Introduction to Algebraic Topology

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Contents

Literature

- Muncres: Topology 2nd edition
- Jänich: Topologie
- Massey: Algebraic Topology: An Introduction
- Stöcker, Zieschang: Algebraische Topologie
- Lickorish: An introduction to knot theory

Introduction

In calculus, you have studied $\mathbb{R}^n$, $n \in \mathbb{N}$, as well as functions $f : \mathbb{R}^k \to \mathbb{R}^\ell$, $k, \ell \in \mathbb{N}$. You have studied notions such as neighbourhood of a point $x \in \mathbb{R}^n$, as well as convergence and continuity of a function f at a point $x \in \mathbb{R}^k$. For this study, you have used the Euclidean metric.
For instance if $f : \mathbb{R} \to \mathbb{R}$, we say that f is continuous at $x \in \mathbb{R}$ if $\forall \varepsilon > 0\ \exists \delta > 0$ such that if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \varepsilon$.
We used the metric $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, with $(x, y) \mapsto |x - y|$. In general the Euclidean metric $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ with $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \mapsto \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$.

In topology, we study notions such as neighbourhood of a point $x \in X$, convergence, continuity for a general set X. For this study a metric is not necessary. What we will use is open sets in X.

In algebraic topology, we use abstract algebra to study topological properties.

1 Topology

1.1. Topological spaces and continuous functions

1.1.1. Topological spaces

Consider a set X and $\mathcal{P}(X) := \{U \mid U \subseteq X\}$.

Definition 1.1 A ***topology*** on X is a collection $\mathcal{T} \subset \mathcal{P}(X)$ such that

1. $\varnothing, X \in \mathcal{T}$.
2. If $U_i \in \mathcal{T}\ \forall i \in I$, then $\bigcup_{i \in I} U_i \in \mathcal{T}$.
3. If $U_i \in \mathcal{T}, i \in \{1, \dots, n\}$, then $\bigcap_{i \in \{1,\dots,n\}} U_i \in \mathcal{T}$.

$(X, \mathcal{T})$ is called a ***topological space*** (sometimes we will only write X).
$U \subseteq X$ is called ***open***, if $U \in \mathcal{T}$.
$U \subseteq X$ is called ***closed***, if $X \backslash U \in \mathcal{T}$.

Example 1.2

1. X some set.
   - $\mathcal{T}_d = \{U \mid U \subseteq X\}$ ***discrete topology***.
   - $\mathcal{T}_t = \{\varnothing, X\}$ ***trivial topology***.
2. $X = \{x_1, x_2, x_3\}$. Then the three collections
   - $\{\varnothing, X\}$
   - $\{\varnothing, \{x_1\}, \{x_1, x_2\}, X\}$
   - $\{\varnothing, \{x_2\}, \{x_1, x_2\}, \{x_2, x_3\}, X\}$

   are topologies on X.
3. X is a set. Then the collections
   - $\mathcal{T}_f = \{U \subseteq X \mid X \backslash U \text{ is finite or } X \backslash U = X\}$ ***finite topology***.
   - $\mathcal{T}_c = \{U \subseteq X \mid X \backslash U \text{ is countable or } X \backslash U = X\}$ ***countable topology***.

   are topologies on X. [Homework]

Definition 1.3 Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are topologies on X. If $\mathcal{T}' \supseteq \mathcal{T}$, then $\mathcal{T}'$ is called ***finer*** than $\mathcal{T}$. If $\mathcal{T}' \supsetneq \mathcal{T}$, then $\mathcal{T}'$ is called ***strictly finer*** than $\mathcal{T}$. If $\mathcal{T}' \supseteq \mathcal{T}$ or $\mathcal{T} \supseteq \mathcal{T}'$, then $\mathcal{T}$ and $\mathcal{T}'$ are ***comparable***.

Definition 1.4 Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$.
The ***interior*** of A is

$$\operatorname{int} A := \bigcup_{U \in \mathcal{T}, U \subseteq A} U.$$

The ***closure*** of A is

$$\overline{A} := \bigcap_{X \setminus U \in \mathcal{T},\ A \subseteq U} U.$$

Basis for a topology

Definition 1.5 A ***basis*** $\mathcal{B}$ for a topological space $(X, \mathcal{T})$ is a collection $\mathcal{B}$ of open subsetes X (i.e. $\mathcal{B} \subseteq \mathcal{T}$), such that $\forall U \in \mathcal{T}\ \exists \{B_i\}_{i \in I}$, $B_i \in \mathcal{B}$, $\forall i \in I$, with $U = \bigcup_{i \in I} B_i$.

Remark 1.6 A basis is not unique.
Properties: $\mathcal{B}$ basis for $(X, \mathcal{T})$ then $\mathcal{B}$ has the following properties

1. $\forall x \in X\ \exists B \in \mathcal{B}$ with $x \in B$.
   Proof $X \in \mathcal{T}$ and $\mathcal{B}$ is a basis $\Longrightarrow X = \bigcup_{i \in I} B_i$, $B_i \in \mathcal{B}$, $i \in I$. So $x \in B_j$ for some $j \in I$.
2. If $x \in B_1 \cap B_2$, $B_1, B_2 \in \mathcal{B}$ then $\exists B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.
   Proof $B_1, B_2 \in \mathcal{B} \subseteq \mathcal{T} \Longrightarrow B_1, B_2 \in \mathcal{T} \Longrightarrow B_1 \cap B_2 \in \mathcal{T} \Longrightarrow B_1 \cap B_2 = \bigcup_{i \in I} B_i, \dots$

Conversely: If a collection $\mathcal{B}$ of subsets of X satisfies properties 1. and 2. then there is a unique topology $\mathcal{T}$ for which $\mathcal{B}$ is a basis.
$\mathcal{T}$ is called the topology generated by $\mathcal{B}$ and it consists of all unions of elements of $\mathcal{B}$.
Proof We prove that $\mathcal{T}$ is a topology

1. $X \in \mathcal{T}$?
   Property 1 $\Longrightarrow$ if $x \in X$ then $\exists B_x \in \mathcal{B}$ with $x \in B_x \subseteq X$. Therefore $X = \bigcup_{x \in X} B_x$, i.e. $X \in \mathcal{T}$.

   $\varnothing \in \mathcal{T}$?
   $\varnothing$ is the empty union of elements in $\mathcal{B}$, so $\varnothing \in \mathcal{T}$.
2. If $U_i \in \mathcal{T}\ \forall i \in I$, does $\bigcup_{i \in I} U_i \in \mathcal{T}$?
   $U_i \in \mathcal{T} \Longrightarrow U_i = \bigcup_{j < inJ} B_{ij}$. So $\bigcup_{i \in I} U_i = \bigcup_{i \in I} \bigcup_{j \in J} B_{ij} \in \mathcal{T}$.
3. If $U_i \in \mathcal{T}$ for $i \in \{1, \dots, n\}$, is $\bigcap_{i \in \{1, \dots, n\}} U_i \in \mathcal{T}$?
   We will show that if $U_1, U_2 \in \mathcal{T}$ then $U_1 \cap U_2 \in \mathcal{T}$.
   $U_1 = \bigcup_{\lambda \in \Lambda} B_\lambda$, $U_2 = U_{k \in K} B_k$. Consider $x \in U_1 \cap U_2$. Then $x \in B_{\lambda_x}$ and $x \in B_{k_x}$, for some $\lambda_x \in \Lambda$ and $k_x \in K$. Thus $x \in B_{\lambda_x} \cap B_{k_x} \Longrightarrow$ [by property 2.] $\exists B_x \in \mathcal{B}$ with $x \in B_x \subseteq B_{\lambda_x} \cap B_{k_x} \subseteq U_1 \cap U_2$.
   Therefore $U_1 \cap U_2 = \bigcup_{x \in U_1 \cap U_2} B_x \in \mathcal{T}$. ■

Example 1.10

1. $X = \mathbb{R}, \mathcal{B} = \{(a,b) \mid a,b \in \mathbb{R},\ a < b\}$. $\mathcal{B}$ satisfies properites 1. and 2. therefore it generates a topology on $\mathbb{R}$. This is the ***standard topology*** on $\mathbb{R}$.

2. $X = \mathbb{R}, \mathcal{B} = \{(a,b) \mid a,b \in \mathbb{Q},\ a < b\}$
   $\mathcal{B}$ satisfies properties 1. and 2.
   Claim [Homework]: The topology $\mathcal{T}$ generated by $\mathcal{B}$ is in fact the standard topology on $\mathbb{R}$.

Lemma 1.11 Let $\mathcal{B}, \mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$ on X. Then the following statements are equivalent:

1. $\mathcal{T}' \supseteq \mathcal{T}$.

2. $\forall x \in X, \forall B \in \mathcal{B}$ with $x \in B$ there is $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

Proof

1.⇒2.
Consider $x \in X$ and $B \in \mathcal{B}$ with $x \in B$. $B \in \mathcal{T} \subseteq \mathcal{T}' \Longrightarrow B \in \mathcal{T}' \Longrightarrow B = \bigcup_{\lambda \in \Lambda} B'_\lambda, B'_\lambda \in \mathcal{B}'$.
So $x \in B$, therefore $x \in B'_\lambda$, for some $\lambda_x \in \Lambda$. i.e. we have $B'_{\lambda_x} \in \mathcal{B}'$, such that $x \in B'_{\lambda_x} \subseteq B$.

2.⇒1.
Consider $U \in \mathcal{T}$ and $x \in U$. Then $x \in U = \bigcup_{\lambda \in \Lambda} B_\lambda, B_\lambda \in \mathcal{B}$. That is $x \in B_{\lambda_x}$, for some $\lambda_x \in \Lambda$. 2. implies that there exists $B'_{\lambda_x} \in \mathcal{B}'$ such that $x \in B'_{\lambda_x} \subseteq B_{\lambda_x}$.
Then $U \subseteq \bigcup_{x \in U} B'_{\lambda_x} \subseteq U \Longrightarrow U = \bigcup_{x \in U} B'_{\lambda_x}$
$\Longrightarrow U \in \mathcal{T}'$. Therefore $\mathcal{T} \subseteq \mathcal{T}'$. ■

Product topology

Let $(X, \mathcal{T}_x)$, $(Y, \mathcal{T}_Y)$ be two topological spaces. The ***product topology*** is the topology with basis the collection $\mathcal{B} = \{U \times V \mid U \in \mathcal{T}_X,\ V \in \mathcal{T}_Y\}$.

Metric topology

X is a set.

Definition 1.13 A ***metric*** on this set is a function $d : X \times X \to \mathbb{R}$ with

1. $d(x,y) \geq 0$, and $d(x,y) = 0$ iff $x = y$.

2. $d(x,z) \leq d(x,y) + d(y,z)\ \forall x,y,z \in X$.

3. $d(x,y) = d(y,x), \forall x,y \in X$.

We call the set $B_d(x,\varepsilon) = \{y \in X \mid d(y,x) < \varepsilon\}$ the ε-***ball*** centered at x.

$\{B_d(x,\varepsilon)\}_{x \in X, \varepsilon > 0}$ is a basis for a topology on X, the ***metric topology***.

Example 1.14 $\mathbb{R}$ with the standard topology ($\mathcal{T}_{\text{stand}}$).
Consider $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$.
Claim: The product topology on $\mathbb{R}^2$ and the metric topology are the same.
For this, one has to show

- $\mathcal{T}_{\text{pr}} \subseteq \mathcal{T}_{\text{m}}$
  By lemma 1.11 one has to show $\forall x \in \mathbb{R}^2$, $\forall B_{\text{pr}} \in \mathcal{B}_{\text{pr}}, x \in B_{\text{pr}}$, $\exists B_{\text{m}} \in \mathcal{B}_{\text{m}}$ such that $x \in B_{\text{m}} \subseteq B_{\text{pr}}$.
- $\mathcal{T}_{\text{m}} \subseteq \mathcal{T}_{\text{pr}}$
  ...

Subspace topology

Definition 1.15 $(X, \mathcal{T})$ is a topological space and $Y \subseteq X$. The collection $\mathcal{T}_Y := \{U \cap Y \mid U \in \mathcal{T}\}$ is a topology on Y, called the ***subspace topology***.

Example 1.16 $Y = [0,1] \cup \{2\} \subseteq X = \mathbb{R}$ with the standard topology.
Then the sets

- (a, b), with $a, b \in [0,1]$,
- $[0, b)$, with $b \in [0,1]$,
- $(a, 1]$, with $a \in [0,1]$,
- $\{2\}$,
- $[0,1]$

are open sets in the subspace Y.

1.1.2. Continuous functions

Definition 1.17 Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$. f is called ***continuous*** if $f^{-1}(V) \in \mathcal{T}_X \ \forall V \in \mathcal{T}_Y$.

Claim $f : \mathbb{R} \to \mathbb{R}$ (with standard topology). The ε–δ definition of continuity is equivalent to the definition above.
PROOF

"$\Leftarrow$" Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous with the definition above.
Consider $x_0 \in \mathbb{R}$ and $\varepsilon > 0$. Then $V = (f(x_0) - \varepsilon, f(x_0) + \varepsilon) \in \mathcal{T}_{\text{stand}}$, so $f^{-1}(V) \in \mathcal{T}_{\text{stand}}$ by our definition.
Now $x_0 \in f^{-1}(V)$, so there exists $(a, b) \in \mathcal{B}_{\text{stand}}$ with $x_0 \in (a, b) \subseteq f^{-1}(V)$.
Take $\delta = \min\{x_0 - a, b - x_0\}$. Clearly $\delta > 0$. Then $|x - x_0| < \delta \implies x \in (a, b) \subseteq f^{-1}(V) \implies f(x) \in V$
$\implies f(x) \in (f(x) - \varepsilon, f(x) + \varepsilon) \implies |f(x) - f(x_0)| < \varepsilon.$ ■

Theorem 1.19 *Let X, Y be topological spaces and $f : X \to Y$. The following are equivalent*

1. *f is continuous.*
2. *for every closed subset of Y the inverse image of it is a closed subset of X.*

Proof

1.⇒2.
Let B be closed in Y, then $X \setminus f^{-1}(B) = f^{-1}(Y \setminus B)$ which is open in X, i.e. $f^{-1}(B)$ is closed in X.

2.⇒1.
…

Lemma 1.21 (the pasting lemma)
Let X, Y be topological spaces and A, B closed subsets of X with $X = A \cup B$. Let $f : A \to Y$, $g : B \to Y$ be continuous with $f(x) = g(x)\ \forall x \in A \cap B$. Then $h : X \to Y$ with

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

is continuous.
Proof Let C be a closed subset of Y. Then $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$ where $f^{-1}(C)$ is closed in A and $g^{-1}(C)$ is closed in B.

(a) $f^{-1}(C)$ is closed in $A \Longrightarrow f^{-1}(C) = A \cap G$, where G closed in X.

(b) $g^{-1}(C)$ is closed in $B \Longrightarrow g^{-1}(C) = B \cap H$, where H closed in X.

(a) $\Longrightarrow f^{-1}(C)$ is closed in X,
(b) $\Longrightarrow g^{-1}(C)$ is closed in X.
$\Longrightarrow h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$ is closed in X. ■

Definition 1.23 Let X, Y be topological spaces. $f : X \to Y$ is called a ***homeomorphism*** if f is bijective, and f, f^{-1} are continuous.

Bijective correspondence not only between X and Y but also between the collection of open sets in X and the collection of open sets Y.
Thus any property of X that is expressed in terms of its open subsets yields via f the corresponding properties for Y. Such a property is called a ***topological property***.

Quotient topology

Example 1.24 Torus

Definition 1.25 Let X, Y be topological spaces, $p : X \to Y$ a surjective map. The map p is called a ***quotient map*** if a subset U of Y is open in Y, if and only if $p^{-1}(U)$ is open in X.

Definition 1.26 Let X be a topological space, Y be some set and $p : X \to Y$ a surjective map. The ***quotient topology*** on Y induced by p is defined as follows:
A subset $U \subseteq Y$ is open if and only if $p^{-1}(U) \subseteq X$ is open.

The fact that this is a topology follows from $p^{-1}(\varnothing) = \varnothing$, $p^{-1}(Y) = X$, $p^{-1}(\bigcup_{a\in J} U_a) = \bigcup_{a\in J} p^{-1}(U_a)$, $p^{-1}(\bigcap_{i=1}^n U_i) = \bigcap_{i=1}^n p^{-1}(U_i)$.

Remark 1.27 The quotient topology on Y is the finest topology that makes p continuous.
Example 1.28

1. $X = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2\pi,\ 0 \leq y \leq 1\} = [0,2\pi] \times [0,1] \subseteq \mathbb{R}^2$. $Y = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1,\ 0 \leq z \leq 1\}$.

   $f : X \to Y$, with $(x,y) \mapsto (\cos x, \sin x, y)$.

   f is surjective. Using f we can define the quotient topology in Y.

2. $p : \mathbb{R} \to \{x_1, x_2, c_3\}$

$$p(x) = \begin{cases} x_1 & x > 0 \\ x_2 & x < 0 \\ x_3 & x = 0 \end{cases}$$

   The quotient topology on $\{x_1, x_2, x_3\}$ induced by p is $\{\varnothing, \{x_1\}, \{x_2\}, \{x_1, x_2\}, \{x_1, x_2, x_3\}\}$

1.2. Conectedness & Compactness

In calculus you have studies functions $f : [a,b] \to \mathbb{R}$ and you have proved three theorems for continuous functions $f : [a,b] \to \mathbb{R}$.

1. Intermediate Value Theorem (IVT),
2. Maximus Value Theorem (MVT),
3. Uniform Continuity Theorem (UCT).

These theorems rely on the continuity of f and some topological properties of $[a,b] \subset \mathbb{R}$. In particular IVT relies on the connectedness of $[a,b]$. MVT and VCT rely on the compactness of $[a,b]$.

1.2.1. Connected spaces

Definition 1.29 Consider $(X, \mathcal{T})$ topological space. A ***separation*** of X is a pair U, V where $U, V \in \mathcal{T}$, $U, V \neq \varnothing$, $U \cap V = \varnothing$, and $X = U \cup V$. X is called ***connected*** if there is no separation of X.

Remark 1.30 Connectedness is a topological property.
Example 1.31

1. $\mathbb{R}$, $\mathcal{T}_{\text{stand}}$,
   $\mathbb{Q} = ((-\infty, a) \cap \mathbb{Q}) \cup ((a, \infty) \cap \mathbb{Q})$, $a \in \mathbb{R} \setminus \mathbb{Q}$.
2. $\mathbb{R}$, $[a,b]$, $[a,b)$, $(a,b]$, (a,b) are connected.

Claim: X is connected iff the only subsets of X that are both open and closed are $\varnothing$ and X.

Proof

"$\Rightarrow$" Suppose, that $A \subseteq X$, $A \neq \varnothing$, $A \neq X$, and $A \in \mathcal{T}_X$, $X \setminus A \in \mathcal{T}_X$. Then $X = A \cup (X \setminus A) \Longrightarrow A, X \setminus A$ is separation of X, contradiction.

"$\Leftarrow$" If U, V is a separation of X. Then $U \in \mathcal{T}_X$, $X \setminus U = V \in \mathcal{T}_x$, $U, X \setminus U \neq \varnothing$, $U \cap (X \setminus U) = \varnothing$. $V \in \mathcal{T}_X$, $X \setminus U \in \mathcal{T}_X$, $U \neq \varnothing$, $U \neq X$, contradiction.
Therefore X is connected. ■

Lemma 1.33 If C, D are a separation of X and Y is a connected subspace of X, then $Y \subseteq C$ or $Y \subseteq D$.

Proof $C \cap Y$, $D \cap Y$ are open subsets of Y. In addition $(C \cap Y) \cap (D \cap Y) = \varnothing$. If $C \cap Y$ and $D \cap Y$ were both nonempty, then they would form a separation of Y. But Y is connected. So $C \cap D = \varnothing$, or $D \cap Y = \varnothing$.
$\Longrightarrow Y \subseteq C$ or $Y \subseteq D$. ■

Theorem 1.35 *Let $\{A_i\}_{i \in I}$ be connected subspaces of X and $p \in \cap_{i \in I} A_i$. Then $\cup_{i \in I} A_i$ is connected subspace of X.*

Proof Assume that $U_{i \in I} A_i$ is not connected, and $\cup_{i \in I} A_i = C \cup D$, C, D a separation of $\cup_{i \in I} A_i$. Then $p \in C \cup D$ and WLOG we can assume that $p \in C$. A_i is a connected subspace.
$\Longrightarrow$ [Lemma] $A_i \subseteq C$ or $A_i \subseteq D$ for any $i \in I$. $p \in \cap_{i \in I} A_i$.
$\Longrightarrow p \in A_i \ \forall i \in I$.
$\Longrightarrow A_i \subseteq C \ \forall i \in I$.
$\Longrightarrow \cup_{i \in I} A_i \subseteq C$.
$\Longrightarrow D = \varnothing$. Contradiction. ■

Theorem 1.37 *Let A be a connected subspace of X and $A \subseteq B \subseteq \overline{A}$. Then B is also connected.*

Proof Assume that C, D is a separation of B. By Lemma $A \subseteq C$ or $A \subseteq D$. WLOG $A \subseteq C$.
$A \subseteq C \Longrightarrow \overline{A} \subseteq \overline{C}$.
$D \subseteq B \subseteq \overline{A} \subseteq \overline{C} \Longrightarrow D \subseteq \overline{C}$.
Claim: $D \cap \overline{C} = \varnothing$.
Proof: The closure of C in B is $\overline{C} \cap B$, where $\overline{C}$ is the closure of C in X. $C = \overline{C} \cap B = \overline{C} \cap (C \cup D) = (\overline{C} \cap C) \cup (\overline{C} \cap D) = C \cup (\overline{C} \cap D)$.
$\Longrightarrow \overline{C} \cap D = \varnothing$.

We have that $D \subseteq \overline{C}$ and we showed that $D \cap \overline{C} = \varnothing$. Therefore $D = \varnothing$, contradiction.
There is no separation of B, i.e. B is connected. ■

Theorem 1.39 *Let $f : X \to Y$ be a continuous function between topological spaces X and Y. If X is connected, then $f(X)$ is connected.*

PROOF Consider $\widetilde{f} : X \to f(X), x \mapsto f(x)$. Then $\widetilde{f}$ is continuous. Indeed, if U open in $f(X)$, then $U = f(X) \cap V$ for some V open in Y. $\widetilde{f}^{1}(U) = \widetilde{f}^{-1}(f(X) \cap V) = \widetilde{f}^{-1}(f(X)) \cap \widetilde{f}^{-1}(V) = X \cap f^{-1}(V) = f^{-1}(V)$ open in X.
Suppose that $f(X) = C \cup D$, C, D separation of $f(X)$. Then $\widetilde{f}^{-1}(C), \widetilde{f}^{-1}(D)$ are open subsets of X, disjoint and nonempty and $X = \widetilde{f}^{-1}(C) \cup \widetilde{f}^{-1}(D)$. Contradcition ($X$ connected). ■

Theorem 1.41 *A finite (cartesian) product of connected spaces (in the product topology) is connected.*

PROOF We start by proving that if X and Y are connected, then $X \times Y$ is connected. Consider $a \times b$ in $X \times Y$, $X \times b$. $X \times b$ is connected. Similarly $x \times Y$ is connected for any $x \in X$. As a result, $T_x = (X \times b) \cup (x \times Y)$ is also connected (by previous theorem, since $x \times b \in (X \times b) \cap (x \times Y)$).
Therefore, $\cup_{x \in X} T_x$ is connected, as the union of connected subspaces, T_x with common point $a \times b$. Now $X \times Y = \cup_{x \in X} T_x$, therefore $X \times Y$ is connected.
By induction, the general proof follows. ■

Theorem 1.43 *(IVT)*
Let $f : X \to \mathbb{R}$ be a continuous function and X be a connected space. If $a, b \in X$ and $r \in (f(a), f(b))$, then $\exists c \in X$ with $f(c) = r$.

PROOF $A = f(X) \cap (-\infty, r)$, $B = f(X) \cap (r, \infty)$. A, B are open in $f(X)$, $A \cap B = \emptyset$, $A \neq \emptyset$, $B \neq \emptyset$ (as $f(a) \in A$, $f(b) \in B$).
If $r \notin f(X)$, then $f(X) = A \cup B$, i.e. A, B is a separation of $f(X)$. However $f(X)$ is connected, because X is connected and f is continuous. I.e. such a separation cannot exist. I.e. $r \in f(X)$, that is, $\exists c \in X$ such that $f(c) = r$. ■

Remark 1.45 One can prove an even more generalised form of IVT, where instead of $\mathbb{R}$ one considers any ordered set.

Definition 1.46 Let X be a topological space and $x, y \in X$. A ***path*** in X from x to y is a continuous map $f : [a, b] \to X$ with $f(a) = x$ and $f(b) = y$. X is called ***path connected***, if every pair of points in X can be joined by a path in x.

Claim: If X is path connected, then X is connected.
PROOF Suppose $X = C \cup D$, C, D a separation of X. Consider $f : [a, b] \to X$ a path. $f([a, b])$ is connected, so $f([a, b]) \subseteq C$ or $f([a, b]) \subseteq D$. This implies that there is no path in X joining a point in C to a point in D. $\Longrightarrow X$ not path connected, contradiction. ■

Example 1.48 (Topologist's sine curve $\overline{S}$)
Consider $S = \{(x, \sin \frac{1}{x}) \mid 0 < x \leq 1\} \subseteq \mathbb{R}^2$.
$f : (0, 1] \to \mathbb{R}^2$, $x \mapsto (x, \sin \frac{1}{x})$. f is continuous, $(0, 1]$ is connected, therefore $S = f((0, 1])$ is connected.
$\overline{S} = S \cup \{0 \times [-1, 1]\}$. S is connected $\Longrightarrow \overline{S}$ is connected.

Claim: $\overline{S}$ is not path connected.
Proof: Suppose $f : [a, c] \to \overline{S}$ continuous with $f(a) = (0, 0)$, and $f(c) \in S$. $f^{-1}(0 \times [-1, 1])$ is closed in $[a, c]$ therefore it has a largest element b. Then $f(b) \in 0 \times [-1, 1]$

and $f((b,c]) \in S$. Let $f(t) = (x(t), y(t))$, $t \in [a,c]$. Then $x(a) = 0$, $x(t) > 0$ and $y(t) = \sin\frac{1}{x(t)}$ for $t > b$. Given $n \in \mathbb{N}$, choose u_n with $0 < u_n < x(b + \frac{1}{n})$ such that $\sin\frac{1}{u_n} = (\pm 1)^n$. By the IVT there exists $t_n \in (b, b + \frac{1}{n})$ with $x(t_n) = u_n$. Then $t_n \to b$ but $y(t_n)$ does not converge. Contradicting the continuity of f.
Remarks on yesterday's example

- If $X = \mathbb{R}^2$ and $S \subseteq X$, then $\overline{S} = \{$limits of convergent sequences of points in $S\}$. More generally, this is true if X is first countable, e.g. X is a metric space.
- If $f : [a,c] \to \overline{S}$ continuous. Of $t \in [a,c]$ $\forall n \in \mathbb{N}$, with $t_n \to b$, $b \in [a,c]$, then $f(t_n) \to f(b)$.
  In general, continuous implies sequentially continuous.

Definition 1.49 Given a topological space X, define an equivalence relation $\sim$ on X, by setting $x \sim y$ $(x, y \in X)$ if there exists a connected subspace of X with $x, y \in Y$. The equivalence classes are called the ***connected component*** of X.
Define another equivalence relation $\sim$ by setting $x \sim y$ $(x, y \in X)$ if there is a path in X from x to y. The equivalence classes are called ***path components*** of X.

1.2.2. Compactness

Definition 1.50 A topological space X is called ***compact*** if for any open covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of X, i.e. U_λ open $\forall \lambda \in \Lambda$ and $X = \cup_{\lambda\in\Lambda} U_\lambda$, there exist finitely many $\lambda_1, \dots, \lambda_n \in \Lambda$ such that $X = \cup_{i=1}^{n} U_{\lambda_i}$.

Compactness is a topological property.
Example 1.51

1. $\mathbb{R}$ is not compact, $\mathbb{R} = \bigcup_{r\in\mathbb{R}}(r - \frac{1}{2}, r + \frac{1}{2})$, or $\mathbb{R} = \bigcup_{n\in\mathbb{N}}(n, n+2)$.
2. $(a,b]$, $a, b \in \mathbb{R}$ is not compact. $(a,b] = \bigcup_{n\in\mathbb{N}}(a + \frac{1}{n}, b]$.
3. Subspaces of X with finitely many points are obviously compact.
4. $X = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ is compact. Consider $\{U_\lambda\}_{\lambda\in\Lambda}$ an open covering of X and consider $\lambda_0 \in \Lambda$ with $0 \in U_{\lambda_0}$. Then there exists N_{λ_0} sucht that $\frac{1}{n} \in U_{\lambda_0}$ $\forall n \geq N_{\lambda_0}$. For each $n \in \{1, \dots, N_{\lambda_0} - 1\}$ choose U_{λ_n} containing it. Then $\bigcup_{i=0}^{N_{\lambda_0}} U_{\lambda_i} = X$.

