APPROXIMATE VERSIONS OF MELAMED'S THEOREM

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Abstract. In 1979, Melamed proved that, in an open migration process, the absence of 'loops' is necessary and sufficient for the equilibrium flow along a link to be a Poisson process. In this paper, we prove approximation theorems with the same flavour: the difference between the equilibrium flow along a link and a Poisson process with the same rate is bounded in terms of expected numbers of loops. The proofs are based on Stein's method, as adapted for bounds on the distance of the distribution of a point process from a Poisson process in Barbour and Brown (1992b). Three different distances are considered, and illustrated with an example consisting of a system of tandem queues with feedback. The upper bound on total variation distance of the process grows linearly with time, and a lower bound shows that this can be the correct order of approximation.

Keywords: Melamed's theorem, loop condition, Poisson approximation, migration process, network flows.

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1. Introduction

In this paper, we are concerned with open Markovian queueing processes (Kelly, 1979, §2.4), in which individuals of a single type arrive at, move among and leave a set of J queues. Arrivals at the different queues occur as independent Poisson streams with rates ν_j, $1 \le j \le J$; the service requirements of the individuals at the queues they visit are independent negative exponential random variables with unit mean; the total service effort at queue j is $\phi_j(n)$ when n customers are in line, where it is assumed that $\phi_j(0) = 0$, $\phi_j(1) > 0$, that each ϕ_j is an increasing function, and that individuals each receive the same service effort; upon leaving queue j, an individual moves to queue k with probability λ_{jk}, $1 \le k \le J$, or leaves the system with probability μ_j: $\mu_j + \sum_k \lambda_{jk} = 1$. The resulting process $N = (N(t),\ t \in \mathbb{R})$ of the numbers of customers in line at the various queues at time t is a pure jump Markov process on $(Z^+)^J$, with transition rates given by

$$\begin{aligned} n &\to n + e_j && \text{at rate} \quad \nu_j; \\ n &\to n - e_j + e_k && \text{at rate} \quad \lambda_{jk}\phi_j(n_j); \\ n &\to n - e_j && \text{at rate} \quad \mu_j\phi_j(n_j), \end{aligned} \tag{1.1}$$

where e_j denotes the j'th coordinate vector in $(Z^+)^J$.

The simplest queueing process of this kind is the $M/M/1$ queue, for which it was proved by Burke (1956) and Reich (1957) that, in the steady state, the output process is Poisson. The generalization of this result which we examine is Melamed's (1979) theorem. In the case where $\phi_j(n) = c_j I[n > 0]$ for all $1 \le j \le J$, he showed that the steady state flow along link (j,k) in an open queueing process is Poisson if and only if a loop condition is satisfied: that no customer can travel along (j,k) more than once. The restriction on the functions ϕ_j was subsequently lifted by Walrand and Varaiya (1981). Here, we relax the loop condition somewhat, and investigate how close to Poisson are the flows from one queue to another.

The argument is based on Stein's method, and in particular on Theorems 2.6 and 3.7 of Barbour and Brown (1992b), where the approximation of point processes by Poisson point processes is discussed. The theorems bound the difference between the distributions of a finite point process and a Poisson point process on a compact, second countable Hausdorff carrier space Γ, with respect to two different metrics, in terms of the difference between the unconditional distribution of the point process and the distribution of the rest of the point process conditional on there being a point at a given position $\gamma \in \Gamma$.

In Section 2, the construction of the open queueing process and its time-reverse is detailed in the way needed for the results here, basic facts about the law of the process conditional on transitions at particular times are derived and the Poisson approximation results are described. Section 3 contains the main results, and discusses in some detail the example of a tandem queueing system with some feedback. in the example, both lower and upper bounds are considered. Section 4 contains improvements and a comparison with previous work in this area of Brown and Pollett (1982). Section 5 contains proofs of a couple of the technical results.

2. Preliminaries

We consider the open queueing process introduced above, and summarize some relevant facts from Kelly (1979), Section 2.4.

Let the *forward customer chain* X be a Markov chain with state space $\{0, \ldots, J\}$, where 0 represents the state before the customer arrives and after the customer leaves and $1, \ldots, J$ the queues. Its transition probabilities $p_{jk}, 0 \le j,k \le J$ are given by

$$p_{0k} = \frac{\nu_k}{\sum_{j=1}^J \nu_j}, \quad 1 \le k \le J; \quad p_{00} = 0;$$

$$p_{j0} = \mu_j, \quad p_{jk} = \lambda_{jk}, \quad 1 \le j,k \le J.$$

A customer arriving into the network in state k chooses a route sampled independently from the forward customer chain starting in k and terminated at the first entrance into 0. The distribution of a customer's initial state is given by $(p_{0k},\ 1 \le k \le J)$. In addition, the arriving customer samples his service requirements at the queues along his route independently from the negative exponential distribution with mean 1.

We assume that the parameters $\lambda_{jk}, \mu_j, \nu_j$ allow an individual to reach any queue from outside the system and to leave the system from any queue, either directly or indirectly via a chain of other queues. Then the forward customer chain is irreducible, and every state is persistent, because the state space is finite. The expected number of visits to queue j between returns to 0 in the forward customer chain is given by $\alpha_j / \sum_k \nu_k$, where $\{\alpha_j\}_{j=1}^J$ is the unique solution to the equations

$$\alpha_j = \nu_j + \sum_{i=1}^J \alpha_i \lambda_{ij} \tag{2.1}$$

(see, for example, Asmussen (1987), Chapter I, Theorem 3.2).

