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LIMIT THEOREMS FOR THE SIMPLE BRANCHING PROCESS ALLOWING IMMIGRATION, II. THE CASE OF INFINITE OFFSPRING MEAN

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Abstract

This paper presents some limit theorems for the simple branching process allowing immigration, $\{X_n\}$, when the offspring mean is infinite. It is shown that there exists a function U such that $\{e^{-n}U(X_n)\}$ converges almost surely, and if $s=\sum b_j \log^+ U(j)<\infty$, where $\{b_j\}$ is the immigration distribution, the limit is non-defective and non-degenerate but is infinite if $s=\infty$.

When $s=\infty$, limit theorems are found for $\{U(X_n)\}$ which involve a slowly varying non-linear norming.

BIENAYMÉ–GALTON–WATSON BRANCHING PROCESS; IMMIGRATION; MARTINGALE CONVERGENCE; LIMIT THEOREMS; REGULAR VARIATION

1. Introduction

Let $\{X_n : n=0,1,\cdots\}$ denote a Bienaymé–Galton–Watson process with immigration, for which the offspring probability generating function, f, satisfies $m=f'(1-)=\infty$ and $\{I_n : n=1,2,\cdots\}$, the number of immigrants into successive generations, are independent and identically distributed with probability generating function b satisfying $b(0)<1$. Suppose also that $X_0=0$.

This paper describes the behaviour of X_n as $n\to\infty$ in terms of appropriate limit theorems. In Section 2 we show that for a certain increasing function U constructed as in [6], the sequence $\{e^{-n}U(X_n)\}$ converges almost surely and that the limit is non-defective if a certain condition on the immigration distribution obtains and is essentially infinite otherwise. It is pointed out that, in the former case, $\{X_n\}$ may be classified in terms of regularity and irregularity exactly as in the case of no immigration [6].

In Section 3 we consider the case where $Y_n=e^{-n}U(X_n)\to\infty$ a.s. Limit theorems are obtained for Y_n which involve a non-linear norming by a slowly

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varying (s.v.) function. Theorem 2 involves no extra assumptions but has a rather strange appearance: neater versions are given in Corollaries 1–3 under conditions expressed in terms of $G(x)=1-b(\exp[-1/V(e^x)])$, where V is the functional inverse of U. These results are analogues of the families of limit theorems for the case $m<\infty$ which were presented in Part I [3], and some examples are given to show that the conditions in Corollaries 1–3 are not vacuous. Finally, it is shown that there is no sequence (c_n) of norming constants such that Y_n/c_n has a limit in distribution which is neither defective nor degenerate at the origin.

To complete the survey in Part I, it suffices to mention that a version of Theorem 1 was first proved in [2] under conditions on f ensuring that U could be chosen as a suitable power of $\log^+$ and only convergence in law was established. The conditions of this result were substantially relaxed in [1] to allow U to be s.v. at infinity and strictly increasing. A restricted and more opaque version of Corollary 1 was proved in [2].

2. Almost sure convergence

Let f_n denote the nth functional iterate of f and h_n the functional inverse of $-\log f_n(e^{-s})$. Let $p_n(s)=\prod_{m=1}^{n} b(\exp(-h_m(s)))$ and $p(s)=\lim_{n\to\infty} p_n(s)$ for $0<s<-\log q$ where $q=f(q)<1$. Since $h_n(s)\downarrow 0 (n\to\infty)$ it is clear that either $p(s)>0$ or $\equiv 0$. In the former case it is clear that $\{[\exp(-h_n(s)X_n)]/p_n(s)\}$ is a martingale and the martingale convergence theorem shows that for each $0<s<-\log q$, $\exp(-h_n(s)X_n)\xrightarrow{\text{a.s.}} Y(s)$, say, and $EY(s)=p(s)$. The behaviour of $\{X_n\}$ can be analyzed in precisely the same way in which that of the corresponding process without immigration was analyzed in [6]. In particular the classification of points in $(0,-\log q)$ as regular and irregular carries over to the present case and if $T=\sup\{s \mid 0<s<-\log q,\ Y(s)>e^{-1}\}$ it follows that T is non-defective and $P(T\geqq s_r)=p(s_r)$, $P(T=s_r)=0$ for every regular point s_r.

