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QUASI–STATIONARY DISTRIBUTIONS IN MARKOV POPULATION PROCESSES

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Abstract

One way of analysing complicated Markov population processes is to approximate them by a diffusion about the deterministic path. This approximation alone may not, however, answer all the questions which might reasonably be asked. Many processes have phases, for example near boundaries, where a different approximation is required; such processes are better described by a succession of diffusion and special approximations alternately.

This paper looks at the special treatment required near a point where the deterministic equations are in equilibrium. When the equilibrium is unstable, the process will eventually wander off, and possibly follow a diffusion around another deterministic path. If the equilibrium is stable, the process will behave as if in stable equilibrium about it for a very much longer time, but may at last be trapped away from it, for instance in an absorbing state. The results presented describe the distributions of the time and place of leaving neighbourhoods of the equilibrium point. The neighbourhoods considered are large enough, in the unstable case, to make possible the link with the next phase of motion. In the stable case, exit times are shown to be so long that the possibility of exit can often be ignored in practice, and the quasi-equilibrium distribution treated as a true equilibrium. A more detailed result, showing how closely the normal approximation holds in this situation, is also provided.

MARKOV POPULATION PROCESS; COMPETITION PROCESS; RECURRENT EPIDEMIC; CENTRAL LIMIT THEOREM; TAIL PROBABILITIES

1. Introduction

It is often useful to approximate the behaviour of a Markov population process, when population sizes are large, by an essentially deterministic motion, with a random diffusion of smaller order superimposed upon it. This approach has been used by, among others, Daley and Kendall (1965), Nagaev and Startsev (1970), Kurtz (1971), Barbour (1972) and McNeil and Schach (1973). The approximation may prove satisfactory on its own, but there will usually be certain phases of a process, for example near boundaries, which need separate treatment. For instance, in the closed epidemic, a diffusion approximation is

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appropriate for all but the initial and final phases, in which the number of infectives is small. A workable approximation to the whole process is then given by combining the diffusion with birth and death approximations at the beginning and end: from this model, useful estimates of the size and duration of the epidemic can be derived.

This paper is concerned with a different type of problem arising from diffusion approximation, that of behaviour near an equilibrium point of the deterministic equations. The deterministic trajectory starting from such a point reduces to the same point for all time, and the stochastic fluctuation around it is approximated by an Ornstein–Uhlenbeck process: many examples of this are given in McNeil and Schach (1973), and limitations of the approximation are mentioned, by D. G. Kendall and V. Barnett, in the discussion of their paper.

If the deterministic equilibrium is unstable, the original population process can be expected to diffuse away from it, and then to follow some deterministic trajectory, with smaller order diffusion around it, before perhaps entering another special phase. The approximating Ornstein–Uhlenbeck process gives an adequate representation of the first phase of the motion. However, in order to carry this approximation far enough to link up with the next phase, it is necessary to extend the time and space scales over which the diffusion limit theorems are applicable: this is the reason for Theorems 1(i), 4, 5 and 7. These theorems describe the distributions of the time and place of leaving neighbourhoods of the equilibrium point, whose boundaries are far enough away from it for subsequent motion to be approximated in the usual way: an illustration of how the link can be accomplished appears, in a different situation, in Barbour (1975). The method used in proving the theorems also, in principle, allows an estimate to be made of the error involved in such a combined approximation.

If the deterministic equilibrium is stable, the original process also enjoys a stable equilibrium behaviour, similar to that of the approximating Ornstein–Uhlenbeck process. It is, however, possible that there exist one or more absorbing states which the process must eventually reach, in which case the apparent equilibrium is a quasi-equilibrium, as described, with several examples, in Bartlett (1960). In view of this, it is of interest to know how closely, and for how long and over what ranges the underlying population process is approximated by the Ornstein–Uhlenbeck process: Theorems 1(iii), 2, 3 and 6 give some answers to these questions.

The main results are stated in Section 2, in the context of one-dimensional processes, and they are applied to the stochastic logistic model of population growth. Section 3 shows how the results carry over into two or more dimensions: a competition process and a recurrent epidemic model are used for illustration. The proofs of the theorems appearing in these sections are indicated in Section 4.
2. One-dimensional processes

Let \( \{x_N(\cdot), N \geq 1\} \) be a sequence of random processes defined as follows. For each \( N \), \( \{x_N(t), t \geq 0\} \) is a continuous-time Markov process on the lattice of points \( \{k/N, k \text{ integral}\} \), with matrix of transition rates given by

\[
(2.1) \quad x \rightarrow x + j/N \quad \text{at rate} \quad Ng_i(x), \quad j \in J,
\]

for some finite \( N \)-independent set \( J \) of integers, where the functions \( g_i(x) \) are assumed to be polynomials in \( x \). Such sequences of processes arise in the derivation of large population approximations for Markov population processes: for details of the procedure, see Barbour (1972). Suppose that \( x_N(0) \rightarrow x_0 \) as \( N \rightarrow \infty \), and take any \( 0 < T < \infty \) such that \( \xi(\cdot) \), the solution of the deterministic equations

\[
(2.2) \quad d\xi/dt = \sum_{j \in J} jg_i(\xi), \quad \xi(0) = x_0
\]

exists and is unique in \([0, T]\). Then, by Kurtz (1971), Theorem (3.5), the sequence \( \{\sqrt{N}[x_N(t) - \xi(t)], 0 \leq t \leq T\} \) converges weakly to a diffusion process \( y(\cdot) \), whose distributions can be deduced from the functions \( g_i(x) \).

When \( x_0 \) is such that \( \Sigma_{j \in J} g_i(x_0) = 0 \), the deterministic trajectory degenerates to \( \xi(t) = x_0 \) for all \( t \), and the limit in distribution as \( N \rightarrow \infty \) of the sequence \( \{\sqrt{N}[x_N(t) - x_0], 0 \leq t \leq T\} \) is an Ornstein–Uhlenbeck process, with drift coefficient \( \beta \) and infinitesimal variance \( \sigma^2 \) given by

\[
\beta = \sum_{j \in J} jg_i'(x_0); \quad \sigma^2 = \sum_{j \in J} j^2g_i(x_0);
\]

where \( g_i'(x) \) denotes \( dg_i(x)/dx \).

