). \( h \) is called **equivalence** between the covering spaces.

**Theorem 2.205** Let \( p: \tilde{X} \to X, p': \tilde{X}' \to X \) be covering maps with \( p(\tilde{x}_0) = p'(\tilde{x}_0') = x_0 \).

There is an equivalence \( h: \tilde{X} \to \tilde{X}' \) with \( h(\tilde{x}_0) = \tilde{x}_0' \iff p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = p'_*(\pi_1(\tilde{X}', \tilde{x}_0')) \). If such an \( h \) exists, then it is unique.

**Proof**

\[ \Rightarrow \] \( h \) homeomorphism \( \iff h_*(\pi_1(\tilde{X}, \tilde{x}_0)) = \pi_1(\tilde{X}', \tilde{x}_0') \).

\[ \Rightarrow \] \( p'_*(h_*(\pi_1(\tilde{X}, \tilde{x}_0))) = p'_*(\pi_1(\tilde{X}', \tilde{x}_0')) \).

\[ \Rightarrow \] \( p_*(\pi_1(\tilde{X}, \tilde{x}_0')) = p'_*(\pi_1(\tilde{X}', \tilde{x}_0')) \).
Let \( \tau : I \to X \) be a path with \( \gamma(0) = \overline{x}_0 \) and \( \gamma(1) = \overline{x}_1 \), and \( \alpha := p \circ \gamma \), then \( [\alpha]H_1[\alpha]^{-1} = H_0 \).

2. Conversely given \( \overline{x}_0 \) and a subgroup \( H \leq \pi_1(X, x_0) \) conjugate to \( H_0 \), there exists a basepoint \( \overline{x}_1 \in p^{-1}(x_0) \) with \( H_1 = H \).

**Proof**

1. “\( \Rightarrow \)" If \( [h] \in H_1 \), then \( [h] = p_*([\overline{h}]) \) for some \( \overline{h} : I \to \overline{X} \) loop based at \( \overline{x}_1 \). Let \( \overline{k} = (\gamma \cdot h) \cdot \overline{\pi} \). Then \( p_*([\overline{k}]) = p_*([\overline{\gamma} \cdot \overline{h} \cdot \overline{\pi}]) = [\alpha \cdot h \cdot \overline{\pi}] = [\alpha]h[\alpha]^{-1}. \)

That is \( [\alpha]H_1[\alpha]^{-1} = H_0 \).

“\( \Leftarrow \)" Apply the above for \( \overline{\pi} \) and \( \overline{\pi} = p \circ \overline{\pi} \). We that that \( [\overline{\pi}]H_0[\overline{\pi}]^{-1} \subseteq H_1. \Rightarrow H_0 \subseteq [\alpha]H_1[\alpha]^{-1}.

2. Let \( \overline{x}_0 \in \overline{X} \) and \( H \leq \pi_1(X, x_0) \) conjugate to \( H_0 \). Then \( H_0 = [\alpha]H_1[\alpha]^{-1} \) for some \( \alpha : I \to X \) based at \( x_0 \). Let \( \gamma \) be the lift of \( \alpha \) to \( \overline{X} \) with \( \gamma(0) = \overline{x}_0 \). Let \( \gamma(1) = \overline{x}_1 \). Then by 1. \( \Rightarrow H_0 = [\alpha]H_1[\alpha]^{-1} \).

**Theorem 2.209** Let \( p : \overline{X} \to X, p' : \overline{X}' \to X \) be covering maps with \( p(\overline{x}_0) = p'(\overline{x}_0') = x_0 \). \( p, p' \) are equivalent \( \iff H_0 := p_*(\pi_1(\overline{X}, \overline{x}_0)) \) and \( H_0' := p'_*(\pi_1(\overline{X}', \overline{x}_0')) \) are conjugate.