Order topology

Definition 1.52 A relation C on a set X is an ***order relation*** if it has the following properties:

1. If $x, y \in X$, $x \neq y$ then xCY or yCx.
2. For no $x \in X$ does xCx hold.
3. If xCy, yCz then xCz, $x, y, z \in X$.

Example 1.53 $X = \mathbb{R}$, $C = <$.
If X is a set with an order relation $<$ and $a, b \in X$, $a < b$.

$$(a,b) = \{x \in X \mid a < x < b\},$$
$$(a,b] = \{x \in X \mid a < x \le b\},$$
$$[a,b) = \{x \in X \mid a \le x < b\},$$
$$[a,b] = \{x \in X \mid a \le x \le b\}.$$

Definition 1.54 X is a set with an order relation and more than two elements. Let $\mathcal{B}$ be the collection of all sets of the following types:

- (a,b), $a, b \in X$,
- $(a, b_0]$, $a \in X$, b_0 the largest element (if any) in X,
- $[a_0, b)$, $b \in X$, a_0 the smallest element (if any) in X,

$\mathcal{B}$ is a basis for a topology, and we call the topology generated by $\mathcal{B}$ the ***order topology***.

Example 1.55 $\mathbb{R}^2$ with the dictionary order, i.e. $a \times b < c \times d$ if $a < c$ or $a = c$ and $b < d$. $\mathbb{R} \times \mathbb{R}$ has no largest or smallest element. [Graph .. open sets ...]

An ordered set X has the ***least upper bound property***, if every subset of X that is bounded above has a least upper bound.

Theorem 1.56 *Let X an ordered set with the least upper bound property. Then any closed interval $[a,b]$ in X is compact.*

[Without proof]
Corollary 1.57 In $\mathbb{R}$ ($\mathcal{T}_{\text{stand}} = \mathcal{T}_{\text{ordered}}$), $[a,b]$ is compact for any $a, b \in \mathbb{R}$.

Theorem 1.58 *A closed subspace Y of a compact space X is compact.*

Proof Consider $\{U_\lambda\}_{\lambda\in\Lambda}$ an open covering of Y. Then $U_\lambda = Y \cap U'_\lambda$ for some U'_λ open in X, $\forall \lambda \in \Lambda$. Denote $U' = \{U'_\lambda\}_{\lambda\in\Lambda}$. Now $U'' = U' \cup \{X \setminus Y\}$ is an open covering of X. X is compact, so a finite subcollection of U'' covers X. If this contains $X \setminus Y$, discard it. If not I leave the subcollection unchanged. What we obtain is a finite subcollection of U', say $\{U'_\lambda\}_{i\in\{1,\dots,n\}}$ with $Y \subseteq \bigcup_{i=1}^n U'_{\lambda_i}$. This implies, that $\{U_{\lambda_i}\}_{i\in\{1,\dots,n\}}$ is a finite subcovering of Y. ∎

Theorem 1.60 *X compact topological spaces. Y topological space and $f : X \to Y$ continuous. Then $f(X)$ is compact.*

Proof Let $U = \{U_\lambda\}_{\lambda\in\Lambda}$ be an open covering of $f(X)$. Then $\forall \lambda \in \Lambda$ $U_\lambda = U'_\lambda \cap f(X)$ for some U'_λ open in Y. Denote $U' = \{U'_\lambda\}_{\lambda\in\Lambda}$. $\{f^{-1}(U'_\lambda)\}_{\lambda\in\Lambda}$ is an open covering of X. X is compact, hence it has a finite subcovering $\{f^{-1}(U'_{\lambda_i})\}_{i\in\{1,\dots,n\}}$. Then $\{U_{\lambda_i}\}_{i\in\{1,\dots,n\}}$ is a finite covering of X. ∎

Theorem 1.62 *The product of finitely many compact spaces is compact.*

[Proof omitted]

Theorem 1.63 *(Extreme value theorem)*
X is a compact topological space and Y is an ordered set in the order topology. If $f : X \to Y$ is continuous then $\exists c, d \in X$ with $f(c) \leq f(x) \leq f(d)$ $\forall x \in X$.

Proof X is compact and f is continuous, so $f(X)$ is compact. Define $(-\infty, y) := \{a \in Y \mid a < y\}$ and note that if Y has no smallest element, then $(-\infty, y) = \bigcup_{\substack{a \in Y \\ a < y}} (a, y)$ and if Y has a smallest element a_0, then $(-\infty, y) = [a_0, y)$.
$(-\infty, y)$ is an open set in Y.
If $f(X)$ has no largest element, then $\{(-\infty, y) \cap f(X)\}_{y \in f(X)}$ is open covering of $f(X)$. Since $f(x)$ is compact, there exist y_i, $i \in \{1, \dots, n\}$, such that $\{(-\infty, y_i) \cap f(X)\}_{i \in \{1,\dots,n\}}$ covers $f(X)$. If $y_j = \max\{y_i\}_{i \in \{1,\dots,n\}}$, then $y_j \in f(X)$, but $y_j \notin \{(-\infty, y_i) \cap f(X)\}$, $i \in \{1, \dots, n\}$, contradiction, since $\{(-\infty, y_i) \cap f(X)\}_{i \in \{1,\dots,n\}}$ is an open covering of $f(X)$.
Similarly we can prove, that $f(X)$ has a smallest element.f ∎

Hausdorff spaces

Terminology: X topological space, $x \in X, U$ open set in X with $x \in U$, then U is called a ***open neighbourhood*** of x.

Definition 1.65 A topological space X is called ***Hausdorff*** if for every pair $x_1, x_2 \in X$ with $x_1 \neq x_2$ there exist open neighbourhoods U_1, U_2 of x_1, x_2 respectively with $U_1 \cap U_2 = \emptyset$.

Facts:

- Every ordered set with the order topology is a Hausdorff space.
- If X, Y are Hausdorff spaces, then $X \times Y$ is Hausdorff.
- A subspace of a Hausdorff space is Hausdorff.
- Every finite point set in a Hausdorff space is closed.

Theorem 1.66 *Every compact subspace of a Hausdorff space is closed.*

Proof X Hausdorff space, $Y \subseteq X$ compact $\Longrightarrow Y$ is closed.
We will prove, that $X \setminus Y$ is open.
Consider $x \in X \setminus Y$. $\forall y \in Y$ chose U_y, V_y open in X with $x \in U_y, y \in V_y$ and $U_y \cap V_y = \emptyset$ (this can be done, because X is Hausdorff). Then $\{Y \cap V_y\}_{y \in Y}$ is an open covering of Y. Y is compact, so $\exists \{Y \cap V_{y_i}\}_{i \in \{1,\dots,n\}}$ a finite subcovery of Y. Note that $Y \subseteq V_{y_1} \cup \dots \cup V_{y_n} =: V$. $U_x := U_{y_1} \cap \dots \cap U_{y_n}$. Then $U_x \cap V = \emptyset$. Indeed, if $z \in U_x \cap V$, then $z \in U_{y_i}$ $\forall i \in \{1, \dots, n\}$ and $z \in V_{y_j}$ for some $j \in \{1, \dots, n\}$ but $V_{y_i} \cap U_{y_j} = \emptyset$. So $U_x \cap V = \emptyset$. We have constructed U_x open in X with $x \in U_x$ and $U_x \cap Y \subseteq U_x \cap V = \emptyset$, i.e. $U_x \subseteq X \setminus Y$.
So $X \setminus Y = \cup_{x \in X \setminus Y} U_x$ and $X \setminus Y$ is open. ∎

Remark 1.68 $(X, \mathcal{T})$ topological space, $x \in X$. Some authors (e.g. Munkres) define: A neighbourhood of X is a set $U \in \mathcal{T}$ such that $x \in U$. Other authors (e.g. Janich) define: A neighbourhood of X is a set $U \subseteq X$ such that $\exists V \in \mathcal{T}$ with $x \in V \subseteq U$.

Theorem 1.69 *Let $f : X \to Y$ be a continuous bijective function . If X is compact, and Y is Hausdorff, then f is a homeomorphism.*

PROOF We have to prove that f^{-1} is continuous. Equivalently tat if U is open in X, then $(f^{-1})^{-1}(U)$ is open in Y. Equivalently, that if U is closed in X, then $f(X)$ is closed in Y.
Indeed, U is closed in X and X compact, so $f(U) \subset Y$ is compact. Y is Hausdorff, therefore $f(U)$ is closed in Y. ∎

Local connectedness and local path connectedness

Definition 1.71 X is called ***locally (path) connected*** at $x \in X$ if for every open neighbourhood U of x there is a (path) connected open neighbourhood V of x with $V \subseteq U$. If Y is locally (path) connected at every $x \in X$, then X is called ***locally (path) connected***.

2 Algebraic topology

Determining whether two spaces are homeomorphic and studying continuous functions between topological spaces are two of the central problems in topology.
To show that X, Y are homemorphic, we need to construct $f : X \to Y$ bijective, continuous, with f^{-1} continuous.
To show that X, Y are not homeomorphic, we can for instance show that X has a topological property (e.g. connectendes or compactness) but Y does not have this property.
Any continuous map between topological spaces induces a homeomorphism between there fundamental groups.
$x_0 \in X \rightsquigarrow \pi_1(X, x_0)$ fundamental group. We will show that if $X \simeq Y$ then their fundamental group is isomorphic.

2.1. Fundamental group

2.1.1. Path homotopy

Notation: $I := [0, 1]$.

Definition 2.1 Let $f : X \to Y$, $f' : X \to Y$ be continuous maps. f, f' are called ***homeotopic*** if there exists a continuous map $F : X \times I \to Y$ with $F(x, 0) = f(x)$ $\forall x \in X$ and $F(x, 1) = f'(x)$ $\forall x \in X$.
F is called a ***homotopy*** between f and f'.
If f, f' are homotopic, we write $f \simeq f'$.
If f is homotopic to a constant map then f is called ***nullhomotopic***.

We focus on the special case $X = [a, b]$.
Then if $f : [a, b] \to Y$ is a continuous function, f is called a ***path***. $f(a)$ is called the ***initial point***, and $f(b)$ the ***final point***.
WLOG we can assume, that the domain of f is $[0, 1]$.

Definition 2.2 Let $f : I \to X$, $f' : I \to X$ two paths in X. f, f' are called ***path homotopic*** if $f(0) = f'(0)$, $f(1) = f'(1)$ and there exists continuous map $F : I \times I \to X$, with $F(x, 0) = f(x)$ $\forall x \in I$ and $F(x, 1) = f'(x)$ $\forall x \in I$ and $F(0, t) = f(0)$ $\forall t \in I$, $F(1, t) = f(1)$ $\forall t \in I$.
If f, f' are path homotopic, we write $f \simeq_p f'$.

Claim: $\simeq$ and $\simeq_p$ are equivalence relations.
PROOF We prove this for $\simeq_p$.

1. $f \simeq_p f$:
   $F(x, t) := f(x)$.

2. $f \simeq_{\mathrm{p}} f' \implies f' \simeq_{\mathrm{p}} f$:
   If F is a path homotopy from f to f', then $G(x,t) := F(x, 1-t)$ is a path homotopy from f' to f.

3. $f \simeq_{\mathrm{p}} f', f' \simeq_{\mathrm{p}} f'' \implies f \simeq_{\mathrm{p}} f''$:
   If F is a path homotopy from f to f' and F' is a path homotopy from f' to f'', then
   $$G(x,t) := \begin{cases} F(x,2t) & t \in [0,1/2] \\ F'(x,2t-1) & t \in [1/2,1] \end{cases}$$
   is a path homotopy from f to f''. $G(x,1/2) = F(x,1) = F'(x,0) = f'(x)$. i.e. G is well defined. Is G continuous?

   G is continuous in $I \times [0,1/2]$ and G is continuous in $[1/2,1] \implies$ [pasting lemma (lemma 1.21] G is continuous.

   $$G(0,t) = \begin{cases} F(0,2t) & t \in [0,1/2] \\ F'(0,2t-1) & t \in [1/2,1] \end{cases} = \begin{cases} f(0) = f'(0) \\ f'(0) = f''(0) \end{cases}$$
   $G(1,t) = f(1)$,
   $G(x,0) = f(x)\ \forall x \in I$ (since $G(x,0) = F(x,0) = f(x)\ \forall x \in I$.
   $G(x,1) = f''(x)\ \forall x \in I$ (since $G(x,1) = F'(x,1) = f''(x)\ \forall x \in I$. ■

If f is a path, we denote its homotopy class by $[f]$.

Example 2.4

1. Let $f : I \to \mathbb{R}^2$, $g : I \to \mathbb{R}^2$, paths with $f(0) = g(0)$ and $f(1) = g(1)$. Then $F : I \times I \to \mathbb{R}^2$ with $(x,t) \mapsto (1-t)f(x) + tg(x)$ is a path homotopy between f and g.

   F is called a linear homotopy.

2. $f : I \to \mathbb{R}^2$, $s \mapsto (\cos(\pi s), \sin(\pi s))$, $g : I \to \mathbb{R}^2$, $t \mapsto (\cos(\pi t), 3\sin(\pi t))$.

   $f(0) = (1,0) = g(0)$, and $f(1) = (-1,0) = g(1)$.

   f, g are path homotopic. Indeed, consider the linear homotopy F described in 1.

3. $f : I \to \mathbb{R}^2 \setminus \{(0,0)\}$ with $s \mapsto (\cos(\pi s), \sin(\pi s))$, and $g : I \to \mathbb{R}^2 \setminus \{(0,0)\}$ with $t \mapsto (\cos(\pi t), \sin(\pi t)$.

   Are f, g path homotopic? Yes. Prove using the linear homotopy.

4. $f : I \to \mathbb{R}^2 \setminus \{(0,0)\}$, with $s \mapsto (\cos(\pi s), \sin(\pi s))$, and $g : I \to \mathbb{R}^2 \setminus \{(0,0)\}$ with $t \mapsto (\cos(\pi t), -3\sin(\pi t))$.
   The linear homotopy is not a path homotopy between f and g.
   Inf fact, there exists no path homotopy between f and g.

We define the following operation:

Definition 2.5 Let $f : I \to X$ path in X with $f(0) = x_0$, $f(1) = x_1$, $x_0, x_1 \in X$. $g : I \to X$ path in X with $g(0) = x_1$, $g(1) = x_2$, $x_2 \in X$.
We define the ***product*** $f \cdot g$ of f, g as the path h with
$$h(s) = \begin{cases} f(2s) & s \in [0,1/2] \\ g(2s-1) & s \in [1/2,1]. \end{cases}$$

Remark 2.6 h is well defined, since $f(1) = g(0) = h(1/2)$.
h is continuous by the pasting lemma (lemma 1.21).
The product $f \cdot g$ induces a well defined product on path homotopy equivalce classes defined by $[f] \cdot [g] = [f \cdot g]$.
Indeed, if f, f' paths in X with $[f] = [f']$, and g, g' paths in X with $[g] = [g']$. Consider path homotopies F, G between f and f', g and g' respectively. Then

$$H(s,t) := \begin{cases} F(2s,t) & t \in [0,1/2] \\ G(2s-1,t) & s \in [1/2,1] \end{cases}$$

is a path homotopy between $f \cdot g$ and $f' \cdot g'$ (homework).
Therefore $[f \cdot g] = [f' \cdot g']$, i.e. $[f] \cdot [g] = [f'] \cdot [g']$ and the induced product is well-defined.

Reparametrisation

Define a ***reparametrisation*** of a path f to be a composition $f \circ \varphi$, where $\varphi : I \to I$ continuous map, such that $\varphi(0) = 0$, and $\varphi(1) = 1$.
Reparametrising a path preserves its homotopy class (i.e. $[f \circ \varphi] = [f]$). Indeed, consider the path homotopy $f \circ \varphi_t$ where$\varphi_t(s) := (1-t)\varphi(s) + ts$.
$\varphi_0(s) = \varphi(s) \rightsquigarrow f \circ \varphi_0 = f \circ \varphi$, $\varphi_1(s) = s \rightsquigarrow f \circ \varphi_1 = f$.
Note that $(1-t)\varphi(s) + ts$ lies between $\varphi(s)$ and s, hence $(1-t)\varphi(s) + ts$ lies in I, so $f \circ \varphi_t$ is defined.

2.2. The fundamental group

We restrict our attention to paths $f : I \to X$ with $f(0) = f(1)$. Call $x_0 := f(0) = f(1)$. Such paths are called ***loops*** in X at the ***basepoint*** x_0. The set $\{[f] \mid f : I \to X,\ f(0) = f(1) = x_0\}$ is denoted by $\pi_1(X, x_0)$.
If $[f], [g] \in \pi_1(X, x_0)$, then f, g are loops in X at the basepoint x_0, so $f \cdot g$ is defined.
The induced multiplication $[f] \cdot [g] = [f \cdot g]$ is well-defined.
Proposition 2.7 $(\pi_1(X, x_0), \cdot)$ is a group.
Proof

1. associativity: $[f] \cdot ([g] \cdot [h]) = ([f] \cdot [g]) \cdot [h],\ [f], [g], [h] \in \pi_1(X, x_0)$.
   $\iff [f] \cdot [g \cdot h] = [f \cdot g] \cdot [h]$
   $\iff [f \cdot (g \cdot h)] = [(f \cdot g) \cdot h]$.
   Claim: $f \cdot (g \cdot h)$ is a reparametrisation of $(f \cdot g) \cdot h$ by the piecewise linear function φ: where

   $$\varphi(t) = \begin{cases} \frac{1}{2}t & t \in [0,1/2] \\ t - \frac{1}{4} & t \in [1/2,3/4] \\ 2t-1 & t \in [3/4,1] \end{cases}$$

   Proof:

   $$(f \cdot (g \cdot h))(t) = \begin{cases} f(2t) & t \in [0,1/2] \\ (g \cdot h)(2t-1) & t \in [1/2,1] \end{cases} = \begin{cases} f(2t) & t \in [0,1/2] \\ g(4t-2) & t \in [1/2,3/4] \\ h(4t-3) & t \in [3/4,1] \end{cases}$$

   Analogously

   $$((f \cdot g) \cdot h)(s) = \begin{cases} (f \cdot g)(2s) & s \in [0,1/2] \\ h(2s-1) & s \in [1/2,1] \end{cases} = \begin{cases} f(4s) & s \in [0,1/4] \\ g(4s-1) & s \in [1/4,1/2] \\ h(2s-1) & s \in [1/2,1] \end{cases}$$

So if one easily checks, that $((f \cdot g) \cdot h)(\varphi(t)) = (f \cdot (g \cdot h))(t)$.

2. Identity: Consider $c : I \to X$ with $c(s) = x_0 \ \forall s \in I$. Then $f \cdot c$ is a reparametrisation of f via φ:
$$\varphi(t) = \begin{cases} 2t & t \in [0, 1/2] \\ 1 & t \in [1/2, 1] \end{cases}$$
I.e. $f \cdot c = f \circ \varphi$.
$[f] = [f \circ \varphi] = [f \cdot c]$.
$[f] = [f] \cdot [c]$.

3. inverse: Consider $[f] \in \pi_1(X, x_0)$ and $\overline{f}(s) = f(1-s)$.
Claim: $f \cdot \overline{f} \simeq c \simeq \overline{f} \cdot f$.
Proof: Consider the homotopy
$$H(s,t) = \begin{cases} f(2s) & s \in [0, (1-t)/2] \\ f(1-t) & [(1-t)/2, (1+t)/2] \\ f(2-2s) & [(1+t)/2, 1] \end{cases}$$
$H(0,t) = f(0) = x_0$,
$H(1,t) = f(0) = x_0$.
$$H(s,0) = \begin{cases} f(2s) & s \in [0,1/2] \\ f(2-2s) & s \in [1/2,1] \end{cases} = f \cdot \overline{f}$$
$H(s,1) = x_0$.

H is a homotopy between $f \cdot \overline{f}$ and c i.e. $[f \cdot \overline{f}] = [c]$,
i.e. $[f] \cdot [\overline{f}] = [c]$.
Analogously $[\overline{f}] \cdot [f] = [c]$.

$\Longrightarrow [\overline{f}]$ is the inverse of $[f]$. ∎

Definition 2.9 $\pi_1(X, x_0)$ is called the ***fundamental group*** of X at x_0.

Example 2.10 Consider $X \subseteq \mathbb{R}^n$ convex. $x_0 \in X$.
$\pi_1(X, x_0) = \{[\overline{f}] \mid f : I \to X \text{ path}, f(0) = f(1) = x_0\}$.
If f_0, f_1 are two paths in X at the basepoint x_0, then $f_t(s) = (1-t)f_0(s) + tf_1(s)$ is a linear homotopy between f_0 and f_1.
Let $x_0, x_1 \in X$, $\pi_1(X, x_0), \pi_1(X, x_1)$.
Proposition 2.11 If x_0, x_1 are in the same path component of X, then $\pi_1(X, x_0), \pi_1(X, x_1)$ are isomorphic.
Proof $x_0, x_1 \in X$. Consider $h : I \to X$ a path with $h(0) = x_0$, $h(1) = x_1$ and $\overline{h} : I \to X, \overline{h}(s) = h(1-s)$.
Define $\widehat{h} : M_1(X, x_1) \to \pi_1(X, x_0)$ where $[f] \mapsto [h \cdot f \cdot \overline{f}]$.

- $\widehat{h}$ is well-defined: If $[f] = [g] \in \pi_1(X, x_1)$ and f_t is a homotopy between f and g then $h \cdot f_t \cdot \overline{h}$ is a homotopy between $h \cdot f \cdot \overline{h}$ and $h \cdot g \cdot \overline{g}$ i.e. $[h \cdot f \cdot \overline{h}] = [h \cdot g \cdot \overline{h}]$.
- $\widehat{h}$ is a homomorphism: $\widehat{h}([f] \cdot [g])\widehat{h}([f \cdot g]) = [h \cdot f \cdot g \cdot \overline{h}] = [h \cdot f \cdot \overline{h} \cdot h \cdot g \cdot \overline{h}] = [hf\overline{h}][hg\overline{h}] = \widehat{h}([f])\widehat{h}([g])$.

- $\widehat{h}$ is bijective: $\widehat{\overline{h}} = (\widehat{h})^{-1}$.
  $(\widehat{h}\widehat{\overline{h}})[f] = \widehat{h}(\widehat{\overline{h}}([f])) = \widehat{h}([\overline{h}fh]) = \cdots = [f]$. ∎

Definition 2.13 X is ***simply connected*** if it is path connected and $\pi_1(X)$ is trivial.

Proposition 2.14 X is simply connected iff there is a unique homotopy class of paths connecting any two points in X.

Proof Path connectedness is the existence of paths connecting every pair of points in X. So we only need to check uniqueness.

"⇒" Suppose $\pi_1(X) = 0$. Let $x_0, x_1 \in X$. If f, g are two paths from x_0 to x_1, then $f \simeq f\overline{g}g \simeq g$. $\Longrightarrow [f] = [g]$.

"⇐" If there is a unique homotopy class of paths connecting any two points, then there is a unique homotopy class of paths connecting a point to itself. $\Longrightarrow \pi_1(X) = 0$. ∎

Induced homomorphisms

X, Y topological spaces, $x_0 \in X$, $y_0 \in Y$ and $h : X \to Y$ a continuous map with $h(x_0) = y_0$. We will denote this by $h : (X, x_0) \to (Y, y_0)$.
If $f : I \to X$ is aloop in X, based at x_0, then $h \circ f$ is a loop in Y based at y_0.

Definition 2.16 Let $h : (X, x_0) \to (Y, y_0)$ continuous. Define $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ where $[f] \mapsto [h \circ f]$. h_* is called the ***homomorphism induced by*** h relative to x_0.

Remark 2.17 h_* is well-defined.
If f, g loops in X based at x_0 with $[f] = [g]$. Consider H a homotopy from f to g. Then $h \circ H$ is a homotopy from $h \cdot f$ to $h \cdot g$.
I.e. $[h \cdot f] = [h \cdot g] \Longleftrightarrow h_*([f]) = h_*([g])$.
Remark 2.18 h_* is a group homomorphism.
Just calculate and compare the following two expressions:

$$h_*([f] \cdot [g]) = h_*([f \cdot g]) = [h \circ (f \cdot g)]$$
$$h_*([f]) \cdot h_*([g]) = [h \circ f] \cdot h[\cdot g] = [(h \circ f) \cdot (h \circ g)].$$

Remark 2.19 h_* depends on the basepoint x_0. So strictly speaking we should have written $(h_{x_0})_*$.

Properties of induces homomorphisms

1. If $h : (X, x_0) \to (Y, y_0), g : (Y, y_0) \to (Z, z_0)$ continuous, then $(g \circ h)_* = g_* \circ h_*$.
2. If $1_X : (X, x_0) \to (X, x_0)$, then $(1_x)_* = 1_{\pi_1(X, x_0)}$.

Proposition 2.20 If $H : (X, x_0) \to (Y, y_0)$ a homoeomorphism,t hen $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is a grou isomorphism.

PROOF Consider $h^{-1}(Y, y_0) \to (X, x_0)$ and $(h^{-1})_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$
$(h^{-1})_* \circ h_* = (h^{-1} \circ h)_* = 1_* = 1.$
$h_* \circ (h^{-1})_* = (h \circ h^1)_* = 1_* = 1.$ ■

π_1 gives a covariant functor from the category with objects topological spases with basepoint and morphism basepoint preserving continuous maps to the category with objects groups and morphisms group homomorphisms.

A ***category*** $\mathcal{D}$ consists of

1. a collection of objects $\mathrm{Ob}(\mathcal{D})$,
2. sets of morphisms $\mathrm{Mor}(X, Y)$ for each $X, Y \in \mathrm{Ob}(\mathcal{D})$ with a distinguished identity morphism 1_X in $\mathrm{Mor}(X, X)$,
3. a composition of morphisms $\circ : \mathrm{Mor}(X, Y) \times \mathrm{Mor}(Y, Z(\to \mathrm{Mor}(X, Z)$ for each tripel $X, Y, Z \in \mathrm{Ob}(\mathcal{D})$ with the properties $f \circ 1 = 1 \circ f = f$ and $(f \circ g) \circ h = f \circ (g \circ h)$.

Example 2.22

1. $\mathrm{Ob}(\mathcal{D}) = \{G \mid G \text{ group}\}$.
   $\mathrm{Mor}(G, H) = \{f : G \to H \mid f \text{ group homomorphism}\}$.
2. $\mathrm{Ob}(\mathcal{D}) = \{(X, x_0) \mid X \text{ topological space, } x_0 \text{ basepoint}\}$.
   $\mathrm{Mor}((X, x_0), (Y, y_0)) = \{f : (X, x_0) \to (Y, y_0) \mid f \text{ continuous}\}$.

A ***coveriant functor*** F from a categoriy $\mathcal{C}$ to a category $\mathcal{D}$ assigns to each $X \in \mathrm{Ob}(\mathcal{C})$ an $F(X) \in \mathrm{Ob}(\mathcal{D})$ to each $f \in \mathrm{Mor}(X, Y)$ a $F(f)$ such that $F(1_X) = 1_{F(X)}$ and $F(f \circ g) = F(f) \circ F(g)$.

Example 2.23 π_1 is a covariant functor from category $\mathcal{C}_2$ in example 2 to the category $\mathcal{C}_1$ in example 1.
$(X, x_0) \in \mathrm{Ob}(\mathcal{C}_2) \overset{\pi_1}{\rightsquigarrow} \pi_1(X, x_0) \in \mathrm{Ob}(\mathcal{C}_1)$.
$f \in \mathrm{Mor}((X, x_0), (Y, y_0)) \overset{\pi_1}{\rightsquigarrow} f_* \in \mathrm{Mor}(\pi_1(X, x_0), (Y, y_0))$
$(1_{(X, x_0)})_* = 1_{\pi_1(X, x_0)}$.
$(f \circ g)_* = f_* \circ g_*$.

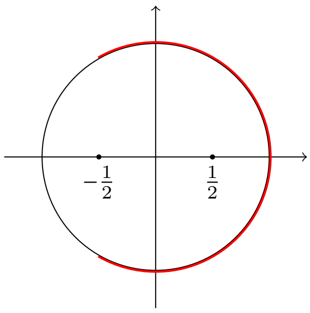
2.3. Covering spaces

- useful for computing π_1,
- algebraic features of π_1 can be translated into geometric features of the spaces.

Definition 2.24 Consider $p : \widetilde{X} \to X$ a continuous and surjective map. If there exists an open cover $\{U_a\}_{a \in A}$ of X such that $\forall a \in A \; p^{-1}(U_a) = \cup_{b \in B_a} V_a^b$, where $V_a^b \cap V_a^{b'} = \varnothing$ for any $b, b' \in B_a$, $b \neq b'$. $p : V_a^b \to U_a$ is a homeomorphism $\forall b \in B_a$ and V_a^b open in $\widetilde{X}$ $\forall b \in B_a$, then p is called a ***covering map***. $\widetilde{X}$ is called a ***covering space*** of X.

Example 2.25

1. $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. $p : \mathbb{R} \to S^1$ where $s \mapsto (\cos 2\pi s, \sin 2\pi s)$.
   p is a covering map, p is continuous (calculus), and p is surjective.
   $U_1 := \{(x, y) \in S^1 \mid x > -\sqrt{2}/2\}$
   $U_2 := \{(x, y) \in S^1 \mid x < \sqrt{2}/2\}$.

