Surface waves

Wasilij Barsukow, mail@sturzhang.de
June 20, 2016

1 Equations

The dynamics of an interface defined as the zero set of $F(x,y,z,t)$ is obtained through the equation

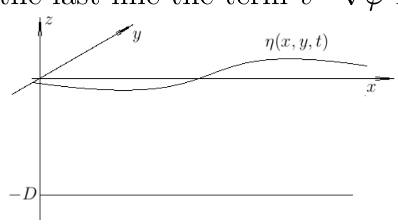
$$\left.\frac{DF}{Dt}\right|_{F=0} = 0 \tag{1}$$

This means that fluid particles are trapped at the boundary. Consider the case of water waves. The pressure can be taken as such a function F, as it is uniformly constant at the boundary and given by the outside (atmospheric) pressure whose variation with height is much less important. Taking the fluid flow to be potential, i.e. $\mathbf{v} = -\nabla\varphi$ and linearising, one obtains

$$\frac{p}{\rho} = \partial_t \varphi - gz \tag{2}$$

$$\frac{Dp}{Dt} = \partial_t p + v_z \partial_z p = \partial_t^2 \varphi + g\partial_z \varphi = 0 \tag{3}$$

In the last line the term $v \cdot \nabla\varphi$ has been negelected as higher order.



The very same equation can be obtained by arguing that the surface elevation $z = \eta(x,y,t)$ is given by

$$\eta = \frac{1}{g}\left.\partial_t\varphi\right|_{z=\eta} \simeq \frac{1}{g}\left.\partial_t\varphi\right|_{z=0} \tag{4}$$

(having set the constant p to zero at the position of the boundary). The last equality sign is true in the sense of a linearization around the state of $\eta = 0$. The condition of the surface speed being identical to the speed of the fluid at that position is simplified by assuming the main part of the motion to be vertical:

$$\left.\partial_t\eta\right|_{z=0} = v_z = -\left.\partial_z\varphi\right|_{z=0} \tag{5}$$

Eliminating η yields again

$$\left.(\partial_t^2 + g\partial_z)\varphi\right|_{z=0} = 0 \tag{6}$$

2 Dispersion relation

Together with the divergence-free condition $\Delta\varphi = 0$, one thus is left with a linear system. The solution ansatz is taken to be that of traveling waves in x-direction with a multiplicative factor associated to the depth:

$$\varphi(t,x,z) = P(z)\exp(ikx - i\omega t) \tag{7}$$

$$\Delta\varphi = (P''(z) - k^2 P(z))\exp(ikx - i\omega t) = 0 \tag{8}$$

such that the general solution is

$$P(z) = A\exp(kz) + B\exp(-kz) \tag{9}$$

with A, B some constants. In multiple dimensions k is to be replaced by $|\mathbf{k}|$.

The equations have to be augmented by a boundary condition at the bottom. The fluid shall have no vertical velocity at $z = -D$. This imitates a solid bottom with an equilibrium depth $D > 0$ of the fluid. Then

$$-v_z|_{z=-D} = \partial_z\varphi|_{z=-D} = P'(-D)\exp(\ldots) = k\Big(A\exp(-kD) - B\exp(kD)\Big)\exp(\ldots) = 0 \tag{10}$$

which allows to eliminate B:

$$\varphi(t,x,z) = A\Big(\exp(kz) + \exp(-2kD)\exp(-kz)\Big)\exp(\ldots) = 2A\exp(-kD)\cosh(kz + kD)\exp(\ldots) \tag{11}$$

Rename $\varphi_0 := 2A\exp(-kD)$:

$$\boxed{\varphi(t,x,z) = \varphi_0\cosh(kz + kD)\exp(ikx - i\omega t)} \tag{12}$$

Condition

$$(\partial_t^2 + g\partial_z)\varphi\Big|_{z=0} = 0 \tag{13}$$

finally can be evaluated as

$$\boxed{\omega^2 = gk\tanh(kD)} \tag{14}$$

Such a relation is called *dispersion relation*. It has, in function of the depth, two important limits:

$$\omega = \begin{cases} k\sqrt{gD} & D \to 0 \text{ (shallow water)} \\ \sqrt{gk} & D \to \infty \text{ (deep water)} \end{cases} \tag{15}$$

In the case of shallow water the waves are dispersion-free, i.e. they travel at the same *phase speed* $c := \frac{\omega}{k} = \sqrt{gD}$.

