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BROWNIAN MOTION AND A SHARPLY CURVED BOUNDARY

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Abstract

Daniels (1974) reduced the problem of approximating the distribution of the maximum size of a closed epidemic to that of finding the distribution of $\max_{0\leq t\leq 2}\{W(t)-N^{\frac{1}{2}}c(t)\}$, where c is a smooth function with a unique minimum of 0 at $t=1$, and he derived an approximation to this distribution which he showed to be accurate to order $N^{-\frac{1}{4}}$. In this paper, his approximation is shown to be accurate to order $N^{-\frac{1}{3}}$, and a refined approximation is given which is accurate to order $N^{-\frac{1}{2}}\log N$. The new approximation is still normal, and its accuracy is similar to that of the original approximation of a discrete process by the Wiener process.

CLOSED EPIDEMIC; CURVED BOUNDARY; BROWNIAN MOTION

1. Introduction

Daniels (1974) considered the problem of finding a useful approximation to the distribution of the maximum number of infectives in a closed epidemic, as the initial number ξ of susceptibles and the removal rate ρ became large. In such circumstances, the behaviour of the epidemic process is dominated by the contribution of order N from the deterministic drift, where $N=\rho\{\log(\xi/\rho)+1-\xi/\rho\}$, and a first approximation to the maximum of the epidemic is that of the corresponding deterministic curve. However, even for numbers of susceptibles of the order of 1000, the stochastic variation about the maximum is appreciable, and Daniels found it necessary to proceed to the next approximation to the epidemic process, a diffusion of order $N^{\frac{1}{2}}$ around the deterministic path. Using the results of Nagaev and Startsev (1970), he was able to standardize the diffusion representing the number of infectives to a Wiener process by appropriate normalizations and a change of time, and he found that, at this level of approximation, the problem reduced to that of finding the distribution of

$$Q=\max_{0\leq t\leq 2}\{W(t)-N^{\frac{1}{2}}c(t)\},$$

where c was a smooth function with a unique minimum of 0 at $t=1$. $W(1)$

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clearly provides a first estimate of Q, but $W(1) \leqq Q$ always, and Daniels was able to correct for this bias by obtaining the next approximation, showing that $Q \sim \mathcal{N}(\lambda[c''(1)]^{-\frac{1}{3}}N^{-\frac{1}{6}}, 1)$ with error of order at most $N^{-\frac{1}{4}}$, where $\lambda \simeq 0{\cdot}996$ is a constant independent of the choice of the function c. Barbour (1975) gave a rather shorter proof of this result, and showed that the error in the approximation was at most of order $N^{-\frac{1}{3}}$.

The approximation actually appeared to be more accurately centred than an error of order $N^{-\frac{1}{3}}$ would suggest, but the estimated standard deviation, although close to the values obtained by exact computation for the epidemic, did not fit quite so precisely. In this paper, the reasons for this are made clear. It is shown that there is indeed a correction of order $N^{-\frac{1}{3}}$ to be made, and that the correction improves the estimate of the spread of the distribution while leaving the mean unchanged. It is also shown thay any further errors are no larger than the error of order $N^{-\frac{1}{2}} \log N$ that must be expected to arise from making a diffusion approximation.

2. An improved approximation

The derivation of the next approximation depends upon a number of subsidiary estimates, which are set out in a series of lemmas. Let W denote a standard Brownian motion, and define $Z(t) = W(t) - \frac{1}{2}t^2$; let $F_y(t) = P[\max_{0 \leqq s \leqq t} Z(s) \geqq y]$, and let $U(t)$ denote the earliest time at which Z attains its maximum value on the interval $[0, t]$. Henceforth c is used to denote a generic constant, not necessarily the same at each appearance.

Lemma 1. The following estimates may be made:

(i) $F_y(\infty) \leqq c \exp\{-\frac{7}{8}y^{\frac{3}{2}}\}$,

(ii) $F_y(\infty) - F_y(t) \leqq c \min\{\exp\{-\frac{7}{8}y^{\frac{3}{2}}\}, \exp\{-\frac{1}{8}t^3\}\}$;

(iii) $P[U(\infty) > t] = P[U(\infty) \neq U(t)] \leqq c \exp\{-\frac{1}{8}t^3\}$.

Proof. Since, for each $z > 0$, $s^2/2 \geqq \max\{0, sz - z^2/2\}$, it follows that

$$\begin{aligned} F_y(\infty) &\leqq P\left[\max_{0 \leqq s \leqq z} W(s) \geqq y\right] + P\left[\max_{z \leqq s < \infty} \left\{W(s) - zs + \frac{z^2}{2}\right\} \geqq y\right] \\ &= 2P[\mathcal{N}(0, 1) \geqq yz^{-\frac{1}{2}}] + P[\mathcal{N}(0, 1) \geqq z^{-\frac{1}{2}}(y + z^2/2)] \\ &\quad + \int_{-\infty}^{y + \frac{1}{2}z^2} (2\pi z)^{-\frac{1}{2}} \exp\left\{-\frac{w^2}{2z} - 2z\left(y + \frac{z^2}{2} - w\right)\right\} dw \\ &\leqq 3(z/2\pi y^2)^{\frac{1}{2}} \exp\{-y^2/2z\} + \exp\{z^3 - 2zy\}. \end{aligned}$$

Estimate (i) follows by taking $z = 4y^{\frac{1}{2}}/7$.

In a similar way,

$$P\left[\max_{s\geqq t} Z(s)\geqq 0\right]\leqq P\left[\max_{s\geqq t}\{W(s)-ts+\tfrac{1}{2}t^2\}\geqq 0\right]$$
$$\leqq P[\mathcal{N}(0,1)\geqq \tfrac{1}{2}t^{\frac{3}{2}}]+\exp\{t^3\}P[\mathcal{N}(2t^2,t)\leqq \tfrac{1}{2}t^2]$$
$$\leqq c\exp\{-t^3/8\}.$$

Since both $F_y(\infty)-F_y(t)$ and $P[U(\infty)>t]$ are no larger than $P[\max_{s\geqq t} Z(s)\geqq 0]$, estimates (ii) and (iii) are now established.