Finally, for the function U constructed in [6], the arguments used in this reference show that $e^{-n}U(X_n)\xrightarrow{\text{a.s.}} U(T^{-1})$. In the complementary case, when $p(s)\equiv 0$, the martingale convergence theorem shows that $\exp(-h_n(s)X_n)\xrightarrow{\text{a.s.}} 0$ for each $0<s<-\log q$. To proceed further we need the following details about U. It is shown in [6] that by starting with a fixed $s_0\in(0,-\log q)$ it is possible to construct U so that it is continuous on $[0,\infty)$, identically zero on $[0, 1/(-\log q)]$, positive and strictly increasing to infinity on $(1/(-\log q),\infty)$ and satisfies the relations

$$
\begin{aligned}
&U(1/h_n(s_0))=e^n,\\
&U(1/h_n(s))/U(1/h_n(s_0))=U(1/s) \qquad (n=0,1,\cdots).
\end{aligned}
\tag{2.1}
$$

Choose a sequence $\{s_m : m = 1, 2, \cdots\}$ such that $-\log q > s_m \downarrow 0 (m \to \infty)$. Let Ω_m be a subset of the basic probability space such that $P(\Omega_m) = 1$ and $h_n(s_m)X_n(\omega) \to \infty$ $(n \to \infty; \omega \in \Omega_m)$. Thus on $\Omega' = \bigcap_{m \geqq 1} \Omega_m$, $h_n(s_m)X_n(\omega) \to \infty$ for each m and this implies that eventually $X_n(\omega) \geqq 1/h_n(s_m)$ and hence, from (2.1), eventually

$$e^{-n}U(X_n(\omega)) \geqq U(1/s_m) \qquad (\omega \in \Omega').$$

Letting $n \to \infty$ and them $m \to \infty$, we obtain all except the final assertion of the following theorem.

Theorem 1. *If* $p(s) > 0$ *then* $e^{-n}U(X_n) \xrightarrow{\text{a.s.}} U(1/T)$ *where* T *is defined above, and if* $p(s) \equiv 0$, $e^{-n}U(X_n) \xrightarrow{\text{a.s.}} \infty$. *Furthermore,* $p(s) > 0$ *iff*

$$\sum_{j=1}^{\infty} b_j \log^+ U(j) < \infty, \tag{2.2}$$

where $\{b_j\}$ *denotes the immigration distribution.*

The last assertion follows from these observations. Since U is strictly increasing, the relations (2.1) can be solved to obtain an explicit expression

$$h_n(s) = 1/V(e^n U(s^{-1})) \tag{2.3}$$

for h_n, and the proof of Lemma 2 in [1] needs, with one exception, only trivial changes to show that (2.2) holds iff

$$\prod_{n=1}^{\infty} b(\exp(-1/V(e^n U(s^{-1})))) > 0$$

where V is the functional inverse of U, whence the assertion in this case. The exception is that the proof given in [1] assumes that $\log U$ is s.v.: that this is so follows from the following lemma.

Lemma 1. *If* L *is s.v. at infinity then so is* $L(U(\cdot))$.

Proof. By virture of the uniform convergence theorem for s.v. functions [7] it suffices to show that for each $\lambda > 0$, $U(\lambda x)/U(x)$ is bounded away from zero and infinity for all sufficiently large x. It suffices to consider the case $1 \leqq \lambda < \infty$. Choose $s_0 \in (0, -\log q)$ as in Construction 2.3.1 of [6], and note that, from Equation (3) of [6], it follows that

$$h_{n+1}(s_0)/h_n(s_0) \to 0 \quad \text{as} \quad n \to \infty. \tag{2.4}$$

Choose n_0 such that, for all $n \geqq n_0$, $h_{n+1}(s_0)/h_n(s_0) < 1/\lambda$. Then, for any $x > 1/h_{n_0}(s_0)$, if n is chosen so that

$$1/h_n(s_0) \leqq x < 1/h_{n+1}(s_0),$$

it follows that

$$1/h_n(s_0) \leqq x < \lambda x < \lambda/h_{n+1}(s_0) < 1/h_{n+2}(s_0),$$

and so $1 < U(\lambda x)/U(x) < e^2$. The lemma now follows.