Let $\{b_j\}_{j=1}^J$ be defined by the equations

$$b_j^{-1} = \sum_{n=0}^\infty \frac{\alpha_j^n}{\prod_{r=1}^n \phi_j(r)}.$$

We assume that the sum is finite for each j. This assumption implies that the process N has a unique stationary distribution and is ergodic. In the stationary state, for each $t > 0$, the numbers $N_j(t), j = 1, \ldots, J,$ in the queues in the stationary state are independent, and the number in queue j is n with probability

$$b_j \frac{\alpha_j^n}{\prod_{r=1}^n \phi_j(r)}. \tag{2.2}$$

Equilibrium realizations of $(N(t),\ t \ge 0)$ can be constructed in the following way. First, the values $N_j(0)$, $j = 1, \ldots, J$, are sampled from the stationary distribution (2.2). Then the customers present at time 0 are ordered in some arbitrary way, the i'th in order being in queue $j(i)$. The i'th customer now samples his route $X^{(i)}$ through the system independently from the chain X started in $j(i)$, and his service requirements are then determined by the independently sampled negative exponential variables with mean 1. The arrivals of the customers after time 0 are determined by a Poisson process of rate $\sum_j \nu_j$, and each of these customers receives a route sampled from X with initial distribution $(p_{0k},\ 1 \le k \le J)$. Standard theory then assures us that the exogenous arrivals at the queue j are Poisson with rate ν_j, independently of the other exogenous arrival streams. The customers arriving after time 0 also sample their corresponding independent exponential service requirements.

To construct the system before time 0, we use Theorem 2.5 of Kelly (1979), which describes the time–reversed process as another open queueing network, whose transition rates are given by:

$$\begin{aligned} n &\to n - e_k && \text{at rate} \quad \frac{\nu_k \phi_k(n_k)}{\alpha_k}; \\ n &\to n + e_j - e_k && \text{at rate} \quad \frac{\lambda_{jk}\phi_k(n_k)\alpha_j}{\alpha_k}; \\ n &\to n + e_j && \text{at rate} \quad \mu_j \alpha_j. \end{aligned} \tag{2.3}$$

The forward customer chain for this time–reversed process we call the *backward customer chain* X^*. The backward customer chain can be seen to be the time–reversal of the forward customer chain, and it has transition probabilities:

$$\begin{aligned} p^*_{0j} &= \frac{\mu_j \alpha_j}{\sum_{l=1}^J \mu_l \alpha_l}, \quad 1 \le j \le J; \quad p^*_{00} = 0; \\ p^*_{k0} &= \frac{\nu_k}{\alpha_k}, \quad p^*_{kj} = \frac{\lambda_{jk}\alpha_j}{\alpha_k}, \quad 1 \le j,k \le J. \end{aligned} \tag{2.4}$$

The system before time 0 can then be constructed in an exactly analogous way to the system after time 0, with the exception that the Poisson process of exogenous "arrivals" (which in reality represent departures) has rate $\sum_{j=1}^J \mu_j \alpha_j$, and that routes are sampled independently from paths of X^* stopped at 0.

It will be useful to use conditional intensities (see, for example, Brémaud (1981)). The history $\mathcal{F}$ that we use is that generated by all arrivals, departures and transfers between queues, or equivalently by the state of the process N at the current and all previous instants. For each $1 \le j,k \le J$, let M_{jk} denote the point process consisting of transitions from queue j to queue k; set $M = \{M_{jk}, 0 \le j,k \le J\}$, where departures are interpreted as transitions to 0 and arrivals as transitions from 0. Then the conditional intensity for M_{jk} ($1 \le j,k \le J$) at time t is $\phi_j(N_j(t-))\lambda_{jk}$, for M_{j0} it is $\phi_j(N_j(t-))\mu_j$ and for M_{0j} just ν_j. From this, the form of the stationary distribution and the fact that the expected integral of the conditional intensity over $[0,1]$ is the rate of a stationary point process, it can be seen that the rate of M_{jk} is

$$\rho_{jk} = \lambda_{jk}\alpha_j.$$

Palm distributions figure largely in the approximations which follow, and in particular the reduced Palm distribution. The Palm distribution $P^{(j,k)}$ of M is the distribution of M, considered as a random measure on $\mathbb{R} \times \{(j,k),\ 0 \le j \ne k \le J\}$, conditional on there being a transition from queue j to queue k at 0, and the reduced Palm distribution $P_0^{(j,k)}$ is the distribution obtained from it by removing from the random measure the deterministic atom at $(0,(j,k))$. The reduced Palm distribution for the networks considered here can be particularly simply expressed, as the following lemma shows.