**Proof**

“\( \Rightarrow \)" If \( h : \overline{X} \to \overline{X}' \) is an equivalence, let \( \overline{x}_1 = h(\overline{x}_0) \) and \( H_1' = p_*(\pi_1(\overline{X}', \overline{x}_1')) \). By the previous theorem \( H_0 = H_1' \). And by the lemma \( H_0', H_1' \) are conjugate. \( \Rightarrow H_0, H_0' \) are conjugate.

“\( \Leftarrow \)" If \( H_0, H_0' \) are conjugate, then according to the lemma there exists \( \overline{x}_1' \in \overline{X}' \) with \( H_1' = H_0 \). Then by previous theorem there is an equivalence \( h : \overline{X} \to \overline{X}' \) with \( h(\overline{x}_0) = \overline{x}_1' \).

**Example 2.211** \( X = S^1, x_0 \in S^1, \pi_1(S^1, x_0) \cong \mathbb{Z}. \)

Subgroups of \( \mathbb{Z} \): \( n\mathbb{Z}, n \in \mathbb{N} \).

Covering spaces of \( S^1 \): \( p : \mathbb{R} \to S^1 \) where \( t \mapsto (\cos 2\pi t, \sin 2\pi t) \).

\( p_*(\pi_1(\mathbb{R})) \) is trivial. That is this covering space of \( S^1 \) corresponds to the trivial subgroup
of $\pi_1(S^1)$.

An other covering map we saw was $p_n : S^1 \to S^1, z \mapsto z^n$. Then $(p_n)_*(\pi_1(S^1)) = n\mathbb{Z}$.

Question: Is there an a covering map $p : \tilde{X} \to S^1$, that is not equivalent to any of the above.

No, as $p_* (\pi_1(\tilde{X})) \leq \pi_1(S^1)$.

The universal covering space

**Definition 2.212** Let $p : \tilde{X} \to X$ be a covering map. If $\tilde{X}$ is simply connected, then $\tilde{X}$ is called a **universal covering space** of $X$.

**Remark 2.213** $p_*(\pi_1(\tilde{X}))$ trivial $\implies$ any two universal covering spaces of $X$ are equivalent. We can therefore speak of the universal covering space of $X$.

**Lemma 2.214** Let $p, q, r$ be continuous maps with $p = r \circ q$. If $p, r$ are covering maps, then $q$ is also a covering map.

**Proof** Let $x_0 \in X, x_0 \in p(x_0), y_0 = q(x_0)$.

**Claim 1:** $q$ is surjective.

Let $y \in Y$. Choose a path $\overline{\alpha} : I \to Y$ with $\overline{\alpha}(0) = y_0$ and $\overline{\alpha}(1) = y$. Then $\alpha = r \circ \overline{\alpha}$ is a path in $Z$ with $\alpha(0) = r(\overline{\alpha}(0)) = r(y_0) = r(q(x_0)) = p(x_0) = y_0$. Let $\overline{\alpha}$ be the lift of $\alpha$ to a path in $X$ with $\overline{\alpha}(0) = x_0$. Then $q \circ \overline{\alpha}$ is a lift of $\alpha$ to $Y$ with $q \circ \overline{\alpha}(0) = y_0$. By uniqueness of path liftings $q \circ \overline{\alpha}(1) = \overline{\alpha}(1) = y \implies y = q(\overline{\alpha}(1)) = q(1)$ is surjective.

**Claim 2:** Given $y \in Y$, there exists n open neighbourhood $V$ of $y$ with $q^{-1}(V) = \cup_\gamma W_\gamma$, where $W_\gamma$ pairwise disjoint open sets with $q|_{W_\gamma} : W_\gamma \to V$ homeomorphism $\forall \gamma$. So, let $y \in Y$. Set $z := r(y)$. Since $p, r$ are covering maps we can find an open path-connected neighbourhood of $z$ such that $p^{-1}(U) = \cup_\alpha U_\alpha, U_\alpha$ disjoint open with $p|_{U_\alpha} : U_\alpha \to U$ homeomorphism $\forall \alpha$. $r^{-1}(U) = \cup_\beta V_\beta, V_\beta$ open, $r|_{V_\beta} : V_\beta \to U$ homeomorphism $\forall \beta$.