The limit distribution of the time and position of the first exit of \( x_N \) from \( |x - x_0| < cN^{-1/2} \) can thus be deduced, from the convergence theorem of Kurtz, upon replacing \( N^{1/2}\{x_N(\cdot) - x_0\} \) by the Ornstein–Uhlenbeck process. However, as mentioned in the introduction, it is useful to have similar information about exit distributions when \( c \) is allowed to depend on \( N \), in such a way that \( c_N \rightarrow \infty \) as \( N \rightarrow \infty \). Theorem 1, below, shows that, under some growth restrictions on \( (c_N) \), the Ornstein–Uhlenbeck process still gives the correct approximation.

Suppose, therefore, that \( \Sigma_{j \in J} g_i(x_0) = 0 \), and that \( \beta \) and \( \sigma^2 \) are as above. Define \( z_N(t) = \sqrt{N}[x_N(t) - x_0] \), and let

\[
\tau = \tau(N, c_N) = \inf \{t \geq 0: |z_N(t)| \geq c_N\},
\]

\[
p_N = p_N(c_N, z_N(0)) = P[z_N(\tau) \geq c_N | z_N(0)],
\]

where \( (c_N) \) is a sequence of real numbers, and \( c_N \rightarrow \infty \) as \( N \rightarrow \infty \).
**Theorem 1.** Suppose that \( \lim_{N \to \infty} z_N(0) = z \). Then, as \( n \to \infty \),

(i) if \( \beta > 0 \) and \( c_N = o(\sqrt{N}/\log N) \):

\[
\lim_{N \to \infty} p_N = \Phi(z \sqrt{(2\beta)}/\sigma),
\]

and

\[
\tau_+ - \beta^{-1} \log (c_N \sqrt{\beta/\sigma}) \sim W(z),
\]

where

\[
P[W(z) \leq - \beta^{-1} \log (x/\sqrt{2})] = \Phi(z \sqrt{(2\beta/\sigma^2)} - x)/\Phi(z \sqrt{(2\beta/\sigma^2)});
\]

(ii) if \( \beta = 0 \) and \( c_N = o(N^{1/6}) \):

\[
\lim_{N \to \infty} p_N = 1/2, \quad \text{and} \quad \sigma \tau_+ / c_N^{1/2} \sim X,
\]

where \( P[X \leq x] = P[\sup_{0 \leq s \leq x} |B(s)| \geq 1] \), with \( B(\cdot) \) standard Brownian motion;

(iii) if \( \beta < 0 \) and \( c_N = o(N^{1/6}) \):

\[
\lim_{N \to \infty} p_N = 1/2,
\]

and

\[
-2\beta c_N \sqrt{(-\beta/\pi \sigma^2)} \exp(\beta c_N^2/\sigma^2). \tau_+ \sim E,
\]

where

\[
P[E \geq x] = e^{-x};
\]

where \( \tau_+ \) denotes a random variable distributed as \( \tau \) conditional on \( z_N(\tau) \geq c_N \) and \( z_N(0) \), and where \( \Phi \) is the distribution function of a standard normal random variable.

Theorem 1(i) is adequate for linking the first and second phases of motion in the unstable case. In the stable case, in addition, it is useful to have a better measure of the fit of the diffusion approximation. The next theorem provides this: without real loss of generality, \( z_N(0) \) is taken to be zero.

**Theorem 2.** Choose any \( \delta, T_0 > 0 \), and define

\[ v(T) = -\sigma^2(1 - e^{2\beta T})/2\beta. \]

Then, if \( \beta < 0 \),

\[
P[z_N(T) \geq x] = \{1 - \Phi(x/\sqrt{v(T)})\} \{1 + O(X^4 N^{-1/2} \log N)\},
\]

uniformly in \( 0 \leq x \leq X \) and in \( T_0 \leq T \leq \exp(\delta X^2) \), for any \( X \) such that \( \sqrt{(\log N)} \leq X \leq N^{1/6} \).

By way of example, consider the stochastic logistic model of self-regulating population growth in a single species. This is a Markov process on the non-negative integers, with transition rates given by

\[
X \to X + 1 \quad \text{at rate} \quad \alpha_1 X - \gamma_1 X^2/N
\]

\[
X \to X - 1 \quad \text{at rate} \quad \alpha_2 X + \gamma_2 X^2/N
\]
where $\alpha_1 > \alpha_2 > 0$ and $\gamma_1$ and $\gamma_2 > 0$. $X(t)$ denotes the population size at time $t$, and $N$ can be interpreted as a measure of the typical population size.

To derive large population approximations for $X(\cdot)$, put $x_N(t) = X(t)/N$, and consider a sequence $\{x_N(\cdot)\}$ of such processes as $N \to \infty$, the remaining constants being held fixed. This yields a model of the form of (2.1), with $J = \{-1, +1\}$,

$$g_1(x) = \alpha_1 x - \gamma_1 x^2, \quad g_{-1}(x) = \alpha_2 x + \gamma_2 x^2.$$

The point $x_0 = (\alpha_1 - \alpha_2)/(\gamma_1 + \gamma_2)$ is an equilibrium point of the deterministic equations

$$\dot{x} = (\alpha_1 - \alpha_2)x - (\gamma_1 + \gamma_2)x^2,$$

and the drift and variance coefficients of the Ornstein–Uhlenbeck approximation to $N^{1/2}\{x_N(\cdot) - x_0\}$ are given by

$$\beta = - (\alpha_1 - \alpha_2); \quad \sigma^2 = 2(\alpha_1 - \alpha_2)(\alpha_1 \gamma_2 + \alpha_2 \gamma_1)/(\gamma_1 + \gamma_2)^2.$$

Since $\alpha_1 > \alpha_2$, the deterministic equilibrium is stable, and Theorem 2 can be applied, to show how closely, even in the tails, the normal approximation holds.

The state $x = 0$ is an absorbing state, and it is not difficult to show that $x_N$ must eventually reach 0, and the population become extinct, with probability one. It is therefore important, if using the model in a practical context, to know how long the quasi-equilibrium will persist, before extinction takes place. Theorem 1(iii) implies that the time to first leaving $|x - x_0| < N^{-1/2}c_N$ is of order $c_N^{-1} \exp\{(\alpha_1 - \alpha_2)c_N^2/\sigma^2\}$ whenever $c_N = o(N^{1/8})$, and hence asymptotically larger than any power of $N$ if $c_N = N^{1/8}/\log N$, say. Although this may well be enough, for practical purposes, to justify treating the normal approximation around $x_0$ as a true equilibrium distribution, it does not yield the distribution of the time until extinction, which would require $c_N = O(N^{1/2})$.