3 Surface and particle motion

Exercise:

Recall that the surface elevation $\eta(t,x)$ of a gravity wave is approximately given as

$$\eta = \frac{1}{g}\,\partial_t\varphi|_{z=0} \tag{16}$$

and the potential has been obtained as

$$\varphi(t,x,z) = \varphi_0 \cosh(kz+kD)\exp(ikx - i\omega t) \tag{17}$$

with the dispersion relation $\omega^2 = gk\tanh(kD)$.

i) Show that $\eta(t,x) = \eta_0 \exp(ikx - i\omega t)$ and determine η_0 as function of D and other constants. In contrast to φ_0, the amplitude η_0 of surface waves is directly measurable and will from now on be taken bounded in the limit $D \to \infty$.

ii) Compute the velocity $\begin{pmatrix} v_x \\ v_z \end{pmatrix}$ inside the fluid and keep only its real part. (Recall that, as the equations were linear, we had taken the imaginary part with us only for computational convenience.)

iii) Consider now a particle performing periodic motion at some depth z. Its position is given by $(\bar{x}, z+\bar{z})$ with $\bar{x}, \bar{z}$ considered as small quantities. Solve for the particle's path lines given by $\begin{pmatrix} v_x \\ v_z \end{pmatrix} = \begin{pmatrix} \dot{\bar{x}} \\ \dot{\bar{z}} \end{pmatrix}$ in the limit $D \to \infty$. Comment on the form of the path lines.

Solution:
Using

$$\eta = \frac{1}{g}\,\partial_t\varphi|_{z=0} \tag{18}$$

one obtains the equation for the surface elevation:

$$\eta = \frac{-i\omega}{g}\varphi_0 \cosh(kD)\exp(ikx - i\omega t) =: \eta_0 \exp(ikx - i\omega t) \tag{19}$$

with $\eta_0 = \frac{-i\omega}{g}\varphi_0\omega\cosh(kD)$. In the following we choose η_0 to be bounded even if $D \to \infty$. Then

$$\varphi_0 = \frac{i\eta_0 g}{\omega\cosh(kD)} \tag{20}$$

The velocity inside the fluid is

$$\begin{pmatrix} v_x \\ v_z \end{pmatrix} = -k\varphi_0 \begin{pmatrix} i\cosh(kz+kD) \\ \sinh(kz+kD) \end{pmatrix} \exp(ikx - i\omega t) \tag{21}$$

$$= -\eta_0 \frac{kg}{\omega\cosh(kD)} \begin{pmatrix} -\cosh(kz+kD) \\ i\sinh(kz+kD) \end{pmatrix} \exp(ikx - i\omega t) \tag{22}$$

Consider now a particle performing periodic motion at some depth z. Its position is given by $(\bar{x}, z+\bar{z})$ with $\bar{x}, \bar{z}$ considered as small quantities. Then its path lines fulfill to first order

$$\begin{pmatrix} \dot{\bar{x}} \\ \dot{\bar{z}} \end{pmatrix} = -\eta_0 \frac{kg}{\omega \cosh(kD)} \begin{pmatrix} -\cosh(kz+kD)\cos(\omega t) \\ \sinh(kz+kD)\sin(\omega t) \end{pmatrix} \tag{23}$$

where also now the real part has been taken. In deep water, $\cosh(kz+kD) \simeq \sinh(kz+kD) \simeq \frac{1}{2}\exp(kz+kD)$ such that

$$\begin{pmatrix} \dot{\bar{x}} \\ \dot{\bar{z}} \end{pmatrix} = -\eta_0 \frac{kg}{\omega} \begin{pmatrix} -\exp(kz)\cos(\omega t) \\ \exp(kz)\sin(\omega t) \end{pmatrix} \tag{24}$$

$$\begin{pmatrix} \bar{x} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}\Bigg|_{t=0} - \eta_0 \exp(kz) \begin{pmatrix} \sin(\omega t) \\ \cos(\omega t) \end{pmatrix} \tag{25}$$

This shows that the particles are moving along circles with radii decreasing exponentially downwards $z \to -\infty$. Particles at the surface ($z=0$) move on circles with radii equal to the amplitude of the wave.