Lemma 1. *For the open queueing network, the reduced Palm distribution for the network given a transition from queue j to queue k at time 0 ($1 \le j,k \le J$) is the same as that for the original network, save that the network on $(0,\infty)$ behaves as if there were an extra individual at queue k at time 0 and the network on $(-\infty,0)$ behaves as if there were an extra individual in queue j at time 0: i.e. if $A \in \mathcal{F}(0-), B \in \sigma(N(z), z > 0))$ and $n_0 \in Z^J$ then*

$$P_0^{(j,k)}(A, N(0-) = n_0 + e_j, B) = P(A|N(0-) = n_0 + e_j)P(N(0) = n_0)P(B|N(0) = n_0 + e_k). \tag{2.5}$$

Proof. Let $\theta_s, s \geq 0$ denote the shift transformation on the underlying probability space which translates each arrival, departure or transition in the queueing network by s to the left. The definition of Palm probabilities for stationary point processes given by Brémaud, Kannurpatti and Mazumdar (1992) [BKM], p. 383, and the fact that the rate of M_{jk} is ρ_{jk} combine to give the probability on the left of (2.5) as

$$\begin{aligned}\rho_{jk}^{-1} E\Big(\int_0^1 I[A, N(0-) = n_0 + e_j, B] \circ \theta_s \, dM_{jk}(s)\Big) \\ = \rho_{jk}^{-1} E\Big(\sum_{n\geq 1} I[A, N(0-) = n_0 + e_j, B] \circ \theta_{T_n} I[0 < T_n \leq 1]\Big) \qquad (2.6) \\ = \rho_{jk}^{-1} E\Big(\sum_{n\geq 1} P([A, N(0-) = n_0 + e_j, B] \circ \theta_{T_n} \mid \mathcal{F}_{T_n}) I[0 < T_n \leq 1]\Big),\end{aligned}$$

where $0 < T_1 < T_2 < \cdots$ are the times of (j,k) transitions after time 0. By the strong Markov property, we now have

$$\begin{aligned}P\big([A,N(0-) = n_0 + e_j, B] \circ \theta_{T_n} \mid \mathcal{F}_{T_n}\big) I[T_n < \infty] \\ = P\big(B \mid N(0) = n_0 + e_k\big) I[A, N(0-) = n_0 + e_j] \circ \theta_{T_n} I[T_n < \infty]\end{aligned}$$

and reversing the above argument then gives

$$P_0^{(j,k)}(A, N(0-) = n_0 + e_j, B) = \rho_{jk}^{-1} P[B \mid N(0) = n_0 + e_k] E\Big(\int_0^1 I[A, N(0-) = n_0 + e_j] \circ \theta_s \, dM_{jk}(s)\Big).$$

Note that $I[A, N(0-) = n_0 + e_j] \circ \theta_s$ is predictable by (7.1.4) of Baccelli and Brémaud(1987). Applying the stochastic intensity formula ([BKM], (2.11) p. 385),

$$\begin{aligned}P_0^{(j,k)}(A,N(0-) = n_0 + e_j, B) \\ = \rho_{jk}^{-1} P[B \mid N(0) = n_0 + e_k] E\Big(\int_0^1 I[A, N(0-) = n_0 + e_j] \circ \theta_s \lambda_{jk} \phi_j((n_0 + e_j)_j) ds\Big) \\ = \rho_{jk}^{-1} P[B \mid N(0) = n_0 + e_k] P[A \mid N(0-) = n_0 + e_j] P[N(0-) = n_0 + e_j] \lambda_{jk} \phi_j((n_0 + e_j)_j),\end{aligned}$$

and using the form (2.2) of the stationary distribution gives the lemma.

The terminology and results necessary for the Poisson approximations are now introduced. The space of finite non–negative integer valued measures on the Borel sets of the carrier space Γ is denoted $\mathcal{X}$. Elements of this space can be thought of as finite *configurations* of points in Γ. A *point process* Ξ is a random element of $\mathcal{X}$, with mean measure (assumed finite) denoted by π. The distribution of the *Poisson process* on Γ with mean measure π is denoted by $\mathrm{Po}(\pi)$.

We use two metrics to describe how well $\mathrm{Po}(\pi)$ approximates $\mathcal{L}(\Xi)$. The first is the total variation distance between probability measures,

$$d_{TV}(P,Q) = \sup_A |P(A) - Q(A)| = \frac{1}{2}||P - Q||,$$

where $||\cdot||$ denotes the total variation norm on finite signed measures and the supremum is taken over the Borel sets of $\mathcal{X}$. The second is a metric d_2 introduced in Barbour and Brown (1992b), which might be described as an average variation metric: see also Barbour, Holst and Janson (1992), Chapter 10. To define this metric, suppose that d_0 is a metric on Γ bounded by 1. We first define a metric d_1 between pairs of configurations in $\mathcal{X}$ and then use this to define a metric d_2 between pairs of probability measures over $\mathcal{X}$. Let $\mathcal{K}$ denote the set of functions $k : \Gamma \to \mathbb{R}$ such that

$$\sup_{y_1 \neq y_2 \in \Gamma} |k(y_1) - k(y_2)|/d_0(y_1, y_2) \leq 1,$$

and define a distance d_1 between finite measures $\boldsymbol{\rho}, \boldsymbol{\sigma}$ over Γ by

$$d_1(\boldsymbol{\rho}, \boldsymbol{\sigma}) = \begin{cases} 1 & \text{if } \boldsymbol{\rho}(\Gamma) \neq \boldsymbol{\sigma}(\Gamma), \\ m^{-1} \sup_{k \in \mathcal{K}} \{ |\int k \, d\boldsymbol{\rho} - \int k \, d\boldsymbol{\sigma}| \} & \text{if } \boldsymbol{\rho}(\Gamma) = \boldsymbol{\sigma}(\Gamma) = m > 0. \end{cases}$$

