On Laslett’s Transform for the Boolean Model

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22.01.01

Abstract
Consider the Boolean model in \( \mathbb{R}^2 \), where the germs form a homogeneous Poisson point process with intensity \( \lambda \) and the grains are convex compact random sets. It is known (see e.g. Section 9.5.3 in Cressie (1993)) that Laslett’s rule transforms the exposed tangent points of the Boolean model into a homogeneous Poisson process with the same intensity \( \lambda \). In the present paper, we give a simple proof of this result, which is based on a martingale argument. We also consider the cumulative process of uncovered area in a vertical strip and show that a (linear) Poisson process with intensity \( \lambda \) can be embedded in it.

Keywords  STOCHASTIC GEOMETRY, RANDOM CLOSED SETS, BOOLEAN MODEL, TANGENT POINTS, POISSON PROCESS, MARTINGALE APPROACH

Classification Primary 60D05; secondary 60G55, 60G57, 60G44

1 Laslett’s Theorem

Let \( X \) be a stationary Poisson process on \( \mathbb{R}^2 \) with finite positive intensity \( \lambda \), and let \( \{ X_n, n \geq 1 \} \) be an enumeration of its points. Let \( \{ M_n, n \geq 1 \} \) be an independent sequence of random closed sets that are independent and identically distributed, nonempty, compact and convex with probability 1, and such that

\[
\mathbb{E}(\| M_1 \|^2) < \infty ,
\]

(1.1)

where \( \| M_1 \| := \sup \{ \| x \| : x \in M_1 \} \) and \( \| x \| \) denotes the norm of \( x \). Then the sequence \( \{(X_n, M_n), n \geq 1 \} \) is a marked point process; the \( X_n \) are called germs, and the \( M_n \) grains. Using (1.1), it can be shown that the union of shifted grains

\[
\Xi := \bigcup_n (M_n + X_n)
\]

(1.2)

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is a closed set with probability 1. Thus $\Xi$ itself is a random closed set, and
\[ \Xi \overset{d}{=} \Xi + x \]  
(1.3)
for each $x \in \mathbb{R}^2$, because $X$ is stationary. This $\Xi$ is called the Boolean germ-grain model.

Writing $x = (x^1, x^2) \in \mathbb{R}^2$, the random vector $X'_n = (X'^1_n, X'^2_n)$ with
\[ X'^1_n := \inf \{ x^1 : (x^1, x^2) \in M_n + X_n \text{ for some } x^1 \} \]
and
\[ X'^2_n := \inf \{ x^2 : (x^1, x^2) \in M_n + X_n \} \]
is called the (left-lower) tangent point of the shifted grain $M_n + X_n$. For simplicity, we assume that
\[ X'^n = \sup \{ x^1 : (x^1, x^2) \in M_n + X_n \} \]
(1.4)
with probability 1. Notice that condition (1.4) is fulfilled if the distribution of $M_n$ is isotropic, i.e. its distribution is invariant under non-random rotations.

Furthermore, without loss of generality, we can (and shall) assume that
\[ X'_n = X_n \]
(1.5)
for all $n$; that is, we identify the tangent points with the germs; see pp. 66/67 of Stoyan, Kendall and Mecke (1995).

The tangent point $X'_n$ is said to be exposed if
\[ X'_n \in \overline{\Xi^c} = \overline{\mathbb{R}^2 \setminus \Xi} ; \]
that is, $X'_n$ belongs to the closure of the complement $\Xi^c$ of the Boolean model $\Xi$ given in (1.2).

The idea of Laslett’s transform is to consider the restriction $\Xi_+ := \Xi \cap \mathbb{R}^2_+$ of $\Xi$ to the halfplane $\mathbb{R}^2_+ := \{ (x^1, x^2) : x^1 \geq 0 \}$, and to ‘remove’ $\Xi_+$ from the two-dimensional image, except for the exposed tangent points, then closing up, along the $x^1$-direction, the gaps in the background left by the removed set $\Xi_+$. More formally, let $\{Y_1, Y_2, \ldots \}$ denote the set of all exposed tangent points of $\Xi_+$, where $Y_n := (Y^1_n, Y^2_n)$. Then the closed segment connecting the points $(0, Y^2_n)$ and $(Y^1_n, Y^2_n)$ contains a finite number $k$, say, of entrance points $(Y^1_{i_n}, Y^2_{i_n}), \ldots, (Y^1_{k_n}, Y^2_{k_n})$ from $\Xi_+^c$ into the set $\Xi_+$, and $k - 1$ exit points $(Y^1_{1_n}, Y^2_{1_n}), \ldots, (Y^1_{k_n}, Y^2_{k_n})$ from $\Xi_+^c$ to $\Xi_+$, where we put $Y^1_{1_n} = 0$ if $(0, Y^2_n) \in \Xi_+$. Notice that $k \geq 1$ and $Y^1_{k_n} = Y^1_{1_n}$. Furthermore, $Y^1_{i_n} < Y^1_{i+1} < \ldots < Y^1_{k_n} < Y^1_{k+1} > Y^1_{k_n} = Y^1_{1_n}$ if $k > 1$. The transformed tangent point $Z_n := (Z^1_n, Z^2_n)$ corresponding to the exposed tangent point $Y_n$ is now given by
\[ Z^1_n := Y^1_n - \sum_{j=1}^{k-1} (Y^2_{j+1} - Y^2_{j}) ; \quad Z^2_n := Y^2_n . \]

Let $Z$ denote the random counting measure associated with the transformed tangent points $\{Z_n, n \geq 1 \}$. Formally, the Laslett transform $L$ is the mapping which transforms the random counting measure $Y$ associated with the exposed tangent points $\{Y_n\}$ into the random counting measure $Z$.  