   $\{U_1, U_2\}$ is an open cover of S^1
   $p^{-1}(U_1) = \cup_{n\in\mathbb{Z}} V_n$, where $V_n := (n - 3/8, n + 3/8)$.
   V_n is open in $\mathbb{R}$ $\forall n \in \mathbb{Z}$.
   $V_n \cap V_n = \varnothing$, for any $n, m \in \mathbb{Z}$, $n \neq m$.
   $p : V_n \to U_1$ is a homeomorphism $\forall n \in \mathbb{Z}$.
   We can make the analogous construction for U_2. So p is a covering map.
   Remark 2.26 You can prove that p is a covering by using any open cover of S^1 by two open subsets of S^1 ($\neq S^1$).

2. $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, $n \in \mathbb{N}$, $n \geq 1$.
   $p_n : S^1 \to S^1$, $z \mapsto z^n$. p_n is a covering map (problem sheet 5).
   $f : X \to Y$ is called an ***embedding*** if $f : X \to f(X)$ is a homeomorphism.

   Consider the solid torus $S^1 \times D^2$ where $D^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$, and its boundary $\partial(S^1 \times D^2) = S^1 \times S^1$.
   Consider $f : S^1 \to \partial(S^1 \times D^2)$ an embedding such that $f(S^1)$ wraps around the first S^1 three times.
   Lastly consider the projection $\pi : S^1 \times D^2 \to S^1 \times \{(0, 0)\}$ and restrict it to $f(S^1)$.

3. $f : \mathbb{R}_+ \to S^1$, where $s \mapsto (\cos 2\pi s, \sin 2\pi s)$ is not a covering map.
   Consider an open cover $\{U_a\}_{a\in A}$ of S^1. Then $\exists U \in \{U_a\}_{a\in A}$ with $x \in U$. $p^{-1}(U) = V_0 \cup (\cup_{b\in B_a} V_a^b)$.
   $p : V_0 \to U$ is not a homeomorphism.

Theorem 2.27 *If $p : \widetilde{X} \to X$ is a covering map, $X_0 \subseteq X$, and $\widetilde{X}_0 = p^{-1}(X_0)$, then $p_0 : \widetilde{X}_0 \to X_0$ obtained by restricting p to $\widetilde{X}_0$ is a covering map.*

Proof p_0 is continuous (restricting the domain or the range of a continuous function gives a continuous function). p_0 is surjective.
Consider an open cover $\{U_a\}_{a\in A}$ of X with the properties in the definition of covering map.

Then $\forall a \in A\; p^{-1}(U_a) = \cup_{b \in B_a} V_a^b$.
If $x_0 \in X_0$, then there exists $a \in A$ with $x_0 \in U_a$. Now $U_a \cap X_0$ is an open set in X_0.
$p_0^{-1}(U_a \cap X_0) = p_0^{-1}(U_a) \cap p_0^{-1}(X_0) = p^{-1}(U_a) \cap \widetilde{X}_0 = (\cup_{b \in B_a} V_a^b) \cap \widetilde{X}_0 = \cup_{b \in B_a}(V_a^b \cap \widetilde{X}_0)$.
We have

- $V_a^b \cap \widetilde{X}_0$ open in $\widetilde{X}_0$,
- $(V_a^b \cap \widetilde{X}_0) \cap (V_a^{b'} \cap \widetilde{X}_0) = \varnothing$ for any $b, b' \in B_a$, $b \neq b'$.
- $p_0 : V_a^b \cap \widetilde{X}_0 \to U_a \cap X_0$ is a homeomorphism.

$\{U_a \cap X_0\}_{a \in A}$ is an open cover of X_0 with the desired properties $\Longrightarrow p_0 : \widetilde{X}_0 \to X_0$ is a covering map. ■

Theorem 2.29 *If $p : \widetilde{X} \to X$, $p' : \widetilde{X}' \to X'$ are covering maps, then $p \times p' : \widetilde{X} \times \widetilde{X}' \to X \times X'$ is a covering map.*

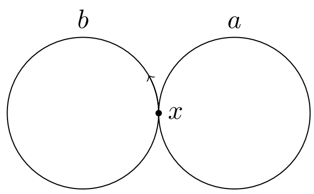
Example 2.30

1. Consider $p : \mathbb{R} \to \mathrm{S}^1$, where $s \mapsto (\cos 2\pi s, \sin 2\pi s)$.

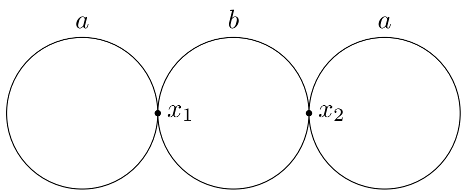
   The map $p \times p : \mathbb{R} \times \mathbb{R} \to \mathrm{S}^1 \times \mathrm{S}^1$ is a covering map.

2. $X_0 = (\mathrm{S}^1 \times p(0)) \cup (p(0) \times \mathrm{S}^1)$.
   $\widetilde{X}_0 = (p \times p)^{-1}(X_0) = (\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R})$.
   $(p \times p)_0 : \widetilde{X}_0 \to X_0$ is a covering map (by theorem).
   Two circles S^1 with a one common point is called wedge of two circles (notation $\mathrm{S}^1 \vee \mathrm{S}^1$).

3. Other covering spaces of $\mathrm{S}^1 \vee \mathrm{S}^1$.

   Consider $\widetilde{X}$ as in the following picture

   Then $p : \widetilde{X} \to \mathrm{S}^1 \vee \mathrm{S}^1$, where $x_1 \mapsto x$ and $x_2 \mapsto x$. p maps each edge of $\widetilde{X}$ to the edge of X with the same label by a map that is a homeomorphism and preserves the orientation.

4. Consider $p \times 1_{\mathbb{R}_+} : \mathbb{R} \times \mathbb{R}_+ \to S^1 \times \mathbb{R}_+$ (where $p : \mathbb{R} \to S^1$: $s \mapsto (\cos 2\pi s, \sin 2\pi s)$) and $f : S^1 \times \mathbb{R}_+ \to \mathbb{R}^2 \smallsetminus \{(0,0)\}$, where $(x,t) \mapsto t \cdot x$.

   f is a homeomorphism.

   $f \circ (p \times 1_{\mathbb{R}^+}) : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^2 \smallsetminus \{(0,0)\}$ is a covering map.

Proof $p \times p'$ is continuous (problem sheet 3). $p \times p'$ is surjective (if $(x,x') \in X \times X'$, then $x \in X$ and $x' \in X' \Longrightarrow$ [p, p' is surjective] $\exists \widetilde{x} \in \widetilde{X}$, $\widetilde{x}' \in \widetilde{X}'$, such that $p(\widetilde{x}) = x$ and $p'(\widetilde{x}') = x'$. $\Longrightarrow \exists (\widetilde{x}, \widetilde{x}') \in \widetilde{X} \times \widetilde{X}'$ such that $(p \times p')(\widetilde{x}, \widetilde{x}') = (x, x')$).
Consider $\{U_a\}_{a \in A}$, $\{U'_c\}_{c \in C}$ open covers of X and X' as in the definition of covering maps.
If $(x,x') \in X \times X'$ then $x \in X$ and $x' \in X'$ i.e. $\exists U \in \{U_a\}_{a \in A}$ with $x \in U$ and $\exists U' \in \{U_c\}_{c \in C}$ with $x' \in U'$.
$p^{-1}(U) = \cup_{b \in B} V_b$, V_b open $V_b \cap V_{b'} = \varnothing$ for $b \neq b'$ and $p : V :_b \to U$ homeomorphism $\forall b \in B$. $(p')^{-1}(U') = \cup_{d \in D} V'_d$, V'_d open, $V'_d \cap V'_{d'} = \varnothing$ for $d \neq d'$ and $p' : V'_d \to U'$ homeomorphism $\forall d \in D$.
$(x,x') \in U \times U'$, $U \times U'$ is open in $X \times X'$,
$(p \times p')^{-1}(U \times U') = \cup_{\substack{b \in B \\ d \in D}} (V_b \times V'_d)$, where $V_b \times V'_d$ open in $X \times X'$, $V_b \times V'_d$ disjoint,
$p \times p' : V_b \times V'_d \to U \times U'$ is a homeomorphism.
I.e. $\{U_a \times U'_c\}_{\substack{a \in A \\ c \in C}}$ is the open cover of $X \times X'$. That shows that $p \times p'$ is a covering map.■

2.4. Lifting properties

We will discuss two properties of covering spaces.

Definition 2.32 Let $p : \widetilde{X} \to X$ be a map. If $f : Y \to X$ is continuous a ***lifting*** of f is a map $\widetilde{f} : Y \to \widetilde{X}$ such that $p \circ \widetilde{f} = f$.

Example 2.33 $p : \mathbb{R} \to S^1$, $s \mapsto (\cos 2\pi s, \sin 2\pi s)$, $f : I \to S^1$ the path $f(s) = (\cos \pi s, \sin \pi s)$.
Then $\widetilde{f} : I \to \mathbb{R}$, $\widetilde{f}(s) = \frac{s}{2}$ is a lifting of f.

Proposition 2.34 (path lifting property)
Let $p : \widetilde{X} \to X$ be a covering map with $p(\widetilde{x}_0) = x_0$. If $f : I \to X$ is a path with $f(0) = x_0$, then $\exists!$ lifting of f to a path $\widetilde{f} : I \to \widetilde{X}$ with $\widetilde{f}(0) = \widetilde{x}_0$.

Proof

1. Consider the open cover $\{U_a\}_{a \in A}$ of X (as in the defintion of covering map). Let $t \in (0,1)$, and $ft) \in X \Longrightarrow f(t) \in U_a$ for some $a \in A$. $f : I \to X$ is continuous $\Longrightarrow \exists (a_t, b_t) \subseteq (0,1)$ with $t \in (a_t, b_t)$ and $f([a_t, b_t]) \subseteq U_a$.

   Analogously for $t = 0$ $\exists [0, b_0) \subseteq [0,1]$ with $f([0,b_0]) \subseteq U_a$, and for $t = 1$ $\exists (a_0, 1] \subseteq [0,1]$ with $f([a_0, 1]) \subseteq U_a$.

   I is compact, and $\{(a_t, b_t)\}_{t \in (0,1)}$ together with $[0, b_0)$ and $(b_1, 1]$ is an open cover of I. We can choose a finite subcover, say $[0, b_0), (a_0, 1], (a_1, b_1), \dots, (a_m, b_m)\}$. Consider $\{a_i, b_i\}_{i \in \{0, \dots, m\}}$ and orther its elements. (Possibly) rename the elements as follows: $0 < t_1 \leq \dots \leq t_{2m+2} < 1$. this gives a subdivision $0 = s_0 < s_1 < \dots < s_n = 1$ of I with the property, that $\forall i \in \{0, \dots, n-1\}$ $f([s_i, s_{i+1}]) \subseteq U_a$ for some $a \in A$.

2. Define $\widetilde{f}(0) = \widetilde{X}_0$. Suppose that $\widetilde{f}$ is defined in $[0, s_i]$. Define $\widetilde{f}$ in $[s_i, s_{i+1}]$ as follows: $f([s_i, s_{i+1}]) \subseteq U$ for some U in $\{U_a\}_{a\in A}$.
   Let $p^{-1}(U) = \cup_{b\in B} V_b$. Now $(p \circ \widetilde{f})(s_i) = f(s_i) \in U$.
   $\implies \widetilde{f}(s_i) \in p^{-1}(U) = \cup_{b\in B} V_b$
   $\implies \widetilde{f}(S_i) \in V_0$ for some $V_0 \in \{V_b\}_{b\in B}$.
   Define $\widetilde{f}(s) = (p_{|V_0})^{-1}(f(s))$, $s \in [s_i, s_{i+1}]$.

   $\widetilde{f}$ is continuous in $[s_i, s_{i+1}]$, since $p_{|V_0}$ is a homeomorphism. Thus we have defined $\widetilde{f} : [0,1] \to \widetilde{X}$ continuous with $\widetilde{f}(0) = \widetilde{x}_0$.

3. Suppose $\widetilde{\widetilde{f}}$ is another lifting of f with $\widetilde{\widetilde{f}}(0) = x_0$. Then $\widetilde{\widetilde{f}}(0) = \widetilde{f}(0) = x_0$.

   Suppose that $\widetilde{\widetilde{f}}(s) = \widetilde{f}(s)$ in $[0, s_i]$. $p \circ \widetilde{\widetilde{f}}([s_i, s_{i+1}]) = f([s_i, s_{i+1}]) \subseteq U_a$ for some $a \in A$.
   $\implies \widetilde{f}([s_i, s_{i+1}]) \subseteq p^{-1}(U_a) = \cup_{b\in B} V_b$.
   $\widetilde{\widetilde{f}}([s_i, s_{i+1}])$ is connected and $\widetilde{\widetilde{f}}(s_i) = \widetilde{f}(s_i) \in V_0$.
   $\implies \widetilde{\widetilde{f}}([s_i, s_{i+1}]) \subseteq V_0$.

   For $s \in S$, $p \circ \widetilde{\widetilde{f}}(s) = f(s)$
   $\implies \widetilde{\widetilde{f}}(s) \in p^{-1}(f(s))$.

   Also $\widetilde{\widetilde{f}} \in V_0$. So $\widetilde{\widetilde{f}}(s) = (p_{|V_0})^{-1}(f(s)) = \widetilde{f}(s)$.

   I.e. $\widetilde{\widetilde{f}}(s) = \widetilde{f}(s)$. ∎

Proposition 2.36 Let $p : \widetilde{X} \to X$ be a covering map with $p(\widetilde{x}_0) = x_0$. Let $F : I \times I \to X$ be continuous with $F(0,0) = x_0$ Then $\exists!$ lifting of F to a continuous map $\widetilde{F} : I \times I \to \widetilde{X}$ with $\widetilde{F}(0,0) = \widetilde{x}_0$. If F is a path homotopy then $\widetilde{F}$ is a path homotopy.

Proof Consider $\{U_a\}_{a\in A}$ an open cover of X with the properties as in the definition of covering map.

1. Define $\widetilde{F}((0,0)) = \widetilde{x}_0$.
   Extend $\widetilde{F}$ to $I \times \{0\}$ and $\{0\} \times I$ using the proposition from last time. Extend ~ to $I \times I$, as follows: Choose subdivision $s_0 < s_1 < \cdots < s_m$, $t_0 < t_1 < \cdots < t_n$ of I with the property that for each such rectangle $I_i \times J_j = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$, $F(I_i \times J_j) \subseteq U_a$ for some $a \in A$. Define $\widetilde{F}$ first in $I_1 \times I_j$, then $I_2 \times J_1, \ldots, I_m \times J_1$, then $I_1 \times J_2, I_2 \times J_2$ and so on.

   Consider $I_{i_0} \times J_{j_0}$ and suppose that $\widetilde{F}$ is defined on the union A of the rectangles $I_i \times J_j$ with $j < j_0$ or $j = j_0$ and $i < i_0$.

   Consider $C = A \cap (I_{i_0} \times J_{j_0})$. Choose $U \in \{U_a\}_{a\in A}$ with $F(I_{i_0} \times J_{j_0}) \subseteq U$. $p^{-1}(U)) \cup_{b\in B} V_b$, $\widetilde{F}$ is defined in C, C is connected. So, $\widetilde{F}(C)$ is connected, i.e. $\exists V_0$ such that $\widetilde{F}(C) \subseteq V_0$.

   If $p_0 = p_{|x_0} : V_0 \to U$, then $p_0 \circ \widetilde{F}(x) = p \circ \widetilde{F}(x) = F(x) \implies F(x) = (p_0^{-1})(F(x))$.
   Define $\widetilde{F}(x) = p_0^{-1}(F(x))$ $\forall x \in I_{i_0} \times J_{j_0}$.

   $\widetilde{F}$ is continuous, according to the pasting lemma.

2. Check that at every step, there is a unique way to define $\widetilde{F}$.

3. If F is a path homotopy, then $F(\{0\} \times I) = x_0$. Also, $\widetilde{F}(\{0\} \times I) \subseteq p^{-1}(\{x_0\})$. $p^{-1}(\{x_0\})$ has the discrete topology as a subspace of $\widetilde{X}$. Anlose $\widetilde{F}(\{0\} \times I)$ is connected. Thus $\widetilde{F}(\{0\} \times I) = \{\widetilde{x}_0\}$.

   Similarly $\widetilde{F}(\{1\} \times I)$ is a set with one point.

   Thus, $\widetilde{F}$ is a path homotopy. ■

Proposition 2.38 (Homotopy lifting property)
$p : \widetilde{X} \to X$ covering map with $p(\widetilde{x}_0) = x_0$. Consider f, g the paths in X from x_0 to x_1 and $\widetilde{f}, \widetilde{g}$ their liftings to paths in $\widetilde{X}$ with $\widetilde{f}(0) = \widetilde{g}(0) = \widetilde{x}_0$. If $f \simeq g$, then $\widetilde{f} \simeq \widetilde{g}$.

Proof Consider F the path homotopy between f and g. Then $F((0,0)) = x_0$. Let $\widetilde{F} : I \times I \to \widetilde{X}$ the lifting of f to $\widetilde{X}$ with $\widetilde{F}((0,0)) = \widetilde{x}_0$. Then $F(\{0\} \times I) = \{\widetilde{x}_0\}$ and $\widetilde{F}(\{1\} \times I) = \{\widetilde{x}_1\}$.
$\widetilde{F}_{|I \times \{0\}}$ is a path i $\widetilde{X}$ starting at $\widetilde{x}_0$ that is a lifting of $F_{|I \times \{0\}}$. I.e. $\widetilde{F}_{I \times \{0\}} = \widetilde{f}$ (uniqueness of path liftings).
$\widetilde{F}_{|I \times \{1\}} = \widetilde{g}$.
So $\widetilde{f}(1) = \widetilde{g}(1) = \widetilde{x}_1$ and $\widetilde{F} : I \times I \to \widetilde{X}$ is a path homotopy from $\widetilde{f}$ to $\widetilde{g}$. ■

Definition 2.40 Let $p : \widetilde{X} \to X$ be a covering map with $p(\widetilde{x}_0) = x_0$. Given a loop f in X based x_0, let $\widetilde{f}$ be the lifting of f in $\widetilde{X}$ with $\widetilde{f}(0) = \widetilde{x}_0$. Define

$$\phi : \pi_1(X, x_0) \to p^{-1}(\{x_0\}),$$
$$[f] \mapsto \widetilde{f}(1).$$

ϕ is called the ***lifting correspondence*** derived from the map p

Claim: ϕ is well-defined.
Proof If $[f] = [f']$, then f, f' are path homotopic. We can lift this homotopy to a path homotopy between $\widetilde{f}$ and $\widetilde{f'} \implies \widetilde{f}(1) = \widetilde{f'}(1)$.

Theorem 2.42

1. *If $p : \widetilde{X} \to X$ be covering map with $p(\widetilde{x}_0) = x_0$. If $\widetilde{X}$ is path connected, then ϕ is surjective.*
2. *If $\widetilde{X}$ is simply connected, then ϕ is bijective.*

Proof

1. Let $\widetilde{x}_1 \in p^{-1}(\{x_0\})$. Then since $\widetilde{X}$ is path connected, there is a path $\widetilde{f} : I \to \widetilde{X}$ with $\widetilde{f}(0) = \widetilde{x}_0$ and $\widetilde{f}(1) = \widetilde{x}_1$. Then $f = p \circ \widetilde{f}$ is a loop in X based at x_0, since $f(0) = p(\widetilde{x}_0) = x_0$, $f_(1) = p(\widetilde{x}_1) = x_0$ and $\phi([f]) = \widetilde{f}(1) = \widetilde{x}_1$.

2. We only need to check that ϕ is injective. Indeed consider ä$[f], [g] \in \pi_1(X, x_0)$ with $\phi([f]) = \phi([g])$. Then $\widetilde{f}(1) = \widetilde{g}(1)$, where $\widetilde{f}, \widetilde{g}$ are the liftings of f, g with $\widetilde{f}(0) = \widetilde{g}(0) = \widetilde{x}_0$. Since $\widetilde{X}$ is simply connected, we have that $[\widetilde{f}] = [\widetilde{g}]$, i.e. there is a path homotopy $\widetilde{F}$ between $\widetilde{f}$ and $\widetilde{g}$. Then $F = p \circ \widetilde{F}$ is a path homotopy between f and g, i..e $[f] = [g]$. ■

2.5. The fundamental group of the circle and applications

Remark 2.44 S^1 is path connected, so we will write $\pi_1(\mathrm{S}^1)$.

Theorem 2.45

$$(\pi_1(\mathrm{S}^1), \cdot) \simeq (\mathbb{Z}, +).$$

Proof Let $p : \mathbb{R} \to \mathrm{S}^1$, $p(t) = (\cos 2\pi t, \sin 2\pi t)$. Denote $p(0) = (1,0) = x_0$. Then $p^{-1}(\{x_0\}) = \mathbb{Z}$. Consider $\phi : \pi_1(\mathrm{S}^1) \to \mathbb{Z}$, where $[f] \mapsto \widetilde{f}(1)$.
By exercise 1 of sheet 3 $\mathbb{R}$ is simply connected. By the second point of the theorem ϕ is bijective.
It remains to show that ϕ is a group homomorphism,
i.e. $\phi([f] \cdot [g]) = \phi([f]) + \phi([g])$.
$\iff \phi([f \cdot g]) = \phi([f]) + \phi([g])$
$\iff \widetilde{f \cdot g}(1) = \widetilde{f}(1) + \widetilde{g}(1)$,
where $\widetilde{f}$ is the lift of f with $\widetilde{f}(0) = 0 \in \mathbb{R}$ and $\widetilde{g}$ the lift of g with $\widetilde{g}(0) = 0 \in \mathbb{R}$ and $\widetilde{f \cdot g}$ the lift of $f \cdot g$ where $\widetilde{f \cdot g}(0) = 0 \in \mathbb{R}$.
Let $\widetilde{g}' : I \to \mathbb{R}$ be the path $\widetilde{g}'(s) = \widetilde{g}(s) + \widetilde{f}(1)$. Then $(p \circ \widetilde{g}')(s) = p(\widetilde{g}(s) + \widetilde{f}(1)) = (p \circ \widetilde{g})(s) = g(s)$.
$\implies \widetilde{g}'$ is a lifting of g with $\widetilde{g}'(0) = \widetilde{f}(1)$.
Furthermore, $p((\widetilde{f} \cdot \widetilde{g}')(s)) = (f \cdot g)(s) \implies \widetilde{f} \cdot \widetilde{g}'$ is a lifting of $f \cdot g$ with $\widetilde{f} \cdot \widetilde{g}'(0) = 0$.
$(\widetilde{f} \cdot \widetilde{g}')(+) = \widetilde{f}(1) + \widetilde{g}(1)$
$\implies \widetilde{f \cdot g}(1) = \widetilde{f}(1) + \widetilde{g}(1)$. ■

Retractions

Definition 2.47 A ***retraction*** of a X onto $A \subset X$ is a continuous map $r : X \to A$ with $r(a) = a \; \forall a \in A$.
If such a map exists A is called a ***retract*** of X.

Claim: If A is a retract of X and $i : A \hookrightarrow X$ is the inclusion map, then i_* is injective.
Proof $r \circ i : A \to A$. $r \circ i = 1_A$
$\implies (r \circ i)_* = (1_A)_*$.
$\implies r_* \circ i_* = 1_{\pi_1(A, x_0)}$
$\implies i_*$ is injective. ■

Proposition 2.49 There is no retraction of D^2 onto S^1.

Proof If S^1 was a retract of D^2 and $i : \mathrm{S}^1 \to \mathrm{D}^2$ the inclusion. Then $i_* : \pi_1(\mathrm{S}^1) \to \pi_1(\mathrm{D}^2)$ would be injective. But $\pi_1(\mathrm{S}^1) \simeq \mathbb{Z}$ and $\pi_1(\mathrm{D})^2$ is trivial as a convex subset of $\mathbb{R}^2$. ■

Lemma 2.51 Let $h : \mathrm{S}^1 \to X$ be a continuous map. The following are equivalent:

1. h is nullhomotopic.
2. There exists a continuous map $\overline{h} : \mathrm{D}^2 \to X$ with $\overline{h}_{|\mathrm{S}^1} = h$.
3. h_* is trivial.

Proof

1. "1.⇒2.": Consider the homotopy $H : S^1 \times I \to X$ between h and the constant map c. Let $q : S^1 \times I \to D^2$ with $(x,t) \mapsto t \cdot x$.

   q is continuous, q is open and surjective.

2.

3.

Vorlesung von Di 17. März fehlt noch

Remark 2.53 Ha the corollary: If $S^2 = A_1 \cup A_2 \cup A_3$, A_1 closed $\forall i \in \{1,2,3\}$, then $\exists x \in S^2, i \in \{1,2,3\}$ such that $\{x, -x\} \subseteq A_i$.

Inscribe a sphere in the tetrahedron. Project each face of the tetrahedral radially onto the sphere. Then one obtains four closed sets $A_i, i \in \{1,\dots,4\}$ with $S^2 = A_1 \cup A_2 \cup A_3 \cup A_4$, but none of the A_i contains a pair of antipodal points.

2.5.1. The Fundamental Theorem of Algebra

(Any non-constant polynomial in $\mathbb{C}[x]$ has a root in $\mathbb{C}$).

- proof in algebra,
- proof in complex analysis (corollary of Liouville's theorem),
- proof in topology.

Theorem 2.54 *(Fundamental Theorem of Algebra)*
If $a_n x^n + \cdots + a_1 + a_0 \in \mathbb{C}[x]$, $n > 0$, *then* $\exists x_0 \in \mathbb{C}$ *with* $a_n x_0^n + \cdots + a_1 x_0 + a_0 = 0$.

PROOF Let $a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{C}[x]$ and $n > 0$.

1. We can assume $a_n = 1$ $(a_n(x^n + \frac{a_{n-1}}{a_n} x^{n-1} + \cdots + \frac{a_0}{a_n}))$.
   We can assume $\sum_{i=0}^{n-1} |a_i| < 1$. Indeed choose $c \in \mathbb{R}_{>0}$ and set $x = cy$ This gives $c^n y^n + \cdots + a_1 cy + a_0 = c^n(y^n + \frac{a_{n-1}}{c} y^{n-1} + \cdots + \frac{a_0}{c^n})$. Choose c large enough such that $\left|\frac{a_{n-1}}{c}\right| + \cdots + \left|\frac{a_0}{c^n}\right| < 1$. Now if y_0 is a root of $y^n + \frac{a_{n-1}}{c} y^{n-1} + \cdots + \frac{a_0}{c^n}$, then cy_0 is a root of $x^n + \cdots + a_0$.

2. Consider $f : S^1 \to S^1$, $z \mapsto z^n$ and $p : I \to S^1$, $s \mapsto (\cos 2\pi s, \sin 2\pi s) = e^{2\pi i s}$. Then $f_* : \pi_1(S^1,(1,0)) \to \pi_1(S^1,(1,0))$. $f_*([p]) = [f \circ p]$, where $f \circ p(s) = (\cos 2\pi ns, \sin 2\pi ns)$.

   This shows that f_* is injective. Also if $\iota : S^1 \to \mathbb{R}^2 \setminus \{(0,0)\}$ (inclusion), then ι_* is injective, since S^1 is a retract of $\mathbb{R}^2 \setminus \{(0,0)\}$.

   I.e. $\iota_* \circ f_* = (\iota \circ f)_*$ is injective, which implies (Lemma **??**) that $\iota \circ f$ is not nullhomotopic (a).

3. Suppose, that our polynomial has no root in D^2. Consider $\overline{h} : D^2 \to \mathbb{R}^2 \setminus \{(0,0)\}$, $z \mapsto z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$. By lemma **??** $\rightsquigarrow \overline{h}_{|S^1}$ is nullhomotopic (b). Furthermore: $F : S^1 \times I \to \mathbb{R}^2 \setminus \{(0,0)\}$ where $(z,t) \mapsto z^n + t(a_{n-1} z^{n-1} + \cdots + a_1 z + a_0)$. This is a homotopy between $\overline{h}_{|S^1}$ and $\iota \circ f$ (c).

$|F(z,t)| = |z^n + t(a_{n-1}z^{n-1} + \cdots + a_0)| \geq |z^n| - |t||a_{n-1}z^{n-1} + \cdots + a_0| \geq 1 - |t|\sum_{i=0}^{n-1}|a_i| > 0$.

(a),(b),(c) contradiction, i.e. our polynomial has a root in D^2.

Exercise: Any root of such a polynomial $z^n + \cdots + a_0$ with $\sum |a_i| < 1$ is in D^2.

2.5.2. Deformation retracts and homotopy type

Covering spaces $\rightsquigarrow$ fundamental group computation (e.g. $\pi_1(\mathrm{S}^1)$).