For a general sequence of processes $\{x_N\}$, the asymptotic distribution of exit times from larger regions than those specified in Theorem 1(iii) involves more than the local diffusion parameters, and depends on the discrete nature of the state space. The appropriate limit distribution can often be shown, on general grounds, to be negative exponential; but, even then, the difference equations for the normalising factor are usually intractable. However, for birth and death processes, of which the stochastic logistic process is a particular case, the equations are soluble (Bartlett (1960)): here, if $T_N$ denotes the time to extinction, we have

$$\lim_{N \to \infty} P[k_N T_N \geq x] = e^{-x},$$

where

$$k_N = \left\{ N \left( \frac{\gamma_1 + \gamma_2}{\alpha_1 + \alpha_2} \right)^{1/2} \frac{(\alpha_1 - \alpha_2)^2}{\gamma_1 + \gamma_2} \left\{ \frac{\alpha_1 \gamma_2 + \alpha_2 \gamma_1}{\alpha_1 (\gamma_1 + \gamma_2)} \right\}^{Na_1^y \gamma_1} \left\{ \frac{\alpha_1 \gamma_2 + \alpha_2 \gamma_1}{\alpha_2 (\gamma_1 + \gamma_2)} \right\}^{Na_2^y \gamma_2} \right\}.$$
3. Two-dimensional processes

Suppose now that, in direct extension of Section 2, \( x_n(\cdot), N \geq 1 \) is a sequence of continuous-time Markov processes, such that \( x_n(t) \) takes values on the two-dimensional lattice \( \{(k/N, m/N); k, m \text{ integral}\} \) and has transition rates given by

\[
x \rightarrow x + j/N \quad \text{at rate} \quad Ng_i(x), \quad j \in J
\]

for some finite \( N \)-independent set \( J \) of integer pairs. The functions \( g_i(x) \) are assumed to be multinomial expressions in \( x_1 \) and \( x_2 \). Suppose also that \( \sqrt{N} [x_n(0) - x_0] \rightarrow z_0 \) as \( N \rightarrow \infty \), where

\[
\sum_{j \in J} jg_i(x_0) = 0,
\]

and define matrices \( B \) and \( S \) by

\[
B_{km} = \sum_{j \in J} jg_{ij}^{(m)}(x_0); \quad S_{km} = \sum_{j \in J} j^2g_{ij}^{(m)}(x_0),
\]

where \( g_{ij}^{(m)}(x_0) \) denotes \( \partial g_i(x)/\partial x_m \) evaluated at \( x_0 \).

At any point \( x \) near \( x_0 \), the deterministic equations

\[
dx/\, dt = \sum_{j \in J} jg_i(x)
\]

can approximately be expressed by the linear system

\[
dx/\, dt = B(x - x_0),
\]

which suggests that behaviour near \( x_0 \) will be strongly influenced by the eigenvalues and eigenvectors of \( B \): note that, since \( B \) is not symmetric, the eigenvalues need not be real, and \( B \) need not have diagonal Jordan form. For brevity, only two of the possible cases are considered, that of distinct real eigenvalues and that of a complex pair.

Consider first the case where the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) of \( B \) are both real. As in one dimension, the signs of the eigenvalues are critical in determining the behaviour to be expected. For instance, the point \( x_0 \) gives rise to a stable quasi-equilibrium only if both eigenvalues are negative. In this case, the diffusion approximation to \( z_N(\cdot) = \sqrt{N} [x_n(\cdot) - x_0] \) is a stable bivariate Ornstein–Uhlenbeck process, with local drift matrix \( B \) and local covariance matrix \( S \); its stationary distribution is \( N(0, \Sigma) \), where

\[
B\Sigma + \Sigma B' = -S.
\]

A more convenient reformulation, which facilitates translation from one dimension, is to change coordinates by putting
\[ w_N = Mz_N, \]

where the \( j \)th row of \( M, j = 1, 2, \) is a left eigenvector of \( B \) with eigenvalue \( \lambda_j \).

The Ornstein-Uhlenbeck approximation to \( w_N \) then has local drift matrix \( \Lambda = \text{diag}(\lambda_1, \lambda_2) \), local covariance matrix \( T = \text{MSTM}' \) and stationary distribution \( N(0, C) \), where

\[ C_{ij} = -T_{ij}/(\lambda_i + \lambda_j); \]

the individual components of this process are themselves one-dimensional Ornstein-Uhlenbeck processes.

The length of time for which \( w_N \) remains faithful to this approximation is indicated by the following theorem. Let \( \tau(N, c_N) \) denote the time until \( w_N \) first crosses the contour

\[ (3.7) \quad \sum_{k=1}^2 \sqrt{2C_{kk}/w_k^2} \exp(w_k^2/2C_{kk}) = c_N \exp(c_N^2), \]

which, to order \( c_N^3 \), delimits the rectangle \( |w_k| \leq c_N \sqrt{2C_{kk}}, \ k = 1, 2. \)

**Theorem 3.** As \( N \to \infty \), if \( c_N = o(N^{1/8}) \),

\[ -\tau(N, c_N) \cdot 2(\lambda_1 + \lambda_2) \pi^{-1/2} c_N \exp(-c_N^2) \to E. \]

A similar argument applied to the exit times \( \tau_m(N, c_N), m = 1, 2, \) from regions approximating \( |w_k| \leq (c_N + 1 - \delta_{km}) \sqrt{2C_{kk}}, \ k = 1, 2, \) where \( \delta_{km} \) is the Kronecker delta, yields the negative exponential limit in distribution with normalising factor

\[ -2\lambda_m \pi^{-1/2} c_N \exp(-c_N^2). \]

Since \( \tau = \min(\tau_1, \tau_2) \), this indicates also that \( \tau_1 \) and \( \tau_2 \) are asymptotically independent, and hence that large excursions in the different directions away from \( x_0 \) occur asymptotically as independent Poisson streams.

Results for other combinations of signs of \( \lambda_1 \) and \( \lambda_2 \) can be derived in a similar way. Two examples which describe the escape distributions from unstable equilibria are stated below.