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Theorem 1.1 If $X$ is Poisson with intensity $\lambda$, then $Z$ is the $\mathbb{R}_+^2$-restriction of a Poisson process with the same intensity $\lambda$.

This theorem is due to G. M. Laslett. It has a number of applications in the statistical analysis of the Boolean model; see, for example, Section 9.5.3 in Cressie (1993), Section 9.2 in Molchanov (1997), and Section 5.4 in Schneider and Weil (2000).

2 Martingale Approach

We now give a simple proof of Theorem 1.1, which is based on a martingale argument.

Consider the restriction

$$
\Phi_t := \{(X_n, M_n) : X_n^2 \leq t\}
$$

of the germ-grain process $\{(X_n, M_n) n \geq 1\}$ to germs from the set $\mathbb{R} \times (-\infty, t]$, and let $\mathcal{F}_t := \sigma(\Phi_t)$ denote the $\sigma$-algebra of subsets of $\Omega$ generated by $\Phi_t$; $t \geq 0$. Notice that $\{\mathcal{F}_t, t \geq 0\}$ is a right-continuous filtration. A random measure $\nu$ on $\mathbb{R} \times (0, \infty)$ is called adapted if the process $\{\nu(B \times (0,t)), t \geq 0\}$ is $\{\mathcal{F}_t\}$-adapted for each bounded Borel set $B \in \mathcal{B} = \mathcal{B}(\mathbb{R})$.

Let $\Psi_t$ be the restriction of the point process $Y$ of exposed tangent points to the set $\mathbb{R} \times (0, t]$. Then, using our assumption that $X_n$ is the tangent point of $M_n + X_n$, it is not difficult to see that $\Psi_t$ is $\mathcal{F}_t$-measurable for all $t \geq 0$. Thus $\Psi_\infty := \lim_{t \to \infty} \Psi_t$ is adapted.

A random measure $\nu$ on $\mathbb{R} \times (0, \infty)$ is called predictable if $\{\nu(B \times (0,t)), t \geq 0\}$ is $\{\mathcal{F}_t\}$-predictable for each bounded $B \in \mathcal{B}$. Following Kallenberg (1997), p. 422, we say that a process on $\mathbb{R} \times [0, \infty)$ is predictable if it is $\mathcal{B} \otimes \mathcal{P}$-measurable, where $\mathcal{P}$ denotes the predictable $\sigma$-algebra in $[0, \infty) \times \Omega$.

Let $\nu$ be any locally integrable, adapted random measure on $\mathbb{R} \times (0, \infty)$. It is well known (see, for example, Theorem 22.22 in Kallenberg (1997)) that there exists an a.s. unique predictable random measure $\tilde{\nu}$ on $\mathbb{R} \times (0, \infty)$ such that

$$
\mathbb{E} \int \xi(x)\nu(dx) = \mathbb{E} \int \xi(x)\tilde{\nu}(dx)
$$

for each nonnegative predictable process $\xi$ on $\mathbb{R} \times [0, \infty)$. The random measure $\tilde{\nu}$ in (2.1) is called a compensator of $\nu$. If there exists a nonnegative predictable process $\eta$ on $\mathbb{R} \times [0, \infty)$ such that

$$
\tilde{\nu}(B) = \int_B \eta(x) dx
$$

for all $B \in \mathcal{B}(\mathbb{R} \times (0, \infty))$, then $\eta$ is called a stochastic intensity of $\nu$.

Theorem 2.1 If $X$ is Poisson with intensity $\lambda$, then a compensator of $\Psi_\infty$ is

$$
\tilde{\Psi}_\infty(dx^1 \times dx^2) := \lambda 1((x^1, x^2) \not\in \Xi_{x^2}) dx^1 dx^2
$$

(2.3)
for \( x^1 \in \mathbb{R} \) and \( x^2 > 0 \), where
\[
\Xi_{x^2} := \bigcup_{n: x_n^2 < x^2} (M_n + X_n).
\]

**Proof** Let \( \xi \) be a nonnegative predictable process on \( \mathbb{R} \times [0, \infty) \). It is not difficult to see that the nonnegative process \( \zeta \) given by
\[
\zeta(x^1, x^2) := 1((x^1, x^2) \not\in \Xi_{x^2})
\]
is predictable. Thus, the product \( \xi \cdot \zeta \) is also predictable. On the other hand, it is well known (see, for example, Corollary 22.25 in Kallenberg (1997)) that the compensator \( \widetilde{X} \) of the Poisson counting measure \( X \) is given by
\[
\widetilde{X}(B) = \lambda|B|.
\]
Thus, we have
\[
\mathbb{E} \int \xi(x) \psi \infty(dx) = \mathbb{E} \int (\xi \cdot \zeta)(x) X(dx)
= \lambda \mathbb{E} \int (\xi \cdot \zeta)(x) dx
= \mathbb{E} \int \xi(x) \tilde{\psi} \infty(dx).
\]
This completes the proof, because the random measure \( \tilde{\psi} \infty \) given in (2.3) is predictable. \( \Box \)

Using Theorem 2.1, we can now give a simple proof of Theorem 1.1.