3. The case $p(s) \equiv 0$

The description of the behaviour of X_n in the case $p(s) \equiv 0$ is through distributional, rather than almost sure, limit theorems. The first step in deriving them is to note that, from (2.3),

$$E(\exp(-h_n(t)X_n)) = p_n(t) = \prod_{m=1}^{n} b[\exp(-1/V(e^m U(t^{-1})))].$$

Using integral test comparisons, as in [3], it is not difficult to show that

$$p_n(t) = \Delta_n(t) \exp \int_0^n \log b[\exp(-1/V(e^y U(t^{-1})))]\, dy$$

where $\Delta_n(t_n) \to 1$ if $t_n \to 0$ as $n \to \infty$. Denoting above, with t_n substituted for t, the integral by J_n and making the change of variable $e^x = e^y U(t_n^{-1})$, it follows that if $t_n \to 0$ then

$$-J_n \sim j_n = \int_{\log U(1/t_n)}^{n+\log U(1/t_n)} G(x)\, dx \tag{3.1}$$

where $G(x) = 1 - b(\exp(-1/V(e^x)))$. We are now in a position to approach the main limit theorem, by making a suitable choice of t_n, and by using the following version of Reuter's Lemma 1 in [1].

Lemma 2. *Let U be constructed according to Construction* 2.3.1 *in* [6] *and in addition suppose it is strictly increasing on* $(1/(-\log q), \infty)$ *Let* y_n *be positive, increasing and satisfy the properties that, for some fixed* $0 < c < d \leqq \infty$,

$$\text{(a)}\quad y_n(u)/y_n(v) \to \infty (n \to \infty) \qquad \textit{if} \quad c < v < u < d,$$

and

$$\text{(b)}\quad e^n y_n(u) \to \infty (n \to \infty;\ c < u < d).$$

If, for a sequence of non-negative random variables $\{W_n\}$ *there is a continuous function* $a(\cdot)$ *such that*

$$E[\exp(-W_n/V(e^n y_n(u)))] \to a(u) \qquad (c < u < d;\ n \to \infty)$$

then

$$P(e^{-n} U(W_n) \leqq y_n(u)) \to a(u).$$

Proof. Let $A_n = \{e^{-n}U(W_n) \leqq y_n(u)\} = \{W_n \leqq V(e^n y_n(u))\}$. Choose $u \in (c, d)$ and $c < u_1 < u < u_2 < d$. If $Y_n^{(i)} = \exp(-W_n/V(e^n y_n(u_i)))(i = 1, 2)$ then arguing as in [1] we obtain

$$EY_n^{(1)} - \exp(-\lambda_n^{(1)}) \leqq P(A_n) \leqq (EY_n^{(2)}) \exp \lambda_n^{(2)}$$

where $\lambda_n^{(i)} = V(e^n y_n(u))/V(e^n y_n(u_i))(i = 1, 2)$. If we can show that $\lambda_n^{(1)} \to \infty$ and $\lambda_n^{(2)} \to 0$, the assertion follows upon then letting $u_i \to u$. We shall prove only that $\lambda_n^{(1)} \to \infty$. For this, it is enough to note that, if $\lambda_n^{(1)} \not\to \infty$, there exist

$$s_k \equiv V(e^{n_k} y_{n_k}(u)) \quad \text{and} \quad t_k \equiv V(e^{n_k} y_{n_k}(u_1))$$

such that $s_k, t_k \to \infty$ (by (b)), s_k/t_k remains bounded, and $U(s_k)/U(t_k) = y_{n_k}(u)/y_{n_k}(u_1) \to \infty$ (by (a)), as $k \to \infty$: but this, as in the proof of Lemma 1, is impossible.

Now, for $x \geqq 1$, let $\Lambda(x) = \exp \int_0^{\log x} G(y)\, dy$, which is s.v. at infinity and strictly increasing. Furthermore, the ratio

$$\Lambda(x)/\Lambda(xe^n) = \exp\left\{-\int_{\log x}^{n+\log x} G(y)\, dy\right\}$$

increases with x on $(1, \infty)$ from m_n say, to unity, and since $p(t) \equiv 0$ iff $\int_0^\infty G(x)\, dx = \infty$, it follows that $m_n \to 0$ $(n \to \infty)$; moreover, for each fixed $x \geqq 1$, $\Lambda(x)/\Lambda(xe^n)$ decreases to zero as $n \to \infty$. Hence, for any $0 < u < 1$ and all n sufficiently large, we may define $y_n(u) > 0$ uniquely in such a way that