**Theorem 4.** If \( \lambda_1 > \lambda_2 > 0 \) and \( c_N = o(N^{1/2}/\log N) \), then, as \( N \to \infty \),

\[ (c_N^{-1}w_N^{(1)}(t_N), c_N^{-1}w_N^{(2)}(t_N)) \to N(Mz_0, C), \]

where \( t_N = \lambda_1^{-1} \log c_N \) and \( C_{ij} = T_{ij}/(\lambda_i + \lambda_j). \)

**Theorem 5.** If \( \lambda_1 > 0 > \lambda_2 \) and \( c_N = o(N^{1/2}/\log N) \), then, as \( N \to \infty \),

\[ (c_N^{-1}w_N^{(1)}(t_N), w_N^{(2)}(t_N)) \to N(I_1Mz_0, D), \]

where \( t_N = \lambda_1^{-1} \log c_N, I_1 \) is the matrix

\[
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix},
\]

and \( D = \text{diag}(T_{11}/2\lambda_1, -T_{22}/2\lambda_2). \)
Suppose now that $B$ has complex eigenvalues. In this case, the underlying deterministic motion is oscillatory, and rotating rather than fixed coordinates are appropriate. Let the eigenvalues be denoted by $\alpha \pm i\eta$, and corresponding left eigenvectors by $u \pm iv$; define a matrix $M(t)$ by

$$
M(t) = e^{\alpha t} \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} e^{-\beta t}
$$

(3.8)

$$
= \begin{pmatrix} u_1 \cos \eta t + v_1 \sin \eta t & u_2 \cos \eta t + v_2 \sin \eta t \\ -u_1 \sin \eta t + v_1 \cos \eta t & -u_2 \sin \eta t + v_2 \cos \eta t \end{pmatrix}.
$$

Then earlier methods are applicable, without essential change, to $w_N(t) = M(t)z_N(t)$. For the stable case, $\alpha < 0$, define

$$
T(t) = M(t)SM(t),
$$

(3.9)

$$
\sigma_i^2 = 2\alpha \{1 - \exp(-2\alpha \pi / \eta)\}^{-1} \int_0^{\pi/\eta} T_u(x) e^{-2\alpha x} dx,
$$

and

$$
U_i(t) = -(2\alpha)^{-1} \log \left\{1 - 2\alpha \sigma_i^{-2} \int_0^t T_u(x) e^{-2\alpha x} dx\right\}.
$$

(3.10)

let $\tau(N, c_n)$ denote the time until $w_N$ first crosses the variable contour

$$
\sum_{k=1}^{2} \sqrt{(2C_{kk}(t)w_k^2)} \exp\left(w_k^2/2C_{kk}(t)\right) = c_n^{-1} \exp\left(c_n^{-1}\right),
$$

where

$$
C_{kk}(t) = (\sigma_i^2/2\alpha) \exp\{2\alpha(t - U_k(t))\}.
$$

**Theorem 6.** As $N \to \infty$, if $c_n = o(N^{1/8}),$

$$
- \tau(N, c_n) \cdot 4\alpha \pi^{-1/2} c_n e^{-c_n^2} \overset{D}{\to} E.
$$

In this case, as earlier, the stationary distribution of the Ornstein–Uhlenbeck process approximating $z_N$ is $N(0, \Sigma)$, where $\Sigma$ satisfies (3.6).

For the unstable case, there is a counterpart to Theorem 4.

**Theorem 7.** If $\alpha > 0$ and $c_n = o(N^{1/2} / \log N)$, then, as $N \to \infty,$

$$
c_n^{-1} w_N(t_n) \overset{D}{\to} N(Mz_0, C),
$$

where $t_n = \alpha^{-1} \log c_n$, $M = M(0)$ and $C_{ij} = T_{ij}(0)/2\alpha$.

Similar techniques can be used in more than two dimensions, and most of the notation in this section carries over unchanged. In general, the transformation
\( w = M(t)z \), where \( M \) satisfies \( dM/dt + MB = 0 \), reduces the problem to a form in which one-dimensional methods can be applied, and it is essentially this transformation which is used above in two dimensions.

As a first application, consider the model for competition between two species, described in Reuter (1961), Example 1. For large population approximations, take a sequence of two-dimensional processes \( \{x_n\} \) with transition rates, in the notation of (3.1), determined by the choices \( J = \{(1,0), (0,1), (-1,0), (0,-1)\} \), and

\[
\begin{align*}
g_{(1,0)}(x) &= \alpha x_1; & g_{(0,1)}(x) &= \beta x_2; \quad (3.11) \\
g_{(-1,0)}(x) &= \gamma x_1 x_2; & g_{(0,-1)}(x) &= \delta x_1 x_2.
\end{align*}
\]

\( N \) is interpreted as a typical population size, and \( N x_n(t) \) as the numbers \((X_1, X_2)\) of each species alive at time \( t \). Reuter shows that, with probability one, one or other of the species eventually becomes extinct: we use the methods described above to give approximations to the probability that it is species 1 that dies out, for various choices of initial conditions.

The deterministic equations (3.4) for the process are

\[
\begin{align*}
\dot{x}_1 &= x_1(\alpha - \gamma x_2); & \dot{x}_2 &= x_2(\beta - \delta x_1); \quad (3.12)
\end{align*}
\]

and they integrate to give paths

\[
\begin{align*}
k(x_1, x_2) &= \alpha \log x_2 - \gamma x_2 - \beta \log x_1 + \delta x_1 = \text{constant},
\end{align*}
\]

some of which are illustrated, together with the sense in which they are described, in Figure 1. The methods of Barbour (1974) applied to the function \( h(x, t) = k(x_1, x_2) \) show that, as \( N \to \infty \), the extinction probability for species 1 with initial condition \( x_n(0) \), \( p_n(x_n(0)) \), tends to one as \( N \to \infty \) if either of the following three conditions holds:

\[
\begin{align*}
(i) \quad & \sqrt{N}k(x_n(0)) \to +\infty \quad \text{and} \quad \limsup x_n^{(0)}(0) < \beta/\delta \\
(ii) \quad & \sqrt{N}k(x_n(0)) \to -\infty \quad \text{and} \quad \liminf x_n^{(0)}(0) > \alpha/\gamma \\
(iii) \quad & \sqrt{N}\{x_n^{(0)}(0) - \beta/\delta\} \to -\infty \quad \text{and} \quad \sqrt{N}\{x_n^{(0)}(0) - \alpha/\gamma\} \to +\infty.
\end{align*}
\]

This naturally implies a complementary result for \( p_n \to 0 \). The critical initial conditions, therefore, are those which lie near the converging arms of the curves \( k(x_1, x_2) = k(\beta/\delta, \alpha/\gamma) \).