Lemma 2. For any $b\leqq 1$ and $T>3$,

$$F_y(T+1)-F_{y+b}(T+1)=P\left[y\leqq \max_{0\leqq s\leqq T+1} Z(s)\leqq y+b\right]$$
$$\leqq cb(1+y^{\frac{1}{2}})e^{-7y^{\frac{3}{2}}/8}+e^{-Ty/2}.$$

Proof. For any $k\leqq 1$,

$$P\left[\max_{t\leqq s\leqq t+k} Z(s)>y+b\mid Z(t)=y\right]$$
$$\geqq P\left[\max_{t\leqq s\leqq t+k}\{W(s)-\tfrac{1}{2}t^2-(s-t)(t+1)\}>y+b\mid W(t)=y+\tfrac{1}{2}t^2\right]$$
$$=P\left[\max_{0\leqq u\leqq k}\{W(u)-u(t+1)\}>b\right]$$
$$=1-P[|\mathcal{N}(k(t+1),k)|\leqq b]-(1-\exp\{-2(t+1)b\})P[\mathcal{N}(k(t+1),k)\geqq b].$$

The event

$$\left\{y\leqq \max_{0\leqq s\leqq T+1} Z(s)\leqq y+b\right\}$$

is contained in the event

$$\left\{\max_{0\leqq s\leqq T+1} Z(s)\geqq y\right\}\cap\left\{\max_{T_y\leqq s\leqq \min(T_y,T)+1} Z(s)\leqq y+b\right\},$$

where T_y denotes the time at which Z first reaches y. Hence it follows that

$$P\left[y\leqq \max_{0\leqq s\leqq T+1} Z(s)\leqq y+b\right]\leqq \int_0^T dF_y(s)\{b(2/\pi)^{\frac{1}{2}}+2b(s+1)\}$$
$$+\int_T^{T+1} dF_y(s)\{b(2/\pi(T+1-s))^{\frac{1}{2}}+2b(s+1)\}$$
$$\leqq 2b\left\{\int_0^\infty((2\pi)^{-\frac{1}{2}}+s+1)\,dF_y(s)\right.$$
$$\left.+\int_T^{T+1}(2\pi(T+1-s))^{-\frac{1}{2}}\,dF_y(s)\right\}.$$

For the first integral, integrate by parts to get

$$2b\left\{[(1+s+(2\pi)^{-\frac{1}{2}})(F_y(s)-F_y(\infty))]_0^\infty-\int_0^\infty (F_y(s)-F_y(\infty))\,ds\right\},$$

which, by Lemma 1(ii), is no larger than

$$2b\left\{(1+(2\pi)^{-\frac{1}{2}})F_y(\infty)+c\int_0^{\bar{y}} \exp\left(-\tfrac{7}{8}y^{\frac{3}{2}}\right) ds+c\int_{\bar{y}}^\infty \exp\left(-s^3/8\right) ds\right\},$$

where $\bar{y}=\max(2y^{\frac{1}{2}}, 1)$. The first term in braces is, by Lemma 1(i), no larger than $c\exp(-\frac{7}{8}y^{\frac{3}{2}})$; the second is no larger than $c(1+y^{\frac{1}{2}})\exp(-\frac{7}{8}y^{\frac{3}{2}})$; and the third is no larger than

$$c\int_0^\infty \exp\{-\tfrac{1}{8}(\bar{y}^3+3\bar{y}^2u)\}\,du=c\exp(-\tfrac{1}{8}\bar{y}^3)\bar{y}^{-2}\leqq c\exp(-\tfrac{7}{8}y^{\frac{3}{2}}),$$

so that the whole of the first integral contributes at most $cb(1+y^{\frac{1}{2}})\exp\{-\frac{7}{8}y^{\frac{3}{2}}\}$. For the second integral, note that if Z crosses the level y for the first time in the interval $[s, s+h]$, then W must also cross the level $y+s^2/2$ for the first time during the same interval, so that

$$dF_y(s)\leqq ds(2\pi s^3)^{-\frac{1}{2}}(y+s^2/2)\exp\{-(y+s^2/2)^2/2s\}.$$

Hence the second term is no greater than

$$\frac{2T^{-\frac{3}{2}}}{\pi}\max(2y, T^2)\exp\left\{-\frac{y^2}{2(T+1)}--\frac{T^3}{8}-\frac{Ty}{2}\right\},$$

and the lemma now follows.

Now, for any $N>0$ and for any w such that $|w|\leqq\frac{1}{6}N^{\frac{1}{6}}$, define $Z_{Nw}(s)$ to be $Z(s)-N^{-\frac{1}{6}}ws$, and let

$$\Delta_{Nw}(t, y)=P\left[\max_{0\leqq s\leqq t} Z_{Nw}(s)\geqq y\right]-P\left[\max_{0\leqq s\leqq t} Z(s)\geqq y\right].$$

Lemma 3.

$$|\Delta_{Nw}(t, y)+wN^{-\frac{1}{6}}k(y)|\leqq N^{-\frac{1}{3}}w^2h(y)+cN^{-1},$$

where

$$k(y)=\int_0^\infty (y+s^2/2)\,dF_y(s);\ h(y)=c(1+y^3)e^{-\frac{7}{16}y^{\frac{3}{2}}};$$

and where the estimate is uniform in $t\geqq 3(\log N)^{\frac{1}{3}}$ and in $|w|\leqq\frac{1}{6}N^{\frac{1}{6}}$.