Similarly, let $\mathcal{G}$ denote the set of functions $f : \mathcal{X} \to \mathbb{R}$ such that

$$s_2(f) := \sup_{\xi_1 \neq \xi_2 \in \mathcal{X}} |f(\xi_1) - f(\xi_2)|/d_1(\xi_1, \xi_2) \leq 1,$$

and define a distance d_2 between probability measures over $\mathcal{X}$ by

$$d_2(Q, R) = \sup_{f \in \mathcal{G}} \left| \int f \, dQ - \int f \, dR \right| / s_2(f).$$

The metric d_1, when restricted to probability measures over (Γ, d_0), is variously known as the Dudley, Fortet–Mourier, Kantorovich $D_{1,1}$ or Wasserstein metric (see Rachev (1984)) induced by d_0. When considered as a distance between configurations $\xi_1, \xi_2 \in \mathcal{X}$ with $|\xi_1| = |\xi_2| = n$, $d_1(\xi_1, \xi_2)$ can be interpreted in dual form as

$$\min_{\pi \in S_n} \{ n^{-1} \sum_{i=1}^{n} d_0(y_{1i}, y_{2\pi(i)}) \},$$

the average distance between the points $(y_{11}, \ldots, y_{1n})$ of ξ_1 and $(y_{21}, \ldots, y_{2n})$ of ξ_2 under the closest matching (Rachev (1984), Section 2.2). The metric d_2 is then the corresponding metric induced by d_1 over the probability measures on $\mathcal{X}$. Note that the distances d_1 and d_2, like d_0, are bounded.

The metric d_2 is considerably weaker than the total variation metric. However, a small value of d_2 implies that the laws of the empirical distribution functions of the interpoint distances of the two processes are correspondingly close, and the same is true if the joint empirical distributions of any k successive interpoint distances are considered. Work is in progress on relating such information to other quantities of interest, such as waiting times. In the following theorem, as is to be expected, the bounds (a) for total variation distance are less good than those (b) for the metric d_2: this is expressed in the bound (b) by the additional factor $k_2(\pi(\Gamma))$, which is small like $\pi(\Gamma)^{-1} \log(\pi(\Gamma))$ when $\pi(\Gamma)$ is large. Estimates (a) and (b) follow from Theorems 2.6 and 3.7 of Barbour and Brown (1992b); an even tighter bound (c) for the distribution of the total number of points in the process is based on Theorem 3.1 of Barbour and Brown (1992a).

Theorem 1. *Suppose Ξ is a finite point process on Γ with mean measure π. Suppose that, for each $\gamma \in \Gamma$, a finite point process $\Xi^{(\gamma)}$ on Γ is realized on the same probability space as Ξ, in such a way that $\Xi^{(\gamma)}$ has the distribution of the reduced Palm process at γ for Ξ. Then*

$$(a) \quad d_{TV}(\mathcal{L}(\Xi), \mathrm{Po}\,(\pi)) \leq \int_\Gamma E\|\Xi - \Xi^{(\gamma)}\| \pi(d\gamma),$$

$$(b) \quad d_2(\mathcal{L}(\Xi), \mathrm{Po}\,(\pi)) \leq k_2(\pi(\Gamma)) \int_\Gamma E\|\Xi - \Xi^{(\gamma)}\| \pi(d\gamma),$$

and

$$(c)\quad d_{TV}\left(\mathcal{L}(\Xi(\Gamma)), \mathrm{Po}\,(\pi(\Gamma))\right) \le k_1(\pi(\Gamma)) \int_\Gamma E\|\Xi - \Xi^{(\gamma)}\|\pi(d\gamma),$$

where the functions k_1 *and* k_2 *satisfy*

$$\lambda k_1(\lambda) \le 1; \qquad \lambda k_2(\lambda) \le \frac{5}{2}\Big(1 + 2\log^+\Big(\frac{2\lambda}{5}\Big)\Big).$$

3. Around Melamed's theorem

In order to apply Theorem 1 to the open migration process, we need some further notation. Throughout, C denotes a subset of $T = \{(j,k), 1 \le j \ne k \le J\}$, and the carrier space Γ has the form $[0,t]\times C$, for some fixed $t > 0$. A location in Γ is then interpreted as a time s and a directed link from j to k, and a point of the process at (s,j,k) represents a departure from queue j of a person who immediately joins queue k at time s, with $(s,0,j)$ used to denote an arrival at queue j and $(s,k,0)$ used to denote a departure from queue k. With M as defined preceding Lemma 1, let M_C denote the process $\{M_{jk},\ (j,k) \in C\}$, M_C^t its restriction to $[0,t]$. Let ν^t be the restriction of Lebesgue measure ν to $[0,t]$, ρ the measure on T with atoms ρ_{jk}, the steady state flow rates along the links (j,k), and ρ_C its restriction to C. Finally, let $X^{(0)}$ be a realization of X started in state k, and let $X^{*(0)}$ be a realization of X^* started in state j; set

$$\varepsilon_C^k = E\sum_{i=0}^{\infty} I[(X_i^{(0)}, X_{i+1}^{(0)}) \in C],$$

and

$$\eta_C^j = E\sum_{i=0}^{\infty} I[(X_{i+1}^{*(0)}, X_i^{*(0)}) \in C].$$

Thus ε_C^k denotes the expected number of transitions along links in C yet to be made by an individual currently in state k, and η_C^j denotes the expected number of transitions along links in C already made by an individual currently in state j: note that these quantities do not involve the ϕ_j's. Then the following result complements those of Melamed (1979) and Walrand and Varaiya (1981).