For initial conditions of the form \( \sqrt{N}\{x_n(0) - x^*\} \to z \) as \( N \to \infty \), where \( k(x^*) = k(\beta/\delta, \alpha/\gamma) \), the methods of Barbour (1974) can again be adapted to provide a result, whenever \( x^* \neq x_0 = (\beta/\delta, \alpha/\gamma) \). In the exceptional case, the situation is as discussed above. In particular, the matrices \( B \) and \( S \) are given by

\[
\begin{align*}
B &= \begin{pmatrix} 0 & -\gamma \beta/\delta \\
-\delta \alpha/\gamma & 0 \end{pmatrix}; \\
S &= \begin{pmatrix} 2\alpha \beta/\delta & 0 \\
0 & 2\alpha \beta/\gamma \end{pmatrix}.
\end{align*}
\]
and \( B \) has eigenvalues \( \pm \sqrt{(\alpha \beta)} \), real, distinct and of opposite sign. A suitable choice for \( M \) is, with the corresponding \( T \),

\[
M = \begin{pmatrix} -\delta \sqrt{\alpha} & \gamma \sqrt{\beta} \\ \delta \sqrt{\alpha} & \gamma \sqrt{\beta} \end{pmatrix}; \quad T = 2\alpha \beta \begin{pmatrix} \alpha \delta + \beta \gamma & -\alpha \delta + \beta \gamma \\ -\alpha \delta + \beta \gamma & \alpha \delta + \beta \gamma \end{pmatrix}.
\]

It then follows, by way of Theorem 4, that, if \( \sqrt{N} (x_N(0) - x_0) \to z \) as \( N \to \infty \),

\[
p_N(x_N(0)) \to \Phi(\sigma^{-1}(z_2 \gamma \sqrt{\beta} - z_1 \delta \sqrt{\alpha})),
\]

where \( \sigma^2 = \sqrt{(\alpha \beta)(\alpha \delta + \beta \gamma)} \). See also Ridler-Rowe (1964), where a contrasting asymptotic expression for the extinction probability is found, under a limiting procedure in which the birth rates \( \alpha \) and \( \beta \) become relatively negligible as \( N \to \infty \).
The competition model described above gives an example of an unstable quasi-equilibrium point. The Hamer–Soper model of a recurrent epidemic, described in Bartlett (1960), Chapters VI and VII, correspondingly gives an example of quasi-stable equilibrium. Consider a sequence of two-dimensional processes \( \{x_n\} \) as in (3.1), with transition rates given by \( J = \{(-1,1), (0,-1), (1,0)\} \) and

\[
g_{(-1,1)}(x) = \alpha x_1 x_2; \quad g_{(0,-1)}(x) = \beta x_2; \quad g_{(1,0)}(x) = \gamma.
\]

\( N \) represents a typical population size, and \( N x_n(t) \) gives the numbers \((X_1, X_2)\) of susceptibles and infected persons at time \( t \).

The deterministic equations (3.4) are

\[
\dot{x}_1 = \gamma - \alpha x_1 x_2; \quad \dot{x}_2 = x_2(\alpha x_1 - \beta);
\]

and the point \( x_0 = (\beta/\alpha, \gamma/\beta) \) satisfies (3.2). The matrices \( B \) and \( S \) are given by

\[
B = \begin{pmatrix} -\tau & -\beta \\ \tau & 0 \end{pmatrix}; \quad S = \begin{pmatrix} 2\gamma & -\gamma \\ -\gamma & 2\gamma \end{pmatrix};
\]

where \( \tau = \alpha \gamma / \beta \): \( B \) has eigenvalues

\[
\lambda_\pm = -\left(\tau/2\right) \left\{ 1 \pm \sqrt{1 - 4\beta/\tau} \right\}
\]

which are real or complex depending on the sign of \( 1 - 4\beta / \tau \), but which both always have negative real part, indicating stable equilibrium. The quasi-stationary distribution of \( z_n(t) = \sqrt{N(x_n(t) - x_0)} \) is found, from (3.6), to be \( N(0, \Sigma) \), where

\[
\Sigma = \frac{\beta}{\alpha} \begin{pmatrix} 1 + \beta / \tau & -1 \\ -1 & 1 + \tau / \beta \end{pmatrix}.
\]

Ridler-Rowe (1964) proves that the epidemic process must eventually reach the axis \( x_2 = 0 \) and the disease die out; and that the expected time until this happens is finite. The importance of Theorems 3 and 6 in this context is to show that the time scale for which quasi-equilibrium persists is typically large enough to be taken as infinite for practical purposes. The diffusion model may then be taken as an effective description of endemic disease.

A feature of particular interest in endemic diseases is the periodicity they exhibit. In the diffusion approximation, the critical parameter affecting periodicity is \( r = \beta^2 / \alpha \gamma \), which can be interpreted as the equilibrium ratio of susceptibles to infected persons. If \( r < 1/4 \), no periodic behaviour is predicted. If \( r > 1/4 \), an underlying period of \( 4\pi / \tau \sqrt{4r - 1} \) is predicted, together with a damping factor of \( \exp \left\{ -2\pi / \sqrt{4r - 1} \right\} \) per period, for the oscillations of the mean of \( z_n(t) \) about zero. However, the covariance matrix of \( z_n \), under the same approxima-
tion, after evolving for one period from any given starting point, is 
\[1 - \exp \left( -4\pi/\sqrt{(4r - 1)} \right) \] 
indicating that regular periodicity becomes dominant as \( r \) increases, whereas random fluctuation is more important for \( r \) near 1/4.

Note also that large \( r \) may, when approximating a realistic situation, equally indicate that the equilibrium number of infected persons is small, in which case the typical population size is not really large in the sense required for the approximation. An alternative model, involving extinction of infection subsequently regenerated by immigrant infectives, may then be more suitable.

4. Proofs

4(a). Proof of Theorem 1. Let \( \tau \) denote \( \min(t, \tau) \), and let \( \phi(x, t) : R \times R^+ \to R \) be differentiable with respect to \( t \) for each \( x \) in \( R \). Then, as in Barbour (1973), Theorem 1.2,

\[ \phi(x_\tau(\tau), \tau) - \int_0^\tau Q_\phi(x_N(u), u) \, du \]

is a martingale, where

\[ Q_\phi(x, t) = \partial \phi(x, t)/\partial t + N \sum_{j \in J} g_j(x) \{ \phi(x + j/N, t) - \phi(x, t) \}. \]