Proof. To simplify the notation, write $d=N^{-\frac{1}{6}}w$, $\Delta=\Delta_{Nw}$ and $m(d, s, y)=d(y+\frac{1}{2}s^2)+\frac{1}{2}d^2s$. Then, using the Radon–Nikodym derivative of the measure of Brownian motion with drift $-d$ with respect to that of X on the interval

$[0, T]$, it follows that

$$\Delta(t, y) = \int_0^t dF_y(s) E[\exp\{-dW(t) - \tfrac{1}{2}d^2 t\} - 1 \mid A_s],$$

where A_s is the event that $s = \inf\{t\colon W(t) \geqq y + \frac{1}{2}t^2\}$, and so that

$$\Delta(t, y) = \int_0^t dF_y(s)[\exp\{-m(d, s, y)\} - 1].$$

Hence

$$\begin{aligned} &\left|\Delta(t, y) + \int_0^t dF_y(s) m(d, s, y)\right| \\ &\leqq \int_0^\infty dF_y(s) \tfrac{1}{2} m^2(d, s, y)(1 + \exp\{-m(d, s, y)\}). \end{aligned} \tag{1}$$

For $-\frac{1}{6} < d < 0$, the integrand may be replaced by

$$d^2(y + \tfrac{1}{2}s^2 + \tfrac{1}{12}s)^2 \exp\{\tfrac{1}{12}(y + \tfrac{1}{2}s^2)\},$$

and for $0 \leqq d < \frac{1}{6}$ by the same expression but without the exponential factor. The integral on the right-hand side is then estimated by integrating by parts and applying Lemma 1(ii), much as in the proof of Lemma 2. In either case, it follows easily that the expressions in (1) are no larger than $d^2 h_1(y)$, where $h_1(y) = c(1 + y^3)\exp\{-\frac{7}{16}y^{\frac{3}{2}}\}$. Thus

$$|\Delta(t, y) + dk(y)| \leqq d^2 h_1(y) + |d| \int_t^\infty dF_y(s)(y + s^2/2) + \tfrac{1}{2} d^2 \int_0^\infty s\, dF_y(s).$$

The last term is easily estimated, once again integrating by parts and using Lemma 1(ii), to be less than $d^2 h_2(y)$, where again $h_2(y) = c(1 + y^3)\exp\{-\frac{7}{16}y^{\frac{3}{2}}\}$. The remaining term is bounded above in similar fashion by $c\exp\{-\frac{7}{64}t^3\}$, which, for the range of t considered, does not exceed cN^{-1}.

The next lemma establishes some useful properties of the function k.

Lemma 4.

(i) $k(y) \leqq cy \exp\{-\frac{7}{8}y^{\frac{3}{2}}\}$ in $y \geqq 1$;

(ii) $|k(y + b) - k(y)| \leqq cb\left\{\left(\log\frac{1}{b}\right)^{\frac{2}{3}} + y\right\}\{1 + y^{\frac{1}{2}}\}\exp\{-\frac{7}{8}y^{\frac{3}{2}}\}$ in $b \leqq 1$.

Proof. For Part (i), integrate by parts and use Lemma 1, as in the proof of Lemma 2. For Part (ii), integration by parts gives

$$k(y + b) - k(y) = -y\{F_y(\infty) - F_{y+b}(\infty)\} + \int_0^\infty s\{F_{y+b}(\infty) - F_y(\infty) - F_{y+b}(s) + F_y(s)\}\, ds.$$

From Lemma 2, the integrand is bounded by

$$csb\{(1+y^{\frac{1}{2}})\exp\{-\tfrac{7}{8}y^{\frac{3}{2}}\}+\exp\{-(s-1)y/2\}\};$$

on the other hand, a simple probabilistic argument shows that it cannot exceed $sP[U(\infty)\geqq s]$. Hence, performing the integration using the first estimate in the range $s\leqq 7^{\frac{1}{3}}y^{\frac{1}{2}}+2(\log b^{-1})^{\frac{1}{3}}$ and that of Lemma 1(iii) for all larger values of s, the lemma is established.

With Z_{Nw} and Z as already specified, define

$$Y_{Nw}(s)=Z_{Nw}(s)(1+sN^{-\frac{1}{3}})^{-1}+N^{-\frac{1}{3}}g_N(s)$$

and

$$Y_N(s)=Z(s)+N^{-\frac{1}{3}}g_N(s),$$

where g_N is any function satisfying $(1+sN^{-\frac{1}{3}})\,|g_N(s)|\leqq Gs^3$ in $s\geqq 0$, and where $G>0$. Let $U_{Nw}(t)$, $V_{Nw}(t)$ and $V_N(t)$ be, respectively, the times at which the processes Z_{Nw}, Y_{Nw} and Y_N first attain their maxima in the interval $[0, t]$.

Lemma 5. If $|w|\leqq\frac{1}{6}N^{\frac{1}{6}}$,

(i) $P[U_{Nw}(\infty)>t]=P[U_{Nw}(\infty)\neq U_{Nw}(t)]\leqq c\exp\{-t^3/12\}$, $t\geqq 2$;

(ii) $P[V_{Nw}(T)\geqq t]\leqq c\exp\{-t^3/24\}$, $t\geqq 2$, $T\leqq N^{\frac{1}{3}}/12G$;

(iii) $P[V_N(T)\geqq t]\leqq c\exp\{-t^3/12\}$, $T\leqq N^{\frac{1}{3}}/12G$.

Proof. Again, for simplification, write $d=N^{-\frac{1}{6}}w$ and $D=N^{-\frac{1}{3}}G$.