$$\Lambda(y_n(u)) = u\Lambda(e^n y_n(u)),$$

where it follows that y_n is increasing on $(0, 1)$, that $y_n(u) \to \infty$ $(n \to \infty)$ and that if $0 < v < u < 1$,

$$\frac{y_n(u)}{y_n(v)} = \frac{\Lambda^{-1}(u\Lambda(e^n y_n(u)))}{\Lambda^{-1}(v\Lambda(e^n y_n(v)))} \geqq \frac{\Lambda^{-1}(u\Lambda(e^n y_n(v)))}{\Lambda^{-1}(v\Lambda(e^n y_n(v)))} \to \infty$$

as $n \to \infty$, since Λ is s.v. at infinity. Choosing $U(t_n^{-1}) = y_n(u)$ in (3.1) we see that $j_n = -\log u$ and hence $E[\exp(-X_n/V(e^n y_n(u)))] \to u (0 < u < 1)$, and that the other conditions of Lemma 2 are satisfied. Since also $e^{-n}U(X_n) \leqq y_n(u)$ if and only if $\Lambda(e^{-n}U(X_n)) \leqq u\Lambda(U(X_n))$, the proof of the following theorem is complete.

Theorem 2. *If* $p(s) \equiv 0$, *then*

$$\Lambda(e^{-n}U(X_n))/\Lambda(U(X_n)) \xrightarrow{d} W,$$

where W *is uniformly distributed on* $[0, 1]$.

Analogues of Theorem 2 also obtain when $m < 1$ and $1 < m < \infty$. When $1 < m < \infty$, U could be constructed by the method of [6] exactly as for $m = \infty$. However, it is more natural to consider instead a strictly increasing solution U

to the equation

$$U(1/k(s)) = m^{-1}U(1/s), \qquad 0 < s < -\log q,$$

where $k(s) = -\log f(e^{-s})$, which is equivalent to (2.1) but for having e replaced by m. A suitable choice for U is given by

$$U(x) = 1/\varphi^{-1}(e^{-1/x}), \qquad -1/\log q < x < \infty,$$

where, if $(Z_n)_{n \geqq 0}$ denotes the corresponding process without immigration starting with $Z_0 = 1$, and if $(c_n)_{n \geqq 0}$ are its Seneta constants, φ is the Laplace transform of $\lim_{n \to \infty} Z_n/c_n$. Now, defining $V(y) = U^{-1}(y)$ as before, and setting

$$G(x) = 1 - b(\exp[-1/V(m^x)]); \qquad \Lambda(x) = \exp\left[\int_0^{\log x} G(y)\, dy\right],$$

it follows, from an argument similar to the one above, that $\Lambda(m^{-n}U(X_n))/\Lambda(U(X_n))$ converges in law to the uniform distribution on $[0, 1]$.

For $0 < m < 1$, one can again consider increasing solutions U to the equation $U(1/k(s)) = m^{-1}U(1/s)$, this time for all $s > 0$, one such is given by

$$U(x) = 1/[1 - Q(e^{-1/x})],$$

where Q is the probability generating function of the limiting distribution of Z_n conditional on $Z_n > 0$, where Z_n again denotes the process without immigration starting with $Z_0 = 1$. Defining

$$V(y) = U^{-1}(y); \qquad G(x) = 1 - b(\exp[-1/V(m^{-x})]);$$
$$\Lambda(x) = \exp\left[\int_0^{\log x} G(y)\, dy\right],$$

it can be shown that $\Lambda(U(X_n))/\Lambda(m^{-n}U(X_n))$ converges in law to the uniform distribution on $[0, 1]$.

The proof of Theorem 2 has been carried out under the condition $X_0 = 0$. However

$$\begin{aligned} P_i^{(n)}(t) &= E[\exp(-X_n h_n(t)) \mid X_0 = i] \\ &= [f_n(\exp(-h_n(t)))]^i P_0^{(n)}(t) \\ &= e^{-it} P_0^{(n)}(t). \end{aligned}$$

and since t was chosen to converge to zero, we see that Theorem 2 is valid for any initial state. Furthermore, $\Lambda(e^{-n}U(\cdot))/\Lambda(U(\cdot))$ is strictly increasing and continuous and hence the weakly convergent sequence in Theorem 2 is a Markov chain and, in addition, is mixing ([5], Theorem 2). Thus [4] we cannot have convergence in probability in Theorem 2.