Theorem 2. *Let* $\theta_C^{jk} = \varepsilon_C^k + \eta_C^j$. *Then*

$$\text{(a)}\quad d_{TV}(\mathcal{L}(M_C^t), \mathrm{Po}\,(\nu^t \times \rho_C)) \;\le\; t \sum_{(j,k)\in C} \rho_{jk}\theta_C^{jk};$$

$$\text{(b)}\quad d_2(\mathcal{L}(M_C^t), \mathrm{Po}\,(\nu^t \times \rho_C)) \;\le\; tk_2(t\rho(C)) \sum_{(j,k)\in C} \rho_{jk}\theta_C^{jk};$$

$$\text{(c)}\quad d_{TV}\left(\mathcal{L}(M_C\{[0,t]\times C\}), \mathrm{Po}\,(t\rho(C))\right) \;\le\; tk_1(t\rho(C)) \sum_{(j,k)\in C} \rho_{jk}\theta_C^{jk}.$$

Remark 1. The bound (c) for the total number of C–transitions is particularly attractive, being uniform for all t. The d_2 process bound (b) is almost as satisfactory, in that it grows only logarithmically with t. By contrast, the total variation process bound (a) grows linearly with t.

Remark 2. It is possible to include arrival processes and departure processes in the point process being approximated. However, this complicates the notation and argument, and the details, which are straightforward, are omitted.

Remark 3. The quantity θ_C^{jk} can be interpreted as the expected number of past and future transitions along links in C for a customer currently travelling along the link from queue j to queue k.

Remark 4. If $\theta_C = \max_{(j,k)\in C} \theta_C^{jk} = 0$, as is the case when Melamed's 'absence of loops' condition is satisfied, the distribution of M_C^t is thus Poisson, and the theorem contains the direct half of Melamed's theorem. Here, the interest lies more in the fact that the effects of departures from his stringent condition are explicitly quantified: see Example 1 below.

Proof. In view of Theorem 1, it is enough to estimate the norm $\|M_C - M_C^{(j,k)}\|$ for each $(j,k) \in C$, where M_C, as before, is the steady state process of C–transitions, and $M_C^{(j,k)} + \delta_0^{(j,k)}$ is constructed on the same probability space, and has the conditional distribution of M_C given the occurrence of a (j,k) transition at time 0, so that $M_C^{(j,k)}$ has the corresponding reduced Palm distribution of M_C. We achieve the construction in such a way that

$$M^{(j,k)} \geq M \quad \text{on} \quad \{R \times T\} \setminus \{(0,(j,k))\}, \tag{3.1}$$

and then show that

$$E\|M_C^{(j,k)} - M_C\| = \varepsilon_C^k + \eta_C^j. \tag{3.2}$$

Construct the process N as indicated in Section 2. To realize the corresponding $\{N^{(j,k)}(t),\ t \geq 0\}$, add an individual to queue k at time 0, and use $X^{(0)}$ to give the route taken by the extra individual. To get the extra time evolution, use independently sampled exponential service requirements and let its instantaneous service rate be $\phi_l(r+1) - \phi_l(r)$ whenever it is in queue l together with r other individuals. Note that, in this construction, it is essential that individuals are indistinguishable. The same procedure is used looking backwards in time, starting $N^{(j,k)}$ this time with an extra individual in queue j at time 0, and using the same $N(0)$ as before. We use $X^{*(0)}$ as the sequence of queues that the extra customer in queue j has visited in the past, with the corresponding extra service requirements and service rates as for the extra individual in forward time. Claims (3.1) and (3.2) are now immediate and the theorem follows.

Example 1. [Tandem $M/M/1/\infty$ queues, with occasional feedback into the second queue]. Suppose that $J = 2$, $\phi_1(n) = \phi_2(n) = 1$ for all $n \geq 1$, $\lambda_{12} = 1$, $\lambda_{22} = 1 - \mu_2 = p$, $\nu_2 = 0$ and $\nu_1 < 1 - p$. The choice $p = 0$ would give a simple tandem queue, and the input into the second queue would be precisely a Poisson stream. Here, we describe what happens if p is positive, but small. If we take $C = \{(1,2),(2,2)\}$, so that M_C records the flows into queue 2, we have the following results, proved in Section 5:

$$\text{(a)} \quad \frac{(1-\nu_1-p)^2\nu_1 pt}{14(1-p)^2} \;\leq\; d_{TV}(\mathcal{L}(M_C^t), \mathrm{Po}\,(\nu^t \times \rho_C)) \;\leq\; \frac{2\nu_1 pt}{(1-p)^2};$$