The first estimate is derived by observing that

$$\begin{aligned}
P[U_{Nw}(\infty)\geqq t]&\leqq P\Big[\max_{s\geqq t} Z_{Nw}(s)\geqq 0\Big]\\
&\leqq P[\mathcal{N}(0,1)\geqq\tfrac{1}{2}t^{\frac{3}{2}}-dt^{\frac{1}{2}}]\\
&\quad+\int_{-\infty}^{\frac{1}{2}t^2-dt}(2\pi t)^{-\frac{1}{2}}\exp\left\{-\frac{x^2}{2t}-2(t-d)\left(\frac{t^2}{2}-dt-x\right)\right\}dx\\
&=\Phi(dt^{\frac{1}{2}}-\tfrac{1}{2}t^{\frac{3}{2}})+\exp\{t^2(t-d)\}\Phi(dt^{\frac{1}{2}}-\tfrac{3}{2}t^{\frac{3}{2}})\\
&\leqq ce^{-t^3/12}.
\end{aligned}$$

For the second estimate, note that

$$P[V_{Nw}(T)\geqq t]\leqq P\Big[\max_{t\leqq s\leqq T}\{W(s)-\tfrac{1}{2}s^2+ds+Ds^3\}\geqq 0\Big].$$

The function $\frac{1}{2}s^2-ds-Ds^3$ has increasing derivative in $0\leqq s\leqq\frac{1}{12}D$, and so this probability can be estimated as not exceeding

$$P\Big[\max_{s\geqq t}\{W(s)-\tfrac{1}{2}t^2+dt+Dt^3-(s-t)(t-d-3Dt^2)\}\geqq 0\Big],$$

which in turn can be estimated, as above, as less than

$$\Phi(-t^{\frac{3}{2}}/3)+\exp\{t^3-4d^2t-dt^2+4dDt^3-7Dt^4-24D^2t^5\}\Phi(-\tfrac{3}{2}t^{\frac{3}{2}}+dt^{\frac{1}{2}}+5Dt^{\frac{5}{2}})$$
$$\leqq c\exp\{-t^3/24\}.$$

Finally,

$$P[V_N(T)\geqq t]\leqq P\left[\max_{t\leqq s\leqq T}\{Z(s)+Ds^3\}\geqq 0\right],$$

and the earlier estimates can now be used, taking $w=0$.

Lemma 6. If $|w|\leqq\frac{1}{6}N^{\frac{1}{6}}$ and $\delta\leqq\frac{1}{12}G$, then

$$|P[Y_{Nw}(V_{Nw}(\delta N^{\frac{1}{3}}))\geqq y]-P[Z(U(\infty))\geqq y]+wk(y)N^{-\frac{1}{6}}|$$
$$\leqq c\{N^{-\frac{1}{3}}(\log N+w^2)+N^{-\frac{1}{2}}|w|(\log N)^{\frac{5}{3}}\}h^*(y)+O(N^{-1}),$$

where

$$h^*(y)=(1+y^3)\exp\{-y^{\frac{3}{2}}/7\}.$$

Proof. Write $\underline{Y}$, $\underline{Z}$, $\underline{U}$ and $\underline{V}$ for Y_{Nw}, Z_{Nw}, U_{Nw} and V_{Nw}, and also d for $N^{-\frac{1}{6}}w$.

From the definitions of $\underline{Y}$, $\underline{Z}$, $\underline{U}$ and $\underline{V}$, it is immediate that

$$\begin{aligned}&\underline{Z}(\underline{U}(T))\{1+N^{-\frac{1}{3}}\underline{U}(T)\}^{-1}+N^{-\frac{1}{3}}g_N(\underline{U}(T))\\&\quad\leqq\underline{Y}(\underline{V}(T))\\&\quad\leqq\underline{Z}(\underline{U}(T))\{1+N^{-\frac{1}{3}}\underline{V}(T)\}^{-1}+N^{-\frac{1}{3}}g_N(\underline{V}(T)),\end{aligned}$$

and so

$$\begin{aligned}P[\underline{Z}(\underline{U}(T))\geqq(y-N^{-\frac{1}{3}}g_N(\underline{V}(T)))(1+N^{-\frac{1}{3}}\underline{V}(T))]&\\\geqq P[\underline{Y}(\underline{V}(T))\geqq y]&\\\geqq P[\underline{Z}(\underline{U}(T))\geqq(y-N^{-\frac{1}{3}}g_N(\underline{U}(T)))(1+N^{-\frac{1}{3}}\underline{U}(T))]&.\end{aligned}$$

Taking the right-hand member with $T=T_N=\delta N^{\frac{1}{3}}$, it is no smaller than

$$P[\underline{Z}(\underline{U}(T_N))\geqq y_N]-cN^{-1},$$

where

$$y_N=y(1+3N^{-\frac{1}{3}}(\log N)^{\frac{1}{3}})+27GN^{-\frac{1}{3}}\log N,$$

by Lemma 5(i); and this, in turn, can be estimated by Lemma 3 to be no larger than

$$P[Z(U(T_N))\geqq y_N]-dk(y_N)-d^2h(y_N)-cN^{-1};$$

or, this time by Lemma 1(iii), no larger than

$$P[Z(U(\infty))\geqq y_N]-dk(y_N)-d^2h(y_N)-cN^{-1}.$$

Lemma 2 then shows that replacing y_N by y in $P[Z(U(\infty))\geqq y_N]$ introduces an error of at most

$$c\{3yN^{-\frac{1}{3}}(\log N)^{\frac{1}{3}}+27GN^{-\frac{1}{3}}\log N\}(1+y^{\frac{1}{2}})\exp\{-\tfrac{7}{8}y^{\frac{3}{2}}\};$$

Lemma 4 permits the replacement of y_N by y in $dk(y_N)$ at a cost of at most

$$c\,|d|\,\{3yN^{-\frac{1}{3}}(\log N)^{\frac{1}{3}}+27GN^{-\frac{1}{3}}\log N\}\{c(\log N)^{\frac{2}{3}}+y\}\,.\,(1+y^{\frac{1}{2}})\exp\{-\tfrac{7}{8}y^{\frac{3}{2}}\};$$

and $h(y_N)$ may be replaced for all large enough N by $h(\frac{1}{2}y-1)$, $y\geqq 2$. Similar estimates also hold for the left-hand inequality. Collecting the various errors together, the statement of the lemma is obtained.

Lemma 7. If $\delta\leqq\frac{1}{12}G$,

$$|P[Y_N(V_N(\delta N^{\frac{1}{3}}))\geqq y]-P[Z(U(\infty))\geqq y]|\leqq cN^{-\frac{1}{3}}\log Nh^+(y)+O(N^{-1}),$$

where $h^+(y)=(1+y^{\frac{1}{2}})\exp\{-\frac{7}{8}y^{\frac{3}{2}}\}$.