The quantity converging in law in Theorem 2 has a rather odd appearance

since $U(X_n)$ is present in both numerator and denominator. Tidier versions can be obtained by making estimates of

$$\Lambda(e^{-n}U(X_n))/\Lambda(U(X_n)) = \exp\left\{-\int_{Y_n-n}^{Y_n} G(y)\,dy\right\}; \tag{3.2}$$

under appropriate assumptions on the behaviour of G, where, from now on, we write $Y_n \equiv \log U(X_n)$ throughout.

Corollary 1. If $p(s) \equiv 0$ and $\lim_{x\to\infty} xG(x) = 0$, then

$$\Lambda(e^{-n}U(X_n))/\Lambda(e^n) \xrightarrow{d} W,$$

where W is uniformly distributed on $[0, 1]$.

Proof. Since $xG(x) \to 0$, we have

$$\int_{Y_n-n}^{Y_n} G(y)\,dy = o\{-\log(1-n/Y_n)\}; \tag{3.3}$$

combining Theorem 2, (3.2) and (3.3), it follows that, as $n \to \infty$, $n/Y_n \xrightarrow{d} 1$. Hence, as $n \to \infty$,

$$\begin{aligned}\Lambda(U(X_n))/\Lambda(e^n) &= \exp\left\{\int_n^{Y_n} G(y)\,dy\right\}\\ &= \exp\{o(\log[Y_n/n])\} \xrightarrow{d} 1,\end{aligned}$$

and the result follows.

Corollary 2. If $p(s) \equiv 0$ and $\lim_{x\to\infty} xG(x) = a$, $0 < a < \infty$, then

$$n^{-1}\log U(X_n) - 1 \xrightarrow{d} W_1,$$

where $P[W_1 \leqq u] = \{u/(1+u)\}^a$.

Proof. Immediately, from (3.2), we have

$$\Lambda(e^{-n}U(X_n))/\Lambda(U(X_n)) \overset{\text{a.s.}}{\sim} \{1-n/Y_n\}^a,$$

and the result now follows from Theorem 2.

Corollary 3. If $p(s) \equiv 0$, $\lim_{x\to\infty} xG(x) = \infty$, and G is regularly varying at infinity, then $nG(\log U(X_n)) \xrightarrow{d} W_2$, where W_2 is a standard negative exponential random variable.

Proof. Here, from (3.2), we have

$$\Lambda(e^{-n}U(X_n))/\Lambda(U(X_n)) = \exp\left\{-Y_n\int_{1-n/Y_n}^{1} G(zY_n)\,dz\right\}; \tag{3.4}$$

combining (3.4) with Theorem 2 and $\lim_{x\to\infty} xG(x)=\infty$, it follows that, as $n\to\infty$, $n/Y_n \xrightarrow{d} 0$. Hence, since G varies regularly,

$$\Lambda(e^{-n}U(X_n))/\Lambda(U(X_n))\sim\exp\{-nG(Y_n)\},$$

and the result follows.

If in Corollary 3 G has a positive index Δ i.e. if $\lim_{x\to\infty} G(\lambda x)/G(x)=\lambda^{-\Delta}$ for all $\lambda>0$, then $0<\Delta\leqq 1$ and the conclusion can be transformed to the form

$$a_n^{-1}\log U(X_n)\xrightarrow{d} W'$$

where W' has the extreme value distribution function $\exp(-x^{-\Delta})$ and $G(a_n)=n^{-1}$.