Call $V$ the member of the family $V_\beta$ that contains the point $y$. $q(U_\alpha) \subseteq r^{-1}(U) \forall \alpha$ and $U_\alpha$ is connected, hence $q(U_\alpha) \subseteq V_\beta$ for some $\beta$. Also $q^{-1}(V) \subseteq \cup_\alpha U_\alpha$. Indeed let $x \in q^{-1}(V)$ then $q(x) \in V \implies r(q(x)) \in r(V) = U \implies p(x) \in U \implies x \in p^{-1}(U) \implies x \in \cup_\alpha U_\alpha$.

Therefore $q^{-1}(V)$ is the union of those $U_\alpha$’s with $q(U_\alpha) \subseteq V$.

In fact, $q|_{U_\alpha} : U_\alpha \to V$ is a homeomorphism for each such $\alpha$.

Since $p|_{U_\alpha}, r|_{V_\beta}$ are homeomorphisms and $q|_{U_\alpha} = (r|_{V_\beta})^{-1} \circ p|_{U_\alpha}$.

Call this family of $U_\alpha$’s $W_\gamma$.

**Theorem 2.216** Let $p : \tilde{X} \to X$ be a covering map, and $\tilde{X}$ simply connected. If $r : Y \to X$ is a covering map. Then there exists a covering map $q : \tilde{X} \to Y$ with $r \circ q = p$.

**Proof** Let $x_0 \in X$. Choose $\overline{x_0} \in \tilde{X}$ and $y_0 \in Y$ with $p(\overline{x_0}) = x_0, r(y_0) = x_0, p_*(\pi_1(\tilde{X}, \overline{x_0})) \subseteq r_*(\pi_1(Y, y_0))$. So we can apply the lifting lemma for the covering map $r : Y \to X$. That is $\exists q : \tilde{X} \to Y$ continuous with $q(\overline{x_0}) = y_0$ and $r \circ q = p$. $p, r$ are covering maps $\implies q$ is covering map.

**Remark 2.218** This theorem justifies the use of the term the universal covering space.

**Question:** Does every space $X$ have a universal covering space?

**Answer:** No.

**Lemma 2.219** Let $p : \tilde{X} \to X$ be a covering map with $p(\overline{x_0}) x_0$ and $\tilde{X}$ is simply connected. Then $x_0$ has an open neighbourhood $U$ such that the map $i$ induced by the inclusion $i : (U, x_0) \to (X, x_0)$ is trivial.
2.12 Classification of covering spaces

Proof Let $U$ be an open neighbourhood of $x_0$ with $p^{-1}(U) = \bigcup \alpha U_\alpha$, $U_\alpha$ pairwise disjoint open sets, $p|U_\alpha : U_\alpha \to U$ homeomorphism $\forall \alpha$. Let $U_0$ be the $U_\alpha$, that contains $x_0$.

Consider $f : I \to U$ a loop based at $x_0$. And let $\bar{f} = (p|U_0)^{-1}(f)$ be the lift of $f$ in $\bar{X}$.

$\bar{f}$ is a loop based at $\bar{x}_0$, $\pi_1(\bar{X}, \bar{x}_0)$ trivial $\implies$ $\exists$ a path homotopy $\bar{F}$ between $\bar{f}$ and the constant loop at $\bar{x}_0$. Then $\bar{F} = p \circ \bar{F}$ is a path homotopy in $X$ between $p \circ \bar{f} = f$ and the constant path at $p(\bar{x}_0) = x_0$. That is, $i_{*}$ is trivial.

Example 2.221 $X = \bigcup n \in \mathbb{N} C_n$ (Hawaiian earrings). Let $r_n : X \to C_n$ mapping every $C_i$ with $i \neq n$ to $(0,0)$ with $r_n C_n = 1 C_n$.