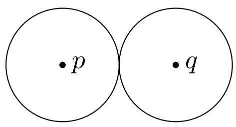
Definition 2.56 Let $A \subseteq X$, A is called a ***deformation retract*** of X, if there is a continuous map $H : X \times I < raX$ with $H(x,0) = x\ \forall x$, $H(x,1) \in A\ \forall x \in X$, $H(a,t) = a$ $\forall a \in A$, $\forall i \in I$. The homotopy H is called a ***deformation retraction*** of X onto A.

Remark 2.57

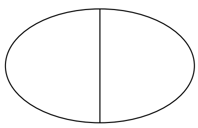
1. $r : X \to A$, $r(x) = H(x,1)$ is a retraction of X onto A.
2. H is a homotopy between 1_X and $\iota \circ r$, where $\iota : A \hookrightarrow X$ inclusion map.

Example 2.58

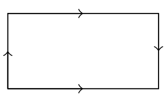
1. $\{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \neq (0,0),\ z = 0\} =: A$ is a deformation retract of $\{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \neq (0,0)\} =: X$.
   $H : X \times I \to X$, $H(x,y,z,t) := (x,y,(1-t)z)$.
   - H is continuous,
   - $H(x,y,z,0) = (x,y,z)\ \forall (x,y,z) \in X$,
   - $H(x,y,z,1) = (x,y,0) \in A\ \forall (x,y,z) \in X$,
   - if $(x,y,z) \in A$, then $z = 0$ and thus $H(x,y,z,t) = (x,y,0) \in A$.
2. Let $p,q \in \mathbb{R}^2$, $p \neq q$. Then $\mathrm{S}^1 \vee \mathrm{S}^1$ is a deformation retract of $\mathbb{R}^2 \setminus \{p,q\}$.



3. $p,q \in \mathbb{R}^2$, $p \neq q$, $\mathbb{R}^2 \setminus \{p,q\}$ deform retracts onto the theta space:



   (Homeomorph to $\mathrm{S}^1 \cup (0 \times [-1,1])$).
4. Möbius band



Theorem 2.59 *Let A be a deformation retract of X, $x_0 \in A$, and $\iota : (A, x_0) \to (X, x_0)$ the inclusion map. Then $\iota_* : (A, x_0) \to \pi_1(X, x_0)$ is an isomorphism.*

Proof Let $r : (X, x_0) \to (A, x_0)$, $r(x) = H(x, 1)$ Then $r \circ \iota : (A, x_0) \to (A, x_0)$ is the id map 1_A and $(r \circ \iota)_* = (1_A)_*$. $\Longrightarrow r_* \circ \iota_* = 1_{\pi_1(A,x_0)}$.
Furthermore $\iota_* \circ r_* = (\iota \circ r)_*$. $\iota \circ r : (X, x_0) \to (X, x_0)$ and there is a homotopy H between 1_X and $i \circ r$. $H(x_0, t) = x_0 \; \forall t \in I$.
If $f : I \to X$ loop , $f(0) = f(1) = x_0$, then $H \circ (f \times 1_I)) : I \times I \to X$ is a path homotopy between $1_X \circ f$ and $(\iota \circ r) \circ f$. (path homotopy: $H \circ (f \times 1_I)(0, t) = H(x_0, t) = x_0$ $\forall t \in I$, $H \circ (f \times 1_I)(1, t) = H(x_0, t) = x_0 \; \forall t \in I$).
So $[1_X \circ f] = [(\iota \circ r) \circ f] \Longrightarrow (1_*(f) = (\iota \circ r)_*(f)$.
$\Longrightarrow 1_{\pi_1(X,x_0)} = (\iota \circ r)_* \Longrightarrow \iota_* r_* = 1_{\pi_1(X,x_0)}$. ∎

Corollary 2.61 If $\iota : S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ the inclusion map, $x_0 \in S^n$, then $\iota_* : \pi_1(S^n, x_0) \to \pi_1(\mathbb{R}^{n+1} \setminus \{0\})$ is isomorphism.

Proof Define $H : (\mathbb{R}^{n+1} \setminus \{0\}) \times I \to \mathbb{R}^{n+1} \setminus \{0\}$, $H(x, t) := (1 - t)x + t\frac{x}{\|x\|}$.

- H is continuous,
- $H(x, 0) = x \; \forall x \in \mathbb{R}^{n+1} \setminus \{0\} =: X$,
- $H(x, 1) = \frac{x}{\|x\|} \in S^n =: A$.
- If $x \in S^n$, then $H(x, t) = (1 - t)x + tx = x \; \forall t \in I$.

H is a deformation retraction of $\mathbb{R}^{n+1} \setminus \{0\}$ onto S^n. Now use theorem. ∎

Example 2.63 We can look at example 2.58 again:

1. $\pi_1(X) \overset{\text{Thm}}{\simeq} \pi_1(A) \overset{\text{Cor}}{\simeq} \pi_1(S^1) \simeq \mathbb{Z}$.

2.,3. "figure eight" space is deformaton retraction of $\mathbb{R}^2 \setminus \{p, q\} \overset{\text{Thm}}{\Longrightarrow} \pi_1(A_1) \simeq \pi_1(\mathbb{R}^2 \setminus \{p, q\})$,
"theta" space is deformation retraction of $\mathbb{R}^2 \setminus \{p, q\} \overset{\text{Thm}}{\Longrightarrow} \pi_1(A_2) \simeq \pi_1(\mathbb{R}^2 \setminus \{p, q\})$.
$\Longrightarrow \pi_1(A_1) \simeq \pi_1(A_2)$.
Comment: "figure eight" and "theta" space have isomorphic fundamental group, but none is a deformation retract of the other (exercise).

Definition 2.64 Let $f : X \to Y$ be a continuous map. If there exists a continuous map $g : Y \to X$ such that $g \circ f : X \to X$ is homotopic to $1_X : X \to X$ and $f \circ g : Y \to Y$ is homotopic to $1_Y : Y \to Y$, then f is called a ***homotopy equivalence***.
g is called ***homotopy inverse*** of f. X,Y are called ***homotopy equivalent***.
If X, Y are homotopy equivalent, we say that X, Y have the same ***homotopy type***.

Check: Homotopy equivalence is an equivalence relation.

Remark 2.65 Homotopy equivalence is more general than deformation retraction.

- Suppose that $H : X \times I \to X$ is a deformation retraction of X onto A. Let $\iota : A \to X$ the inclusion map and $r : X \to A$, $r(x) = H(x, 1)$. Then $r \circ \iota : A \to A$ is the id map 1_A, and H is a homotopy between $\iota \circ r$ and 1_X.

Lemma 2.66 Let $h, k : X \to Y$ homotopic continuous maps with $h(x_0) = y_0$, $k(x_0) = y_1$. If $H : X \times Y \to Y$ is the homotopy between h and k and if $a(t) = H(x_0, t)$, then $k_* = \widehat{a} \circ h_*$.

$$\begin{array}{ccc} \pi_1(X,x_0) & \xrightarrow{h_*} & \pi_1(Y,y_0) \\ & \searrow^{k_*} & \downarrow \widehat{a} \\ & & \pi_1(Y_1,y_1) \end{array}$$

Proof ***…der Beweis ist noch sehr fehlerhaft, noch fehlerhafter als gewohnt :P …***
Let $f : I \to X$ a loop based at x_0.
We want to show that $k_*([f]) = \widehat{a} \circ h_*([f])$
$\iff [k \circ f] = \widehat{a}([h \circ f])$
$\iff [k \circ f] = [\widehat{a}][h \circ f][a]$
$\iff [a][k \circ f] = [h \circ f][a]$
$\iff [a(k \circ f)] = [(h \circ f)a]$.

$h(x) = H(x,0)\ \forall x \in X$, so $h(f(s)) = H(f(s),0) = (H \circ f_0)(s)$ where $f_0 : I \to X \times I$, $f_0(s) = (f(s),0)$. I.e. $h \circ f = H \circ f_0$.
$k(x) = H(x,1)\ \forall x \in X$, so $k(f(s)) = H(f(s),1) = (H \circ f_1)(s)$ where $f_1 : I \to X \times I$, $f_1(s) = (f(s),1)$. I.e. $k \circ f = \circ f_1$.
Consider $c : I \to X \times I$, $c(t) = (x_0, t)$. Then $H(c(t)) = a(t)$.
$F : I \times I \to X \times I$, $F(s,t) := (f(s),t)$.
$G_0\gamma_1$ is homotopic to $\gamma_0 B_1$, call G the path homotopy between them.
Then $F \circ G : I \times I \to X \times I$, continuous, $(F \circ G)(s,0) = F(G(s,0)) = \begin{cases} F(\beta_0(s),0), s \in [0,\frac{1}{2}] \\ F(1,\gamma_1(s)) & s \in [\frac{1}{2},1] \end{cases} = \begin{cases} (f \circ \beta_0(s),0) & s \in [0,\frac{1}{2}] \\ (f(1),\gamma_1(s)) & s \in [\frac{1}{2},1] \end{cases} = \begin{cases} f_0(s) & s \in [0,\frac{1}{2}] \\ (x_0,s) & s \in [\frac{1}{2},1] \end{cases}$.
$(F \circ G)(s,1) = \cdots = (c \cdot f_1)(s)$, $(F \circ G)(0,t) = F((0,0)) = (x_0,0)$, $(F \circ G)(1,t) = F((1,1)) = (x_0,1)$
$\implies (F \circ G$ is a path homotopy from $c \cdot f_1$ to $c \cdot f_1$.
Lastly, $H \circ (F \circ G)$ is a path homotopy from $H(f_0 \cdot c) = H(f_0) \cdot H(c) = (h \circ f) \cdot a$ to $H(c \cdot f_1) = H(c) \cdot H(f_1) = a \cdot (k \circ f)$ ■

Corollary 2.68 Let $h, k : (X, x_0) \to (Y, y_0)$ homotopic continuous maps. If $H : X \times I \to Y$ is the homotopy between h and k and $H(x_0,t) = y_0\ \forall t \in [0,1]$, then $k_* = h_*$.

Theorem 2.69 *Let $f : X \to Y$ homotopy equivalence with $f(x_0) = y_0$. Then $f_* : \pi_1(X,x_0) \to \pi_1(Y,y_0)$ is an isomorphism.*

Proof Let $g : Y \to X$ be a homotopy inverse of f, and let $g(y_0) = x_1$ and $f(x_1) = y_1$.

$$(X,x_0) \xrightarrow{f} (Y,y_0) \xrightarrow{g} (X,x_1) \xrightarrow{f} (Y,y_1)$$

$$\pi_1(X,x_0) \xrightarrow{(f_{x_0})_*} \pi_1(Y,y_0) \xrightarrow{g_*} \pi_1(X,x_1) \xrightarrow{(f_{x_1})_*} \pi_1(Y,y_1)$$

$g \circ f_{x_0}$ is homotopy to $1_X \implies$[Lemma] there is a path $a : I \to X$, such that $(g \circ f_{x_0})_* = \widehat{a} \circ (1_X)_* = \widehat{a}$.

$\widehat{\alpha}$ is a group isomorphism, so $g_* \circ f_*$ is a group isomorphism.
$\Longrightarrow g_*$ is surjective (1).
Similarly $(f_{x_1} \circ g)_*$ is a group isomorphism.
$\Longleftrightarrow (f_{x_1})_* \circ g_*$ is a group isomorphism.
$\Longrightarrow g_*$ is injective (2).

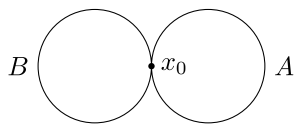
(1),(2) $\rightsquigarrow g_*$ is a group isomorphism, $\rightsquigarrow (f_{x_0})_* = (g_*)^{-1} \circ \widehat{\alpha}$ is a group isomorphism.

Theorem 2.71 *X, Y have the same homotopy type iff X, Y are homeomorphic to to deformation retracts of a space Z.*

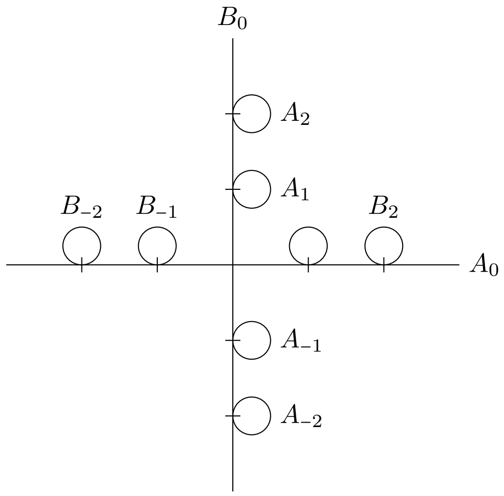
[Proof: hard, we omit it]

Theorem 2.72 *$\pi_1(\mathrm{S}^1 \vee \mathrm{S}^1)$ is not abelian.*

Proof $X := \mathrm{S}^1 \vee \mathrm{S}^1$.



Consider the following covering space



$p : \widetilde{X} \to X$, p maps each A_i homeomorphically onto A $\forall i \in \mathbb{Z} \setminus \{0\}$. and B_i homeomorphically to B $\forall i \in \mathbb{Z} \setminus \{0\}$. p maps all tangency points to X_0. p wraps A_0 around A, B_0 around B.
p is a covering map (check). Consider $\widetilde{f} : I \to \widetilde{X}, f(s) = (s, 0)$. $\widetilde{g} : I \to \widetilde{X}, g(s) = (0, s)$ and $f = p \circ \widetilde{f}, g = p \circ \widetilde{g}$.
Also consier $f \cdot g$ and $g \cdot f$ and lift these two paths to paths $\widetilde{f \cdot g}, \widetilde{g \cdot f}$ in $\widetilde{X}$ both starting at $(0,0)$. $\widetilde{f \cdot g}$ is a path in $\widetilde{X}$ that goes from $(0,0)$ to $(1,0)$ along x-axis and then once around B_1. $\widetilde{g \cdot f}$ is a path in $\widetilde{X}$ that goes from $(0,0)$ to $(0,1)$ along the y-axis and then once around A_1.
$\widetilde{f \cdot g}(1) \neq \widetilde{g \cdot f}(1)$. So $f \cdot g$ and $g \cdot f$ are not path-homotopic. $[f \cdot g] \neq [g \cdot f]$ i.e. $[f] \cdot [g] \neq [g] \cdot [f]$
$\Longrightarrow \pi_1(\mathrm{S}^1 \vee \mathrm{S}^1)$ is not abelian. ■

Theorem 2.74 $\pi_1(X \times Y, x_0 \times y_0) \simeq \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

PROOF Consider the projections $p : X \times Y \to X, x \times y \mapsto x$ and $q : X \times Y \to Y, x \times y \mapsto y$, and $p_* : \pi_1(X \times Y, x_0 \times y_0) \to \pi_1(X_1, x_0$ and $q_* : \pi_1(X \times Y, x_0 \times y_0) \to \pi_1(Y, y_0)$.
Define $\Phi : \pi_1(X \times Y, x_0 \times y_0) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$, $[f] \mapsto (p_*([f]), q_*([f]))$.
Claim: Φ is a group homomorphism.
Proof: $\Phi([f][g]) = \Phi([fg]) = (p_*([fg]), q_*([fg])) = ([p \circ (fg)], [g \circ (fg)]) = ([(p \circ f][p \circ g], [g \circ f][q \circ g]])$.
$\Phi([f])\Phi([g]) = ([p \circ f], [q \circ f])([p \circ g][q \circ g]) = ([p \circ f][p \circ g], [g \circ f][q \circ g])$.

Claim: Φ is surjective.
Proof: Consider $g : I \to X$ Loop based at x_0, and $h : I \to Y$ loop based at y_0. Define $f : I \to X \times Y$, $f(s) = (g(s), h(s))$. Then f is a loop based at (x_0, y_0) and $\Phi([f]) = (p_*([f]), q_*([f])) = ([g], [h])$.

Claim: Φ is injective.
Proof: Suppose that $f : I \to X \times Y$ is a loop based at (x_0, y_0) with $\Phi([f])$ is the identity element in $\pi_1(X, x_0) \times \pi_1(Y, y_0)$. Then $p \circ f$ is path homotopic to c_{x_0} via a path homotopy G and $q \circ f$ is path homotopic to c_{y_0} via a path homotopy H. Define $F : I \times I \to X \times Y$ by $F(s,t) := (G(s,t), H(s,t))$. F is a path homotopy from f to $c_{(x_0,y_0)}$. $\Longrightarrow [f]$ trivial. ∎

Corollary 2.76 Consider the torus $\mathrm{T} \simeq \mathrm{S}^1 \times \mathrm{S}^1$. $\Longrightarrow \pi_1(\mathrm{T}) \simeq \mathbb{Z} \times \mathbb{Z}$.

Definition 2.77

1. Let $\{X_i\}_{i \in I}$ an indexed collection of sets. The ***disjoint union*** of these sets is $\sqcup_{i \in I} X_i = \cup_{i \in I} X_i \times \{i\}$.
2. Suppose that X_i is a topological space $\forall i \in I$. The ***disjoint union topology*** on $\sqcup_{i \in I} X_i$ is the finest topology, such that all of the following maps are continuous: $\varphi_i : X_i \to \sqcup_{i \in I} X_i, x \mapsto x \times \{i\}$.
   Explicitly, U is open in $\sqcup_{i \in I} X_i$ iff $\varphi^{-1}(U)$ is open $\forall i \in I$.
3. Let X, Y be topological spaces, $A \subseteq X$, and $f : A \to Y$ continuous. The ***adjunction space*** $X \cup_f Y$ (adjunction of X to Y along f (along A)) is the quotient space
   $$X \sqcup Y \big/ \sim,$$
   where $a \sim f(a)\ \forall a \in A$. f is called the ***attaching map***.

Remark 2.78 If $X_i \cap X_j = \varnothing$ for any $i, j \in I$, $i \neq j$, then $\sqcup_{i \in I} X_i \simeq \cup_{i \in I} X_i$.
Example 2.79 $X_1, X_2 \subset \{R, \mathrm{T}, \mathrm{S}^2\}$, $X_1 \sqcup X_2$.

Definition 2.80 The ***wedge sum*** of X and Y is

$$X \vee Y := X \sqcup Y \big/ x_0 \sim y_0,$$

for $x_0 \in X, y_0 \in Y$.

If X_i are topological spaces, $x_i \in X_i$, $i \in I$, then

$$\vee_{i\in I} X_i = {\sqcup_{i\in I} X_i} \big/ {\{x_i \sim x_j,\ i,j \in I\}}.$$

Definition 2.81 Let X, Y n-manifolds. Consider U, V, open discs D^2 in X, Y respectively, and $f : \partial U \to \partial V$ a homeomorphism. The ***connected sum*** of X and Y is

$$X\#Y := {(X \setminus U) \sqcup (Y \setminus V)} \big/ {\sim},$$

where $x \sim y$ if $x \in \partial U$, $y \in \partial V$ and $f(x) = y$.

Remark 2.82 Although this construction involves the choice of the discs U, V the resulting space is unique up to homeomorphism.

Claim: $Y := \mathrm{S}^1 \vee \mathrm{S}^1$ is a retract of $X := \mathrm{T}\#\mathrm{T}$.
Proof (hier bräuchte es ne Graphik…)
Consider $f : X \to Y$, where f maps the dotted circle to y, and f restricted to $\mathrm{S}^1 \vee \mathrm{S}^1 \subseteq X$ is a homeomorphism h, and f $1:1$ everywhere else. Further consider r, where r retracts Y onto $\mathrm{S}^1 \vee \mathrm{S}^1$ by mapping each cross-sectional circle to the point where it intersets $\mathrm{S}^1 \vee \mathrm{S}^1$. Use h^{-1} to map $\mathrm{S}^1 \vee \mathrm{S}^1$ in Y to $\mathrm{S}^1 \vee \mathrm{S}^1$ in X.
$\rightsquigarrow$ retraction of $\mathrm{T}\#\mathrm{T}$ onto $\mathrm{S}^1 \vee \mathrm{S}^1$. ∎

Corollary 2.84 $\pi_1(\mathrm{T}\#\mathrm{T})$ is not abelian.

Proof If $\iota : \mathrm{S}^1 \vee \mathrm{S}^1 \to \mathrm{T}\#\mathrm{T}$ the inclusion map, then ι_* is injective (follows from claim).

$$\iota_* : \pi_1(\mathrm{S}^1 \vee \mathrm{S}^1) \to \pi_1(\mathrm{T}\#\mathrm{T}).$$

So $\pi_1(\mathrm{T}\#\mathrm{T})$ is not abelian. ∎

Corollary 2.86 T, $\mathrm{T}\#\mathrm{T}$ are not homotopy equivalent.

Proof obvious.

Lemma 2.88 If $P = (0, \dots, 0, 1) \in \mathrm{S}^n \subset \mathbb{R}^{n+1}$, then $\mathrm{S}^n \setminus \{P\}$ is homeomorphic to $\mathbb{R}^n$.

Proof Define $f : \mathrm{S}^n \setminus \{P\} \to \mathbb{R}^n$ (stereographic projection) , where $(x_1, \dots, x_{n+1}) \mapsto \frac{1}{1-x_{n+1}}(x_1, \dots, x_n)$.
f is obviously continuous and furthermore $g : \mathbb{R}^n \to \mathrm{S}^n \setminus \{P\}$ where $(y_1, \dots, y_n) \mapsto (\frac{2}{1+\|y\|^2} y_1, \dots, \frac{2}{1+\|y\|^2} y_n, 1 - \frac{2}{1+\|y\|^2})$ is continuous and $f \circ g = 1_{\mathbb{R}^n}$ and $g \circ f = \mathrm{id}_{\mathrm{S}^n \setminus \{P\}}$. ∎

Proposition 2.90 $\pi_1(\mathrm{S}^n) = 0$ $\forall n \in \mathbb{N}, n \geq 2$.

Proof Consider $f : I \to \mathrm{S}^n$ a loop based at $x_0 \in \mathrm{S}^n$. Let $x \in \mathrm{S}^n \setminus \{x_0\}$ and B a small open ball in $\mathrm{S}^n \setminus \{x_0\}$ with $x \in X$.
Then $f^{-1}(B) \subseteq (0,1)$ and $f^{1-}(B)$ is open., So $f^{-1}(B) = \cup_{i\in I}(a_i, b_i)$.
$f^{-1}(\{x\})$ is a closed subset of the compact space I, i.e. $f^{-1}(\{x\})$ is compact.
$f^{-1}(\{x\}) \subseteq f^{-1}(B) = \cup_{i\in I}(a_i, b_i)$, $\Longrightarrow \exists (a_1, b_1), \dots, (a_n, b_n)$ such that $f^{-1}(\{x\}) \subseteq (a_1, b_1) \cup \dots \cup (a_n, b_n)$.
Consider $f_i = f_{|[a_i,b_i]}$. Then $f((a_i, b_i)) \subseteq B$ and $f([a_i, b_i]) = f(\overline{(a_i, b_i)}) \subseteq \overline{f((a_i, b_i))} \subseteq \overline{B}$.
$f(a_i), f(b_i) \in \partial B$.
Since $n \geq 2$, choose $g_i : I \to \partial B$ path, such that $g_i(0) = f(a_i)$ and $g_i(1) = f(b_i)$.
Furthermore, $\overline{B}$ is convex, so $\pi_1(\overline{B}) = 0$. $\Longrightarrow f_i \simeq g_i$.

Repeating this process for all intervals $[a_i, b_i]$, $i \in \{1, \dots, n\}$, we obtain a loop $g : I \to S^n$, homotopic to f, and with the property that $g(I) \subseteq S^n \setminus \{x\}$, $[f] = [g]$.
The lemma implies that $[g] = [c_{x_0}]$, so $[f] = [c_{x_0}]$ and $\pi_1(S^n, x_0) = 0$. ■

Corollary 2.92 $\mathbb{R}^2$ and $\mathbb{R}^n$ are not homeomorphic for any $n \neq 2$.

Proof Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^n$ a homeomorphism ($n \neq 2$). Then

- $n = 1$: $\mathbb{R}^2 \setminus \{0\} \simeq \mathbb{R} \setminus \{f(0)\}$, but $\mathbb{R}^2 \setminus \{0\}$ is path connected, but $\mathbb{R} \setminus \{f(0)\}$ is not.
- Recall $\pi_1(\mathbb{R}^2 \setminus \{P\}) \simeq \pi_1(S^1) \simeq \mathbb{Z}$, $\pi_1(\mathbb{R}^n \setminus \{f(p)\}) \simeq \pi_1(S^{n-1}) = 0$ for $n \geq 3$. $f_{|\mathbb{R}^2 \setminus \{P\}} : \mathbb{R}^2 \setminus \{P\} \to \mathbb{R}^n \setminus \{f(P)\}$ homeomorphism, contradiction. ■

Recall:

space	π_1	using
$X \subseteq \mathbb{R}^n$ convex, $x_0 \in X$	0	definition
S^1	$\mathbb{Z}$	covering spaces
$\mathbb{R}^2 \setminus \{(0,0)\}$	$\mathbb{Z}$	deformation retracts
$\mathbb{R}^3 \setminus \{z\text{-axis}\}$	$\mathbb{Z}$	deformation retracts
$S^1 \vee S^1$	$\pi_1(S^1 \vee S^1) \simeq \pi_1(\mathbb{R}^2 \setminus \{p,q\})$	deformation retracts
θ-space	$\pi_1(\theta) \simeq \pi_1(\mathbb{R}^2 \setminus \{p,q\})$	deformation retracts
Möbius-band M	$\mathbb{Z}$	deformation retracts
$S^1 \vee S^1$	not abelian	covering spaces
$T \# T$	not abelian	retraction
S^n, $n \in \mathbb{N}, n \geq 2$	0	
T	$\mathbb{Z} \times \mathbb{Z}$	

2.6. Seifert von Kampen Theorem

2.6.1. Direct sums of abelian groups

Definition 2.94 Let G be an abelian group and $\{G_a\}_{a \in A}$ a family of subgroups of G. We say that the groups G_a ***generate***s G if every element x of G can be written as a finite sum of elements of the groups G_a.
Since G is abelian, we can always write this sum in the form $X = X_{a_1} + \dots + X_{a_n}$, $X_a \in G_a$, $a_i \neq a_j$, if $i \neq j$.
In this case we often write $x = \sum_{a \in A} X_a$.
If the groups G_a generate G, then G is called the ***sum*** of G_a and we write $G = \sum_{a \in A} G_a$ or $G = G_1 + \dots + G_n$.
If G_a generates G and $\forall x \in G$ $\exists! A$-tuple $(x_a)_{a \in A}$ with $x_a = 0$, for all but finitely many a in A, and $G = \sum_{a \in A} X_a$ then G is called the ***direct sum*** of G_a, and we write $G = \oplus_{a \in A} G_a$, or $G = G_1 \oplus \dots \oplus G_n$.

Example 2.95

1. $\mathbb{R}^2$, $G_1 = \{(x, 0) \mid x \in \mathbb{R}\}$, $G_2 = \{(0, y) \mid y \in \mathbb{R}\}$, $\mathbb{R}^2 = G_1 \oplus G_2$.
2. $\mathbb{R}^\infty$: the set of all sequences of real numbers that are eventually zero. $\mathbb{R}^\infty$ is a group under coordinate addition. $G_n = \{(0, \dots, 0, x_n, 0, \dots \mid x_n \in \mathbb{R}\}$ are subgroups of $\mathbb{R}^\infty$ $\forall n \in \mathbb{N}$ and $\mathbb{R}^\infty = \oplus_{n \in \mathbb{N}} G_n$.

Lemma 2.96 (Extension condition)
Let G be an abelian group and $\{G_a\}_{a\in A}$ be a family of subgroups. If $G = \oplus_{a\in A} G_a$, then G staisfies the following condition:

(∗) If H is any abelian group and $h_a : G_a \to H$ is a family of homomorphisms, then there exists a homomorphism $h : G \to H$, such that $h_{|G_a} = h_a \; \forall a \in A$.

Furthermore, this is unique. Conversely if G_a generate G and (∗) holds, then $G = \oplus_{a\in A} G_a$.

Proof Suppose that $G = \oplus_{a\in A} G_a$. Consider H abelian group., and $h_a : G_a \to H$ family of homomorphisms. Define $h : G \to H$ in the following way: If $x \in G$, then $x = \sum_{i=1}^n x_{a_i}$ and $h(x) = \sum_{i=1}^n h_{a_i}(X_{a_i})$. h is well-defined, because there is a unique way of writing x as $\sum_{i=1}^n x_{a_i}$. h is a homomorphism and $h_{|G_a} = h_a$.
Suppose that $h' : G \to H$ with $h'_{G_a} = h_a$, $\forall a \in A$. Then $h'(x) = h'(\sum_{i=1}^n x_{a_i}) = \sum_{i=1}^n h'(x_{a_i}) = \sum_{i=1}^n h_{a_i}(x_{a_i}) = h(x)$.