Proof. Much as in the proof of Lemma 6, but using Lemmas 5(iii) and 1(iii),

$$\begin{aligned}P[Z(U(\delta N^{\frac{1}{3}}))\geqq y+27GN^{-\frac{1}{3}}\log N]+O(N^{-1})\\ \leqq P[Y_N(V_N(\delta N^{\frac{1}{3}}))\geqq y]\\ \leqq P[Z(U(\delta N^{\frac{1}{3}}))\geqq y-27GN^{-\frac{1}{3}}\log N]+O(N^{-1}).\end{aligned}$$

Taking the right-hand inequality, and applying Lemma 1(iii) again, gives an upper estimate of

$$P[Z(U(\infty))\geqq y-27GN^{-\frac{1}{3}}\log N]+O(N^{-1}),$$

which may be converted into

$$P[Z(U(\infty))\geqq y]+cN^{-\frac{1}{3}}\log N(1+y^{\frac{1}{2}})\exp\{-\tfrac{7}{8}y^{\frac{3}{2}}\}+O(N^{-1})$$

by Lemma 2. A similar argument operates for the left-hand inequality.

We now turn more directly to the problem at hand. For W a standard Brownian motion, consider the maximum of $W(t)-N^{\frac{1}{2}}c(t)$ in the interval $0\leqq t\leqq 2$, where c satisfies $|c(t)-\frac{1}{2}(t-1)^2|\leqq K\,|t-1|^3$ in some range $|t-1|\leqq\delta<1$, δ also being chosen to be smaller than $1/(12K+6)$, and where

$$\eta\equiv\inf_{\delta\leqq|t-1|\leqq 1}c(t)>0.$$

Lemma 8. Let V_1 denote the time at which $W(t)-N^{\frac{1}{2}}c(t)$ first attains its maximum over the interval $[0,1]$, V_2 the time at which it last attains its

maximum in $[1, 2]$. Then, if $|w| \leqq \frac{1}{2}N^{\frac{1}{2}}\eta$, $P[V_1 \leqq 1-\delta \mid W(1)=w] \leqq \exp\{-\frac{1}{2}N\eta^2\}$. Similarly, for all w, $P[V_2 \geqq 1+\delta \mid W(1)=w] \leqq \exp\{-\frac{1}{2}N\eta^2\}$.

Proof. Write $H = N^{\frac{1}{2}}\eta$. A simple argument shows that

$$P[V_1 \leqq 1-\delta \mid W(1)=w] \leqq P\left[\max_{0 \leqq t \leqq 1-\delta} W(t) \geqq H+w \mid W(1)=w\right]$$
$$= P\left[\max_{0 \leqq t \leqq 1-\delta} \{W_0(t)+wt\} \geqq H+w\right],$$

where W_0 denotes a Brownian bridge. Using the transformation $s = t/(1-t)$, this becomes

$$P\left[\max_{0 \leqq s \leqq (1/\delta)-1} \{(\tilde{W}(s)+ws)/(s+1)\} \geqq H+w\right],$$

where $\tilde{W}$ is standard Brownian motion, and it can be rewritten as

$$P\left[\max_{0 \leqq s \leqq (1/\delta)-1} \{\tilde{W}(s)-Hs-H-w\} \geqq 0\right]$$
$$\leqq \exp\{-2H(H+w)\} \leqq \exp\{-\tfrac{1}{2}H^2\}.$$

In a similar way,

$$P[V_2 \geqq 1+\delta \mid W(1)=w] \leqq P\left[\max_{\delta \leqq s \leqq 1} \tilde{W}(s) \geqq H\right] \leqq \exp\{-\tfrac{1}{2}H^2\}$$

irrespective of w.

The preparations for the theorem are now complete. Let R_1 and R_2 be independent random variables with the same distribution as $Z(U(\infty))$, and let $R = \max(R_1, R_2)$; set $\lambda = ER$ and $\sigma^2 = \operatorname{Var} R$, and define

$$\rho^2 = 2\int_0^\infty k(y)P[Z(U(\infty)) \leqq y]\,dy.$$

Theorem. For any function c with the properties stated above,

$$\left|P\left[\sup_{0 \leqq t \leqq 2} \{W(t)-N^{\frac{1}{2}}c(t)\} \geqq z\right] - P[\mathcal{N}(N^{-\frac{1}{6}}\lambda, 1+N^{-\frac{1}{3}}(\sigma^2-\rho^2)) \geqq z]\right| \leqq cN^{-\frac{1}{2}}\log N,$$

for all z.

Proof. Let

$$Q^N = \sup_{0 \leqq t \leqq 2} \{W(t)-N^{\frac{1}{2}}c(t)\} = \sup_{0 \leqq t \leqq 2} \{W(t)-W(1)-N^{\frac{1}{2}}c(t)\}+W(1).$$

Conditional on $W(1)=w$, Q^N is distributed as $w+\max\{Q_0^N(w), Q_1^N\}$, where

$Q_0^N(w)$ and Q_1^N are independent,

$$Q_0^N(w) \overset{D}{=} \sup_{0 \leqq s \leqq 1} [W_0(s) - sw - N^{\frac{1}{2}}c(1-s)],$$

where W_0 is a Brownian bridge, and

$$Q_1^N \overset{D}{=} \sup_{0 \leqq v \leqq 1} [W(v) - N^{\frac{1}{2}}c(1+v)].$$