Let $U$ be an open neighbourhood of $x_0 \in X$. Choose $n$ large enough such $C_n \subseteq U$.

$$j : C_n \to X, k : C_n \to U, i : U \to X.$$ $$r_n \circ j = 1 C_n \implies (r_n)* \circ j_* = 1_{\pi_1 C_n, x_0} \implies j_*$ is injective.

Now $j_* = i_* \circ k_*$, hence $i_* \circ k_* : \pi_1 (C_n, x_0) \to \pi_1 (X, x_0)$ is injective. I.e. $i_*$ is not trivial.

Lemma $\implies$ The shrinking of circles has no universal covering space.

Existence of covering spaces

Definition 2.222 $X$ is called semilocally simply connected if $\forall x \in X \exists$ open neighbourhood $U$ of $x$ with $i_* : \pi_1 (U, x) \to \pi_1 (X, x)$, induced by the inclusion $i : (U, x) \to (X, x)$, is trivial.

Remark 2.223

1. If $U$ is an open neighbourhood of $x$ with $i_* : \pi_1 (U, x) \to \pi_1 (X, x)$ trivial and $V$ is an open neighbourhood of $x$ with $V \subseteq U$, then obviously $j_* : \pi_1 (V, x) \to \pi_1 (X, x)$ is trivial.

2. If $X$ is locally simply connected, i.e. $\forall x \in X$ and $\forall$ open neighbourhoods $U$ of $x$ there exists a simply connected open neighbourhood $V$ of $x$ with $V \subseteq U$, then $X$ is semilocally simply connected.

Semilocal simply connectedness is in fact a necessary and sufficient condition for the correspondance

$$(\text{Covering maps of } X) \not\sim \overset{1:1}{\sim} \text{ conjugacy classes of subgroups of } \pi_1 (X, x_0)$$

to be surjective.

Theorem 2.224 Let $X$ be path connected, locally path and semilocally simply connected. Let $x_0 \in X$ and $H \leq \pi_1 (X, x_0)$. Then $\exists p : \bar{X} \to X$ covering map with $p_*(\pi_1 (\bar{X}, \bar{x}_0)) = H$.

Corollary 2.225 $X$ has a universal covering space iff $X$ is path connected, locally path connected and semilocally simply connected.

Proof (of the theorem)

We do this in seven steps:

Step 1 Construction of $\bar{X}$;

Define $P := \{ \gamma : I \to Y \text{ path with } \gamma(0) = x_0 \}$. For $\alpha, \beta \in P$ define $a \sim \beta$ if $a(1) = \beta(1)$ and $\alpha \beta \in H$. Now define $\bar{X} := P \not\sim = \{ \gamma \# | \gamma : I \to X \text{ path with } \gamma(0) = x_0 \}$, and define $p : \bar{X} \to X$ by $p(\gamma \#) = \gamma(1)$.

Note:
Algebraic topology

- $p$ is surjective, since $X$ is path connected.

We will define a topology on $\tilde{X}$ so that $p: \tilde{X} \to X$ is a covering map.

**Remark 2.227** If $[\alpha] = [\beta]$, then $\alpha^\# = \beta^\#$. Further, if $\alpha^\# = \beta^\#$ and $\delta: I \to X$ with $\delta(0) = \alpha(1)$, then $(\alpha\delta)^\# = (\beta\delta)^\#$.

**Step 2** Let $P \subseteq U$ and $U$ a path connected neighbourhood of $\alpha(1)$. Define $B(U, \alpha) := \{(\alpha\delta)^\# \mid \delta: I \to U \text{ with } \delta(0) = \alpha(1)\}$. Then $\alpha^\# \in B(U, \alpha)$.

Claim: The sets $B(U, \alpha)$ form a basis for a topology on $\tilde{X}$.