Converse: Suppose that $x = \sum_{a\in A} x_a = \sum_{a\in A} y_a$ for some $x \in G$. Choose $b \in A$ and let $H = G_b$. Define $h_a : G_a \to H$ by $h_a = 1_{G_b}$ if $a = b$, and h_a trivial if $a \neq b$. Let $h : G \to H$ be the extension of the homomorphisms h_a. Then $h(x) = \sum_{a\in A} h(x_a) = \sum_{a\in A} h_a(x_a) = X_b$, $h(x) = \sum_{a\in A} h(y_a) = \sum_{a\in A} h_a(y_a) = Y_b$.
$\implies X_b = Y_b, b \in A$,
$\implies X_a = Y_a \; \forall a \in A$. ∎

Corollary 2.98 Let $G = G_1 \oplus G_2$ and $G_1 = \oplus_{a\in A} H_a$, and $G_2 = \oplus_{b\in B} H_b$, where the index sets A,B are disjoint. Then $G = \oplus_{\gamma\in A\cup B} H_\gamma$.
[Proof: Problem Set 7, Exercise 4.a)]

Corollary 2.99 If $G = G_1 \oplus G_2$, then ${}^{G}\!/\!{}_{G_2} \simeq G_1$.
[Proof: Problem Set 7, Exercise 4.b)]

Given a family of abelian groups $\{G_a\}_{a\in A}$, find a group G that contains subgroups $G'_a \simeq G_a \; \forall a \in A$ and $G = \oplus_{a\in A} G'_a$.

Definition 2.100 Let $\{G_a\}_{a\in A}$ be a family of abelian groups, G be an abelian group, and $i_a : G_a \to G$ a family of monomorphisms such that $G = \oplus_{a\in A} \iota_a(G_a)$. Then G is called the ***external direct sum*** of the groups G_a, relative to monomorphisms i_a.

Notation: ≤: Subgroup.

Theorem 2.101 *If $\{G_a\}_{a\in A}$ is family of abelian groups, then there exists an abelian group G, and a family of monomorphisms $i_a : G_a \to G$, such that $G = \oplus_{a\in A} i_a(G_a)$.*

Proof Consider the product $\prod_{a\in A} G_a$ (with coordinate wise addition), which is an abelian group. Let $G = \{(x_a)_{a\in A} \mid x_a = 0$ for all but finitely many indices in $A\} \leq \prod G_a$. Given $\beta \in A$, define $i_\beta : G_\beta \to G$, by letting $i_\beta(x)$ the tuple with x as its β-th coordinate and 0 in every other coordinate. i_β is a monomorphism (follows from the definition), and if $x \in G$, then x has finitely many non zero coordinates, i.e. $x = (x_1, \dots, x_n, 0, \dots)$ and $x = i_1(x_1) + \cdots + i_n(x_n)$. Furthermore, this expression is unique. ∎

Lemma 2.103 (extension condition)
Let $\{G_a\}_{a\in A}$ be a family of abelian groups, G be an abelian group, $i_a : G_a \to G$ family of homomorphisms. If i_a is a monomorphism $\forall a \in A$, and $G = \oplus_{a\in A} i_a(G_a)$, then the following condition holds:

($*$) Given any abelian group H and any family of homomorphisms, $h_A : G_a \to H$, then there exists a homomorphism $h : G \to H$, with $h \circ i_a = h_a \ \forall a \in A$.

Futhermore h is unique.
Conversely if $i_a(G_a)$ generate G and ($*$) holds, then i_a is a monomorphism $\forall a \in A$, and $G = \oplus_{a\in A} i_a(G_a)$.
[Proof: Problem Set 7, Exercise 4.c)]

Theorem 2.104 *(Uniqueness of direct sums)*
Let $\{G_a\}_{a\in A}$ be a family of abelian groups, G, G' be abelian groups, $i_a : G_a \to G$, $i'_a : G_a \to G'$ families of monomorphisms, such that $G = \oplus_{a\in A} i_a(G_a)$ and $G' = \oplus_{a\in A} i'_a(G_a)$. Then $\exists! \varphi : G \to G'$ isomorphism, such that $\varphi \circ i_a = i'_a \ \forall a \in A$.

Proof

- Lemma 2.103 ($H = G'$) $\Longrightarrow \exists! \varphi : G \to G'$, such that $\varphi \circ i_a = i'_a \ \forall a \in A$.
- Lemma 2.103 ($H = G$) $\Longrightarrow \exists! \psi : G' \to G$, such that $\psi \circ i'_a = i_a \ \forall a \in A$.

Now (Lemma 2.103) $\psi \circ \varphi : G \to G$ satisfies $(\psi \circ \varphi) \circ i_a = i_a \ \forall a \in A$.
$\Longrightarrow \psi \circ \varphi = 1_G$, analogously $\varphi \circ \psi = 1_{G'}$. ■

2.7. Free abelian groups

Definition 2.106 Let G be an abelian group and $\{x_a\}_{a\in A}$ a family of elements in G. Let $G_a = \langle x_a \rangle$, i.e. the subgroup generated by x_a. If G_a generate G, we also say that x_a generate G.
If $G_a = \langle x_a \rangle$ is infinite cyclic $\forall a \in A$, and $G = \oplus_{a\in A} \langle x_a \rangle$, then G is called a ***free abelian group*** with basis $\{x_a\}_{a\in A}$.

Lemma 2.107 (extension condition)
Let G be an abelian group and $\{x_a\}_{a\in A}$ a family of elements of G that generates G. Then G is a free abelian group with basis $\{x_a\}_{a\in A}$ iff for any abelian group H and any family $\{y_a\}_{a\in A}$ of elements in H, there is a homomorphism $h : G \to H$, such that $h(x_a) = y_a$ $\forall a \in A$. In such a case h is unique.
Proof

"$\Rightarrow$" Given H, $\{y_a\}_{a\in A}$ define homomorphisms $h_a : G_a \to H$, with $h_a(x_a) = y_a$. (G_a is cyclic, so the condition $h(x_a) = y_a$ determines uniquely a homomorphism $h_a : G_a \to H$). By lemma 2.96 $\exists h : G \to H$ homomorphism, with $h_{|G_a} = h_a$ $\forall a \in A$. Furthermore h is unique.

"$\Leftarrow$" Suppose that for some $\beta \in A$, $\langle x_\beta \rangle$ is finite. Then for $H = \mathbb{Z}$ and $y_a = 1 \ \forall a \in A$, there is no homomorphism $h : G \to H$ with $h(x_a) = 1 \ \forall a \in A$.

By lemma 2.96, we get that $G = \oplus_{a\in A} G_a$, i.e. G is a free abelian group with basis $\{x_a\}_{a\in A}$. ■

Remark 2.109 Let G be free abelian with basis $\{x_1, \dots, x_n\}$. $G = \langle x_1 \rangle \oplus \cdots \oplus \langle x_n \rangle \simeq \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$.

Proposition 2.110 If G is free abelian with basis $\{x_1, \dots, x_n\}$, then n is uniquely determined by G.

Proof $G \simeq \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \geq 2\mathbb{Z} \oplus \cdots \oplus 2\mathbb{Z}$,

$$\mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \big/ 2\mathbb{Z} \oplus \cdots \oplus 2\mathbb{Z} \simeq \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2.$$

$|\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2| = 2^n$. ∎

Definition 2.112 If G is a free abelian group with basis $\{x_1, \dots, x_n\}$, then n is called the ***rank*** of G.

2.8. Free products of groups

Definition 2.113 Let G be a group and $\{G_a\}_{a \in A}$ be a family of subgroups of G. We say that G_a ***generate*** G, if $\forall x \in G\ \exists (x_1, \dots, x_n)$ with x_i element of some G_a $\forall i \in \{1, \dots, n\}$, and $x = x_1 \dots x_n$.
Such a sequence is called a ***word*** of ***length*** n in the groups G_a that represents x.

If $x_i, x_{i+1} \in G_a$ for some $a \in A$, we group them together to obtain the word

$$(x_1, x_2, \dots, x_{i-1}, x_i x_{i+1}, x_{i+2}, \dots, x_n)$$

of length $n - 1$. If $x_i = 1$, we delete x_i from our sequence.
Repeating these reduction operations, we obtain $(y_1, \dots, y_m)$ representing x, such that no group G_a contains y_i and y_{i+1} and, $y_i \neq 1$ $\forall i \in \{1, \dots, m\}$. We call $(y_1, \dots, y_m)$ a ***reduced word***.
Convention: $\varnothing$ is a reduced word representing $1 \in G$.

Definition 2.114 Let G be a group and $\{G_a\}_{a \in A}$ a family of subgroup that generates G and $G_a \cap G_b = \{1\}$ $\forall a, b \in A$, $a \neq b$. If $\forall x \in G$ there is only one reduced word in the group $\{G_a\}_{a \in A}$, that represents that element, then G is called the ***free product*** of the groups $\{G_a\}_{a \in A}$.
$G = *_{a \in A} G_a$, or $G = G_1 * \cdots * G_n$.

Example 2.115 $G_1 = \{1, x_1\}$, $G_2 = \{1, x_2\}$. $x \in G = G_1 * G_2$. Elements of G are for example $x_1, x_2, x_1x_2, x_2x_1, x_1x_2x_1, x_2x_1x_2, \dots$

Example 2.116 $G = \{\varphi : \{0, 1, 2\} \to \{0, 1, 2\} \mid \varphi \text{ bijection}\}$. $\varphi_1 : \{0, 1, 2\} \to \{0, 1, 2\}$, $\varphi_1(2) = 2$, $\varphi_1(1) = 0$, $\varphi_1(0) = 1$
$\varphi_2 : \{0, 1, 2\} \to \{0, 1, 2\}$, $\varphi_2(0) = 0$, $\varphi_2(1) = 2$, $\varphi_2(2) = 1$.

$|\langle \varphi_1 \rangle| = 2$, $|\langle \varphi_2 \rangle| = 2$,
$\langle \varphi_1 \rangle, \langle \varphi_2 \rangle$ generate G, but $B \neq \langle \varphi_1 \rangle * \langle \varphi_2 \rangle$.
Indeed the reduced words $(\varphi_1, \varphi_2, \varphi_1)$ and $(\varphi_2, \varphi_1, \varphi_2)$ represent the same element in G.

Claim: Suppose that G_a generate G. $G_a \cap G_b = \{1\}$ $\forall a, b \in A$ with $a \neq b$. If the representation of 1 by the empty word is unique then $G = *_{a \in A} G_a$.

Proof Let $x \in G$ and suppose that $(x_1, \dots, x_n)$, $(y_1, \dots, y_m)$ are two reduced words representing x, where $x_i \in G_{a_i}$, $y_i \in G_{b_j}$, $a_i \in A$, $i \in \{1, \dots, n\}$, $b_j \in A$, $j \in \{1, \dots, m\}$.
$x_1 \dots x_n = x = y_1 \dots y_m \Longrightarrow x_1 \dots x_n y_m^{-1} \dots y_1^{-1}$.
$\Longrightarrow a_n = b_m$ (i.e. $x_n, y_m \in G_{a_n} = G_{b_m}$).
We obtain $(x_1, \dots, x_{n-1}, x_n y_m^{-1}, y_{n-1}^{-1}, \dots, y_1^{-1})$.
Now we must have $x_n y_m^{-1} = 1 \Longrightarrow x_n = y_m$.
Continue this process, to conclue that $n = m$ and $x_i = y_i$, $\forall i \in \{1, \dots, n\}$. ∎

Lemma 2.118 (extension condition / universal mapping property) Let G be a group, $\{G_a\}_{a \in A}$ be a family of subgroups. If $G = *_{a \in A} G_a$, then it satisfies the following extension condition: Given any group H and any family of homomorphisms $h_a : G_a \to H$ thre exists a homomorphism $h : G \to H$, such that $h_{|G_a} = h_A$. In addition h is a unique.

Proof Define $h : G \to H$ as follows:

- $h(1) = 1$,
- If $x \in G$, $x \neq 1$, let $(x_1, \dots, x_n)$ be the (unique) reduced word representing x. Set $h(x) = h_{a_1}(x_1) \dots h_{a_n}(x_n)$, where a_i is the index for which $x_i \in G_{a_i}$.

Then

- h is well-defined,
- $h_{|G_a} = h_a$ $\forall a \in A$,
- h is a group homomorphism (problem sheet 8).

In fact, uniqueness of h follows from the fact that h must statisfy $h(x) = h(x_1, \dots, x_n) = h(x_1) \dots h(x_n) = h_{a_1}(x_1) \dots h_{a_n}(x_n)$. ∎

External free product

Definition 2.120 Let $\{G_a\}_{a \in A}$ be a family of groups. Suppose that G s a group and $\{i_a : G_a \to G\}_{a \in A}$ a family of monomorphisms with $G = *_{a \in A} i_a(G_a)$, then G is called the ***external free product*** of the groups $G_a$relative to the onomorphisms i_a.

Does such a group G exist?

Theorem 2.121 *Let $\{G_a\}_{a \in A}$ be a family of groups. There exists a group G and a family of monomorphisms $i_a : G_a \to G$, such that $G = *_{a \in A} i_a(G_a)$.*

Lemma 2.122 (extension condition)
$\{G_a\}_{a \in A}$ family of groups, G group. $i_a : G_a \to G$ family of homomorphisms. If i_a is a monomorphism $\forall a \in A$ and $G = *_{a \in A} i_a(G_a)$, then the following extension condition holds: Given any group H and a family of homomorphisms $h_a : G_a \to H$, $\exists : G \to H$ homomorphism such that $h \circ i_a = h_a$ $\forall a \in A$. In addition h is unique.

[Proof ommited]

Theorem 2.123 *(uniqueness of free product)*
Let $\{G_a\}_{a\in A}$ be a family of gropus, G, G' be groups, $\{i_a : G_a \to G\}_{a\in A}$, $\{i'_a : G_a \to G'\}_{a\in A}$ families of monomorphism such that $\{i_a(G_a)\}_{a\in A}$, $\{i'_a(G_a)\}$ generate G, G' repsectively.
If G, G' satisfy the extension condition, then $\exists!$ isomorphism $\varphi : G \to G'$ such that $\varphi \circ i_a = i'_a \; \forall a \in A$.

[Proof: analogous to the proof of uniqueness of direct sums (problem sheet 8)]

Lemma 2.124 Let $\{G_a\}_{a\in A}$ be a family of groups, G be a gropu and $i_a : G_a \to G$ be a family of homomorphisms. If $i_a(G_a)$ generate G, and the extension condition holds, then i_a is a monoomorphism $\forall a \in A$, and $G = *_{a\in A} G_a$.

Proof Consider $b \in A$. Set $H = G_b$, $h_a : G_a \to H$ the identity homomorphism 1_{G_b} if $a = b$, and $h_a : G_a \to H$ the trivial homomorphism, if $a \neq b$.
Let $h : G \to H$ be the homomorphism given by the extension condition.
Then $h \circ i_b = h_b \implies h \circ i_b = 1_{G_b}. \implies i_b$ is 1:1.
I.e. i_a is a monomorphism $\forall a \in A$.
Existence theorem $\implies$ there exists a group G' and $\{i'_a : G_a \to G'\}_{a\in A}$ a family of monomorphisms such that $G' = *_{a\in A} i'_a(G_a)$. G, G' both satisfy the extension condition and are generated by $\{i_a(G_a)\}_{a\in A}$, respectively $\{i'_a(G_a)\}_{a\in A}$.
Uniqueness theorem $\implies$ there is an isomorphism $\varphi : G \to G'$ with $\varphi \circ i_a = i'_a$. $G' = *_{a\in A} i'_a(G_a)$.
$\implies G = *_{a\in A} i_a(G_a)$. ∎

Corollary 2.126 If $G = G_1 * G_2$, , $G_1 = *_{a\in A} H_a$, $G_2 = *_{b\in B} H_b$, $A \cap B = \varnothing$, then $G = *_{\gamma \in A\cup B} H_\gamma$.
N normal subgroup of G: $N \trianglelefteq G$.

Theorem 2.127 *Let $G = G_1 * G_2$, $N_i \trianglelefteq G_i$, $i \in \{1, 2\}$. If N is the smallest normal subgroup of G that contains N_1, and $N :$, then*

$$G/N \simeq (G_1/N_1) * (G_2/N_2).$$

Proof Consider the inclusion homomorphism $i : G_1 \to G_1 * G_2$, $i' : G_2 \to G_1 * G_2$, and the projection homomorphism $p : G_1 * G_2 \to G_1 * G2/N$, $g \mapsto gN$.
Let $n_1 \in N_1$. Then $(p \circ i)(n_1) = n_1 N = N$. This implies that $N_1 \leq \ker(p \circ i)$ and $p \circ i$ induces $(p \circ i)' : G_1/N_1 \to G_1 * G_2/N$.
Analogously, $(p \circ i')'; G_2/N_2 \to G_1 * G_2/N$.
We will apply lemma 2.124 for the homomorphisms i_1, i_2.

- Check that the extension condition holds.
  Let $h_1 : G_1/N_1 \to H_1$, $h_2 : G_2/N_2 \to H$ arbitrary homomorphisms and $p_1 : G_1 \to G_1/N_1$, $p_2; G_2 \to G_2/N_2$ the projection homomorphism. Then $h_1 \circ p_1 : G \to H$, $h_2 \circ p_2 : G_2 \to H$ homomorphisms. The extension condition for $G_1 * G_2$ implies that $\exists h : G_1 * G_2 \to H$, with $h_{G_i} = h_i \circ p_i$, $i \in \{1, 2\}$. If $n_i \in N_i$, $i \in \{1, 2\}$, then $h(n_i) = 1_H$. I.e. $N_i \leq \ker h \; \forall i \in \{1, 2\}$.
  $\implies N \leq \ker h$.
  This implies that $h : G_1 * G_2 \to H$ induces a homomorphism $h' : G_1 * G_2/N \to H$

with $h' \circ i_1 = h_1$, and $h' \circ i_2 = h_2$.
$(h' \circ i_1(g_1N_1) = h' \circ (p \circ i)'(g_1N_1) = h'((p \circ i)(g_1)) = h'(p(i(g_1))) = h'(p(g_1)) = h'(g_1N) = h(g_1) = (h_1 \circ p_1)(g_1) = h_1(g_1N_1))$.

- Check that $i_1(G_1/N_1), i_2(G_2/N_2)$ generate $G_1 * G_2/N$.
$i_1(G_1/N_1) = (p \circ i)'(G_1/N_1) = (p \circ i)(G_1) = p(G_1) = G_1/N$, and similarly $i_2(G_2/N_2) = G_2/N_2$.
(Lemma 2.124) $\Longrightarrow i_1, i_2$ are monomorphisms and ${(G_1 * G_2)}\diagup_{N} = i_1(G_1/N_1) * i_2(G_2/N_2)$.

Corollary 2.129 If N is the smallest normal subgroup of $G_1 * G_2$ that contains G_1, then ${G_1 * G_2}\diagup_{N} \simeq G_2$.
Proof $N_1 = G_1, N_2 = \{1_{G_2}\}$. ■
G group, $\{G_a\}_{a\in A}$ family of groups, $\{i_a : G_a \to G\}_{a\in A}$ family of homomorphisms. Then the following statements are equivalent:

1. (i_a is a monomorphism $\forall a \in A$ and) $G = *_{a\in A} i_a(G_a)$.
2. Given any group H and any family of homomorphisms $h_a : G_a \to H$, $\exists!$ homomorphism $h : G \to H$ with $h \circ i_a = h_a$, $\forall a \in A$.

2.8.1. Free groups

Definition 2.131 Let G be a group and $\{x_a\}_{a\in A}$ a family of G with $\langle x_a\rangle$ infinite cylclic $\forall a \in A$. If $G = *_{a\in A}\langle x_a\rangle$, then G is called a ***free group*** and $\{x_a\}_{a\in A}$ is called a system of free generators.
In this case, if $x \in G \setminus \{1\}$, then x can be written uniquely as $x = (x_{a_i})^{n_1} \cdot (x_{a_k})^{n_k}$ where $a_i \neq a_i \neq a_{i+1}$, $\forall i \in \{1, \ldots, k-1\}$ and $n_1 \in \mathbb{Z} \setminus \{0\}$ $\forall i \in \{1, \ldots, k\}$.

Free groups are characterized by the following:
Lemma 2.132 (exension condition)
Let G be a group and $\{x_a\}_{a\in A}$ be a family of elmeents in G. If G is a free group with $\{x_a\}_{a\in A}$ system of free generators, then G satisifies

($*$) Given any group H and any family $\{y_a\}_{a\in A}$ of elements of H there exists a homomorphism $h : G \to H$ with $h(x_a) = y_a$ $\forall a \in A$.

In addition h is unique. Conversely, if $\{x_a\}_{a\in A}$ generates G, and ($*$) holds, then G is a free group with system of free generators $\{x_a\}_{a\in A}$.
Proof Lemma 2.118.
For the converse, consider $b \in A$. Then there exists a homomorphism $h_b : G \to \mathbb{Z}$ with $h_b(x_b) = 1$, and $h_b(x_a) = 0$ $\forall a \in A$, $a \neq b$. ($H = \mathbb{Z}$, $\{y_a\}_{a\in A} = \{0, 1\}$). This implies, that $\langle x_b\rangle$ is infinite cyclic. Therefore $\langle x_a\rangle$ infinite cyclic $\forall a \in A$.
Then lemma 2.124. ■

Theorem 2.134 *Let $G = G_1 * G_2$, G_1, G_2 are free groups with $\{x_a\}_{a\in A}$, $\{x_a\}_{a\in B}$ resepective free sytems of generators with $A \cap B = \varnothing$, then G is a fre group with $\{x_a\}_{a\in A\cup B}$ free system of generators.*

Definition 2.135 Let $\{x_a\}_{a\in A}$ be an arbitrary indexed family and G_a be the set of all symbols x_a^n, $n \in \mathbb{Z}$. Define $x_a^n \cdot x_a^m = x_a^{n+m}$. This makes our set G_a a group.
The external free product of the groups G_a is called the ***free group*** on the elements x_a.

Notation: We will identify the elements x_a^n of G_a with their images $i_a(x_a^n)$ in G.

Relation between free groups and free abelian groups.

Theorem 2.136 *If G is a free group and $\{x_a\}_{a\in A}$ is a system of free generators, then $G \diagup [G,G]$ is the free abelian group with basis the $\{[x_a]\}_{a\in A}$ (where $[x_a]$ is the coset $x_a[G,G]$).*

Proof We wil show, that 1. $\{[x_a]\}_{a\in A}$ generate $G \diagup [G,G]$ and 2. if H is an abelian group and $\{y_a\}_{a\in A}$ is a family of elements in H, then there is a homomorphism $h' : G \diagup [G,G] \to H$, with $h'([x_a]) = y_a \ \forall a \in A$. Lemma 2.107 will them directly imply the proof.

1. $\{x_a\}_{a\in A}$ generate $G \rightsquigarrow \{[x_a]\}_{a\in A}$ generate $G \diagup [G,G]$.
2. Consider an abelian group H and $\{y_a\}_{a\in A}$ a family of elements in H. G free $\Longrightarrow$ $\exists h : G \to H$ homomorphism with $h(x_a) = y_a \ \forall a \in A$.

   H abelian $\Longrightarrow [G,G] \le \ker h. \Longrightarrow$ there exists an induced morphism $h' : G \diagup [G,G] \to H$ with $h'([x_a]) = y_a \ \forall a \in A$.

Corollary 2.138 If G is a free group with n free generators, then an system of free generators of G has n elements.
Proof $G \diagup [G,G]$ free abelian group with basis $\{[x_i]\}_{i=1,\dots,n}$. ■

Definition 2.140 If G is a free group, then

$$\operatorname{rank}(G) := \operatorname{rank}(G \diagup [G,G]).$$

Properties:

G free abelian group	G free group
$H \le G \to H$, is free abelian $\operatorname{rank}(G) = n$, and $H \le G \Longrightarrow \operatorname{rank}(H) \le n$.	$H \le G \rightsquigarrow H$ is free. Consider the free group G and two elements $\{x_1, x_2\}$, $[G,G] \le G$ is freely generated by $[x_1^n, x_2^n]$, $m, n \in \mathbb{Z} \setminus \{0\}$. In fact a free group with rank greater than 1 has subgroups of all countable orders.

Groups up to isomorphism:
G free abelian, then the isomorphism type is determined by the cardinality of the basis.
G free, then the isomorphism type is determined by the cardinality of systems of generators.
G finitely generated abelian group, then $G \simeq H \oplus T$, where H is free-ablian and T is torsion subgroup, so the isomorphismtype is defined by the rank of H and elementary divisors.

Group representation Let G be a group and $\{x_a\}_{a\in A}$ be a family that generates G. Consider the free group F on $\{x_a\}_{a\in A}$. Define a epimorhpishm $h : F \to G$, $x_a \mapsto x_a$. $G \simeq {}^{F}\!\diagup_{\ker h}$.

$N := \ker h$. N is called the ***relation subgroup***. If $n \in N$, then n is called a relation on F. $N \trianglelefteq F$ and we can specify N by specifying a family $\{r_b\}_{b\in B}$ of elements in F such that N is the smalest normal subgroup containing $\{r_b\}_{b\in B}$. (Exercise 4, problem sheet 8). Such a family $\{r_b\}_{b\in B}$ is called a complete set of relations.

Definition 2.141 If G is a group, a ***presentation*** of G is family of generators $\{x_a\}_{a\in A}$ of generators for G together with a complete set of relations $\{r_b\}_{b\in B}$ for G. If both $\{x_a\}_{a\in A}$ and $\{r_b\}_{b\in B}$ are finite, then G is called ***finitely presented***. If $\{x_a\}_{a\in A}$ is finite, then G is called ***finitely generated***.

A presentation determines G uniquelly up to isomorphism. Given two different presentations, it is hard to decide whether they determine isomorphic groups (Unsolvability of the isomorphism problem for groups).

Example 2.142

1. G cyclic group of order n. Let $x \in G$ be generator. $\langle x \mid x^n\rangle$.
2. D_{2n} dihedral group. $\langle r, s \mid r^n,\ s^2, (rs)^2\rangle$.
3. $G := \langle a, b \mid a^3b^{-2}\rangle$, $G' := \langle x, y \mid xyxy^{-1}x^{-1}y^{-1}\rangle$. Is $G \simeq G'$?
   $F = \langle a, b\rangle$, $F' = \langle x, y\rangle$. Define $h : F \to G'$ group homomorphism with $h(a) = xy$ and $h(b) = xyx$.

   $h(a^3b^{-2} = (h(a))^3(h(b))^{-2} = xyxyxyx^{-1}y^{-1}x^{-1}y^{-1}x^{-1} = xyxxyxx^{-1}y^{-1}x^{-1}x^{-1}y^{-1}x^{-1} = 1_G$.

   If N is the smallest normal subgroup containing a^3b^{-2} then (exercise 4, problem sheet 8) $N = \langle ga^3b^{-2}g^{-1}, g \in F\rangle$.

   Then $N \le \ker h$.

   $h' : F \diagup N \to G'$, i.e. $h' : G \to G'$.

   Define $g : F' \to G$ group homomorphism, with $g(x) := a^{-1}b$, and $g(y) := b^{-1}a^2$. Do the same as above, then, g induces a homomorphism $g' : G' \to G$.

   Check that $h' \circ g = 1_{G'} : G' \to G'$ and $g' \circ h' = 1_G : G \to G$.

2.8.2. The Seifert-van Kampen theorem

Let X be a topological space, $x_0 \in X$, and $X = \cup_{\alpha\in A} A_\alpha$, where A_α open path-connected subset of X with $x_0 \in A_\alpha$ $\forall \alpha \in A$. Then we can consider the homomorphism $j_\alpha : \pi_1(A_\alpha) \to \pi_1(X)$ induced by the inclusion $A_\alpha \hookrightarrow X$ (for brevity we write $\pi_1(A_\alpha)$ instead of $\pi_1(A_\alpha, x_0)$ and we write $\pi_1(X)$ instead of $\pi_1(X, x_0)$). These homomomorphism extend to a homomorphism $\Phi : *_{\alpha\in A}\pi_1(U_\alpha) \to \pi_1(X)$ (extension condition for free product).

The Seifert-van Kampen theorem tells us, that Φ is "very often" surjective, but in general not injective.

Consider $i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$ the homomorphism induced by $A_\alpha \cap A_\beta \hookrightarrow A_\alpha$. Then $j_\alpha \circ i_{\alpha\beta} = j_\beta \circ i_{\beta\alpha}$. Let $\omega \in \pi_1(A_\alpha \cap A_\beta)$. Then $\Phi(i_{\alpha\beta}(\omega) \cdot (i_{\beta\alpha}(\omega))^{-1}) =$

$\Phi(i_{\alpha\beta}(\omega))(\Phi(i_{\beta\alpha}(\omega)))^{-1} = (j_\alpha \circ i_{\alpha\beta})(w) \cdot (j_\beta \circ i_{\beta\alpha}(w))^{-1} = 1_{\pi_1(X)}$. I.e. $i_{\alpha\beta}(\omega) \cdot i_{\beta\alpha}^{-1} \in \ker \Phi$.

Theorem 2.143 *(Seifert-van Kampen)*
If $X = \cup_{\alpha\in A} A_\alpha$. A_α open and path connected with $x_0 \in A_\alpha$, $\forall\alpha \in A$, and $A_\alpha \cap A_\beta$ is path connected for any pair $\{\alpha, \beta\}$ in A, then Φ is surjective. If furthermore $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, for any triple $\{\alpha, \beta, \gamma\}$ in A, then $\ker\Phi = N$, where N is the smallest normal subgroup containing all the elements of the form $i_{\alpha\beta}(\omega)i_{\beta\alpha}^{-1}$, so Φ induces an isomorphism ${*_\alpha\pi_1(A_\alpha)}/{N} \simeq \pi_1(X)$.