If, for \( \phi \), we choose a function \( \psi(\sqrt{N}(x - x_0), t) \), where \( \psi(z, t) \) is three times differentiable with respect to \( z \) for each \( t \), and is independent of \( N \), then (4.2), with \( z = \sqrt{N}(x - x_0) \), reduces to

\[ Q_\phi(z, t) = \frac{\partial \psi}{\partial t} + \beta z \frac{\partial \psi}{\partial z} + \frac{1}{2} \sigma^2 \frac{\partial^2 \psi}{\partial z^2} + N^{-1/2} h(\psi, z, t), \]

\[ = L_\phi(z, t) + N^{-1/2} h(\psi, z, t), \]

where, for some constant \( K \) independent of \( \psi, z \) and \( t \),

\[ |h(\psi, z, t)| \leq K \sum_{j \in J} \left\{ z^2 \left[ \frac{\partial \psi}{\partial z} \right]_* \left[ g_j(x) \right]_{**} \right. \]

\[ + z \left[ \frac{\partial^2 \psi}{\partial z^2} \right]_* + \left[ \frac{\partial^3 \psi}{\partial z^3} \right]_* \left[ g_i(x) \right]_{**} \right\}, \]

where

\([u(z)]_* = \sup_{|y - z| \leq \max_{|z|}/\sqrt{N}} |u(y)|, \]

\([f(x)]_{**} = \sup_{|x - y| \leq \sqrt{N}} |f(x)|. \]

The proofs of all the theorems which follow hinge on choosing \( \psi(z, t) \) in such a way that the desired result can be deduced from the martingale property applied to (4.1).
For Case (i), take first
\[ \psi(z, t) = t - 2\sigma^{-2} \int_0^z \exp(-\beta u^2/\sigma^2) \int_0^u \exp(\beta v^2/\sigma^2) \, dv \, du: \]
then \( L_\phi(z, t) = 0 \), and
\[ \sup_{|z| \leq c_N, t \geq 0} |h(\psi, z, t)| \leq Kc_N, \]
for some \( K \). Hence, from (4.1), since \( x_N(\cdot) \) can take only finitely many values in \( t \leq \tau \), it follows that
\[ E\{\psi(z_N(\tau), \tau)\} = \psi(z_N(0), 0)[1 + O(c_N/\sqrt{N})], \]
where the order term is independent of \( t \), and hence that
\[ E(\tau) \leq 2\sigma^{-2} \int_0^{c_N} \exp(-\beta u^2/\sigma^2) \int_0^u \exp(\beta v^2/\sigma^2) \, dv \, du + O(1). \]
Thus, letting \( t \to \infty \),
\[ E(\tau) = O(\log c_N) = O(\log N). \]
Now, for any real \( \theta \), take
\[ \psi(z, t) = e^{i\theta \psi} \{ \Gamma(1/2) i/2\beta \} M(i\theta/2\beta, 1/2, -\beta z^2/\sigma^2) \]
\[ + 2\Gamma(1/2) i/2\beta \sqrt{\beta/\sigma^2} M(1/2, i/2\beta, 3/2, -\beta z^2/\sigma^2), \]
where \( M(a, b, x) \) is the confluent hypergeometric function defined in Abramowitz and Stegun (1965), page 504. Then \( L_\phi(z, t) = 0 \), and the elementary properties of the confluent hypergeometric function show that
\[ \sup_{|z| \leq c_N, t \geq 0} |h(\psi, z, t)| \leq Kc_N; \]
\[ e^{i\theta \psi}(c_N, t) = \psi^*(c_N) = 2\sqrt{\pi} (\beta c_N^2/\sigma^2)^{-1/2} [1 + O(c_N^{-1})]; \]
\[ \psi(-c_N, t) = O(c_N^{-1}). \]
Hence, from (4.1),
\[ E\{\exp(i\theta \tau_\star)\psi^*(c_N) + O(c_N^{-1}) + O(N^{-1/2}c_N E(\tau))\} = \psi(z_N(0), 0), \]
and Part (i) follows from the continuity theorem for characteristic functions.
For Case (ii), pick any \( \theta > 0 \), put
\[ s = s_N(\theta) = -2\beta \sqrt{(-\beta/\pi\sigma^2)c_N \exp(\beta c_N^2/\sigma^2)}, \]
and choose
\[ \psi(z, t) = e^{-s} \{ M(-s/2\beta, 1/2, -\beta z^2/\sigma^2) \]
\[ + \theta^{-1}(1+\theta)\sqrt{(-\pi/\beta\sigma^2)sz} M(1/2, s/\beta, 3/2, -\beta z^2/\sigma^2)\}. \]
then \( L_\psi(z, t) = 0 \), and
\[
| h(\psi, z, t) | \leq K \frac{1}{\sigma^2} \exp \left( - \beta z^2 / \sigma^2 \right);
\]
(4.9) \( e^{\ast \psi}(c_N, t) = 2(1 + \theta) \left[ 1 + O(1) \right] \); \( e^{\ast \psi}(-c_N, t) = O(1) \).
\( \psi(z_N(0), 0) = 1 + o(1) \).

Using (4.1), as above, it follows that
\[
| E\{ \psi(z_N(\tau), \tau) \} - \psi(z_N(0), 0) | \leq \frac{N}{\sqrt{2}} \int_0^\infty E | h(\psi, z_N(u), u) I[\tau \geq u] | \, du,
\]
where \( I[A] \) is the indicator function of the event \( A \). The expectation in the integrand is evaluated using (4.9) and Lemma 4.1, below, with any fixed \( T_0 \); an integration by parts leads to the expression
\[
E | h(\psi, z_N(u), u) I[\tau \geq u] | \leq \frac{N}{\sqrt{2}} K \frac{1}{\sigma^2} \int_{c_N}^{\infty} \exp \left[ - z^2 \left( - \beta / \sigma^2 - 1 / 2 \nu(u, T_0) \right) \right] \, dz \cdot \{ 1 + O(c_N / \sqrt{N}) \},
\]
and the integrand is uniformly bounded because, for all \( u \geq 0 \), \( - 2 \beta \nu(u, T_0) / \sigma^2 < 1 \). The remainder of the proof is immediate.

The proof in Case (ii) is similar to those already given. \( E(\tau) \) is bounded using \( \psi(z, t) \equiv t - z^2 / \sigma^2 \), and the proof is completed using
\[
\psi(z, t) = \exp \left( - \frac{t}{\sigma^2} \right) \left[ \sinh \left( \sqrt{2} \theta \right) \cosh \left( \sqrt{2} \theta \right) / c_N \right] + \cosh \left( \sqrt{2} \theta \right) \sinh \left( \sqrt{2} \theta \right) / c_N \right].
\]

4 (b). \textit{Proof of Theorem 2.} The first step towards proving Theorem 2 is to establish the lemma used during the proof of Theorem 1(ii), which is a somewhat weaker result than the eventual theorem. The notation of Theorem 1 is assumed from here onwards, and \( K \) is used to denote a generic constant, not necessarily the same at each appearance. Without real loss of generality, \( z_N(0) \) is taken to be zero.