Proof: We first prove that if $\beta^\# \in B(U, \alpha)$, then $\alpha^\# \in B(U, \beta)$ and $B(U, \alpha) = B(U, \beta)$. For this consider $\beta^\# \in B(U, \alpha)$. Then $\beta^\# = (\alpha\delta)^\#$, for some $\delta: I \to U$ with $\delta(0) = \alpha(1)$ and $(\beta\delta)^\# = (\alpha\delta)^\# = \alpha^\#$. So $\alpha^\# = (\beta\gamma)^\# \in B(U, \beta)$. Let $(\beta\gamma)^\# \in B(U, \beta)$. Then $(\beta\gamma)^\# = ((\alpha\gamma\delta)^\# = (\alpha(\delta\gamma))^\# \in B(U, \alpha)$. That is $B(U, \beta) \subseteq B(U, \alpha)$, and by analogous argument $B(U, \alpha) = B(U, \beta)$.

Now we can prove that the sets $B(U, \alpha)$ form a basis for a topology on $\tilde{X}$. Indeed:

- Let $\alpha^\# \in \tilde{X}$. Then choose a path connected open neighbourhood $U$ of $\alpha(1)$. Then $\alpha^\# \in B(U, \alpha)$.
- Let $\alpha^\# \in B(U, \alpha_1) \cap B(U, \alpha_2)$. Choose a path connected open neighbourhood $V$ of $\alpha(1)$ with $V \subseteq U_1 \cup U_2$. Then $B(V, \alpha) \subseteq B(U_1, \alpha) \cap B(U_2, \alpha) = B(U_1, \alpha_1) \cap B(U_2, \alpha_2)$.

Therefore the sets $B(U, \alpha)$ form a basis for a topology on $\tilde{X}$.

**Step 3** $p$ is open and continuous:

$p$ is open: We will prove that $p(B(U, \alpha)) = U$, which implies $p$ is open.

"$\subseteq$" Let $x \in U$ and choose a path $\delta: I \to U$ with $\delta(0) = x$, $\delta(1) = x$. Then $(\alpha\delta)^\# \in B(U, \alpha)$ and $p((\alpha\delta)^\#) = (\alpha\delta)(1) = \delta(1) = x$. I.e. $x \in p(B(U, \alpha))$, i.e. $U \subseteq p(B(U, \alpha))$.

"$\supseteq$" $p(B(U, \alpha)) \subseteq U$ (follows form the definition of $p$ and $B(U, \alpha)$.

$p$ is continuous: Let $\alpha^\# \in \tilde{X}$ and $W$ an open neighbourhood of $p(\alpha^\#)$. Choose a path connected open neighbourhood $U$ of $p(\alpha^\#) = \alpha(1)$ with $\subseteq W$. Then $B(U, \alpha)$ is an open neighbourhood of $\alpha^\#$ in $\tilde{X}$ with $p(B(U, \alpha)) = U \subseteq W$.

**Step 4** $\forall x \in X$ an open neighbourhood of $U$ of $x$ with $p^{-1}(U) = \cup_\alpha V_\alpha$ where $V_\alpha$ pairwise disjoint open sets, and $\pi_{\alpha}|_{V_\alpha}: V_\alpha \to U$ homomorphism.

Let $x \in X$. Choose a path connected open neighbourhood $U$ of $x$ such that $i_x: \pi_1(U, x) \to \pi_1(X, x)$ is trivial.

Claim 1: $p^{-1}(U) = \cup_{\alpha: I \to X, \alpha(0)=x} \cup_{\alpha^\#: B(U, \alpha)}^\# B(U, \alpha)$.

Proof: "$\subseteq$" $p(B(U, \alpha)) = U \implies B(U, \alpha) \subseteq p^{-1}(U) \implies \cup_{\alpha: I \to X, \alpha(0)=x} B(U, \alpha) \subseteq p^{-1}(U)$.

"$\supseteq$" Let $\beta^\# \in p^{-1}(U)$. Then $p(\beta^\#) \in U$, i.e. $\beta(1) \in U$. Choose a path $\delta \in U$ from $x$ to $\beta(1)$ and let $\alpha = \beta\delta$. Then $[\beta] = [\alpha\delta] \implies \beta^\# = (\alpha\delta)^\# \in B(U, \alpha)$. I.e. $p^{-1}(U) \subseteq \cup_{\alpha: I \to X, \alpha(0)=x} B(U, \alpha)$.