Proof

- surjectivity of $\Phi : *_\alpha\pi_1(A_\alpha) \to \pi_1(X)$.
  Let f be a loop in $\pi_1(X)$ based at x_0 f is continuous and A_α is open $\forall\alpha \in A$, so $\forall s \in I$ there is an open interval $V_S \subset I$ with $s \in V_s$ and $f(\overline{V}_s) \subseteq A_\alpha$ for some $\alpha \in A$. We have $I = \cup_{s\in I} V_s$ and I is compact, so it is covered by finitely many of the V_s. Then endpoints of these intervalls define a partition $0 = s_0 < s_1 < s_2 < \cdots < s_m = 1$ of I such that for each i there is some subset A_α with $f([s_{i-1}, s_i] \subseteq A_\alpha$.

  Denote the A_α containing $f([s_{i-1}, s_i])$ byA_i and the path $f_{|[s_{i-1},s_i]} = f_i$. Then $f = f_1 \cdot\cdots\cdot f_m$, f_i path in A_i.

  Now $A_i \cap A_{i+1}$ is path-connected, so we can choose a path g_i in $A_i \cap A_{i+1}$ from x_0 to $f(s_i)$.

  Consider the loop $(f_1 \cdot \overline{g_1}) \cdot (g_1 \cdot f_2 \cdot \overline{g_2})\cdots(g_{m-1} \cdot f_m)$, which is homotopic to f.

  This is a composition of loops based at x_0 each lying in a single A_i. Hence $[f] = [f_1 \cdot \overline{g_1}] \cdot [g_1 \cdot f_2 \cdot \overline{g_2}]\cdots[g_{m-1} \cdot f_m] \in \text{im}\Phi$.

  Notation: A factorization of $[f] \in \pi_1(X)$ is a fromal product $[f_1]\cdots[f_k]$, where each f_i is a loop in some A_α based at x_0, and f is homotopic to $f_1\cdots f_k$.
  A factorization of $[f]$ is a word in $*_{\alpha\in A}\pi_1(A_\alpha)$, possibly unreduced, with $\Phi([f_1]\cdots[f_k]) = [f]$.
  Two factorizatons of $[f]$ are equivalent, if they are related by a sequence of the following two kinds of moves or their inverses: 1. Combine adjacent terms $[f_i]\cdot[f_{i+1}]$ into $[f_i \cdot f_{i+1}]$ if $[f_i], [f_{i+1}]$ lie in the same $\pi_1(A_\alpha)$, and 2. regard the term $[f_i] \in \pi_1(A_\alpha)$ as lying in $\pi_1(A_\beta)$, if f_i is a loop in $A_\alpha \cap A_\beta$.

  Let $Q := {*_{\alpha\in A}\pi_1(A_\alpha)}/{N}$. The first mmove, doesn't change the element of $*_{\alpha\in A}\pi_1(A_\alpha)$ defined by the factorization. The second move doesn't change the image of this element in Q.

- $\ker\Phi$.
  Recall that $N \leq \ker\Phi$. If I show that any two factorizations of f are equivalent, this will imply that $\Phi' : Q \to \pi_1(X)$ is injective, hence $\ker\Phi = N$.

  Let $[f_1]\cdots[f_k]$ and $[f_1']\cdots[f_\ell']$ be two factorizations of $[f]$ and let $F : I \times I \to X$ a homotopy from $f_1\cdots f_k$ to $f_1'\cdots f_\ell'$. Then there exists partions $0 = s_0 < \cdots < s_m = 1$ and $0 = t_0 < \cdots < t_n = 1$, such that each rectangle $R_{ij} = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$ is mapped by f into a single A_α, which we label by A_{ij}. These partitions are obtained by

- covering $I \times I$ with finitely many rectangles $[a, b] \times [c, d]$ each mapped into a single A_α,
- partioning $I \times I$ by the union of all horizontal and vertical lines containing edgese of the rectangle above.

We can also assume that the s-partition subdivides the partitions given by the products $f_1 \cdots f_k$, $f'_1 \cdots f'_\ell$. We may also perturb the vertical sides of the rectangles R_{ij}, so that each point in $I \times I$ lies in at most three R_{ij}'s. Relable the new rectangles R_{mn} ordering them as in the picture

R_9	R_{10}	R_{11}	R_{12}
R_5	R_6	R_7	R_8
R_1	R_2	R_3	R_4

Consider γ_r, $r \in \{0, 1, \dots, mn\}$, the path in $I \times I$ from the left edge $\{0\} \times I$ to te right edge $\{1\} \times I$ that separates $R_1, \dots, R_r$ from $R_{r+1}, \dots, R_{mn}$. We will call the corners of the rectangles vertices. For each vertex v with $F(v) \neq x_0$ let g_v be a path from x_0 to $F(v)$ in the intersection of the two or three A_{ij}'s corresponding to the rectangles containing v. Insert into $F_{|\gamma_r}$ the appropriate paths $\overline{g}_v \cdot g_v$ at successive vertices (same idea as in the first part of the proof). This gives us a factorization of $[F_{|\gamma_r}]$ by regarding the loop corresponding to a horizontal or vertical segment between two adjacent vertices as lying in the A_{ij} for either of the rectangles containing the segment.

Diufferent choices of A_{ij} give equivalent factorizations. Also the factorizations associated to successive paths γ_r, γ_{r+1} are equivalent, because pushing γ_r to γ_{r+1} across R_{r+1} changes $F_{|\gamma_r}$ to $F_{|\gamma_{r+1}}$ by the homotopy within A_{ij} corresponding to R_{r+1}.

We can arrange that the factorization associtated to γ_0 is equivalent to $[f_1] \cdots [f_k]$ and by choosing g_v for each vertex v along $I \times \{0\} \subseteq I \times I$, to lie not just in the the two A_{ij}'s corresponding to the rectangles containing v, but also to to lie in the A_α for the f_i containing v in its domain.

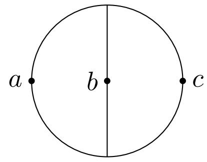
Similarly, the factorization associated to γ_{mn} is equivalent to $[f'_1] \cdots [f'_\ell]$. This gives us that $[f_1] \cdots [f_k]$, $[f'_1] \cdots [f'_\ell]$ are equivalent. ∎

Corollary 2.145 Let $X = U \cup V$, U, V open, $U, V, U \cap V$ path connected. Let $x_0 \in U \cap V$. Then Φ induces an isomorphism $\pi_1(X) \simeq {\pi_1(U) * \pi_1(V)} / {N}$, where N is the normal subgroup generated by all elements $i_1(\omega) \cdot i_2(\omega)^{-1}$, where $i_1 : \pi_1(U \cap V) \to \pi_1(U)$, $i_2 : \pi_1(U \cap V) \to \pi_1(V)$ morphisms induced by the inclusion and $\omega \in \pi_1(U \cap V)$.
[Proof: Immediate from Seifert-van Kampen theorem].

Corollary 2.146 Consider the assumption of corollary 2.145. If in addition $U \cap V$ is simply connected, then Φ induces an isomorphism $\pi_1(X) \simeq \pi_1(U) * \pi_1(V)$.
[Proof: Immediate from corollary 2.145].

Example 2.147

1. Let X be a theta space. Recall $\pi_1(X)$ is not abelian. ($S^1 \vee S^1$ and theta space are deformation retracts of $\mathbb{R}^2 \setminus \{p, q\}$ therfore $\pi(S^1 \vee S^1) \simeq \pi_1(X)$).

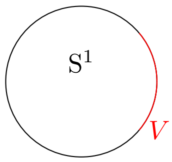


$U := X \setminus \{c\}, V := X \setminus \{a\}$. U, V are open and path connected. $U \cap V = X \setminus \{a, c\}$ is path connected, in addition $U \cap V$ is contractible.
$\implies \pi_1(X) \simeq \pi_1(U) * \pi_1(V) / N$. By corollary 2.146 $\pi_1(X) \simeq \pi_1(U) * \pi_1(V)$.
U, V have the homotopy type of $S^1 \implies \pi_1(U) \simeq \mathbb{Z}, \pi_1(V) \simeq \mathbb{Z}$.
$\implies \pi_1(X) \simeq \mathbb{Z} * \mathbb{Z}$.

2. Let X be the wedge of two circles, i.e. $X = S^1 \vee S^1$. $U := X \setminus \{a\}, V := X \setminus \{c\}$. $\pi_1(X) \simeq \mathbb{Z} * \mathbb{Z}$.

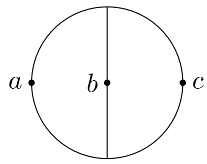
Remark 2.148

1. Necessity of path connectedness of $A_\alpha \cap A_\beta$.
   Consider



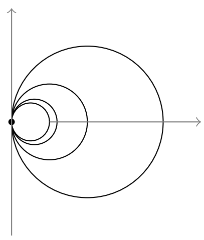
And U analogously by reflection on the vertical through the center. Then U, V open, path connected, but $U \cap V$ is not path connected the morphism induced by Φ is not surjective, $\pi_1(U) * \pi_1(V) \to \pi_1(S^1)$.

2. Necessity of path-connectedness of $A_\alpha \cap A_\beta \cap A_\gamma$.
   Consider the theta space



$A_a := X \setminus \{a\}, A_b := X \setminus \{b\}, A_c := X \setminus \{c\}$, then $A_a \cap A_b \cap A_c = X \setminus \{a, b, c\}$ not path connected, $\pi_1(X) \simeq \mathbb{Z} * \mathbb{Z} *$.

Example 2.149 (The shrinking wedge of circles)
Consider C_n the circle with center $(1/n, 0)$ and radius $1/n$ in $\mathbb{R}^2$. Let $X = \cup_{n \in \mathbb{N}} C_n$ with the subspace topology.



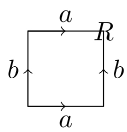
Consider the retractions $r_n : X \to C_n$ collapsing all C_i's except C_n to $(0,0)$. Each r_n induces a surjection $(r_n)_* : \pi_1(X) \to \pi_1(C_n)$ (basepoint $(0,0)$). The product of these surjections (i.e. $\rho[f]) = (r_1[f], r_2[f], \dots,))$ gives a homomorphism $\rho : \pi_1(X) \to \prod_\infty \mathbb{Z}$ (=direct product of countably infinite many copies of $\mathbb{Z}$).
Claim: ρ is surjective ($\Longrightarrow \pi_1(X)$ is uncountable).
Proof: For any sequence of integers $(K_n)_{n\in\mathbb{N}}$ we can construct a loop $f : I \to X$ based at $(0,0)$ and going K_n times around the circle C_n in time $[1 - 1/n, 1 - 1/(n+1)]$.
f is indeed continuous:

- $\forall t \in [0,1)$, f is continuous at t ✓.

- f is contiuous at $1 \in I \Longleftrightarrow$ ($\forall$ neighbourhoods V of $f(1) = (0,0)$ $\exists$ neighbourhood U of 1 such that $f(U) \subseteq V$).
  V contains all but finitely many of the circles C_n, and this proves continuity of f at 1.

2.9. CW complexes (cell complexes)

We have seen, that the torus T can be constructed from



The interior of the rectangle can be though of as an open disc or a 2-cel attached to the union of two circles a, b.
The union of the two circles can be thought of as obtained from their point of intersection P by attachin to it two open arcs, 1-cells.
In general construct a space X by the following procedure:

1. Start with a discrete set X^0, whose points are regarded as 0-cells.

2. Inductively construct the n-skeleton X^n from the $(n-1)$-skeleton X^{n-1} via continuous maps $\varphi_\alpha : S^{n-1} \to X^{n-1}$.
$$X^n = X^{n-1} \coprod_\alpha \overline{e^n_\alpha} \,/\, \sim,$$
where ~ the equivalence relation generated by $x \sim \varphi_\alpha(x)$, $x \in \partial e^n_\alpha$.

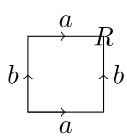
3. Either stop this inductive process after finitely many steps, setting $X = X^n$, or continue indefinitely, setting $X = \cup_{n\in\mathbb{N}} X^n$. In latter case, X is given the weak topology, i.e. $A \subset X$ is open, iff $A \cap X^n$ is open in X^n $\forall n$.

A space constructed in this way is called a CW-complex, where C stands for closure-finiteness: the closure of each cell intersects only finitely many other cells, W stands for weak-topology.
Example 2.150 S^2.

Suppose we attach a collection e^2_α of 2-cells to a path connected space X

$$Y := {X \coprod_\alpha \overline{e^2_\alpha}} \big/ {\sim},$$

where $x \sim \varphi_\alpha(x)$ for $x \in \partial e^2_\alpha$.
If s_0 is a base-point of S^1, then φ_α determines a loop in X based at $\varphi_\alpha(s_0)$.
Proposition 2.151 If $i : X \to Y$ is the inclusion map, then $i_* : \pi_1(X, x_0) \to \pi_1(Y, x_0)$ is surjective and $\ker \Phi_* = N$, i.e. $\pi_1(Y, x_0) \cong {\pi_1(X, x_0)} \big/ {N}$.
X path-connected space, Y space constructed from X by attaching a family of 2-cells e^2_α via maps $\varphi_\alpha : \mathrm{S}^1 \to X$. If s_0 is a basepoint of S^1, then φ_a determines a loop in X based at $\varphi_\alpha(s_0)$. Let $x_0 \in X$ a basepoint and let $\gamma_\alpha : I \to X$ a path with $\gamma(0) = x_0$ and $\gamma(1) = \varphi_\alpha(s_0)$.
Then $\gamma_\alpha : \varphi_\alpha \overline{\gamma_\alpha}$ is a loop in X based at x_0, $\gamma_\alpha \varphi_\alpha \overline{\gamma_\alpha}$ is nullhomotopic in Y $\forall \alpha$.
Let N be the normal subgroup of $\pi_1(X, x_0)$ generated by all elements $\gamma_\alpha \varphi_\alpha \overline{\gamma_\alpha}$.
If $i_* : \pi_1(X, x_0) < ra \pi_1(Y, x_0)$ is the homomorphism induced by $i : X \hookrightarrow Y$, then $N \leq \ker i_*$.
Proposition 2.152 If $i : X \to Y$ is the inclusion map, then $i_* : \pi_1(X, x_0) \to \pi_1(Y, x_0)$ is surjective, and $\ker i_* = N$.
Proof Consider the space Z obtained from Y by attaching rectangular strips $S_\alpha = I \times I$ with $I \times \{0\}$ is attached along γ_α, $I \times \{1\}$ is not attached to anything, $\{0\} \times I$ are identified for all α, $\{1\} \times I$ is attached along an arc in e^2_α.
Z deformation retracts onto Y.
$A := Z \setminus \{y_\alpha\}_\alpha$, where y_α is a point in e^2_α not in the arc along which S_α is attached.
$B := Z \setminus X$.
We apply the Seifert-van Kampen theorem for $Z = A \cup B$. $A = Z \setminus \{y_\alpha\}_\alpha$ deformation retracts onto X. B is contractible. A, B statisfy te conditions of the Seifert-van Kampen theorem.
$\implies \pi_1(Y) \simeq \pi_1(Z) \simeq {\pi_1(A)} \big/ {N}$, where N is the normal subgroup generated by the image of $\pi_1(A \cap B) \to \pi_1(A)$.
Consider the cover $A_\alpha = A \cap B \setminus \cup_{\beta \neq \alpha} e^2_\beta$.
A_α deformation retracts onto a circle in $e^2_\alpha \setminus \{y_\alpha\}$, so $\pi_1(A_\alpha) \simeq \mathbb{Z}$ and $\pi_1(A_\alpha)$ is generated by a loop homotopic to $\gamma_\alpha \varphi_\alpha \overline{\gamma_\alpha}$. So $\pi_1(A \cap B)$ is generated by loops homotopic to $\gamma_\alpha \varphi_\alpha \overline{\gamma_\alpha}$ for all α. ∎
Example 2.154 T has a cell-structure with one 0-cell, two 1-cells and one 2-cell. Let $X = X^1$ be the 1-skeleton of T, $X^1 = \mathrm{S}^1 \vee \mathrm{S}^1$ and attach to X^1 one 2-cell e^2_1 via $\varphi_1 : \mathrm{S}^1 \to X$ with $\varphi_1(\mathrm{S}^1) = a \cdot b \cdot \overline{a} \cdot \overline{b}$, to produce $Y = T$.



It follows from the proof, that $\pi_1(T) \simeq {\pi_1(X^1)} \big/ {N}$, where N is the normal subgroup of $\pi_1(X^1)$ generated by φ_1.
Now $\pi_1(X^1) \simeq \mathbb{Z} * \mathbb{Z}$, with generators $[a], [b]$, so $\pi_1(T) = \langle [a], [b] \mid [a][b][a]^{-1}[b]^{-1} \rangle$. $\alpha := [a]$, $\beta := [b]$. Let $F = \langle \alpha, \beta \rangle$ be the free group on two generators α, β and N be the smallest normal subgropu containing $[\alpha, \beta]$. Then $N \leq [F, F]$. Furthermore ${F} \big/ {N}$ is abelian, then $[F, F] \leq N$. $\implies N = [F, F]$, and $\langle \alpha, \beta \mid [\alpha, \beta] \rangle = {F} \big/ {N} = {F} \big/ {[F, F]} = \mathbb{Z} \oplus \mathbb{Z}$.

2.10. Surfaces (two-dimensional manifolds)

2.10.1. Fundamental group of surfaces

$\mathrm{T}_n := \underbrace{\mathrm{T}\#\ldots\#\mathrm{T}}_{n}$ n-fold Torus.

Example 2.155 $\mathrm{T}\#\mathrm{T}$ has CW-cell structure one 0-cell, four 1-cells, one 2-cell.

Example 2.156 $\mathrm{T}\#\mathrm{T}\#\mathrm{T}$

Example 2.157 n-fold torus: $4n$-gon with labelling $(a_1 b_1 \overline{a}_1 \overline{b}_1)(a_2 b_2 \overline{a}_2 \overline{b}_2)\ldots(a_n b_n \overline{a}_n \overline{b}_n)$

Proposition 2.158

$$\pi_1(\mathrm{T}_n) \simeq \langle \alpha_1, \beta_1, \ldots, \alpha_n, \beta_n \mid [\alpha_1, \beta_1][\alpha_2, \beta_2]\ldots[\alpha_n, \beta_n]\rangle.$$

Proof Completly analogous to the case $n = 1$, i.e. T. T_n has a cell structure with one 0-cell, $2n$ 1-cells and one 2-cell. The attaching map $\varphi_1 : \mathrm{S}^1 \to (\mathrm{T}_n)^1$ of the 2-cell e_1^2 determines a loop in T_n with $[\varphi_1] = [a_1][b_1][a_1]^{-1}[b_1]^{-1}\ldots[a_n][b_n][a_n]^{-1}[b_n]^{-1}$.■

The real projective plane $\mathbb{RP}^2$ **Notation:** $\mathrm{P} := \mathbb{RP}^2$.

$$\mathbb{RP}^2 := {\mathbb{R}^3 \setminus \{0\}}/{\sim},$$

where $x \sim y :\Longleftrightarrow \exists \lambda \neq 0$ with $x = \lambda y$.
We can also look at it as

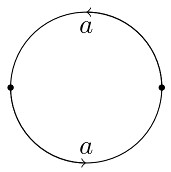
$$\mathrm{S}^2/{\sim}$$

where $p, q \in \mathrm{S}^2, p \sim q :\Longleftrightarrow p = -q$.

$\mathbb{RP}^2$ can also be descried as the quotient of D^2 with antipodal points of $\partial\mathrm{D}^2$ identified.
As $U = \{(x, y, z) \in \mathrm{S}^2 \mid z \geq 0\} \simeq \mathrm{D}^2$.

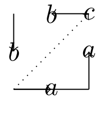
Proposition 2.160 $\pi_1(\mathbb{RP}^2) \simeq \langle \alpha \mid \alpha^2 \rangle \simeq \mathbb{Z}_2$.

Proof Method 1: Using the cell structure of $\mathbb{RP}^2$ given by the picture



■

Method 2: $p : S^2 \to \mathbb{RP}^2$ is a covering map, since $\pi_1(\mathrm{S}^2) = 0$, $\phi : \pi_1(\mathbb{RP}^2, x) \to p^{-1}(x)$ $x \in \mathbb{RP}^2$ is bijective.
Consider $\mathrm{P}_n := \mathrm{P}\#\ldots\#\mathrm{P}$.
$\mathrm{P}_2 = \mathrm{P}\#\mathrm{P}$. has cell structure



Proposition 2.162 $\pi_1(\mathrm{P}_n) \simeq \langle \alpha_1, \ldots, \alpha_n \mid \alpha_1^2 \ldots \alpha_n^2 \rangle$.

Proof Using cell-structure of P_n. ■

Remark 2.164

1. For every group G, there exists a CW-complex X_G with $\pi_1(X_G) \simeq G$: Consider $\langle g_\alpha | r_\beta \rangle$ a representation of G and take $V_\alpha \mathrm{S}^1_\alpha$. Then attach 2-cells e^2_β along the loops specified by hte relations r_β.
2. $\mathbb{R}\mathrm{P}^2 = \mathrm{P}$
   - P cannot be embedded in $\mathbb{R}^3$ (i.e. there does not exist a map $f : \mathrm{P} \to \mathbb{R}^3$, such that $f : \mathrm{P} \to f(P)$ is a homeomorphism).
   - P can be immersed in $\mathbb{R}^3$ (i.e. there exists a differentiable map $f : P \to \mathbb{R}^3$ with $\mathrm{D}_p f : \mathrm{T}_p\mathrm{P} \to \mathrm{T}_{f(p)}\mathbb{R}^3$ is injective $\forall p \in P$.

Question: Can we deduce, that T_n, T_m are not homotopy equivalent, if $n \neq m$?
Answer: Yes, for T and T_2. In general not yet. (It is not a tirivial matter to compare two group presentations).
We turn to study ${}^{\pi_1}/_{[\pi_1, \pi_1]}$.

2.10.2. Homology of surfaces

Let X be a path connected space, $x_0, x_1 \in X$, $a : I \to X$ path from $a(0) = x_0$ to $a(1) = x_1$. Then we can construct an isomorphism

$$\hat{a} : \pi_1(X, x_0) \to \pi_1(X, x_1),$$

where $[f] \mapsto [\overline{a}][f][a]$ and

$$\hat{a}_{\mathrm{ab}} : {}^{\pi_1(X,x_0)}/_{[\pi_1(X,x_0),\pi_1(X,x_0)]} \to {}^{\pi_1(X,x_1)}/_{[\pi_1(X,x_1),\pi_1(X,x_1)]}.$$

If $b : I \to X$ is a path in X with $b(0) = x_0$ and $b(1) = x_1$ and $g = a\overline{b}$, then

$$\hat{g}_{\mathrm{ab}} : {}^{\pi_1(X,x_0)}/_{[\pi_1(X,x_0),\pi_1(X,x_0)]} \to {}^{\pi_1(X,x_1)}/_{[\pi_1(X,x_0),\pi_1(X,x_0)]},$$

where

$$\begin{aligned}[f][\pi_1(X,x_0),\pi_1(X,x_0)] &\mapsto [\overline{g}][f][g][\pi_1(X,x_0),\pi_1(X,x_0)] \\ &= [f][\pi_1(X,x_0),\pi_1(X,x_0)].\end{aligned}$$

So

$$\hat{g}_{\mathrm{ab}} = 1_{\pi_1(X,x_0)/[\pi_1(X,x_0),\pi_1(X,x_0)]}.$$

That is, the isomorphism $\hat{a}_{\mathrm{ab}}$ is independent of the choice of path a.

Definition 2.165 If X is a path connected space and $x_0 \in X$, let

$$H_1(X) := {}^{\pi_1(X,x_0)}/_{[\pi_1(X,x_0),\pi_1(X,x_0)]}.$$

$H_1(X)$ is called the ***first homology group*** of X.

[We omit the basepoint, since there is in fact a unique isomorphism between the corresponding groups of different basepoints].
One can define homology groups $H_n(x)$ $\forall n \in \mathbb{N}$.

Lemma 2.166 Let F be a group and $N \trianglelefteq F$. Then

$$F \diagup N \diagup_{[F \diagup N, F \diagup N]} \simeq F \diagup [F,F] \diagup_{\{n[F,F] \mid n \in N\}}.$$

[Proof omitted].

Let F be a free group with free generators $\alpha_1, \dots, \alpha_n$. Let $x \in F$ and N be the smallest normal subgroup containing x. Finally, let $G = F \diagup N$. $F \diagup [F,F]$ is free abelian gropu with basis $\alpha_1[F,F], \dots, \alpha_n[F,F]$. $N \diagup [F,F]$ is the subgroup generated by $x[F,F]$.
The lemma implies that $G \diagup [G,G]$ is isomorphic to the quotient of a free abelian group with basis $\{\alpha_1[F,F], \dots, \alpha_n[F,F]\}$ by the subgroup $\langle x[F,F] \rangle$.

Theorem 2.167 *$H_0(\mathrm{T}_n)$ is a free abelian group of rank $2n \simeq \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{2n}$.*

Proof $\pi_1(\mathrm{T}_n) = \langle \alpha_1, \beta_1, \dots, \alpha_n, \beta_n \mid [\alpha_1, \beta_1] \cdots [\alpha_n, \beta_n] \rangle$.
$H_1(T_n) = \pi_1(\mathrm{T}_n) \diagup [\pi_1(\mathrm{T}_n), \pi_1(\mathrm{T}_n)$ is the quotient of a free abelian group of rank $2n$ by the group generated by

$$[\alpha_1, \beta_1] \cdots [\alpha_n, \beta_n][\pi_1(\mathrm{T}_n), \pi_1(\mathrm{T}_n)] = 1_{\pi_1(\mathrm{T}_n) \diagup [\pi_1(\mathrm{T}_n), \pi_1(\mathrm{T}_n)]}.$$

I.e. $H_1(\mathrm{T}_n) \simeq \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$. ∎

Theorem 2.169 *$H_1(\mathrm{P}_n)$ has a torsion subgroup $T(\mathrm{P}_n)$ of order 2 and $H_1(\mathrm{P}_n) \diagup T(\mathrm{P}(n)$ is a free abelian group of order $n - 1$.*

Proof $\pi_1(\mathrm{P}_n) = \langle \alpha_1, \dots, \alpha_n \mid \alpha_1^2 \dots \alpha_n^2 \rangle$. $H_1(\mathrm{P}_n) = \pi_1(\mathrm{P}_n) \diagup [\pi_1([\mathrm{P}_n), \pi_1(\mathrm{P}_n)]$ is the quotient of a free abelian group of rank n (basis $\{\overline{\alpha_1}, \dots, \overline{\alpha_n}\}$) by the subgroup generated by $\alpha_1^2 \dots \alpha_n^2[\pi_1(\mathrm{P}_n), \pi_1(\mathrm{P}_n)]$. Since we ompute in an abelian group we can use additive notation and write $2\alpha_1 + \cdots + 2\alpha_n + [\dots]$. Change the basis $\{\overline{\alpha_1}, \dots, \overline{\alpha_n}\}$ to the basis $\{\overline{\alpha_1}, \dots, \overline{\alpha_{n-1}}, \dots, \overline{\alpha_1 + \cdots + \alpha_n}\}$. This shows that $H_1(\mathrm{P}_n)$ is isomorphic to the quotient of the free abelian group with basis $\{\overline{\alpha_1}, \dots, \overline{\alpha_{n-1}}, \overline{\alpha_1 + \cdots + \alpha_n}\}$ by the subgroup $\langle 2(\overline{\alpha_1 + \cdots + \alpha_n}) \rangle$. So $H_1(\mathrm{P}_n) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{n-1} \oplus \mathbb{Z}_2$. ∎

Theorem 2.171 *The surfaces S^2, T, T_2, ..., P, P_2, ...are topologically distinct (i.e. any two of these have different homotopy type).*

[Proof: Follows immediatly from the previous theorems].

2.10.3. Classification of surfaces

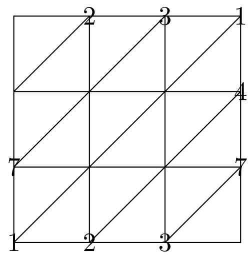
Theorem 2.172 *(Classification of closed connected surfaces)*
Let S be a closed surface (i.e. compact without boundary). Then S is either homeomorphic to S^2, or to T_n for some $n \in \mathbb{N}$, or to P_m for some $m \in \mathbb{N}$.

Ideas of the proof:

Triangulation of a closed surface

Definition 2.173 A ***triangulation*** of a closed surface S is a finite family of closed subsetes $\{T_1, \ldots, T_n\}$ that cover S together with homeomorphisms $\varphi_i : T_i' \to T_i$ where each T_i' is a triangle in $\mathbb{R}^2$. The subsets T_i are called ***triangles***. The subsetes of T_i that are images of vertices and edges are called ***vertices***, respectively ***edges***. We require that if T_i, T_j are distinct, $i, j \in \{1, \ldots, n\}$, then $T_i \cap T_j = \varnothing$, or $T_i \cap T_j$ is vertex or $T_i \cap T_j$ is a common edge.