We now show that for any U the hypotheses of Corollaries 1–3 can be satisfied. Let A be a positive integer-valued random variable, define $I=[V(A)]$ and let b be the probability generating function of I. Since

$$V(A)-1\leqq I\leqq V(A) \tag{3.5}$$

and $\log U$ is s.v. at infinity, it follows that Condition (2.2) is satisfied iff $E\log^+ A<\infty$. In [3], Section 3.1 several examples were given where $E\log^+ A=\infty$, and for each of these $T(x)=P(A>x)$ is s.v. at infinity. We now suppose this to be always the case. It follows then, from Lemma 1 and (3.5), that $P(I>x)\sim T(U(x))(x\to\infty)$ and hence an Abelian theorem for power series yields

$$(1-b(s))/(1-s)=\sum s^j P(I>j)\sim(1-s)^{-1}T(U(\{1-s\}^{-1}))\qquad(s\to 1).$$

Letting $s=\exp(-1/V(e^x))$ and invoking Lemma 1 once again, we obtain

$$G(x)\sim T(e^x). \tag{3.6}$$

Let $\log_1 x=\log x$ and $\log_k x=\log(\log_{k-1}x)(k=2,3,\cdots)$ for all sufficiently large x. In [3] an example was given for which

$$T(x)\sim c\left[\prod_{k=1}^{r}\log_k x\right]^{-1}\qquad(x\to\infty)$$

where $r\geqq 2$ and c is a certain constant. Using (3.6) it is obvious that $\int_0^\infty G(x)\,dx=\infty$ and that $xG(x)\to 0$ $(x\to\infty)$ and hence this example satisfies the conditions of Corollary 1.

Discrete distributions were also constructed in [3] for which

$$T(x)\sim a/\log x\qquad(0<a<\infty);$$
$$T(x)\sim c(\log x)^{-(\delta-1)}\qquad(0<c<\infty,\ 1<\delta<2);$$

and

$$T(x)\sim(c/b)(\log_r x)^{-b}\qquad(0<b,\ c<\infty,\ r\geqq 2).$$

Using (3.6) we see that these examples satisfy the hypotheses of Corollaries 2 and 3.

Finally, we show that the neatest version of Theorem 2 that could be hoped for is in fact impossible.

Theorem 3. There is no sequence of constants (c_n) such that $c_n^{-1}U(X_n)$ has a limit in distribution which is neither defective nor degenerate at zero.

Proof. For any $0<x_1<x_2<\infty$, let $p_n(x_1, x_2) \equiv P[x_1 \leqq c_n^{-1}U(X_n) \leqq x_2]$. Then, since $\Lambda(e^{-n}U(\cdot))/\Lambda(U(\cdot))$ is continuous and strictly increasing, it follows also that

$$p_n(x_1, x_2) = P[r_n(x_1) \leqq \Lambda(e^{-n}U(X_n))/\Lambda(U(X_n)) \leqq r_n(x_2)],$$

where

$$0 < r_n(x) \equiv \Lambda(e^{-n}c_n x)/\Lambda(c_n x) < 1.$$

If (c_n) is such that $c_n^{-1}U(X_n)$ is to converge in distribution, we must have $c_n e^{-n} \to \infty$ because of Theorem 1. Hence, since Λ is s.v., we see that $r_n(x_2) = r_n(x_1)[1+o(1)]$ as $n \to \infty$. Choose any subsequence (n_k) such that $r_{n_k}(x_1) \to r$ for some $r \in [0, 1]$. Then, for any $\varepsilon > 0$, the intervals $(r_{n_k}(x_1), r_{n_k}(x_2))$ belong to $(r-\varepsilon, r+\varepsilon)$ for all k sufficiently large. It follows from Theorem 2 that $p_{n_k}(x_1, x_2) \to 0$, and hence that, if $c_n^{-1}U(X_n)$ converges in distribution, its limit puts no mass on $(0, \infty)$.

It is interesting to note the contrast between the cases $p(x)>0$ and $p(x) \equiv 0$. In the former, the asymptotic behaviour is dominated by the underlying Galton–Watson process, and the effect of immigration, apart from preventing extinction, is seen only in the distribution of the limit of $U(X_n)e^{-n}$: eventually, the contribution of the immigration process becomes negligible. However, when $p(x) \equiv 0$, the immigration distribution has such a broad tail that $U(X_n)e^{-n}$ is pushed off to infinity a.s. by the infinite sequence of occasional, but very large, inflows of immigrants. The character of Theorem 2 and its corollaries, giving limits in distribution but not with probability one, reflects the nature of the immigration process rather than that of the Galton–Watson process. In particular, unlike the case when $p(x)>0$, the limiting distribution appearing in Theorem 2 is the same, whether or not the Galton–Watson process is regular or irregular.

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