Claim 2: Distinct sets $B(U, \alpha)$ are disjoint.

Proof: Suppose $\beta^\# \in B(U, \alpha) \cap B(U, \alpha_2)$. Then $B(U, \alpha_1) = B(U, \beta) = B(U, \alpha_2)$.

Claim 3: $p(B(U, \alpha))$: $B(U, \alpha) \to U$ is bijective.

Proof: We have shown that $p(B(U, \alpha)) = U$, i.e. $p$ is surjective. To check injectivity suppose $p((\alpha\delta_1)^\#) = p((\alpha\delta_2)^\#)$. $\delta_i: I \to U$, $\delta_i(0) = \alpha(1)$, $i \in \{1, 2\}$. Then
2.12 Classification of covering spaces

\((\alpha \delta_1)(1) = (\alpha \delta_2)(1) \implies \delta_1(1) = \delta_2(1)\). So we can define \(\delta_2 : I \to U\) (loop based at \(\delta_1(0) = \delta_2(0) = \delta_2(1) = x\)). Since \(i_* : \pi_1(U, x) \to \pi_1(X, x)\) is trivial, there is a path homotopy in \(X\) between \(\delta_1 \delta_2\) and the constant loop at \(x\). \([\alpha \delta_1] = [\alpha \delta_2]\). Now \(p|B(U, x)\) is bijective, continuous and open. \(\implies p|B(U, x)\) homeomorphism.

**Step 5** Lifting a path in \(X\) to a path in \(\tilde{X}\).

Let \(e_0\) be the equivalence class in \(\tilde{X}\) of the constant path at \(x_0\). Then \(p(e_0) = x_0\). Let \(\alpha : I \to X\) path with \(\alpha(0) = x_0\). We want to compute its lift to a path \(\tilde{\alpha} : I \to \tilde{X}\) with \(\tilde{\alpha}(0) = e_0\) and show that \(\tilde{\alpha}(1) = \alpha^#\). Given \(c \in [0, 1]\), let \(\alpha_c : I \to X\) the path defined by \(\alpha_c(t) = \alpha(ct), t \in I\). Define \(\tilde{\alpha} : I \to \tilde{X}\) by \(\tilde{\alpha}(c) = (\alpha_c)^#\). Then \(p(\tilde{\alpha}(c)) = (p(\alpha_c)^#) = \alpha_c(1) = \alpha(c)\). \(\implies p \circ \tilde{\alpha} = \alpha\) and \(\tilde{\alpha}(0) = (\alpha_0)^# = e_0\), \(\tilde{\alpha}(1) = (\alpha_1)^# = \alpha^#\).

Claim: \(\tilde{\alpha}\) is continuous.

Proof: omitted.

**Step 6** \(p : \tilde{X} \to X\) is covering map.

\(p\) is surjective (step 1), \(p\) satisfies the covering condition (step 4). \(X\) is path and locally path connected by assumption, \(\tilde{X}\) is path connected (step 5). (\(\tilde{X}\) is locally path connected as \(X\) is).

**Step 7** \(H = \pi_1(\tilde{X}, e_0)\):

Let \(\alpha : I \to X\) be a loop based at \(x_0\) and \(\tilde{\alpha} : I \to \tilde{X}\) its lift with \(\tilde{\alpha}(0) = e_0\). Then 
\([\alpha] \in \pi_1(\tilde{X}, e_0) \iff \tilde{\alpha}(1) = \tilde{\alpha}(0) = e_0 \iff \alpha^# = e_0 \iff \alpha \sim \text{constant path at } x_0 \iff [\alpha e_{x_0}] \in H \iff [\alpha] \in H\).
# Index