Example 2.174 Torus:



Theorem 2.175 *(T. Radó, 1925)*
Any closed surface admits a triangulation.

[Proof: Using the Jordan curve theorem].

Cutting and pasting polygonal regions in $\mathbb{R}^2$

Theorem 2.176 *If S is a closed surface, then S is homeomorphic to a space obtained from a polygonal region in $\mathbb{R}^2$ by gluing its edges together in pairs.*

Theorem 2.177 *If X is the quotient space obtained from a polygonal region in $\mathbb{R}^2$ by gluing its edges together, then X is homeomorphic either to S^2 or to T_n, for some $n \in \mathbb{N}$ or to P_m for some $m \in \mathbb{N}$.*

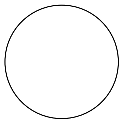
2.11. Knot theory

First attempt to define the notion of a knot A knot is a simple closed curve in $\mathbb{R}^3$ (i.e. $\exists \gamma : \mathrm{S}^1 \to \mathbb{R}^3$ continuous and injective, further, as S^1 is compact and $\mathbb{R}^3$ Hausdorff, we have f is closed) $\iff \gamma : \mathrm{S}^1 \to \gamma(\mathrm{S}^1)$ is a homeomorphism $\iff \gamma$ is an embedding of S^1 in $\mathbb{R}^3$.
Problem: "wild" knots are allowed in this definition.

Definition 2.178 A ***link*** L of m components is a subset of $\mathbb{R}^3$ consisting of m disjoint, piecewise linear, simple closed curves. A link of one component is a ***knot*** K.

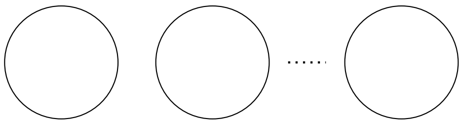
Each of the curves is a union of finitely many line segments attached end to end.
Example 2.179

1. The unknot U:

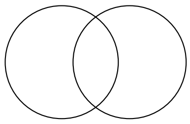


(${U}$ is the only knot bounding a disc embedded in $\mathbb{R}^3$).

2. The trivial knot of m components:



3. Hopf Link



4. Trefoil knot

5. Borromean rings

Definition 2.180 Too links L_1, L_2 in $\mathbb{R}^3$ (respectively in $S^3 = \mathbb{R}^3 \cup \{\infty\}$) are called ***equivalent*** if there is an orientation preserving piecewise linear homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ (respectively $h : S^3 \to S^3$) with $h(L_1) = L_2$.

Question: Given two links L_1, L_2 how can we decide if they are equivalent.

- If L_1, L_2 are equivalent, then try to deform one into the other.
- If L_1 and L_2 are not equivalent, use invariants to distinguish them.

If L_1, L_2 are equivalent, ten $\mathbb{R}^3 \setminus L_1$ and $\mathbb{R}^3 \setminus L_2$ are homeomorphic. The homeomorphism type of $\mathbb{R}^3 \setminus L$ is such an invariant.
How strong an invariant is it?

Theorem 2.181 *(Gordon-Luecue, 1989)*
The knots K_1, K_2 are equivalent $\iff$ $S^3 \setminus K_1$ *and* $S^3 \setminus K_2$ *are homeomorphic.*

[Proof: Very difficult; omited].
Note: The theorem says, that the homeomorphism type of $S^3 \setminus K$ is a complete invariant.

Remark 2.182 The theorem does not hold for links (with more than one component).

Determining the hoeomorphism type of $S^3 \smallsetminus L$ is not so simple. We turn to study $\pi_1(S^3 \smallsetminus L)$.
That is $\pi_1(S^3 \smallsetminus L) \neq \pi_1(S^3 \smallsetminus L_2) \Longrightarrow S^3 \smallsetminus L_1 \neq S^3 \smallsetminus L_2 \Longrightarrow L_1$ and L_2 are not equivalent.

Definition 2.183 If L is a link in S^3, then $\pi_1(S^3 \smallsetminus L)$ is called the ***group of the link*** L. It is sometimes denoted by $\pi_1(L)$.

Remark 2.184 The inclusion $\mathbb{R}^3 \hookrightarrow S^3$ induces an inclusion $i : \mathbb{R}^3 \smallsetminus L \hookrightarrow S^3 \smallsetminus L$ with with $i_* : \pi_1(\mathbb{R}^3 \smallsetminus L) \to \pi_1(S^3 \smallsetminus L)$ an isomorphism. So it is therefore equivalent to consider $L \subset \mathbb{R}^3$ or $L \subset S^3$.

Torus knots Consider $\mathrm{T} \subset \mathbb{R}^3$ (rotat $C_1 = \{(x,y,z) \in \mathbb{R}^3 \mid y = 0,\ (x-1)^2 + z^2 = \frac{1}{9}\}$ about the z-axis). We want to study knots on T.
T can be thought of as the quotient space of $\{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ 0 \le y \le 1\}$ with opposite edges identified or as the quotient space obtained from $\mathbb{R}^2$ by identifying two points (x,y), (x',y') if $x - x' \in \mathbb{Z}$ and $y - y' \in \mathbb{Z}$.
Let $q : \mathbb{R}^2 \to \mathrm{T}$ be the identification map described and $L = \{(x,y) \in \mathbb{R}^2 \mid |y = \frac{m}{n}x,\ m,n \in\in \mathbb{Z}, 1 < m < n,\ (m,n) = 1\}$. Take $\mathrm{T}_{(m,n)} := q(L) \subseteq \mathrm{T}$.
This is a simple closed curve on T that wraps m times around a line through the hole in the torus and n-times around a circle inside T. $\mathrm{T}_{(m,n)}$ is called an (m,n)-torus knot.
$\mathrm{T}_{(m,n)}$ can be given by the following parametrisation:

$$\begin{aligned} x &= (2 + \cos(n\varphi/m))\cos\varphi, \\ y &= (2 + \cos(n\varphi/m))\sin\varphi, \\ z &= \sin(n\varphi/m), \end{aligned}$$

with $\varphi \in [0, 2mn]$. This lies on the torus given by $(r-2)^2 + z^2 = 1$ in cylindrical coordinates.

Remark 2.185

1. $\mathrm{T}_{(1,n)}$ is equivalent to U the unknot.
2. $\mathrm{T}_{(m,n)}, \mathrm{T}_{(n,m)}$ are equivalent.
3. $(m,n) \neq 1$: $T_{(m,n)}$ torus link.

We want to compute $\pi_1(S^3 \smallsetminus K)$, where K is the unknot U, or K is a torus knot.

Decomposition of S^3 $A := \{(x_1,x_2,x_3,x_4) \in S^3 \mid x_1^2 + x_2^2 \le x_3^2 + x_4^2\}$ and $B := \{(x_1,x_2,x_3,x_4) \in S^4 \mid x_1^2 + x_2^2 \ge x_3^2 + x_4^2\}$.
Then A, B are closed subsets of S^3 and $S^3 = A \cup B$. Further $A \cap B = \{(x_1,x_2,x_3,x_4) \in S^3 \mid x_1^2 + x_2^2 = x_3^2 + x_4^2\}$.
So $A \cap B$ is the cartesian products of the circle $\{(x_1,x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1/2\}$ with the circle $\{(x_3,x_4) \in \mathbb{R}^2 \mid x_3^2 + x_4^2 = 1/2\}$, that is $A \cap B$ is a torus.

Now let $\mathrm{D}^2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \le 1/2\}$ and $\mathrm{S}^1 = \{(x_3, x_4) \in \mathbb{R}^3 \mid x_3^2 + x_4^2 = 1/2\}$, and define $f : \mathrm{D}^2 \times \mathrm{S}^1 \to A$ by

$$(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, \sqrt{2}x_3\sqrt{1 - (x_1^2 + x_2^2)}, \sqrt{2}x_4\sqrt{1 - (x_1^2 + x_2^2)}).$$

f is obviously continuous and injective. One can easily verify, that it is also surjective. Further (as $\mathrm{D}^2 \times \mathrm{S}^1$ compact, A Hausdorff) f homeomorphism.
Analogously, we can prove that B is homeomorphic to $\mathrm{D}^2 \times \mathrm{S}^1$.

$$\mathrm{S}^3 = A \cup B.$$

We can now compute:

1. $\pi_1(\mathrm{S}^3 \smallsetminus U)$:
   Take U to be the core circle of the A torus, i.e. $U = \{(x_1, x_2, x_3, x_4) \in A \mid x_1 = x_2 = 0\}$. The boundary $A \cap B$ of A is a deformation retract of $A \smallsetminus U$. So B is a deformation retract of $\mathrm{S}^3 \smallsetminus U = (A \cup B) \smallsetminus U$.
   $\Longrightarrow \pi_1(\mathrm{S}^3 \smallsetminus U) \simeq \pi_1(B) \simeq \mathbb{Z}$.

2. $\pi_1(\mathrm{S}^3 \smallsetminus K)$ where K is an (m, n) torus knot $\mathrm{T}_{(m,n)}$:
   Consider K as a subset of $A \cap B$ in S^3. $\mathrm{S}^3 \smallsetminus K = (A \smallsetminus K) \cup (B \smallsetminus K)$. $(A \smallsetminus K)$, $(B \smallsetminus K)$, $(A \smallsetminus K) \cap (B \smallsetminus K)$ path connected. However $(A \smallsetminus K)$, $(B \smallsetminus K)$, $(A \smallsetminus K) \cap (B \smallsetminus K)$ are not open subsets so we need to modify them in order to apply the Seifert-van Kamepen theorem.

   Choose $\varepsilon > 0$, such that there exists a tubular neighbourhood N of K in S^3 with radius ε. $\mathrm{S}^3 \smallsetminus N$ is a deformation retract of $\mathrm{S}^3 \smallsetminus K$. Then consider $\frac{1}{2}\varepsilon$ open neighbourhoods A', B' of A and B respectively.

   A', B' are homeomorphic to the product of S^1 with an open disk. And $A' \cap B'$ is homemorphic to to $(A \cap B) \times (-\frac{1}{2}\varepsilon, -\frac{1}{2}\varepsilon)$. Lastly A, B are deformation retracts of A' and B' respectively.

   We can apply the Seifert-van Kampen theorem for $\mathrm{S}^3 \smallsetminus N = (A' \smallsetminus N) \cup (B' \smallsetminus N)$ to compute $\pi_1(\mathrm{S}^3 \smallsetminus N) \simeq \pi_1(\mathrm{S}^3 \smallsetminus K)$.

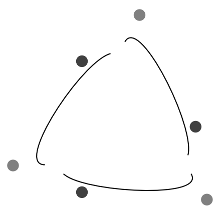
   - $A' \smallsetminus N$ deformation retracts onto the core circle of the torus A i..e $\pi_1(A' \smallsetminus N) \simeq \mathbb{Z}$, $B' \smallsetminus N$ deformation retracts onto the wire cricle of B, i.e. $\pi_1(B' \smallsetminus N) \simeq \mathbb{Z}$.
   - $(A' \smallsetminus N) \cap (B' \smallsetminus N) = (A' \cap B') \smallsetminus N$ has the homotopy type of $(A \cap B) \smallsetminus K$. So $\pi_1((A' \smallsetminus N) \cap (B' \smallsetminus N)) \simeq \pi_1((A \cap B) \smallsetminus K) \simeq \mathbb{Z}$.
   - $\pi_1(A' \cap B' \smallsetminus N) = \langle \gamma \rangle \to \pi_1(A' \smallsetminus N) = \langle \alpha \rangle$, $\pi_2(A' \cap B' \smallsetminus N) = \langle \gamma \rangle \to \pi_1(B' \smallsetminus N) = \langle \beta \rangle$.
     $i_1(\gamma) = a^m$, $i_2(\gamma) = \beta^n$.
   - Applying the Seifert-van Kamen theorem gives

Proposition 2.186 $\pi_1(\mathrm{S}^3 \smallsetminus \mathrm{T}_{(m,n)}) = \langle \alpha, \beta \mid \alpha^m \beta^{-n} \rangle$.

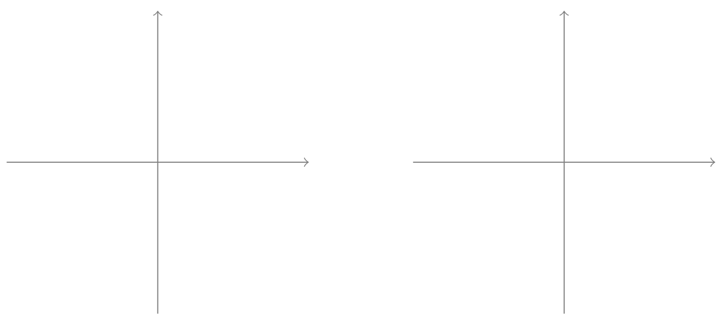
[Proof: Follows from the above analysis].
Proposition 2.187 If the knots $\mathrm{T}_{(m,n)}$, $\mathrm{T}_{(m',n')}$ are equivalent, then $m = m'$ and $n = n'$. Furthermore $\mathrm{T}_{(m,n)}$ and U are not equivalen.t

Proof (Schreier, 1923)
Consider the element $\alpha^m = b^n =: z \in \pi_1(\mathrm{S}^3 \setminus \mathrm{T}_{(m,n)} =: G$, and the subgroup N of G generated by z. $z \in Z(G)$, i.e. $zg = gz \forall g \in G$. So $N \leq Z(G)$.
Then $N \trianglelefteq G$ and we can consider the quotient $G \,/\, N$. $G \,/\, N$ has the following presentation: $\langle \alpha N, \beta N \mid (\alpha N)^m, (\beta N)^n \rangle$. Thus $G \,/\, N \simeq \mathbb{Z}_m * \mathbb{Z}_n$, which in turn implies that $Z(G \,/\, N) = \{1_{G/N}\}$.
Sine $Z(G) \,/\, N \leq Z(G \,/\, N)$, it follows that $N = Z(G)$, so $G \,/\, Z(G) \simeq \mathbb{Z}_m * \mathbb{Z}_n$.
Now if $\mathrm{T}_{(m,n)}, \mathrm{T}_{(m',n')}$ are equivalent, $\pi_(\mathrm{T}_{(m,n)}) \simeq \pi_1(\mathrm{T}_{(m',n')})$.
$G \,/\, Z(G) \simeq G' \,/\, Z(G') \Longrightarrow \mathbb{Z}_m * \mathbb{Z}_n \simeq \mathbb{Z}_{m'} * \mathbb{Z}_{n'}$.
$\Longrightarrow m = m'$ and $n = n'$.
Lastly $\pi_1(U) \simeq \mathbb{Z}$, that is $\pi_1(U) \,/\, Z(\pi_1(U)) \simeq \mathbb{Z} \,/\, \mathbb{Z} \neq \mathbb{Z}_m * \mathbb{Z}_n$ for any $m, n \in \mathbb{Z}$ with $(m, n) = 1$ ■

Wirtinger presentation Procedure for writing a presentation of the group of a knot K starting from a diagram for $K \subset \mathbb{R}^3$.



Label the arcs by $a_1, \dots, a_n$, so that each a_i is ocnnected to a_{i-1} and $a_i + 1 \pmod n$.
Assume that the arcs are oriented compatibly with their labellling.
Draw an arrow x_i passing under each a_i in a right-left direction. Each x_i represents a loop in $\mathbb{R}^3 \setminus K$ as follows. Suppose that the black board is the xy-plane P. Consider as basepoint $*$ the point $(0, 0, 1)$. The loop consists of a segment from $*$ to the tail of x_i, then the arrow, then a segment from the head of x_i to $*$.
At each crossing, there is a ceratain relation:



In total there are n relations.

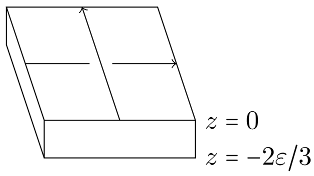
Theorem 2.189 $\pi_1(\mathbb{R}^3 \setminus K) = \langle x_1, \dots, x_m \mid r_1, \dots, r_n \rangle$.

Example 2.190 Trefoil

$\pi_1(\text{trefoil}) = \langle x_1, x_2, x_3 \mid x_3x_1 = x_2x_3,\ x_2x_3 = x_1x_2, x_1x_2 = x_3x_1 \rangle = \langle x_1, x_2 \mid x_1x_2x_1 = x_2x_1x_2 \rangle$.

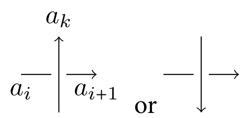
Remark 2.191 In example 3 of group presentations we proved that $\langle x_1, x_2 \mid x_1x_2x_1 = x_2x_1x_2\rangle = \langle \alpha, \beta \mid \alpha^3 = \beta^2\rangle$.

Proof (of theorem 2.189)
We will write f instead of $[f]$.
We will apply the Seifert-van Kampen theorem for $\mathbb{R}^3 \smallsetminus K$. Assume that K lies on the xy-plane except where it goes down by distance $\varepsilon > 0$, at each crossing. Set the basepoint at $(0,0,1)$.



Let $A = \{(x,y,z) \in \mathbb{R}^3 \mid z > -2\varepsilon/3\} - K$. Then $A \simeq_{\text{homeo}}$ open 3-dim Ball $\smallsetminus\{n$ unknotted arcs with endpoints on the boundery of the ball$\}$.

So $\pi_1(A) = \langle x_1, \dots, x_n\rangle$.
For each crossin



let Y_ℓ be an open rectangular box around it at level $-2\varepsilon < z < -\varepsilon/3$, and set $B_\ell = (Y_\ell \smallsetminus K) \cup \{$ open neighbourhood of an arc from a point on ∂Y_ℓ to $*$ $\}$.
Then $B_\ell \simeq_{\text{homeo}}$ open 3-dim ball $\smallsetminus\{$ one unknotted arc with endpoints on the boundary$\}$.
So $\pi_1(B_\ell) = \langle y_\ell\rangle$.
Finally C = open neighbourhood of $(\mathbb{R}^3 \smallsetminus A \smallsetminus \cup_{\ell=1}^n B_\ell) \cup ($ open neighbourhood of an arc to $*$ $)$.

$\mathbb{R}^3 \smallsetminus K = A \cup B_1 \cup \dots \cup B_\ell \cup C$.

$\pi_1(A \cup B_1) =?$.
$A \cap B_1 =?$.
$i_1 : \pi_1(A \cap B_1) \to \pi_1(A), c_1 \mapsto x_k x_i x_k^{-1}, b_1 \mapsto x_{i+1}$,
$i_2 : \pi_1(A \cap B_1) \to \pi_1(B_1), c_1 \mapsto y_1, c_1 \mapsto y_1$.
So we have $i_1(c_1)(i_2(c_1))^{-1} = x_k x_1 x_k^{-1} y_1^{-1}, i_1(b_1)(i_2(b_1))^{1-} = x_{i+1}y_1^{-1}$.
So $\pi_1(A \cup B_1) = \langle x_1, \dots, x_n, y_1 \mid x_{i+1}y_1^{-1}, x_k x_i x_k^{-1} y_1^{-1}\rangle = \langle x_1, \dots, x_n \mid x_k x_i x_k^{-1} x_{i+1}^{-1}\rangle$.
Repeat this process for each B_i to get $\pi_1(A \cup B_1 \cup \dots \cup B_n) = \langle x_1, \dots, x_n \mid r_1, \dots, r_n\rangle$.
Since C and $C \cap (A \cup B_1 \cup \dots \cup B_n)$ are simply connected, applying the Seifert-van Kampen theorem we get that $\pi_1(K) = \langle x_1, \dots, x_n \mid r_1, \dots, r_n\rangle$.

Study $\pi_1(\mathrm{S}^3 \smallsetminus K)$ $(\simeq \pi_1(\mathbb{R}^3 \smallsetminus K))$ to show that any one of the r_i can be ommited.

Let $A' = A \cup \{\infty\}$. $C' = C \cup B_n \cup \{\infty\}$. Then $\pi_1(A') \simeq \pi_1(A)$ and $\pi_1(A' \cup B_1 \cup \dots \cup B_{n-1}) = \langle x_1, \dots, x_n \mid r_1, \dots, r_{n-1}\rangle$.
$\mathrm{S}^3 \smallsetminus K = A' \cup B_1 \cup \dots \cup B_{n-1} \cup C'$.
Now we can compute that $\pi_1(C') = \langle y_n\rangle$ and also that $\pi_1(C' \cap (A' \cup B_1 \cup \dots \cup B_{n-1})) = \langle b_n\rangle$.

$\pi_1(C' \cap (A' \cup B_1 \cup \cdots \cup B_{n-1})) \to \pi_1(C'), b_n \mapsto y_n$.
$\pi_1(C' \cap (A' \cup B_1 \cup \cdots \cup B_{n-1})) \to \pi_1(A' \cup B_1 \cup \cdots \cup B_{n-1}), b_n \mapsto x_{i+1}$.

Applying the Seifert-van Kampen theorem we obtain that $\langle x_1, \dots, x_n \mid r_1, \dots, r_{n-1} \rangle) \pi_1(\mathrm{S}^3 \setminus K) = \pi_1(\mathbb{R}^3 \setminus K)$. ■

Remark 2.193 The above reasoning applies also to compute $\pi_1(\mathrm{S}^3 \setminus L)$ where L is a link and $\pi_1(\mathrm{S}^3 \setminus L) = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$.

Example 2.194

1.

2.

Theorem 2.195 *The group of a knot is not a complete knot invariant.*

2.12. Classification of covering spaces

$p : \widetilde{X} \to X$ covering map with $p(\widetilde{x}_0) = x_0$, $p_* : \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$.
We will study $H := p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \le \pi_1(X, x_0)$ to deduce several statements about covering maps.

Lemma 2.196 Let $p : \widetilde{X} \to X$ be a covering map with $p(\widetilde{x}_0) = x_0$. Then

1. $p_* : \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is a monomorphism.

2. The lifting correspondance $\phi : \pi_1(X, x_0) \to p^{-1}(x_0)$ where $[f] \mapsto \widetilde{f}(1)$ induces an injective map $\Phi : \pi_1(X, x_0) \,/\, H \to p^{-1}(x_0)$, where by $\pi_1(X, x_0) \,/\, H$ we denote all the right cosets of H in $\pi_1(X, x_0)$. If $\widetilde{X}$ is path connected, then Φ is bijective.

3. If $f : I \to X$ is a loop based at x_0, then $[f] \in H$ iff f lifts to a loop in $\widetilde{X}$ based at $\widetilde{x}_0$.

Proof

1. Immediate, it follows from homotopy lifting property.

2. If $f : I \to X$, $g : I \to X$ are loops based at x_0, then let $\widetilde{f} : I \to \widetilde{X}$, $\widetilde{g} : I \to \widetilde{X}$ be their lifts with $\widetilde{f}(0) = \widetilde{g}(0) = \widetilde{x}_0$. If $[f] \in H[g]$, then $[f] = [g]$, where $h \in H := p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ i.e. $h = p \circ \widetilde{h}$ for some loop $\widetilde{h} : I \to \widetilde{X}$ based at $\widetilde{x}_0$. Now $\widetilde{h} \cdot \widetilde{g}$ is defined and it is a lift of $h \cdot g$. Since $[f] = [h \cdot g]$, $\widetilde{f}(1) = \widetilde{h} \cdot \widetilde{g}(1)$.
$\Longrightarrow \widetilde{f}(1) = \widetilde{g}(1)$.
$\Longrightarrow \phi([f]) = \phi([g])$. That shows Φ is well-defined.
Check that Φ is injective: If $\phi([f] = \phi([g])$, then $\widetilde{f}(1) = \widetilde{g}(1)$ and we can define $\widetilde{f} \cdot \overline{\widetilde{g}}$ loop based at $\widetilde{x}_0$. Therefore $[(\widetilde{f} \cdot \overline{\widetilde{g}}) \cdot \widetilde{g}] = [\widetilde{f}]$. If $\widetilde{F}$ is a path homotopy between $(\widetilde{f} \cdot \overline{\widetilde{g}})$ and $\widetilde{f}$, then $F := p \circ \widetilde{F}$ is a path homotopy between $p((\widetilde{f} \cdot \overline{\widetilde{g}}) \cdot \widetilde{g})$ and $p(\widetilde{f})$. That is $[(p(\widetilde{f} \cdot \overline{\widetilde{g}})) \cdot g] = [f]$.
$\Longrightarrow [f] \in H[g]$.
$\Longrightarrow \Phi$ is injective.
If $\widetilde{X}$ is path connected, then ϕ is surjective. So Φ is also surjective.

3. Φ is injective, hence $\phi([f]) = \phi([g]) \iff [f] \in H[g]$. For g the constant loop based at x_0 this gives $[f] \in H \iff \phi([f]) = \widetilde{x}_0 \iff \widetilde{f}(1) = \widetilde{x}_0$. ∎

Lemma 2.198 Let X be path connected and locally path connected. If $\widetilde{X}_a$ is path component of $\widetilde{X}$, then $p_{|\widetilde{X}_a} : \widetilde{X}_a \to X$ is a covering map.

Proof $\widetilde{X}$ is locally homeomorphic to X, so $\widetilde{X}$ is also locally path connected. $\widetilde{X}_a$ is a path component of $\widetilde{X}$, so $\widetilde{X}_a$ is open. Covering maps are open maps, so $p(\widetilde{X}_a)$ is open. Claim: $p(\widetilde{X}_a)$ is closed.
Proof: Let $x \in \overline{p(\widetilde{X}_a)}$ and U be a path connected open neighbourhood of x such that $p^{-1}(U) = \cup_\alpha V_\alpha$, where V_α pairwise disjoint open, $p_{|V_\alpha} : V_\alpha \to U$ homemorphism $\forall \alpha$. Since U contains a point of $p(\widetilde{X}_a)$, $V_\alpha \cap \widetilde{X}_a \neq \varnothing$ for some α. Now V_α is pat hconnected, hence $V_\alpha \subseteq \widetilde{X}_\alpha$. Then $p(V_\alpha) = U \subseteq p(\widetilde{X}_a)$, i.e. $x \in p(\widetilde{X}_a)$. So $p(\widetilde{X}_a) = \overline{p(\widetilde{X}_a)}$. $p(\overline{X}_a) \subseteq X$ is both open and closed
$\implies p(\widetilde{X}_a) = X$
$\implies p_{|\widetilde{X}_a}$ is surjective.
Now let $x \in X$ and chose an open neighbourhood U of x as before. If $V_\alpha \cap \widetilde{X}_a \neq \varnothing$, then $V_\alpha \subseteq \widetilde{X}_a$. Therefore $(p_{|\widetilde{X}_a})^{-1}(U)$ is the union of those V_α's that intersect $\widetilde{X}_\alpha$. Each of these is ope with $p_{|V_\alpha} : V_\alpha \to U$ is a homeomorphism. ∎

$p : \overline{X} \to X$ covering map. We restrict to spaces X thate are locally path connected. Then $\overline{X}$ is also locally path connected. We can also assume that X is path connected. Lastly we assume, that $\widetilde{X}$ is also path connected. So, we can determine all covering spaces of a locally path connected space X by determining all path connected covering spaces of path components of X.
From onw on, whenever we write $p : \widetilde{X} \to X$ is a covering map, we imply that $X, \widetilde{X}$ are locally path connected, and path connected.
Lemma 2.200 (lifting lemma) Let $p : \widetilde{X} \to X$ covering map with $p(\widetilde{x}_0) = x_0$, and $f : Y \to X$ continuos. With Y path connected and locally path connected. $f(y_0) = x_0$. Then f can be lifted to a continous map $\widetilde{f} : Y \to \widetilde{X}$ with $f(y_0) = \widetilde{x}_0$ iff $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$. Also if such a lift exists, it is unique.
Proof

"⇒" If $\widetilde{f}$ exists, then $f_*(\pi_1(Y, y_0)) = (p \circ \widetilde{f})_*(\pi_1(Y, y_0)) = p_*(\widetilde{f}_*(\pi_1(Y, y_0))) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$.

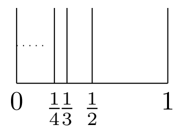
"⇐" Given $y_1 \in Y$, choose a path $\alpha : I \to Y$ with $\alpha(0) = y_0, \alpha(1) = y_1$. Lift $f \circ \alpha : I \to X$ to $\widetilde{f \circ \alpha} : I \to \widetilde{X}$ with $\widetilde{f \circ \alpha}(0) = \widetilde{x}_0$. Define $\widetilde{f} : Y \to \widetilde{X}$ by $\widetilde{f}(y_1) = \widetilde{f \circ \alpha}(1)$.