$$\text{(b)} \quad d_2(\mathcal{L}(M_C^t), \mathrm{Po}\,(\nu^t \times \rho_C)) \;\leq\; \frac{10p}{1-p}\log^+\Big(\frac{2\nu_1 t}{5(1-p)}\Big) + \frac{5p}{1-p};$$

$$\text{(c)} \quad d_{TV}\big(\mathcal{L}(M_C\{[0,t]\times C\}), \mathrm{Po}\,(t\rho(C))\big) \;\leq\; \frac{2p}{1-p},$$

where, for the lower bound in (a), we require that

$$1/2 \leq \nu_1 t \leq (1-p)^2/2p.$$

The upper bound in (a) is small for small p, provided that $\nu_1 t$ is moderate. However, treating the input traffic rate ν_1 as constant and taking the feedback probability p to be smaller than $(1-\nu_1)/2$, we see that we cannot avoid the disappointing growth of the upper bound for the total variation distance in proportion to $\nu_1 t$. This is because the lower bound in (a) is of exactly the same order, with respect to (small) p and to $\nu_1 t$, the latter over the whole range from $1/2$ to the point at which the upper bound no longer gives any useful information. The ratio of upper to lower bounds in (a) is $28/(1-\nu_1-p)^2$, which increases as the input traffic goes towards its maximum level. The upper bound in (c) is uniformly small for small p, whilst that in (b) is small provided t is not very large.

Our proof of the lower bound could be shortened somewhat by using Burke's output theorem for the flow from queue 1 to queue 2. However, the proof given could in principle be extended to more complicated networks with loops, at the expense of much more calculation, suggesting that the linear growth of the total variation distance with time may be a general phenomenon for networks with loops.

Remark 5. Note that if, for any (j,k) and (l,m) in C,

$$E\|M_{lm}^{(j,k)} - M_{lm}\| = \theta^{jk}_{\{(l,m)\}} > 0, \tag{3.3}$$

the distribution of M_C cannot be $\mathrm{Po}\,(\nu \times \rho_C)$, implying the converse half of Melamed's theorem. This is because

$$\begin{aligned} EM_{jk}(t)(M_{jk}(t)-1) &= E\Big(\int_0^t (M_{jk}(t)-1)\,dM_{jk}(s)\Big) \\ &= E\Big(\int_0^t (v_t(s)\circ\theta_s)dM_{jk}(s)\Big), \end{aligned}$$

with $v_t(s) = M_{jk}[-s, t-s] - 1$. Thus, from the Matthes–Mecke formula ([BKM] p. 383), it follows that

$$EM_{jk}^2(t) - EM_{jk}(t) = \rho_{jk} E\int_0^t M_{jk}^{(j,k)}[-s,t-s]\,ds,$$

giving

$$\mathrm{Var}\,M_{jk}(t) - EM_{jk}(t) = \int_0^t \rho_{jk} E\{M_{jk}^{(j,k)}[-s,t-s] - M_{jk}[-s,t-s]\}ds; \tag{3.4}$$

the formula

$$\mathrm{Cov}\,(M_{jk}(t), M_{lm}(t)) = \int_0^t \rho_{jk} E\{M_{lm}^{(j,k)}[-s,t-s] - M_{lm}[-s,t-s]\}ds, \quad (l,m)\neq(j,k), \tag{3.5}$$

is proved similarly. Hence, if (3.3) is true then either (3.4), if $(l,m)=(j,k)$, or otherwise (3.5) is positive, whereas, for independent Poissons, these quantities are all zero. Thus $\theta_C = 0$ is necessary for M_C to have the distribution $\mathrm{Po}\,(\nu\times\rho_C)$. Furthermore, (3.4) could in principle also be used as the basis of an argument for establishing a lower bound on the total variation distance between $\mathcal{L}(M_{jk}^t)$ and $\mathrm{Po}\,(\rho_{jk}t)$, using an analogue of Theorem 3.E of Barbour, Holst and Janson (1992).

There are a number of situations in which the θ_C^{jk} might reasonably be expected to be small but not zero. One such is a hierarchical production process, where items pass through a sequence of stages in their production, but may occasionally be returned to an earlier stage because of faults being present: Example 1 is the most elementary instance. For such simply structured processes, it should be straightforward to estimate the θ_C^{jk}. Another is a 'distributed' service system, in which J is large, the λ_{jk}, $1 \le j \ne k \le J$, are all small, and the μ_j are bounded away from zero: an idealized model of a shopping mall, for instance. Here, the structure is rather more complicated, and a quick estimate of the θ's in terms of the λ's and μ's can prove useful.

For a first version, define the quantities

$$\begin{aligned} \delta^+ &= \min_{1\le j\le J} \mu_j; \quad \psi^+ = \max_{1\le j\le J} \sum_{k=1}^J \lambda_{jk} I[(j,k)\in C]; \\ \delta^- &= \min_{1\le j\le J} \nu_j/\alpha_j; \quad \psi^- = \max_{1\le j\le J} \sum_{k=1}^J (\alpha_k \lambda_{kj}/\alpha_j) I[(k,j)\in C]. \end{aligned} \tag{3.6}$$

Note that ψ^+ is an upper bound for the probability that an individual travels along a link in C at its next transition, whatever its current state, and ψ^- is the corresponding bound in the reversed process: δ^+ is a lower bound for the chance of leaving the system at the next transition, and δ^- the corresponding bound in the reversed process. Then it is immediate that $\varepsilon_C^k \le \psi^+/\delta^+$ for all k and that $\eta_C^j \le \psi^-/\delta^-$ for all j. Hence

$$\theta_C \le \left(\frac{\psi^+}{\delta^+}\right) + \left(\frac{\psi^-}{\delta^-}\right), \tag{3.7}$$

an estimate which can readily be used in conjunction with Theorem 2.