## Symbols

<table>
<thead>
<tr>
<th>Term</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon$-ball</td>
<td>6</td>
</tr>
<tr>
<td>A</td>
<td></td>
</tr>
<tr>
<td>adjunction space</td>
<td>33</td>
</tr>
<tr>
<td>attaching map</td>
<td>33</td>
</tr>
<tr>
<td>B</td>
<td></td>
</tr>
<tr>
<td>basepoint</td>
<td>18</td>
</tr>
<tr>
<td>basis</td>
<td>5</td>
</tr>
<tr>
<td>C</td>
<td></td>
</tr>
<tr>
<td>category</td>
<td>21</td>
</tr>
<tr>
<td>closed</td>
<td>4</td>
</tr>
<tr>
<td>closure</td>
<td>5</td>
</tr>
<tr>
<td>compact</td>
<td>12</td>
</tr>
<tr>
<td>comparable</td>
<td>5</td>
</tr>
<tr>
<td>connected</td>
<td>9</td>
</tr>
<tr>
<td>connected component</td>
<td>12</td>
</tr>
<tr>
<td>connected sum</td>
<td>34</td>
</tr>
<tr>
<td>continuous</td>
<td>7</td>
</tr>
<tr>
<td>countable topology</td>
<td>4</td>
</tr>
<tr>
<td>covariant functor</td>
<td>21</td>
</tr>
<tr>
<td>covering map</td>
<td>21</td>
</tr>
<tr>
<td>covering space</td>
<td>21</td>
</tr>
<tr>
<td>D</td>
<td></td>
</tr>
<tr>
<td>deformation retract</td>
<td>29</td>
</tr>
<tr>
<td>deformation retraction</td>
<td>29</td>
</tr>
<tr>
<td>direct sum</td>
<td>35</td>
</tr>
<tr>
<td>external</td>
<td>36</td>
</tr>
<tr>
<td>discrete topology</td>
<td>4</td>
</tr>
<tr>
<td>disjoint union</td>
<td>33</td>
</tr>
<tr>
<td>disjoint union topology</td>
<td>33</td>
</tr>
<tr>
<td>E</td>
<td></td>
</tr>
<tr>
<td>edges</td>
<td>52</td>
</tr>
<tr>
<td>F</td>
<td></td>
</tr>
<tr>
<td>embedding</td>
<td>22</td>
</tr>
<tr>
<td>equivalence</td>
<td>60</td>
</tr>
<tr>
<td>equivalent</td>
<td>53, 60</td>
</tr>
<tr>
<td>external direct sum</td>
<td>36</td>
</tr>
<tr>
<td>external free product</td>
<td>39</td>
</tr>
<tr>
<td>G</td>
<td></td>
</tr>
<tr>
<td>final point</td>
<td>16</td>
</tr>
<tr>
<td>finer</td>
<td>5</td>
</tr>
<tr>
<td>finite topology</td>
<td>4</td>
</tr>
<tr>
<td>finitely generated</td>
<td>43</td>
</tr>
<tr>
<td>finitely presented</td>
<td>43</td>
</tr>
<tr>
<td>first homology group</td>
<td>50</td>
</tr>
<tr>
<td>free abelian group</td>
<td>37</td>
</tr>
<tr>
<td>free group</td>
<td>41</td>
</tr>
<tr>
<td>free product</td>
<td>38</td>
</tr>
<tr>
<td>fundamental group</td>
<td>19</td>
</tr>
<tr>
<td>G</td>
<td></td>
</tr>
<tr>
<td>generate</td>
<td>35, 38</td>
</tr>
<tr>
<td>group of the link</td>
<td>54</td>
</tr>
<tr>
<td>H</td>
<td></td>
</tr>
<tr>
<td>Hausdorff</td>
<td>14</td>
</tr>
<tr>
<td>homeomorphism</td>
<td>8</td>
</tr>
<tr>
<td>homeotopic</td>
<td>16</td>
</tr>
<tr>
<td>homotopy</td>
<td>30</td>
</tr>
<tr>
<td>homotopy equivalence</td>