If $|w| \leqq \frac{1}{6}N^{\frac{1}{6}} \leqq \frac{1}{2}N^{\frac{1}{2}}$, an application of Lemma 8 and the substitution $v = N^{\frac{1}{3}}s/(1-s)$ yield

$$P\left[\sup_{0 \leqq s \leqq 1} \{W_0(s) - sw - N^{\frac{1}{2}}c(1-s)\} \geqq yN^{-\frac{1}{6}}\right]$$
$$= P\left[\sup_{0 \leqq v \leqq N^{\frac{1}{3}}/(12K-1)} Y_{Nw}(v) \geqq y\right] + O(\exp\{-\tfrac{1}{2}N\eta^2\}),$$

where, in the definition of $Y_{Nw}(v)$,

$$g_N(v) = \tfrac{1}{2}v^3(1+N^{-\frac{1}{3}}v)^{-2} - N[c((1+N^{-\frac{1}{3}}v)^{-1}) - \tfrac{1}{2}v^2N^{-\frac{2}{3}}(1+N^{-\frac{1}{3}}v)^{-2}],$$

and so $(1+vN^{-\frac{1}{3}})\,|g_N(v)v^{-3}|$ is bounded uniformly in N in the range considered by $K+\frac{1}{2} = G$, say. Thus, applying Lemma 6,

$$|P[Q_0^N(w) \leqq yN^{-\frac{1}{6}}] - P[Z(U(\infty)) \leqq y] - wk(y)N^{-\frac{1}{6}}|$$
$$\leqq cN^{-\frac{1}{3}}\{\log N + w^2 + N^{-\frac{1}{6}}|w|(\log N)^{\frac{5}{3}}\}h^*(y) + O(N^{-1}).$$

A similar argument, using Lemma 7, shows that

$$|P[Q_1^N \leqq yN^{-\frac{1}{6}}] - P[Z(U(\infty)) \leqq y]| \leqq cN^{-\frac{1}{3}}\log N + O(N^{-1}).$$

Thus, if $|w| \leqq \frac{1}{6}N^{\frac{1}{6}} \leqq \frac{1}{2}N^{\frac{1}{2}}$,

$$P[\max(Q_0^N(w), Q_1^N) \leqq yN^{-\frac{1}{6}}] - P[Z(U(\infty)) \leqq y]\{P[Z(U(\infty)) \leqq y] + wk(y)N^{-\frac{1}{6}}\} \equiv \varepsilon_N(y, w), \tag{2}$$

where

$$|\varepsilon_N(y, w)| \leqq cN^{-\frac{1}{3}}\{\log N + w^2 + N^{-\frac{1}{6}}|w|(\log N)^{\frac{5}{3}}\}(1+y^3)\exp\{-\tfrac{1}{7}y^{\frac{3}{2}}\} + O(N^{-1}).$$

Now, since $W(1)$ has a standard normal distribution,

$$P[Q^N \leqq z] = \int_{-\infty}^{z} (2\pi)^{-\frac{1}{2}}e^{-w^2/2}P[\max(Q_0^N(w), Q_1^N) \leqq z - w]\,dw.$$

Hence, for any z such that $|z| \leqq \frac{1}{12}N^{\frac{1}{6}}$, writing z_N for $N^{\frac{1}{6}}(z + N^{\frac{1}{6}}/6)$ and $\bar{z}$ for

$z-N^{-\frac{1}{6}}y$, it follows that

$$\begin{aligned}
\Phi(z)-P[Q^N \leqq z] &= \int_{-\infty}^{z} (2\pi)^{-\frac{1}{2}} \exp\{-\tfrac{1}{2}w^2\}\{1-P[\max(Q_0^N(w), Q_1^N) \leqq z-w]\}\, dw \\
&= \int_{-\infty}^{-N^{\frac{1}{6}}/6} (2\pi)^{-\frac{1}{2}} \exp\{-\tfrac{1}{2}w^2\}\{1-P[\max(Q_0^N(w), Q_1^N) \leqq z-w]\}\, dw \\
&\quad + N^{-\frac{1}{6}} \int_0^{z_N} dy (2\pi)^{-\frac{1}{2}} \exp\{-\tfrac{1}{2}\bar{z}^2\} \\
&\quad .\{P[R \geqq y] - \bar{z}P[Z(U(\infty)) \leqq y] k(y) N^{-\frac{1}{6}} - \varepsilon_N(y, \bar{z})\} \\
&= N^{-\frac{1}{6}}(2\pi)^{-\frac{1}{2}} \exp\{-\tfrac{1}{2}z^2\}\{\lambda + \tfrac{1}{2}N^{-\frac{1}{6}} z(\sigma^2+\lambda^2)\} \\
&\quad - N^{-\frac{1}{3}}(2\pi)^{-\frac{1}{2}} z \exp\{-\tfrac{1}{2}z^2\} \int_0^{\infty} k(y) P[Z(U(\infty)) \leqq y]\, dy + \sum_{j=1}^{8} E_{jN},
\end{aligned} \tag{3}$$

where the errors E_{jN} in the approximation are estimated as follows:

$$\begin{aligned}
|E_{1N}| &= \left| \int_{-\infty}^{-N^{\frac{1}{6}}/6} (2\pi)^{-\frac{1}{2}} \exp\{-\tfrac{1}{2}w^2\} P[\max(Q_0^N(w), Q_1^N) \geqq z-w]\, dw \right| \\
&= O(\exp\{-\tfrac{1}{72}N^{\frac{1}{3}}\}) \\
|E_{2N}| &= \int_0^{z_N} (2\pi N)^{-\frac{1}{2}} y k(y) P[Z(U(\infty)) \leqq y] \exp\{-\tfrac{1}{2}\bar{z}^2\}\, dy = O(N^{-\frac{1}{2}}) \\
|E_{3N}| &= \left| N^{-\frac{1}{6}} \int_0^{z_N} \exp\{-\tfrac{1}{2}\bar{z}^2\}(2\pi)^{-\frac{1}{2}} \varepsilon_N(y, \bar{z})\, dy \right| \\
&= O(N^{-\frac{1}{2}} \log N), \\
|E_{4N}| &= \int_{z_N}^{\infty} N^{-\frac{1}{6}}(2\pi)^{-\frac{1}{2}} \exp\{-\tfrac{1}{2}z^2\} P[R \geqq y]\, dy = O(\exp\{-\beta N^{\frac{1}{2}}\}),
\end{aligned}$$