\textit{Lemma 4.1.} If \( \beta < 0 \), then, for any \( T_0 > 0 \),
\[
P[z_N(T) \geq x \cap \tau \geq T] \leq \exp \left\{ - x^2 / 2 \nu(T, T_0) \right\} \left\{ 1 + O(c_N / \sqrt{N}) \right\},
\]
uniformly in \( T \geq 0 \) and \( 0 \leq x \leq c_N \), where \( \nu(T, T_0) = \max \{ \nu(T), \nu(T_0) \} \), and
\[
\nu(T) = - \sigma^2 (1 - e^{-\sigma T}) / 2 \beta.
\]

\textit{Proof.} Choose any \( T \geq T_0 \) and \( x \) in \([0, c_N] \), and put \( \theta = xe^{\beta T} / \nu(T) \). Consider
\[(4.10) \quad \psi(z, t) = \exp(\theta ze^{-\beta t} + \sigma^2 \theta^2 e^{-2\beta t}/4\beta): \]

in the notation of Theorem 1, \( L_\psi(z, t) = 0 \), and

\[ |h(\psi, z, t)| \leq \psi(z, t)K_1(c_N, t), \]

where

\[K_1(c_N, t) = K(\theta^3 e^{-3\beta t} + c_N^2 \theta e^{-\beta t}).\]

From (4.1), for each \( t \in [0, T] \),

\[w_N(t) = |E\{\psi(z_N(\tau), \tau)\} - \exp(\sigma^2 \theta^2/4\beta)| \leq N^{-1/2} \int_0^t E|h(\psi, z_N(\tau), \tau)| \, du \]

\[\leq N^{-1/2} \int_0^t K_1(c_N, u) \{w_N(u) + \exp(\sigma^2 \theta^2/4\beta)\} \, du,\]

whence

\[(4.12) \quad |E\{\psi(z_N(\tau_T), \tau_T)\} \exp(-\sigma^2 \theta^2/4\beta) - 1| = O(N^{-1/2} c_N^3),\]

uniformly in \( T \geq T_0 \). Hence

\[E[\exp\{\theta z_N(\tau_T) e^{-\beta \tau_T}\}] \leq \exp\{-\sigma^2 \theta^2 (e^{-2\beta T} - 1)/4\beta\} \{1 + O(N^{-1/2} c_N^3)\},\]

and so

\[(4.13) \quad P[\sup_{0 \leq t \leq T} |z_N(t)| \geq c_N] \leq \exp(-x^2/2v(T)) \{1 + O(N^{-1/2} c_N^3)\},\]

as required.

For \( 0 \leq T \leq T_0 \), take \( \theta = xe^{\beta T}/v(T_0) \), and argue as before: only (4.13) needs any alteration.

The next lemma provides a bound for the extreme fluctuations of \( z_N \) over an interval \( 0 \leq t \leq T \).

**Lemma 4.2.** If \( \beta < 0 \) and \( c_N = O(N^{-1/8}) \) as \( N \to \infty \), there exists a constant \( K_0 \) such that

\[P\left[\sup_{0 \leq t \leq T} |z_N(t)| \geq c_N \right] \leq K_0 T c_N \exp(\beta c_N^3/\sigma^2),\]

uniformly in \( T > 0 \).

**Proof.** Set \( \psi(z, t) = e^{-uM(-s/2\beta, 1/2, -\beta z^2/\sigma^2)} \): then, as in Theorem 1(iii),

\[|E\{\psi(z_N(\tau_T), \tau_T)\} - 1| \leq KN^{-1/2} c_N^4.\]
Hence, from the asymptotic form of \( M(a, b, x) \) for large \( x \),

\[
P[\tau \leq T] \leq K e^{\gamma \tau} s^{-1} c_N \exp\left(\beta c^2 / \sigma^2\right)\{1 + O(N^{-1/2} c_N)\},
\]

and the result follows by taking \( s = T^{-1} \).

To prove Theorem 2, pick any \( X, \sqrt{\log N} \leq X \leq N^{1/8} \), and set \( c_N = 3X(1 + \alpha)/(1 - e^{2\beta T}) \), where \( \alpha > 0 \) is to be fixed later. Now, choosing any \( 0 \leq x \leq X \), put \( \theta = xe^{\beta T} / v(T) \), and take \( \psi(z, t) \) as in (4.10): Equation (4.12) follows as in Lemma 4.1. Now, from Lemma 4.2 and the choice of \( c_N \), the contribution to \( E\{\psi(z_N(\tau_T), \tau_T)\} \) from the event \( \tau < T \) is of relative order at most

\[
TX(1 + \alpha) \exp\{\alpha X^2 \sigma^2 (2\alpha + 3) / \beta[v(T)]^2\},
\]

and can be made negligible by choosing \( \alpha = \alpha(\delta) \) sufficiently large. Hence it follows that

(4.14) \[ E[\exp\{xz_N(\tau_T) / v(T) - x^2 / 2v(T)\}] = 1 + O(X^3 N^{-1/2}). \]

Now choose \( \theta = (x + i\phi)e^{\beta T} / v(T) \), and define

\[
q(u) = e^u - 1 - u - u^2 / 2,
\]

\[
r_N(\theta, t) = N \sum_{j \in I} g_j(x_0) q(j\theta e^{-\mu t}N^{-1/2})
\]

\[
R_N(\theta, t) = \int_0^t r_N(\theta, u) du:
\]

note that

(4.15) \[ |r_N(\theta, t)| \leq KN^{-1/2} e^{3\beta(T-t)}(X^3 + |\phi|^3), \]

and that, for some suitable constant \( k_0 \),

(4.16) \[ |R_N(\theta, T) - R_N(\theta, t)| \leq -\phi^2 \sigma^2 (1 - e^{2\beta(T-t)}) / 8\beta[v(T)]^2 + KX^3 / \sqrt{N}, \]

for all \( |\phi| \leq k_0 \sqrt{N} \). Then, with the previous choice of \( c_N \), and with any \( \phi \) satisfying \( |\phi| \leq k_0 \sqrt{N} \), take

\[
\psi(z, t) = \exp\left(\theta z e^{-\mu t} + \phi^2 \sigma^2 e^{-2\beta T} / (4\beta - R_N(\theta, t))\right):
\]

much as before, but also using (4.14), (4.15) and (4.16), it follows that

\[
|E[\exp\{[(x + i\phi)z_N(\tau_T) - \frac{1}{2} x^2] / v(T)\}] - \exp\{[ix\phi - \frac{1}{2} \phi^2] / v(T) + R_N(\theta, T)\}]| \leq K N^{-1/2} (1 + \phi^2)^{-1} \{X^2 + X\phi^2 (1 + N^{-1/2} |\phi|)\}.
\]