4 Kelvin ship wake

4.1 Phase velocity

The location of the maximum of the wave $\exp(-i\omega t + i\mathbf{k}(\omega)\cdot\mathbf{x})$ is given by

$$-\omega t + \mathbf{k}(\omega)\cdot\mathbf{x}_{\text{max}} = 2\pi j \qquad j \in \mathbb{N} \tag{26}$$

If $\mathbf{x}_{\text{max}}$ is moving with velocity $\mathbf{v}_{\text{ph}}$ (*phase velocity*), then

$$\mathbf{x}_{\text{max}} = \text{const} + \mathbf{v}_{\text{ph}} t \tag{27}$$

$$\mathbf{v}_{\text{ph}} \cdot \mathbf{x}_{\text{max}} = \text{const} + |\mathbf{v}_{\text{ph}}|^2 t \tag{28}$$

such that, by comparison,

$$\frac{\mathbf{k}(\omega)}{\omega} = \frac{\mathbf{v}_{\text{ph}}}{|\mathbf{v}_{\text{ph}}|^2} \tag{29}$$

In 1-d, this simplifies to $\dfrac{\omega}{k(\omega)} = v_{\text{ph}}$.

4.2 Group velocity

Consider a superposition of waves of different frequencies:

$$\int \mathrm{d}\omega\, A(\omega)\, \exp(-i\omega t + i\mathbf{k}(\omega)\cdot\mathbf{x}) \tag{30}$$

Consider A to be of compact support around ω_0. Then, approximating $\mathbf{k}(\omega) = \mathbf{k}(\omega_0) + (\omega - \omega_0)\mathbf{s}$ to second order with $\mathbf{s} := \partial_w \mathbf{k}|_{\omega_0}$:

$$\int \mathrm{d}\omega\, A(\omega)\, \exp(-i\omega t + i\mathbf{k}(\omega)\cdot\mathbf{x}) = \int \mathrm{d}\omega\, A(\omega)\, \exp(-i\omega t + i\mathbf{k}(\omega_0)\cdot\mathbf{x} + i(\omega-\omega_0)\mathbf{s}\cdot\mathbf{x}) \quad (31)$$

$$= e^{i\mathbf{k}(\omega_0)\cdot\mathbf{x} - i\omega_0 t} \int \mathrm{d}\omega\, A(\omega)\, \exp(-i(\omega-\omega_0)(t - \mathbf{s}\cdot\mathbf{x})) \quad (32)$$

One thus has the wave with the central frequency ω_0 modulated by a function depending only on $t - \mathbf{s}\cdot\mathbf{x}$. The location of its peak is moving with speed $\mathbf{v}_\mathrm{g}$ (group velocity) and fulfills $t - \mathbf{s}\cdot\mathbf{x}_\mathrm{max} = \mathrm{const}$. Consider

$$\mathbf{x}_\mathrm{max} = \mathrm{const} + \mathbf{v}_\mathrm{g} t \quad (33)$$

$$\mathbf{v}_\mathrm{g}\cdot\mathbf{x}_\mathrm{max} = \mathrm{const} + |\mathbf{v}_\mathrm{g}|^2 t \quad (34)$$

$$(35)$$

Again by comparison:

$$\partial_w \mathbf{k} = \mathbf{s} = \frac{\mathbf{v}_\mathrm{g}}{|\mathbf{v}_\mathrm{g}|^2} \quad (36)$$

Again, in 1-d this simplifies to

$$\frac{\mathrm{d}\omega}{\mathrm{d}k} = v_\mathrm{g} \quad (37)$$

In multi-d the phase velocity and the group velocity need not be parallel.

4.3 Wake of a disturbance

Consider an object, whose scales shall be considered as negligible when compared to the length scales we will be interested in, which is moving at speed v in the x-direction and produces a sequence of waves behind it. (We will call this object a *ship*, though in reality, ships have finite size.) This wave system is traveling at the same speed v alongside with the ship and consists of a continuum of frequencies. Each elementary wave with frequency ω is described by

$$\exp(-i\omega t + ik_x x + ik_y y) \quad (38)$$

As the system is moving with the ship's speed, we have $v = \omega/k_x$ such that the elementary wave is given by

$$\exp(-i\omega(t - x/v) + ik_y y) \quad (39)$$

This ensures invariance under

$$\begin{cases} t \mapsto t + \tau \\ x \mapsto x + v\tau \end{cases} \quad (40)$$

Also, in deep water, $\omega = \sqrt{g|\mathbf{k}|}$, such that

$$\omega^2 = g\sqrt{k_x^2 + k_y^2} \tag{41}$$

$$\frac{\omega^4}{g^2} - \frac{\omega^2}{v^2} = k_y^2 \tag{42}$$

Waves will actually travel only if $\omega > \frac{g}{v}$. Otherwise the waves will be damped exponentially in time, and so they are not of interest for the moment. Define $\omega_0 := g/v$ and $\Omega := \omega/\omega_0$.