Let \( \Theta_t \) be the restriction of the point process \( Z \) of transformed tangent points to the set \( \mathbb{R} \times (0, t] \), where \( \mathbb{R} \times := [0, \infty) \). Then \( \Theta_t \) is \( \mathcal{F}_t \)-measurable for all \( t \geq 0 \). Consider the random measure \( \Theta \infty := \lim_{t \to \infty} \Theta_t \) on \( \mathbb{R} \times (0, \infty) \). Notice that
\[
\Theta \infty = \psi \infty \circ L^{-1}
\]
and, by the result of Theorem 2.1, that
\[
\tilde{\psi} \infty \circ L^{-1}(B) = \lambda|B|,
\]
where \( L \) denotes the Laslett transform discussed in Section 1. Since \( L : \mathbb{R}^2_+ \to \mathbb{R}^2_+ \) is a predictable mapping, Theorem 22.24 in Kallenberg (1997) implies that \( \Theta \infty \) is a stationary Poisson process with intensity \( \lambda \). This means that \( Z \) is Poisson in the upper right quadrant. Since the \( x^1 \)-axis can be put at an arbitrary level, the statement of Theorem 1.1 follows.
3 Cumulative Process of Uncovered Area

As a by-product of Theorem 2.1, we get the following interesting result on the cumulative process of uncovered area in a vertical strip.

Let \( c > 0 \) be fixed and let the exposed tangent points \( Y_n := (Y^1_n, Y^2_n) \) in the set \( [0, c] \times (0, \infty) \) be numbered in such a way that

\[
0 < Y^2_1 < Y^2_2 < \ldots .
\]

Furthermore, let

\[
V(t) := \int_0^t \int_0^c 1((x^1, x^2) \not\in \Xi_{x^2}) \, dx^1 \, dx^2
\]

and consider the random measure \( V := \{V(B), B \in \mathcal{B}((0, \infty))\} \) induced by the cumulative process of vacant area \( \{V(t), t \geq 0\} \) defined in (3.4). Then,

\[
V_n = \begin{cases} 
  V(Y^2_1) & \text{if } n = 1 \\
  V(Y^2_n) - V(Y^2_{n-1}) & \text{if } n > 1 
\end{cases}
\]

is the increase of uncovered area between the \((n - 1)\)th and \(n\)th exposed tangent points.

**Theorem 3.1** If \( X \) is Poisson with intensity \( \lambda \), then \( V_1, V_2, \ldots \) are independent and identically distributed random variables with exponential distribution \( \text{Exp}(\lambda) \).

**Proof** By the result of Theorem 2.1, \( \lambda V \) is an compensator of the point process \( Y^2_1, Y^2_2, \ldots \) on \((0, \infty)\). Consider the counting process \( \{Y^2(t), t \geq 0\} \) where

\[
Y^2(t) := \begin{cases} 
  \max\{n : Y^2_n \leq t\} & \text{if } Y^2_1 \leq t, \\
  0 & \text{otherwise.}
\end{cases}
\]

For each \( s \geq 0 \), define the \( \{\mathcal{F}_t\}\)-stopping time \( \tau(s) \) by

\[
\int_0^{\tau(s)} \lambda V(v) \, dv = s .
\]

Then, the time-changed counting process \( \{N(t), t \geq 0\} \) with

\[
N(t) := Y^2(\tau(t))
\]

is a Poisson process with intensity 1; see Corollary 22.26 in Kallenberg (1997). Thus, by rescaling, it follows that the random variables, \( V_1, V_2, \ldots \) defined in (3.5) form a Poisson process on \((0, \infty)\) with intensity \( \lambda \).  \( \Box \)

4 Concluding Remarks

The original proof of Theorem 1.1 given by G. M. Laslett is based on discretization and sequential conditioning; see Cressie (1993), pp. 766–768. This argument is partially
heuristic, though, with some technical difficulty, it could be made rigorous. Our martingale approach is at least formally simpler, and gives more insight into the underlying structure. In particular, the notion of a tangent point and Laslett’s transform can be extended to germ–grain models in \( \mathbb{R}^d \), where \( d \geq 2 \); see Section 2.1 of Molchanov (1995). Using these definitions, \( d \)-dimensional analogues of Theorems 1.1, 2.1 and 3.1 can be proved by arguments similar to those given in Sections 2 and 3 of the present paper.

Acknowledgement

Both authors wish to thank the Department of Mathematics and Statistics at Monash University Melbourne for their warm hospitality.

References