(4.17)

The rest of the proof is similar to that of Cramèr's large deviation theorem, given in Feller (1970), Chapter XVI.7, Theorem 1. Equations (4.14) and (4.17) are used to estimate the distribution of a random variable \( Z \), with
\[ P[Z = y] = e^{s y / u(T)} P[z_N(\tau_T) = y] / E[\exp \{ xz_N(\tau_T)/u(T) \}] \]

and thus with characteristic function \( f_\phi \) satisfying
\[
|f_\phi(\phi/v(T)) - \exp\{(2i \phi - \phi^3)/2v(T)\}| \leq K \phi |\phi|:
\]

then, from (4.14) and (4.17), for \( |\phi| \leq k_0 \sqrt{N} \),
\[
|f_\phi(\phi/v(T)) - \exp\{(2i \phi - \phi^3)/2v(T) + R_N(\theta, T)\}| \\
\leq KN^{-1/2}(1 + \phi^2)^{-1}(X^3 + X\phi^2(1 + N^{-1/2}|\phi|)) \\
= K(X, \phi, N),
\]
say, and so
\[
|f_\phi(\phi/v(T)) - \exp\{(2i \phi - \phi^3)/2v(T)\}| \\
\leq K(X, \phi, N) + \exp\{-\phi^2/2v(T)\}|\exp\{R_N(\theta, T)\} - 1| \\
\leq K(X, \phi, N) + KN^{-1/2}\exp\{-\phi^2/4v(T)\}(|\phi|^3 + X^3).
\]

Writing \( \phi \) for \( \phi/v(T) \) and applying Chung (1968), page 208, Lemma 2, it follows from (4.18) and (4.19) that
\[
\sup_y P[Z = y] - \Phi((y - x)/\sqrt{v(T)}) \\
\leq \frac{2}{\pi} \int_{k_0 \sqrt{N}/v(T)} \phi^{-1}|f_\phi(\phi) - \exp\{ix\phi - v(T)\phi^2/2\}| d\phi + O(N^{-1/2}) \\
= O(X^3N^{-1/2}\log N).
\]

It remains to deduce the distribution of \( z_N(\tau_T) \) from that of \( Z_\phi \). A summation by parts gives
\[
P[z_N(\tau_T) \geq y] / E[\exp\{xz_N(\tau_T)/v(T)\}] \\
= P[Z \geq y] e^{-s y / u(T)} - \frac{x}{u(T)} \int_y^\infty e^{-s y / u(T)} P[Z \geq u] du.
\]

It is now immediate from (4.20) and (4.14) that
\[
P[z_N(\tau_T) \geq y] = e^{x^2/2u(T)}[e^{-x^2/2u(T)}(1 - \Phi(y/\sqrt{v(T)}) \]
\[+ e^{-s y / u(T)}O\{X^3N^{-1/2}\log N\}],
\]
and so, choosing \( y = x \),
\[
P[z_N(\tau_T) \equiv x] = \{1 - \Phi(x/\sqrt{v(T)})\} \{1 + O(X^4N^{-1/2}\log N)\};
\]

the theorem now follows.

4 (c). Proof of Theorems 3–7. The operator \( L \) analogous to that in (4.3) can be defined by
\[
L^x(z, t) = (\partial / \partial t + z'B'De + \frac{1}{2}D'eSDe)\psi,
\]
the right-hand side being evaluated at \((z, t)\), where \( D_e \) denotes the vector of partial differential operators with respect to the components of \( z \). If the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) of \( B \) are real and distinct, the change of variable \( w = Mz \), where the \( j \)th row of \( M \), \( j = 1, 2 \), is a left eigenvector of \( B \) with eigenvalue \( \lambda_j \), reduces the operator to the form
\[
L^x(w, t) = (\partial / \partial t + w'\Lambda D_w + \frac{1}{2}D_w'\Lambda T D_w)\psi,
\]
where \( \Lambda = \text{diag}(\lambda_1, \lambda_2) \) and where \( T = MSM' \). The linear equation \( L^x(w, t) = 0 \) has solutions \( \psi \) which are functions of either variable \( w_i \) and \( t \) alone, when it reduces to the one-dimensional form of (4.3), with \( \beta = \lambda_i \) and \( \sigma^2 = T_{ii} \). Because of this, methods from the one-dimensional case can be adapted without much difficulty.

For instance, in Theorem 3, where both eigenvalues of \( B \) are negative, the appropriate choice for \( \psi \) is
\[
\psi(w, t) = e^{-s} \sum_{k=1}^{2} \lambda_k M(-s/2\lambda_k, 1/2, w_i^k/2C_{kk}),
\]
where \( s = s_N(\theta) = 2\pi^{-1/2}\theta c_n \exp(-c_N^2) \), for any fixed \( \theta > 0 \). The proof follows as for Case (iii) of Theorem 1, including an appropriate extension of Lemma 4.1. Results for other combinations of the signs of \( \lambda_1 \) and \( \lambda_2 \), such as those of Theorems 4 and 5, can be obtained in a similar way.

For Theorems 6 and 7, the change of variable \( w = M(t)z \), where \( M(t) \) is defined in (3.8), reduces (4.21) to
\[
L^x(w, t) = (\partial / \partial t + \alpha w'D_w + \frac{1}{2}D_w'T(t)D_w)\psi,
\]
where \( T(t) = M(t)SM(t)' \). As with (4.22), the equation \( L^x(w, t) = 0 \) has solutions \( \psi \) which are functions of either variable \( w_i \) and \( t \) alone, when it reduces to the form of (4.3), except that \( T_{ii}(t) \), instead of being constant, is periodic in \( t \) with frequency \( \eta/\pi \). This occasions some changes of detail: for example, in the counterparts of expressions such as (4.23), the appropriate basic form is
\[
\exp(-sU_i(t))M(-s/2\alpha, 1/2, -\alpha w_i^2\exp[2\alpha(U_i(t) - t)]/\sigma_i^2),
\]
where \( \sigma_i^2 \) and \( U_i(t) \) are as in (3.9) and (3.10). The essential method, however, remains unchanged.

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