- $\widetilde{f}$ is well-defined:
  Let $\beta : I \to Y$ be a path with $\beta(0) = y_0$ and $\beta(1) = y_1$. Lift $f \circ \overline{\beta} : I \to X$ to a path $\widetilde{f \circ \overline{\beta}} : I \to \widetilde{X}$ with $\widetilde{f \circ \overline{\beta}}(0) = \widetilde{f \circ \alpha}(1)$. Then $\widetilde{f \circ \alpha} \cdot \widetilde{f \circ \overline{\beta}}$ is a lift of the loop $f \cdot (\alpha\overline{\beta})$. By assumption $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$. So $[f \circ (\alpha\overline{\beta})] \in p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$. Lemma 2.196 implies that its lift $\widetilde{f \circ \alpha} \cdot \widetilde{f \circ \overline{\beta}}$ is a loop based at $\widetilde{x}_0$. So $\widetilde{f \circ \alpha}(1) = \widetilde{f \circ \beta}(1)$.
- $\widetilde{f}$ is constinuous:
  Let $y_1 \in Y$ and N an open neighbourhood of $\widetilde{f}(y_1)$. We have to show that there is an open neighbourhood W of y_1 with $\widetilde{f}(W) \subseteq N$. Start by choosing a path connected open neighbourhood U of $f(y_1)$ such that $p^{-1}(U) = \cup_\alpha V_\alpha$, V_α disjoint open sets. $p_{|V_\alpha} : V_\alpha \to U$ homeomorhpism $\forall \alpha$. f is

continuous at y_1 and Y is locally path connected, so there is a path connected open neighbourhood W of y_1 with $f(W) \subseteq U$. We will show that $\widetilde{f}(W) \subseteq V_0$, where V_0 is the subset containing $f(y_0)$. Let $y \in W$, and choose a path $b : I \to W$ with $b(0) = y_1$ and $b(1) = y$. since f is well-defined $\widetilde{f}(y)$ can be obtained by taking the path $\alpha\beta$ from y_0 to y, lifting $f \circ (a\beta)$ to a path $\widetilde{f \circ (a\beta)} : I \to \widetilde{X}$ with $\widetilde{f \circ (a\beta)}(0) = \widetilde{X}_0$ and setting $\widetilde{f}(y) = \widetilde{f \circ (\alpha\beta)}(1)$. Now $\widetilde{f \circ \alpha}$ is a lift of $f \circ \beta$ with $\widetilde{f \circ \alpha}(0) = \widetilde{x}_0$. Since $f \circ \beta(I) \subseteq U$, the path $(p_{|V_0})^{-1} \circ f \circ \beta$ is a lift of $f \circ \beta$ with $((p_{|V_0})^{-1} \circ f \circ \beta)(0) = \widetilde{f}(y_1)$. Then $(\widetilde{f \circ \alpha})((p_{|V_0})^{-1} \circ f \circ \beta)$ is a lift of $f \circ (\alpha\beta)$ with $(\widetilde{f \circ \alpha})((p_{|V_0})^{-1} \circ f \circ \beta)(0) = \widetilde{x}_0$. But $(\widetilde{f \circ \alpha})((p_{|V_0})^{-1} \circ f \circ \beta)(1) = ((p_{|V_0})^{-1} \circ f \circ \beta)(1) \in V_0$, hence $\widetilde{f}(W) \subseteq V_0$.

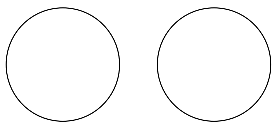
Let $y_1 \in Y$ and $\alpha : U \to Y$, with $\alpha(0) = y_0$, and $\alpha(1) = y_1$. Consider the path $f \circ \alpha : I \to X$ and lift it to a path $\widetilde{f \circ \alpha} : I \to \widetilde{X}$ with $\widetilde{f \circ \alpha}(v) = \widetilde{x}_0$. Then $\widetilde{y}_1 = \widetilde{f}(\alpha(1)$ and $\widetilde{f}(\alpha(1)) = \widetilde{f \circ \alpha}(1)$ since $\widetilde{f} \circ \alpha$ is a lift of $f \circ \alpha$ with $\widetilde{f}(\alpha(0)) = \widetilde{f}(y_0) = x_0$. So we see that if such an $\widetilde{f}$ exists, it is unique. ∎

Example 2.202 Space that is path connected, but not locally path connected.



And consider a point $(0, y)$, where $y > 0$.

Example 2.203 Space that is locally path connected, but not path connected.



Equivalent covering spaces

Definition 2.204 Let $p : \widetilde{X} \to X$, $p' : \widetilde{X}' \to X$ be covering maps, p, p' are called ***equivalent*** if there exists a homeomorphism $h : \widetilde{X} \to \widetilde{X}'$ with $p = p' \circ h$. h is called ***equivalence*** between the covering spaces.

Theorem 2.205 *Let $p : \widetilde{X} \to X$, $p' : \widetilde{X}' \to X$ be covering maps with $p(\widetilde{x}_0) = p'(\widetilde{x}'_0) = x_0$.*
There is an equivalence $h : \widetilde{X} \to \widetilde{X}'$ with $h(\widetilde{x}_0) = \widetilde{x}'_0 \iff p_*(\pi_1(\widetilde{X}, \widetilde{x}_0) = p'_*(\pi_1(\widetilde{X}', \widetilde{x}'_0))$. If such an h exists, then it is unique.

Proof

"⇒" h homeomorphism $\Longrightarrow h_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) = \pi_1(\widetilde{X}', \widetilde{x}'_0)$.
$\Longrightarrow p'_*(h_*(\pi_1(\widetilde{X}, \widetilde{x}_0))) = p'_*(\pi_1(\widetilde{X}', \widetilde{x}'_0))$.
$\Longrightarrow p_*(\pi_1(\widetilde{X}, \widetilde{x}'_0)) = p'_*(\pi_1(\widetilde{X}, \widetilde{x}'_0))$.

"$\Leftarrow$" – p' covering map, p continous, $\widetilde{X}$ path connected and locally path connected $p_*(\pi_1(\widetilde{X},\widetilde{x}_0)) = p'_*(\pi_1(\widetilde{X}',\widetilde{x}'_0))$.
$\implies \exists! h : \widetilde{X} \to \widetilde{X}'$ with $p' \circ h = p$ and $h(\widetilde{x}_0) = \widetilde{x}'_0$.

– p covering map, p' continuos, $\widetilde{X}'$ path connected and locally path conected $p'_*(\pi_1(\widetilde{X}',\widetilde{x}'_0)) = p_*(\pi_1(\widetilde{X},\widetilde{x}_0))$.
$\implies \exists! k : \widetilde{X}' \to \widetilde{X}$ continous with $p \circ k = p'$ and $k(\widetilde{x}'_0) = \widetilde{x}_0$.

– $k \circ h : \widetilde{X} \to \widetilde{X}$ is a lift of p since $p \circ k \circ h = p' \circ h = p$ iwth $k \circ h(\widetilde{x}_0) = \widetilde{x}_0$. $1_{\widetilde{X}} : \widetilde{X} \to \widetilde{X}$ is another such lifting, so by uniqueness of lifts $k \circ h = 1_{\widetilde{X}}$.

– Analogously, we can prove that $h \circ k = 1_{\widetilde{X}'}$. ■

What about equivalences $h : \widetilde{X} \to \widetilde{X}'$ that do not necessarily satisfy $h(\widetilde{x}_0) = \widetilde{x}'_0$?

Lemma 2.207 Let $p : \widetilde{X} \to X$ be a covering map and $\widetilde{x}_0, \widetilde{x}_1 \in p^{-1}(x_0)$ and $H_i := p_*(\pi_1(\widetilde{X},\widetilde{x}_i)), i \in \{0,1\}$.

1. If $\gamma : I \to \widetilde{X}$ is a path with $\gamma(0) = \widetilde{x}_0$ and $\gamma(1) = \widetilde{x}_1$, and $\alpha := p \circ \gamma$, then $[\alpha]H_1[\alpha]^{-1} = H_0$.
2. Conversly given $\widetilde{x}'_0$ and a subgroup $H \leq \pi_1(X, x_0)$ conjugate to H_0, there exists a basepoint $\widetilde{x}_1 \in p^{-1}(x_0)$ with $H_1 = H$.

Proof

1. "$\subseteq$" If $[h] \in H_1$, then $[h] = p_*([\widetilde{h}])$ for some $\widetilde{h} : I \to \widetilde{X}$ loop based at $\widetilde{x}_1$. Let $\widetilde{k} = (\gamma \cdot \widetilde{h}) \cdot \overline{\gamma}$. Then $p_*([\widetilde{k}]) = p_*([\gamma \cdot \widetilde{h} \cdot \overline{\gamma}]) = [\alpha \cdot h \cdot \overline{\alpha}] = [\alpha][h][\alpha]^{-1}$. That is $[\alpha][h][\alpha]^{-1} = p_*([\widetilde{k}]) \in p_*(\pi_1(\widetilde{X},\widetilde{x}_0)) = H_0$.

   "$\supseteq$" Apply the above for $\overline{\gamma}$ and $\overline{\alpha} = p \circ \overline{\gamma}$. We that that $[\overline{\alpha}]H_0[\overline{\alpha}]^{-1} \subseteq H_1$. $\implies$ $H_0 \subseteq [a]H_1[a]^{-1}$.

2. Let $\widetilde{x}_0 \in \widetilde{X}$ and $H \leq \pi_1(X, x_0)$ conjugate to H_0. Then $H_0 = [\alpha]H[\alpha]^{-1}$ for som e$\alpha : I \to X$ based at x_0. Thet let γ be the lift of α to $\widetilde{X}$ with $\gamma(0) = \widetilde{x}_0$. Let $\gamma(1) = \widetilde{x}_1$. Then by 1. $\implies H_0 = [\alpha]H_1[\alpha]^{-1}$. ■

Theorem 2.209 *Let $p : \widetilde{X} \to X$, $p' : \widetilde{X}' \to X$ be covering maps with $p(\widetilde{x}_0) = p'(\widetilde{x}'_0) = x_0$. p, p' are quivalent $\iff H_0 := p_*(\pi_1(\widetilde{X},\widetilde{x}_0))$ and $H'_0 := p'_*(\pi_1(\widetilde{X}',\widetilde{x}'_0))$ are conjugate.*

Proof

"$\Rightarrow$" If $h : \widetilde{X} \to \widetilde{X}'$ is an equivalence, let $\widetilde{x}'_1 = h(\widetilde{x}_0)$ and $H'_1 = p_*(\pi_1(\widetilde{X}',\widetilde{x}'_1))$. By the previous theroem $H_0 = H'_1$. And by the lemma H'_0, H'_1 are conjugate. $\implies$ H_0, H'_0 are conjugate.

"$\Leftarrow$" If H_0, H'_0 are conjugtae, then according to the lemma there exists $\widetilde{x_1}' \in \widetilde{X}'$ with $H'_1 = H_0$. Then by previous theorem there is an equivalence $h : \widetilde{X} \to \widetilde{X}'$ with $h(\widetilde{x}_0) = \widetilde{x}_1 0$. ■

Example 2.211 $X = S^1, x_0 \in S^1, \pi_1(S^1, x_0) \simeq \mathbb{Z}$.
Subgroups of $\mathbb{Z}$: $n\mathbb{Z}, n \in \mathbb{N}$.
Covering spaces of S^1: $p : \mathbb{R} \to S^1$ where $t \mapsto (\cos 2\pi t, \sin 2\pi t)$.
$p_*(\pi_1(\mathbb{R}))$ is trivial. That is this covering space of S^1 corresponds to the trivial subgroup

of $\pi_1(\mathrm{S}^1)$.
An other covering map we saw was $p_n : \mathrm{S}^1 \to \mathrm{S}^1$, $z \mapsto z^n$. Then $(p_n)_*(\pi_1(\mathrm{S}^1)) = n\mathbb{Z}$.
Question: Is there an a covering map $p : \widetilde{X} \to \mathrm{S}^1$, that is not equivalent to any of the above.
No, as $p_*(\pi_1(\widetilde{X})) \leq \pi_1(\mathrm{S}^1)$.

The universal covering space

Definition 2.212 Let $p : \widetilde{X} \to X$ be a covering map. If $\widetilde{X}$ is simply connected, then $\widetilde{X}$ is called a ***universal covering space*** of X.

Remark 2.213 $p_*(\pi_1(\widetilde{X}))$ trivial $\Longrightarrow$ any two universal covering spaces of X are equivalent. We can therefore speak of the universal covering space of X.

Lemma 2.214 Let p, q, r be continuous maps with $p = r \circ q$. If p, r are covering maps, then q is also a covering map.

Proof Let $x_0 \in X$, $z_0 \in p(x_0)$, $y_0 = q(x_0)$.
Claim 1: q is surjective.
Let $y \in Y$. Choose a path $\widetilde{\alpha} : I \to Y$ with $\widetilde{\alpha}(0) = y_0$ and $\widetilde{\alpha}(1) = y$. Then $\alpha = r \circ \widetilde{\alpha}$ is a path in Z iwth $\alpha(0) = r(\widetilde{\alpha}(0)) = r(y_0) = r(q(x_0)) = p(x_0) = z_0$. Let $\widetilde{\widetilde{\alpha}}$ be the lift of α to a path in X with $\widetilde{\widetilde{\alpha}}(0) = x_0$. Then $q \circ \widetilde{\widetilde{\alpha}}$ is a lift of α to Y with $q \circ \widetilde{\widetilde{\alpha}}(0) = y_0$. By uniqueness of path liftings $q \circ \widetilde{\widetilde{\alpha}} = \widetilde{\alpha}$. So $q \circ \widetilde{\widetilde{\alpha}}(1) = \widetilde{\alpha}(1) = y \Longrightarrow y = q(\widetilde{\widetilde{\alpha}}(1))$. $\Longrightarrow q$ is surjective.
Claim 2: Given $y \in Y$, there exists n open neighbourhood V of y with $q^{-1}(V) = \cup_\gamma W_\gamma$, where W_γ pairwise disjoint open sets with $q_{|W_\gamma} : W_\gamma \to V$ homeomorphism $\forall \gamma$. So, let $y \in Y$. Set $z := r(y)$. Since p, r are covering maps we can find an open path-connected neighbourhood of z such that $p^{-1}(U) = \cup_\alpha U_\alpha$, U_α disjoint open with $p_{|U_\alpha} : U_\alpha \to U$ homeomorphism $\forall \alpha$. $r^{-1}(U) = \cup_\beta V_\beta$, V_β open, $r_{|V_\beta} : V_\beta \to U$ homeomorphism $\forall \beta$.
Call V the member of the family V_β that contains the point y. $q(U_\alpha) \subseteq r^{-1}(U)$ $\forall \alpha$ and U_α is connected, hence $q(U_\alpha) \subseteq V_\beta$ for some β. Also $q^{-1}(V) \subseteq \cup_\alpha U_\alpha$. Indeed let $x \in q^{-1}(V)$ then $q(x) \in V \Longrightarrow r(q(x)) \in r(V) = U \Longrightarrow p(x) \in U \Longrightarrow x \in p^{-1}(U)$ $\Longrightarrow x \in \cup_\alpha U_\alpha$.
Therefore $q^{-1}(V)$ is the union of those U_α's with $q(U_\alpha) \subseteq V$.
In fact, $q_{|U_\alpha} : U_\alpha \to V$ is a homeomorphism for each such α.
Since $p_{|U_\alpha}, r_{|V}$ are homeomorphisms and $q_{|U_\alpha} = (r_{|V})^{-1} \cdot p_{|U_\alpha}$.
Call this family of U_α's W_γ. ∎

Theorem 2.216 *Let $p : \widetilde{X} \to X$ be a covering map, and $\widetilde{X}$ simply connected. If $r : Y \to X$ is a covering map. Then there exists a covering map $q : \widetilde{X} \to Y$ with $r \circ q = p$.*

Proof Let $x_0 \in X$. Choose $\widetilde{x}_0 \in \widetilde{X}$ and $g_0 \in Y$ with $p(\widetilde{x}_0) = x_0$, $r(y_0) = x_0$. $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \subseteq r_*(\pi_1(Y, y_0))$. So we can apply the lifting lemma for the covering map $r : Y \to X$. That is $\exists! q : \widetilde{X} \to Y$ continuous with $q(\widetilde{x}_0) = y_0$ and $r \circ q = p$. p, r are covering maps $\Longrightarrow q$ is covering map. ∎

Remark 2.218 This theorem justifies the use of the term the universal covering space.

Question: Does every space X have a universal covering space?
Answer: No.

Lemma 2.219 Let $p : \widetilde{X} \to X$ be a covering map with $p(\widetilde{x}_0)x_0$ and $\widetilde{X}$ is simply connected. Then x_0 has an open neighbourhood U such that the map i_* induced by the inclusion $i : (U, x_0) \hookrightarrow (X, x_0)$ is trivial.

Proof Let U be an open neighbourhood of x_0 with $p^{-1}(U) = \cup_\alpha U_\alpha$, U_α pairwise disjoint open sets, $p_{|U_\alpha} : U_\alpha \to U$ homeomorphism $\forall \alpha$. Let U_0 be the U_α, that contains $\widetilde{x}_0$. Consider $f : I \to U$ a loop based at x_0. And let $\widetilde{f} := (p_{|U_0})^{-1}(f)$ be the lift of f in $\widetilde{X}$. $\widetilde{f}$ is a loop based at $\widetilde{x}_0$. $\pi_1(\widetilde{X}, \widetilde{x}_0)$ trivial $\Longrightarrow \exists$ a path homotopy $\widetilde{F}$ between $\widetilde{f}$ and the constant loop at $\widetilde{x}_0$. Then $F = p \circ \widetilde{F}$ is a path homotopy in X between $p \circ \widetilde{f} = f$ and the constant path at $p(\widetilde{x}_0) = x_0$. That is, i_* is trivial. ∎

Example 2.221 $X = \cup_{n \in \mathbb{N}} C_n$ (Hawaiian earrings). Let $r_n : X \to C_n$ mapping every C_i with $i \neq n$ to $(0,0)$ with $r_{nC_n} = 1_{C_n}$.
Let U be an open neighbourhood of $x_0 \in X$. Choose n large enough such $C_n \subseteq U$. $j : C_n \hookrightarrow X, k : C_n \hookrightarrow U, i : U \hookrightarrow X$.
$r_n \circ j = 1_{C_n} \Longrightarrow (r_n)_* \circ j_* = 1_{\pi(C_n, x_0)} \Longrightarrow j_*$ is injective.
Now $j_* = i_* \circ k_*$, hence $i_* \circ k_* : \pi_1(C_n, x_0) \to \pi_1(X, x_0)$ is injective. I.e. i_* is not trivial.
Lemma $\Longrightarrow$ The shrinking of circles has no universal covering space.

Existence of covering spaces

Definition 2.222 X is called ***semilocally simply connected*** if $\forall x \in X$ $\exists$ open neighbourhood U of x with $i_* : \pi_1(U, x) \to \pi_1(X, x)$, induced by the inclusion $i : (U, x) \hookrightarrow (X, x)$, is trivial.

Remark 2.223

1. If U is an open neighbourhood of x with $i_* : \pi_1(U, x) \to \pi_1(X, x)$ trivial and V is an open neighbourhood of x with $V \subseteq U$, then obviously $j_* : \pi_1(V, x) \to \pi_1(X, x)$ is trivial.
2. If X is locally simply connected, i.e. $\forall x \in X$ and $\forall$ open neighbourhoods U of x there exists a simply connected open neighbourhood V of x with $V \subseteq U$, then X is semilocally simply connected.

Semilocal simply connectedness is in fact a necessary and sufficient condition for the corresopondance

$$\text{“(Covering maps of } X) / \sim \overset{1:1}{\to} \text{conjugacy classes of subgroups of } \pi_1(X, x_0)\text{”}$$

to be surjective.

Theorem 2.224 *Let X be path connected, locally path and semilocally simply connected. Let $x_0 \in X$ and $H \leq \pi_1(X, x_0)$. Then $\exists p : \widetilde{X} \to X$ covering map with $p_*(\pi_1(\widetilde{X}.\widetilde{x}_0)) = H$.*

Corollary 2.225 X has a universal covering space iff X is parth connected, locally path connected and semilocally simply connected.
Proof (of the theorem)
We do this in seven steps:
Step 1 Construction of $\widetilde{X}$:
Define $P := \{\gamma : I \to Y \text{ path with } \gamma(0) = x_0\}$. For $\alpha, \beta \in P$ define $a \sim \beta$ if $\alpha(1) = \beta(1)$ and $\alpha\overline{\beta} \in H$. Now define $\widetilde{X} := P \ / \sim = \{\gamma^\# \mid \gamma : I \to X \text{ path with } \gamma(0) = x_0\}$, and define $p : \widetilde{X} \to X$ by $p(\gamma^\#) = \gamma(1)$.
Note:

- p is surjective, since X is path connected.

We will define a topology on $\widetilde{X}$ so that $p : \widetilde{X} \to X$ is a covering map.
Remark 2.227 If $[\alpha] = [\beta]$, then $\alpha^{\#} = \beta^{\#}$. Furhter, if $\alpha^{\#} = \beta^{\#}$ and $\delta : I \to X$ with $\delta(0) = \alpha(1)$, then $(\alpha\delta)^{\#} = (\beta\delta)^{\#}$
Step 2 Let $\alpha \in P$ and U a path connected neighbourhood of $\alpha(1)$. Define $B(U, \alpha) := \{(\alpha\delta)^{\#} \mid \delta : I \to U \text{ path with } \delta(0) = \alpha(1)\}$. Then $\alpha^{\#} \in B(U, \alpha)$.
Claim: The sets $B(U, \alpha)$ form a basis for a topology on $\widetilde{X}$.
Proof: We first prove that if $\beta^{\#} \in B(U, \alpha)$, then $\alpha^{\#} \in B(U, \beta)$ and $B(U, \alpha) = B(U, \beta)$. For this consider $\beta^{\#} \in B(U, \alpha)$. Then $\beta^{\#} = (\alpha\delta)^{\#}$, for some $\delta : I \to U$ with $\delta(0) = \alpha(1)$ and $(\beta\overline{\delta})^{\#} = (\alpha\delta\overline{\delta})^{\#} = \alpha^{\#}$. So $\alpha^{\#} = (\beta\overline{\delta})^{\#} \in B(U, \beta)$. Not let $(\beta\gamma)^{\#} \in B(U, \beta)$. Then $(\beta\gamma)^{\#} = ((\alpha\delta)\gamma)^{\#} = (\alpha(\delta\gamma))^{\#} \in B(U, \alpha)$. That is $B(U, \beta) \subseteq B(U, \alpha)$, and by analogous argument $B(U, \alpha) = B(U, \beta)$.
Now we can prove that the sets $B(U, \alpha)$ form a basis for a topology on $\widetilde{X}$. Indeed:

- Let $\alpha^{\#} \in \widetilde{X}$. Then choose a path connected open neighbourhood U of $\alpha(1)$. Then $\alpha^{\#} \in B(U, \alpha)$.
- Let $\alpha^{\#} \in B(U, \alpha_1) \cap B(U, \alpha_2)$. Choose a path connected open neighbourhood V of $\alpha(1)$ with $V \subseteq U_1 \cap U_2$. Then $B(V, \alpha) \subseteq B(U_1, \alpha) \cap B(U_2, \alpha) = B(U_1, \alpha_1) \cap B(U_2, \alpha_2)$.

Therefore the sets $B(U, \alpha)$ form a basis for a topology on $\widetilde{X}$.
Step 3 p is open and continuous:
p is open: We will prove that $p(B(U, \alpha)) = U$, which implies p is open.

"$\supseteq$" Let $x \in U$ and choose a path $\delta : I \to U$ with $\delta(0) = x$, $\delta(1) = x$. Then $(\alpha\delta)^{\#} \in B(U, \alpha)$ and $p((\alpha\delta)^{\#}) = (\alpha\delta)(1) = \delta(1) = x$. I.e. $x \in p(B(U, \alpha))$, i.e. $U \subseteq p(B(U, \alpha))$.

"$\subseteq$" $p(B(U, \alpha)) \subseteq U$ (follows form the definition of p and $B(U, \alpha)$.

p is continuos: Let $\alpha^{\#} \in \widetilde{X}$ and W an open neighbourhood of $p(\alpha^{\#})$. Choose a path connected open neighbourhood U of $p(\alpha^{\#}) = \alpha(1)$ with $U \subseteq W$. Then $B(U, \alpha)$ is an open neighbourhood of $\alpha^{\#}$ in $\widetilde{X}$ with $p(B(U, \alpha)) = U \subseteq W$.
Step 4 $\forall x \in X$ $\exists$ an open neighbourhood U of x with $p^{-1}(U) = \cup_\alpha V_\alpha$ where V_α pair-wise disjoint open sets, and $p_{|V_\alpha} : V_\alpha \to U$ homemomorphism.
Let $x \in X$. Choose a path connected open neighbourhood U of X such that $i_* : \pi_1(U, x) \to \pi_1(X, x)$ is trivial.
Claim 1: $p^{-1}(U) = \cup_{\alpha : I \to X, \alpha(0) = x_0, \alpha(1) = x} B(U, \alpha)$.
Proof:

"$\supseteq$" $p(B(U, \alpha)) = U \Longrightarrow B(U, \alpha) \subseteq p^{-1}(U) \Longrightarrow \cup_{\alpha : I \to X, \ldots} B(U, \alpha) \subseteq p^{-1}(U)$.

"$\subseteq$" Let $\beta^{\#} \in p^{-1}(U)$. Then $p(\beta^{\#}) \in U$, i.e. $\beta(1) \in U$. Choose a path $\delta \in U$ from x to $\beta(1)$ and let $\alpha = \beta\overline{\delta}$. Then $[\beta] = [\alpha\delta] \Longrightarrow \beta^{\#} = (\alpha\delta)^{\#} \in B(U, \alpha)$. I.e. $p^{-1}(U) \subseteq \cup_{\alpha : I \to X, \ldots} B(U, \alpha)$.

Claim 2: Distinct sets $B(U, \alpha)$ are disjoint.
Proof: Suppose $\beta^{\#} \in B(U, \alpha) \cap B(U, \alpha_2)$. Then $B(U, \alpha_1) = B(U, \beta) = B(U, \alpha_2)$.
Claim 3: $p_{|B(U, \alpha)} : B(U, \alpha) \to U$ is bijective.
Proof: We have shown that $p(B(U, \alpha)) = U$, i.e. p is surjective. To check injectivity suppose $p((\alpha\delta_1)^{\#}) = p((\alpha\delta_2)^{\#})$. $\delta_i : I \to U$, $\delta_i(0) = \alpha(1)$, $i \in \{1, 2\}$. Then

$(\alpha\delta_1)(1) = (\alpha\delta_2)(1) \Longrightarrow \delta_1(1) = \delta_2(1)$. So we can define $\delta_1\overline{\delta}_2 : I \to U$ (loop based at $\delta_1(0) = \delta_2(0) = \overline{\delta}_2(1) = x$). Since $i_* : \pi_1(U, x) \to \pi_1(X, x)$ is trivial, there is a path homotopy in X between $\delta_1\overline{\delta}_2$ and the constant loop at x. $[\alpha\delta_1] = [\alpha\delta_2]$. Now $p_{|B(U,\alpha)}$ is bijective, continuous and open. $\Longrightarrow p_{|B(U,\alpha)}$ homeomorphism.

Step 5 Lifting a path in X to a path in $\widetilde{X}$.

Let e_0 be the equivalence class in $\widetilde{X}$ of the constant path at x_0. Then $p(e_0) = x_0$. Let $\alpha : I \to X$ path with $\alpha(0) = x_0$. We want to compute its lift to a path $\widetilde{\alpha} : I \to \widetilde{X}$ with $\widetilde{\alpha}(0) = e_0$ and show that $\widetilde{\alpha}(1) = \alpha^{\#}$. Given $c \in [0, 1]$, let $\alpha_c : I \to X$ the path defined by $\alpha_c(t) := \alpha(ct)$, $t \in I$. Define $\widetilde{\alpha} : I \to \widetilde{X}$ by $\widetilde{\alpha}(c) = (\alpha_c)^{\#}$. Then $p(\widetilde{\alpha}(c)) = (p(\alpha_c)^{\#}) = \alpha_c(1) = \alpha(c)$. $\Longrightarrow p \circ \widetilde{\alpha} = \alpha$ and $\widetilde{\alpha}(0) = (\alpha_0)^{\#} = e_0$, $\widetilde{\alpha}(1) = (\alpha_1)^{\#} = \alpha^{\#}$.

Claim: $\widetilde{\alpha}$ is continuous.

Proof: omitted.

Step 6 $p : \widetilde{X} \to X$ is convering map.

p is surjective (step 1), p staisfies the covering condition (step 4). X is path and locally path connected by assumption, $\widetilde{X}$ is path connectec (step 5). ($\widetilde{X}$ is locally path connected as X is).

Step 7 $H = p_*(\pi_1(\widetilde{X}, e_0))$:

Let $\alpha : I \to X$ be a loop based at x_0 and $\widetilde{\alpha} : I \to \widetilde{X}$ its lift with $\widetilde{\alpha}(0) = e_0$. Then $[\alpha] \in p_*(\pi_1(\widetilde{X}, e_0)) \Longleftrightarrow \widetilde{\alpha}(1) = \widetilde{\alpha}(0) = e_0 \Longleftrightarrow \alpha^{\#} = e_0$. $\Longleftrightarrow \alpha \sim$ constant path at $x_0 \Longleftrightarrow [\alpha\overline{c}_{x_0}] \in H \Longleftrightarrow [\alpha] \in H$. ∎

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