from Lemma 1(i), with $\beta = \frac{7}{64}3^{-\frac{3}{2}}$,

$$\begin{aligned}
|E_{5N}| &= \int_{z_N}^{\infty} (8\pi)^{-\frac{1}{2}} N^{-\frac{1}{3}} |z| \exp\{-\tfrac{1}{2}z^2\} y P[R \geqq y]\, dy = O(\exp\{-\beta N^{\frac{1}{2}}\}); \\
|E_{6N}| &= N^{-\frac{1}{3}}(2\pi)^{-\frac{1}{2}} |z| \exp\{-\tfrac{1}{2}z^2\} \int_{z_N}^{\infty} k(y) P[Z(U(\infty)) \leqq y]\, dy \\
&= O(e^{-\beta N^{\frac{1}{2}}}),
\end{aligned}$$

from Lemma 4(i), with the same value of β;

$$\begin{aligned}
|E_{7N}| &\leqq N^{-\frac{1}{3}} \int_0^{z_N} |\exp\{-\tfrac{1}{2}\bar{z}^2\} - \exp\{-\tfrac{1}{2}z^2\}| . (2\pi)^{-\frac{1}{2}} |z|\, k(y) P[Z(U(\infty)) \leqq y]\, dy \\
&\leqq c N^{-\frac{1}{2}} |z| \int_0^{N^{\frac{1}{3}}/3} (y|z| + N^{-\frac{1}{6}} y^2) y \exp\{-\tfrac{1}{2}z^2 + N^{-\frac{1}{6}} y|z| - \tfrac{7}{8}y^2\}\, dy,
\end{aligned}$$

again by Lemma 4(i), and now, since the exponent is no larger than $-\frac{1}{4}z^2-\frac{7}{16}y^{\frac{3}{2}}$ whenever $N^{\frac{1}{3}}\geqq 2^{12}/147$, the contribution to the error from E_{7N} is no more than $O(N^{-\frac{1}{2}})$;

$$|E_{8N}|\leqq(2\pi)^{-\frac{1}{2}}N^{-\frac{1}{6}}\int_0^{z_N}|\exp\{-\tfrac{1}{2}z^2\}(1+N^{-\frac{1}{6}}yz)-\exp\{-\tfrac{1}{2}\bar{z}^2\}|\,.\,P[R\geqq y]\,dy$$

$$\leqq N^{-\frac{1}{6}}\int_0^{N^{\frac{1}{3}}/3}\exp\{-\tfrac{7}{8}y^{\frac{3}{2}}-\tfrac{1}{2}z^2\}(\tfrac{1}{2}N^{-\frac{1}{3}}y^2+N^{-\frac{1}{3}}(y\,|z|+\tfrac{1}{2}N^{-\frac{1}{6}}y^2)^2\exp\{N^{-\frac{1}{6}}y\,|z|\})dy$$

$$\leqq N^{-\frac{1}{2}}\int_0^{\infty}[\tfrac{1}{2}y^2+(y\,|z|+\tfrac{1}{2}N^{-\frac{1}{6}}y^2)^2]\exp\left\{-\frac{z^2}{4}-\frac{7}{16}y^{\frac{3}{2}}\right\}dy=O(N^{-\frac{1}{2}}),$$

again for $N^{\frac{1}{3}}\geqq 2^{12}/147$. In all cases, the error estimates are uniform in $|z|\leqq\frac{1}{12}N^{\frac{1}{6}}$.

Finally, it follows easily from Taylor's expansion that

$$\Phi(z)-\Phi((z-\lambda N^{-\frac{1}{6}})(1+\varphi N^{-\frac{1}{3}})^{-\frac{1}{2}})=(2\pi)^{-\frac{1}{2}}e^{-z^2/2}\{\lambda N^{-\frac{1}{6}}+\tfrac{1}{2}zN^{-\frac{1}{3}}(\lambda^2+\varphi)\}+O(N^{-\frac{1}{2}}),$$

and, comparing this expression with (3), the statement of the theorem is obtained for all z such that $|z|\leqq\frac{1}{12}N^{\frac{1}{6}}$. Since both the probabilities concerned are easily shown to be very small for values of z outside this range, the theorem follows.

3. Discussion

The approximation obtained in the theorem is particularly attractive for a number of reasons. The most satisfying aspects are that the approximating distribution is still normal, and that its mean is the same as that given by Daniels (1974), despite the improved precision. The correction, to the variance, again involves only a universal constant, $\sigma^2-\rho^2$, which can in principle be computed once and for all: none of the properties of the function c, except for its second derivative at 1, are of importance. Finally, the approximation is as accurate as one would naturally require, in that, in most applications, the Wiener process is itself used as an approximation to a discrete process, and the error in this approximation is typically of order $N^{-\frac{1}{2}}\log N$, as in the theorem of Komlós, Major and Tusnády (1975).

The proof is considerably more complicated than that required to fill in the details omitted for clarity from Barbour (1975). Part of the reason is that whereas, in that paper, the estimates were based on comparisons between paths defined on the same probability space, thus making the argument very direct, it has seemed necessary here to argue from a distributional point of view, because the sample-path arguments that one would wish to use were no

longer obviously valid. The place where this approach is of essential importance is in proving Lemma 3, and indeed the way that this lemma is cast influences the whole of the rest of the proof.

