Stationarity preserving schemes for the linearized Euler equations in multiple spatial dimensions

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Euler equations

$$\partial_t \rho + \nabla\cdot(\rho\mathbf{v}) = 0$$
$$\partial_t(\rho\mathbf{v}) + \nabla\cdot\left(\rho\mathbf{v}\otimes\mathbf{v} + \frac{p}{\epsilon^2}\cdot\mathbb{1}\right) = 0$$
$$\partial_t e + \nabla\cdot(\mathbf{v}(e+p)) = 0$$

$\Big\downarrow \epsilon\to 0$

incompressible Euler equations

Acoustic equations

$$\xrightarrow{\text{Linearization}} \quad \partial_t\mathbf{v} + \frac{\nabla p}{\epsilon^2} = 0$$
$$\partial_t p + c^2\nabla\cdot\mathbf{v} = 0$$

$\Big\downarrow \epsilon\to 0$

$\nabla p = 0$ and $\nabla\cdot\mathbf{v} = 0$

Summary

The linearized Euler equations possess a low Mach number limit. This serves as a guideline to the nonlinear case by the following result:

The ability of a numerical scheme to resolve the low Mach number limit is equivalent to preserving a discrete vorticity.

This allows to characterize all low Mach compliant schemes.

The acoustic equations are an example of a

Linear hyperbolic system in multi-d

$$\partial_t q + \mathbf{J}\cdot\nabla q = 0$$

Fourier transform: $q(t,\mathbf{x}) = \hat{q}(t)\exp(\mathbb{i}\mathbf{k}\cdot\mathbf{x})$

$$\Rightarrow \partial_t\hat{q} + \mathbb{i}(\mathbf{J}\cdot\mathbf{k})\hat{q} = 0$$

Continuous statements

Stationary states of linear acoustics:

$$\nabla p = 0 \qquad \nabla\cdot\mathbf{v} = 0$$

Stationary states

- $\det(\mathbf{J}\cdot\mathbf{k}) = 0 \Rightarrow$ stationary states exist
- their Fourier transform is the **right eigenvector** with eigenvalue 0
- If $\det(\mathbf{J}\cdot\mathbf{k}) = 0$ independently of $\mathbf{k}$, then the stationary states are called **nontrivial**.

Vorticity: $\omega = \nabla\times\mathbf{v}$. For linear acoustics,

$$\partial_t\omega = 0$$

Constant of motion

- $\det(\mathbf{J}\cdot\mathbf{k}) = 0 \Rightarrow$ constant of motion exists

Indeed, consider a **left eigenvector** Ω of $\mathbf{J}\cdot\mathbf{k}$ with eigenvalue 0, i.e.

$$\Omega(\mathbf{J}\cdot\mathbf{k}) = 0$$

From $\partial_t\hat{q} + \mathbb{i}(\mathbf{J}\cdot\mathbf{k})\hat{q} = 0$ follows $\partial_t(\Omega\hat{q}) = 0$.

Given initial data $q(0,\mathbf{x}) = \hat{q}\exp(\mathbb{i}\mathbf{k}\cdot\mathbf{x})$ there exist $\hat{q}^0, \hat{q}^\pm$ such that

$$q(t,\mathbf{x}) = \hat{q}^0 + \hat{q}^\pm\exp\left(\mp\mathbb{i}\frac{c|\mathbf{k}|t}{\epsilon} + \mathbb{i}\mathbf{k}\cdot\mathbf{x}\right)$$

By reinterpreting $\frac{c}{\epsilon}\cdot t$ as $c\cdot\frac{t}{\epsilon}$ the **low Mach number limit** is equivalent to the **limit of long times**.

Stationarity preserving $\Leftrightarrow$ **Vorticity preserving** $\Leftrightarrow$ **Low Mach compliant**

In the discrete case, the **evolution matrix** $\mathcal{E}$ (see Box below) plays the role of $\mathbf{J}\cdot\mathbf{k}$:

Discrete stationary states

- $\det\mathcal{E} = 0$ independently of $\mathbf{k} \Rightarrow$ **nontrivial discrete stationary states** exist: **stationarity preserving scheme**.
- their Fourier transform is the **right eigenvector** with eigenvalue 0.
- all the analytic stationary states are captured[a] in the discrete. (Both trivial and nontrivial!)

[a] More precisely: The discrete stationary states are discretizations of all the analytic stationary states, not just of the subset of trivial ones.

Numerical constant of motion

- $\det\mathcal{E} = 0 \Rightarrow$ numerical constant of motion exists.
- its Fourier transform is the **left eigenvector** with eigenvalue 0.

For linear acoustics such schemes are called **vorticity preserving**.

Vorticity preserving schemes for linear acoustics have been studied among others in Morton & Roe 2001, Jeltsch & Torrilhon 2006, Mishra & Tadmor 2011.

- Limit of long time also in the discrete.
- **von Neumann stability**: non-stationary modes decay in time
- Thus all the limit states for $\epsilon\to 0$ are captured in the discrete: **low Mach compliant scheme**.

Discrete statements

Evolution matrix

- rectangular grid with spacing Δx_m in m-th spatial direction, $m = 1,\dots,d$
- Each cell has a unique index $I\in\mathbb{Z}^d$, q_I is the value of q in cell I.
- k_m and I_m are the components of $\mathbf{k}$ and I in m-th direction

Fourier ansatz: $q_I = \hat{q}\exp\left(\mathbb{i}\sum_{m=1}^d I_m k_m\Delta x_m\right) = \hat{q}\prod_{m=1}^d t_m^{I_m}$ with the translation factor $t_m := \exp(\mathbb{i}k_m\Delta x_m)$.

The semidiscrete scheme $\partial_t q_I + \sum_{S\in[-N,N]^d\subset\mathbb{Z}^d}\alpha_S q_{I+S} = 0$ upon the Fourier transform amounts to $\partial_t\hat{q} + \mathbb{i}\mathcal{E}\hat{q} = 0$ with the time evolution governed by the **evolution matrix** $\mathcal{E} = -\sum_{S\in[-N,N]^d}\mathbb{i}\alpha_S\prod_{m=1}^d t_m^{S_m}$.

References:

Wasilij Barsukow: *Stationarity preserving schemes for multi-dimensional linear systems*, 2017, preprint `https://www.mathematik.uni-wuerzburg.de/~barsukow`

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