The system of waves behind a ship is due to the misalignment of the group velocity and the phase velocity. The observable wave crests travel at the phase speed, whereas as a whole they are moving at a different speed (the group speed), in a different direction. The group velocity is given by

$$\mathbf{k} = \begin{pmatrix} \omega/v \\ \sqrt{\frac{\omega^4}{g^2} - \frac{\omega^2}{v^2}} \end{pmatrix} = \frac{g\Omega}{v^2}\begin{pmatrix} 1 \\ \sqrt{\Omega^2 - 1} \end{pmatrix} \tag{43}$$

$$\frac{\mathbf{v}_\mathrm{g}}{|\mathbf{v}_\mathrm{g}|^2} = \partial_\omega \begin{pmatrix} \omega/v \\ \sqrt{\frac{\omega^4}{g^2} - \frac{\omega^2}{v^2}} \end{pmatrix} = \begin{pmatrix} 1/v \\ \left(2\frac{\omega^3}{g^2} - \frac{\omega}{v^2}\right)/\sqrt{\frac{\omega^4}{g^2} - \frac{\omega^2}{v^2}} \end{pmatrix} = \frac{1}{v}\begin{pmatrix} 1 \\ \frac{2\Omega^2 - 1}{\sqrt{\Omega^2 - 1}} \end{pmatrix} \tag{44}$$

Let us assume, that at time $t = 0$ at the location of the disturbance $x = y = 0$ there is a wave maximum. Then the other wave crests will be located at (x, y) fulfilling

$$k_x x + k_y y = 2\pi j \qquad j \in \mathbb{N} \tag{45}$$

They will travel to the right with speed v and with their phase velocity into some other direction.

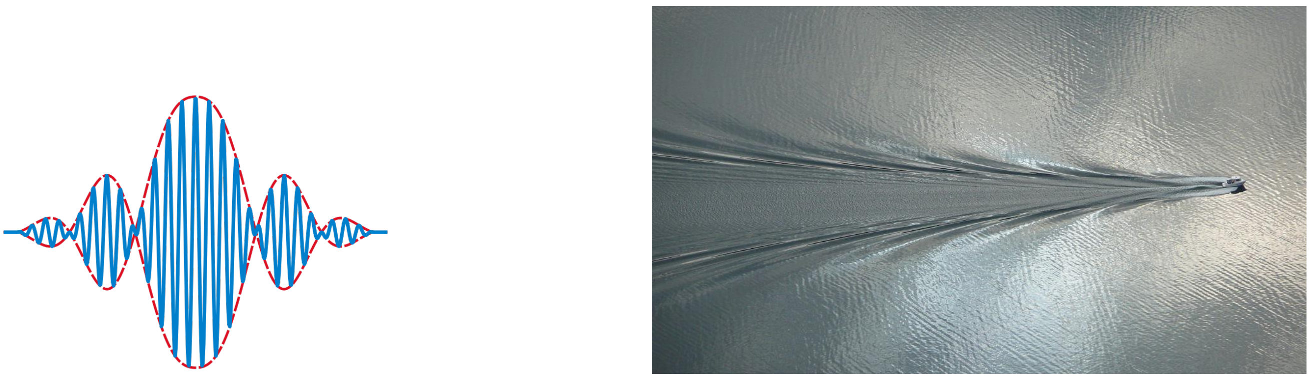


Figure 1: *Left*: The wave crests of the blue oscillation move at the phase speed, whereas the red envelope moves at the group speed. *Right*: The wake of a small boat. Photo by wiki user Edmont.

At any fixed instant of time, the wake of a ship will be a 2-dimensional, and thus a slightly more complicated version of the left image in Fig. 1. There will be short-scale wave crests, modulated by a long-scale function. The short-scale crests are moving at the phase speed, and the peak of the envelope moves at the group speed. In multi-d the situation is more complex because, additionally, they move in different directions.

In order to derive the wake of a ship one would need to know the excitation of the different frequencies. This however is too hard a task. If a given frequency is present one can however find the location of the maximal water elevation associated to it. Assuming that a certain broad continuum of frequencies is actually excited by the moving disturbance, one then, by superposition of the result, can hope to obtain a fair idea of the general picture.