The problem discussed here, approximating the distribution of the maximum of $W(t)-N^{\frac{1}{2}}c(t)$, where $c(t)\sim\frac{1}{2}(t-1)^2$ as $t\to 1$, is equivalent to that analysed by Daniels (1974), who has $c(t)\sim\frac{1}{2}c''(1)(t-1)^2$; if Daniels's N is replaced by $N'=\{c''(1)\}^2N$, the form considered here is recovered. Similarly, a change of time scale allows c to take its minimum value at any time α. It is also possible, though this time with a reworking of the arguments of Section 2, to make a similar approximation to the distribution of

$$\max_{0\leqq t\leqq\beta}\{a(t)W(t)-N^{\frac{1}{2}}c(t)\},$$

where a is a prescribed function of time, and where c takes its unique minimum of 0 at $\alpha<\beta$. The reason for extending the approximation to cover this more complicated problem is that many one-dimensional population processes have a large population behaviour given approximately by $N\xi(t)+N^{\frac{1}{2}}U(t)$, where ξ is a deterministic function, and U is typically an Ornstein–Uhlenbeck process, rather than a Wiener process. In such cases, U can be transformed to a Wiener process with a time-dependent multiplying factor, giving rise to extremal problems for which the above generalization is necessary. The theorem now takes the following form. If $|c(t)-\frac{1}{2}c''(\alpha)(t-\alpha)^2|\leqq K|t-\alpha|^3$ and $|a(t)-a(\alpha)-a'(\alpha)(t-\alpha)|\leqq K'|t-\alpha|^2$ in some range $|t-\alpha|\leqq\delta$, where $a(\alpha)>0$ and $c''(\alpha)>0$, and if

$$\eta=\inf_{\delta\leqq|t-\alpha|,0\leqq t\leqq\beta}c(t)/a(t)>0,$$

then

$$P\left[\sup_{0\leqq t\leqq\beta}\{a(t)W(t)-N^{\frac{1}{2}}c(t)\}\geqq y\right]$$
$$-P[\mathcal{N}(N^{-\frac{1}{6}}\mu\lambda,\ \alpha a^2(\alpha)+N^{-\frac{1}{3}}\mu^2(\sigma^2-\rho^2))\geqq y]=O(N^{-\frac{1}{2}}\log N),$$

where $\mu=a(\alpha)\{a(\alpha)/c''(\alpha)\}^{\frac{1}{3}}$, and where λ, σ^2 and μ^2 are as above. Curiously, the only significant terms reflecting the fact that a varies with time cancel out in the equivalent of (2).

Smith (1979) has also discussed a very similar problem, arising from the analysis of the strength of fibre bundles, in which it is important to be able to approximate the distribution of

$$\sup_{0\leqq t\leqq 1}\{a(t)W_0(t)-N^{\frac{1}{2}}c(t)\},$$

where W_0 denotes the Brownian bridge. In this setting, he proves a result

similar to that of Daniels, and extends it to cover the probabilities of large deviations. An analysis similar to that in Section 2 would show that

$$P\left[\sup_{0\leqq t\leqq 1}\{a(t)W_0(t)-N^{\frac{1}{2}}c(t)\}\leqq y\right]$$
$$-P[\mathcal{N}(N^{-\frac{1}{6}}\mu\lambda,\ \alpha(1-\alpha)a^2(\alpha)+N^{-\frac{1}{3}}\mu^2(\sigma^2-\rho^2))\leqq y]=O(N^{-\frac{1}{2}}\log N)$$

where μ, λ, σ^2 and ρ^2 are defined as above. The constants are the same as before, and the only significant terms reflecting the time dependence of a again cancel out.

It is natural to ask how good the approximation actually is in practice, since some of the estimates used in Section 2 were rather crude: they were not intended to be the most economical estimates possible, but merely to be precise enough to get the right rate of convergence. In order to give some flavour of the accuracy of the approximation, numerical comparisons were made between the true distribution of the maximum of the closed epidemic and the new normal approximation, for the values of the underlying parameters used for illustration in Table 1 of Daniels (1974). In his notation, the new estimate of the distribution of the maximum number of infectives in a major epidemic is normal, with the same mean as in his approximation, $\hat{y}+\lambda(4\rho)^{\frac{1}{3}}$, and with standard deviation $\{N+\varphi(4\rho)^{\frac{2}{3}}\}^{\frac{1}{2}}$, instead of his $N^{\frac{1}{2}}$, where, to avoid confusion of notation, φ is used here to denote the constant '$\sigma^2-\rho^2$'. The exact values of λ and φ are not known, since they have to be estimated by computer. Present calculations suggest that $\lambda=0{\cdot}996$ and $\varphi=-0{\cdot}317$, and these values have been used to compute the following version of Daniels' Table 1. With the exception of the entry under σ in row 4, all the figures have been recomputed, and some slight differences detected. The new estimate of σ is seen to be a clear improvement.

TABLE 1
The maximum of a serious epidemic

ξ	ρ	σ	$N^{\frac{1}{2}}$	$\{N+\varphi(4\rho)^{\frac{2}{3}}\}^{\frac{1}{2}}$
50	10	4·34	4·91	4·52
100	10	5·21	5·66	5·32
100	25	6·66	7·31	6·82
100	50	6·9	7·72	6·99
1000	200	21·23	21·95	21·32

ξ denotes the initial number of susceptibles, ρ the removal rate and σ true standard deviation of the maximum number of infectives, and the last two columns give Daniels' estimate of σ and the new estimate.

TABLE 2
The maximum of a serious epidemic

ξ	ρ	μ	$\hat{\mu}$	σ	$\hat{\sigma}$	$\mu-\hat{\mu}$	$\hat{\sigma}-\sigma$
50	12·5	24·65	23·84	4·52	4·73	0·81	0·21
100	25	45·77	44·97	6·66	6·82	0·80	0·16
200	50	87·30	86·51	9·66	9·80	0·79	0·14
400	100	169·50	168·71	13·90	14·02	0·79	0·12
800	200	332·78	331·99	19·90	20·00	0·79	0·10

Legend as in Table 1, and now μ denotes the true expectation of the maximum number of infectives, and $\hat{\mu}$ and $\hat{\sigma}$ denote the new estimates of μ and σ.

Table 2 shows rather more clearly how the errors in the approximation behave as functions of N. The values of ξ and ρ are chosen in such a way that N doubles from line to line. Theoretically, the absolute differences between the approximations to μ and σ given by the theorem and their true values should be of order 1 as N increases, give or take the factor of $\log N$. The figures below give reasonable confirmation of this, over the range of N considered.

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