The fact that (3.7) uses both forward and backward estimates is a drawback if, for example, $C = \{(k, j_0) : 1 \le k \le J\}$ consists of all links into state j_0, since then ψ^- is likely to be big, because of the contribution to the maximum from $j = j_0$. However, the quantity $\sum_{(j,k)\in C} \rho_{jk}\theta_C^{jk}$ appearing in the sharper estimates of Theorem 2 has an interpretation which can be analysed using just one time direction.

To see this, let $\ldots, Z_{-1}, Z_0, Z_1, \ldots$ be a stationary Markov chain whose transition matrix is that of the forward customer chain. As previously stated, this chain is irreducible and persistent, and it has a stationary distribution which is proportional to $\{\sum_{j=1}^J \nu_j, \alpha_1, \ldots, \alpha_J\}$. By Theorem 7.16 of Breiman (1968), this process has no non-trivial invariant events. Let S be the space of doubly infinite sequences of elements of $\{0, 1, \ldots, J\}$, let $f : S \to \mathbb{R}$ and for l be an integer, let θ_l denote the shift transformation on S which shifts a sequence so that its lth element is in the zero position. Then, since the invariant events of the stationary process $\{f(Z\circ\theta_l)\}_{l=0}^\infty$ are contained in those of Z, the ergodic theorem gives

$$\lim_{n\to\infty} \frac{\sum_{l=0}^n f(Z\circ\theta_l)}{n} = E\Big(f(Z)\Big). \tag{3.8}$$

For $(j,k)\in C$ and $z \in S$ we can apply (3.8) to $f_1(z) = I[z_0 = j, z_1 = k]$ and
$f_2(z) = I[(z_0, z_1)\in C]$ to get

$$\lim_{n\to\infty} \frac{\sum_{l=0}^n I[Z_l = j, Z_{l+1} = k]}{\sum_{l=0}^n I[(Z_l, Z_{l+1})\in C]} = \frac{\rho_{jk}}{\rho(C)}. \tag{3.9}$$

For $z \in S$, let $m_1(z) < m_2(z) < \ldots$ be all the indices > 0 at which z takes the value 0. Let $m_0(z)$ be the greatest strictly negative index for which z takes the value 0. For $r = 1, 2, \ldots$, let $K_r(z)$ denote the number of l such that $m_{r-1}(z) < l < m_r(z)$ and $(z_l, z_{l+1}) \in C$, so that $K_r(z)$ represents the number of transitions in C between successive visits to 0. For $l = 1, 2\ldots$, let $r(l,z)$ denote the unique integer such that $m_{r(l,z)-1}(z) < l \leq m_{r(l,z)}(z)$. For z in S, let $f_3(z) = (K_{r(0,z)} - 1)I[z_0 = j, z_1 = k]$. For notational convenience we adopt the convention that if the argument $z \in S$ in a function is omitted from now on, it is to be replaced by the Markov chain Z. Applying (3.8) to f_3 and f_1 gives

$$\begin{aligned} \lim_{n\to\infty} \frac{\sum_{l=0}^{n}(K_{r(l)} - 1)I[Z_l = j, Z_{l+1} = k]}{\sum_{l=0}^{n} I[Z_l = j, Z_{l+1} = k]} &= \frac{E\left(f_3\right)}{\rho_{jk}} \\ &= \theta_C^{jk}. \end{aligned} \tag{3.10}$$

For $n = 1, 2, \ldots$, let $A(n)$ be the random collection of indices between 0 and m_1 together with those between $m_{r(n)-1}$ and n. Thus, from (3.9) and (3.10),

$$\begin{aligned} \frac{\sum_{(j,k)\in C} \rho_{jk}\theta_C^{jk}}{\rho(C)} &= \lim_{n\to\infty} \frac{\sum_{l=0}^{n}\sum_{(j,k)\in C}(K_{r(l)} - 1)I[Z_l = j, Z_{l+1} = k]}{\sum_{l=0}^{n} I[(Z_l, Z_{l+1}) \in C]} \\ &= \lim_{n\to\infty} \frac{\sum_{p=2}^{r(n)-1}(K_p - 1)K_p + \sum_{l\in A(n)}(K_{r(l)} - 1)I[(Z_l, Z_{l+1}) \in C]}{\sum_{p=2}^{r(n)-1} K_p + \sum_{l\in A(n)} I[(Z_l, Z_{l+1}) \in C]} \\ &= \lim_{q\to\infty} \frac{\sum_{p=1}^{q}(K_p - 1)K_p}{\sum_{p=1}^{q} K_p} \\ &= \frac{E\{(K_1 - 1)K_1\}}{EK_1}, \end{aligned}$$

the last two steps following from the strong law of large numbers, in view of the fact that $\{K_p\}_{p=1}^{\infty}$ is a sequence of independent and identically distributed random variables. Now