The envelope of the group originates at the location of the ship. Assume the maximum of the wave package to be coinciding with the disturbance. Whether this is true depends on the structure of the ship, and therefore the analysis here cannot but assume idealized values which might seem unmotivated by reality. The maximum of the envelope of the group is then, at any given fixed time t, a straight line from the current position of the ship on each side, such that it is orthogonal to $\mathbf{v}_{\mathrm{g}}$. Along this line, the amplitude of the actual wave crests is maximal.

Exercise

The following analysis happens at a fixed time t.

i) Show that

$$|\mathbf{v}_{\mathrm{g}}| = \frac{v}{\Omega}\frac{\sqrt{\Omega^2-1}}{\sqrt{4\Omega^2-3}} \tag{46}$$

ii) The direction $\mathbf{n}$, $|\mathbf{n}| = 1$ along the maximum of the group's envelope is perpendicular to $\mathbf{v}_{\mathrm{g}}$. Calculate it and take it to point towards positive y.

iii) The wave crests are highest along the maximum of their group's envelope. The distance d between consecutive wave crests along the straight line pointing towards $\mathbf{n}$ from the instantaneous location of the ship is given by

$$\mathbf{k}\cdot\mathbf{n}d = 2\pi \tag{47}$$

Show that

$$d = 2\pi\frac{v^2}{g}\frac{\sqrt{4\Omega^2-3}}{\Omega^2} \tag{48}$$

Calculate the location $\begin{pmatrix} x \\ y \end{pmatrix} = dj\begin{pmatrix} n_x \\ \pm n_y \end{pmatrix}$ of the wave crests for a given Ω and by superposing the locations for different $\Omega \in [0,3] \subset \mathbb{R}$ make a plot of the wave system produced by the traveling disturbance (take $j \in [0,10] \subset \mathbb{N}$). Compare to the image above.

iv) Calculate the half-opening angle of the complete disturbance assuming all $\Omega > 1$ to be present and show it to be a universal constant.

Solution

The absolute value of the group velocity is given by

$$|\mathbf{v}_{\mathrm{g}}| = \frac{g}{\omega}\frac{\sqrt{\Omega^2-1}}{\sqrt{4\Omega^2-3}} \tag{49}$$

Then

$$\frac{\mathbf{v}_\mathrm{g}}{|\mathbf{v}_\mathrm{g}|} = \frac{1}{\Omega\sqrt{4\Omega^2-3}} \begin{pmatrix} \sqrt{\Omega^2-1} \\ 2\Omega^2-1 \end{pmatrix} \tag{50}$$

$$\mathbf{n} = \frac{1}{\Omega\sqrt{4\Omega^2-3}} \begin{pmatrix} 1-2\Omega^2 \\ \sqrt{\Omega^2-1} \end{pmatrix} \tag{51}$$

$$\mathbf{k}\cdot\mathbf{n} = \frac{1}{\Omega\sqrt{4\Omega^2-3}} \begin{pmatrix} 1-2\Omega^2 \\ \sqrt{\Omega^2-1} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \sqrt{\Omega^2-1} \end{pmatrix} \frac{g\Omega}{v^2} = -\frac{\Omega^2}{\sqrt{4\Omega^2-3}}\frac{g}{v^2} \tag{52}$$

The distance between consecutive wave crests along the line of maximum amplitude in the group is

$$d = \frac{2\pi}{|\mathbf{k}\cdot\mathbf{n}|} \tag{53}$$

Then the location of the wave crests is

$$\begin{pmatrix} x \\ y \end{pmatrix} = dj\mathbf{n} = \frac{2\pi j}{\Omega^3}\frac{v^2}{g} \begin{pmatrix} 1-2\Omega^2 \\ \sqrt{\Omega^2-1} \end{pmatrix} \tag{54}$$

With increasing values of ω the value of x experiences a change of sign:

$$\partial_\Omega x \propto \partial_\Omega \frac{1-2\Omega^2}{\Omega^3} \propto \Omega^3(-4\Omega) - (1-2\Omega^2)3\Omega^2 = 0 \tag{55}$$

$$\Omega^2 = \frac{3}{2} \tag{56}$$

This gives

$$\arctan\frac{y}{x} = -\arctan 2^{-3/2} \simeq -19.47^\circ \tag{57}$$

Photo by wiki user Arpingstone.

5 Literature

- Lamb, Hydrodynamics
- Howard Georgi, "The Physics of Waves", Chapter 14
  `http://www.people.fas.harvard.edu/~hgeorgi/onenew.pdf`