<td>30</td>
</tr>
<tr>
<td>homotopy equivalent</td>
<td>30</td>
</tr>
<tr>
<td>homotopy inverse</td>
<td>30</td>
</tr>
<tr>
<td>homotopy type</td>
<td>30</td>
</tr>
<tr>
<td>I</td>
<td></td>
</tr>
<tr>
<td>induced homomorphism</td>
<td>20</td>
</tr>
<tr>
<td>initial point</td>
<td>16</td>
</tr>
<tr>
<td>interior</td>
<td>5</td>
</tr>
<tr>
<td>K</td>
<td></td>
</tr>
<tr>
<td>knot</td>
<td>53</td>
</tr>
<tr>
<td>L</td>
<td>S</td>
</tr>
<tr>
<td>--------------------</td>
<td>--------------------</td>
</tr>
<tr>
<td>least upper bound property</td>
<td>semilocally simply connected</td>
</tr>
<tr>
<td>length</td>
<td>separation</td>
</tr>
<tr>
<td>lifting</td>
<td>simply connected</td>
</tr>
<tr>
<td>lifting correspondence</td>
<td>standard topology</td>
</tr>
<tr>
<td>link</td>
<td>stereographic projection</td>
</tr>
<tr>
<td>locally connected</td>
<td>strictly finer</td>
</tr>
<tr>
<td>locally path connected</td>
<td>subset</td>
</tr>
<tr>
<td>loop</td>
<td>closed</td>
</tr>
<tr>
<td></td>
<td>open</td>
</tr>
<tr>
<td></td>
<td>subspace topology</td>
</tr>
<tr>
<td></td>
<td>sum</td>
</tr>
<tr>
<td>M</td>
<td>T</td>
</tr>
<tr>
<td>metric</td>
<td>topological property</td>
</tr>
<tr>
<td>metric topology</td>
<td>topological space</td>
</tr>
<tr>
<td>N</td>
<td>topology</td>
</tr>
<tr>
<td>nullhomotopic</td>
<td>countable</td>
</tr>
<tr>
<td>O</td>
<td>discrete</td>
</tr>
<tr>
<td>open</td>
<td>finite</td>
</tr>
<tr>
<td>open neighbourhood</td>
<td>metric</td>
</tr>
<tr>
<td>order relation</td>
<td>product</td>
</tr>
<tr>
<td>order topology</td>
<td>subspace</td>
</tr>
<tr>
<td>P</td>
<td>trivial</td>
</tr>
<tr>
<td>path</td>
<td>triangles</td>
</tr>
<tr>
<td>path components</td>
<td>triangulation</td>
</tr>
<tr>
<td>path connected</td>
<td>trivial topology</td>
</tr>
<tr>
<td>path homotopic</td>
<td>vertices</td>
</tr>
<tr>
<td>presentation</td>
<td>U</td>
</tr>
<tr>
<td>product</td>
<td>universal covering space</td>
</tr>
<tr>
<td>product topology</td>
<td>V</td>
</tr>
<tr>
<td>Q</td>
<td>W</td>
</tr>
<tr>
<td>quotient map</td>
<td>wedge</td>
</tr>
<tr>
<td>quotient topology</td>
<td>wedge sum</td>
</tr>
<tr>
<td>R</td>
<td>word</td>
</tr>
<tr>
<td>rank</td>
<td>word sum</td>
</tr>
<tr>
<td>reduced word</td>
<td>word</td>
</tr>
<tr>
<td>relation subgroup</td>
<td>38</td>
</tr>
<tr>
<td>reparametrisation</td>
<td>38</td>
</tr>
<tr>
<td>retract</td>
<td>43</td>
</tr>
<tr>
<td>retraction</td>
<td>18</td>
</tr>
<tr>
<td></td>
<td>27</td>
</tr>
<tr>
<td></td>
<td>27</td>
</tr>
</tbody>
</table>