$$P[K_1 \geq i \,|\, K_1 \geq 1] \leq \{\psi_C^+/(\psi_C^+ + \delta^+)\}^{i-1},$$

where ψ^+ and δ^+ are as in (3.21), and

$$\frac{E\{(K_1 - 1)K_1\}}{EK_1} = \frac{E\{(K_1 - 1)K_1 \,|\, K_1 \geq 1\}}{E\{K_1 \,|\, K_1 \geq 1\}} \leq E\{(K_1 - 1)K_1 \,|\, K_1 \geq 1\},$$

giving a simple bound

$$\frac{1}{\rho(C)} \sum_{(j,k)\in C} \rho_{jk}\theta_C^{jk} = \frac{E\{(K_1 - 1)K_1\}}{EK_1} \leq 2\Big(\frac{\psi_C^+}{\delta^+}\Big)\Big(1 + \frac{\psi_C^+}{\delta^+}\Big), \tag{3.11}$$

for use in conjunction with Theorem 2. The neater bound

$$\frac{1}{\rho(C)} \sum_{(j,k)\in C} \rho_{jk}\theta_C^{jk} \leq 2\psi_C^+/\delta^+, \tag{3.12}$$

matching (3.7), follows if $\psi_C^+ \leq \delta^+$ from Lemma 2 in Section 5. Note that, under the suggested conditions of small λ's and δ^+ not too close to 0, ψ_C^+/δ^+ is small unless C contains many of the links out of a given state. Note also that, by reversing time, (3.11) and (3.12) could be replaced by the same formulae with $-$ for $+$ in the indices.

4. Bounds for moderate t.

In Brown and Pollett (1982), bounds on the total variation distance between the distributions of the arrival process at a queue and of a Poisson process are also derived. These bounds are quite different in nature. They do not make any appeal to loop structure or its absence, and do not vanish if Melamed's loop condition is satisfied. On the other hand, they can be constructed in systems, such as closed migration processes, where an individual may make many C-transitions during its time in the system. Broadly speaking, their bounds are tight when the arrival process at a queue can be thought of as a superposition of thinned point processes, a typical scenario for good Poisson approximation.

For example, in the distributed service system, taking $C = \{(k, j_0) : 1 \le k \le J\}$, Theorem 2 and (3.11) give a bound

$$d_{TV}\big(\mathcal{L}(M_C^t), \mathrm{Po}\,(\nu^t \times \rho_C)\big) \le 2t\rho(C)\Big(\frac{\psi_C^+}{\delta^+}\Big)\Big(1 + \frac{\psi_C^+}{\delta^+}\Big) \tag{4.1}$$

for the joint distributions of all flows into j_0, with

$$\psi_C^+ = \max_{1 \le k \le J} \lambda_{kj_0}.$$

The corresponding bound given in Brown and Pollett (1982), for the single server case $\phi_k(r) = c_k I[r > 0]$ for all k, is

$$t\Big[\sum_{k=1}^{J} c_k \alpha_k \lambda_{kj_0}^2 (1 - \alpha_k/c_k)\Big]^{1/2}, \tag{4.2}$$

and is for the aggregate flow into j_0: note that stationarity requires $\alpha_k < c_k$. By way of comparison between the bounds, Cauchy–Schwarz makes (4.2) bigger than $t\rho(C)(2J)^{-1/2}$ if $2\alpha_k < c_k$ for all k, ensuring that (4.1) is smaller if

$$4 \max_{1 \le k \le J} \lambda_{kj_0} \le \delta^+ (2J)^{-1/2};$$

on the other hand, if $\delta^+ = 0$, (4.1) is of no use, while (4.2) may be very small.

To obtain a somewhat closer comparison between the results of the two methods, observe that the bounds given in Theorem 2 can be improved for moderate values of t. This is because, in the proof of the theorem, $\|M_C^{(j,k)} - M_C\|$ is used as an upper bound for $\|M_C^{(j,k)} - M_C\|_{[-s,t-s]}$, $0 \le s \le t$, where the latter expression denotes the total variation norm of the difference of the two measures on $[-s, t-s] \times C$, and may be much smaller. Thus, if $\varepsilon_C^k(s)$ denotes the expected number of C-transitions made in $[0, s]$ by the 'extra' individual in state k at time 0, and if $\eta_C^j(s)$ denotes the analogous quantity in the reversed process, the proof of Theorem 2 can be directly modified to yield the following estimate.

Theorem 3. *Let* $\theta_C^{jk}(s) = \varepsilon_C^k(s) + \eta_C^j(s)$. *Then*

$$d_{TV}\big(\mathcal{L}(M_C^t), \mathrm{Po}\,(\nu^t \times \rho_C)\big) \le \int_0^t \sum_{(j,k) \in C} \rho_{jk} \theta_C^{jk}(s)\, ds.$$

As a result, if $\max_{r \ge 0}(\phi_j(r+1) - \phi_j(r)) = c_j$ for each $1 \le j \le J$, we have

$$\begin{aligned} &d_{TV}\big(\mathcal{L}(M_C^t), \mathrm{Po}\,(\nu^t \times \rho_C)\big) \\ &\le \frac{1}{2} t^2 \rho(C) \Big\{ \max_{1 \le j \le J} c_j \sum_{k=1}^{J} \lambda_{jk} I[(j,k) \in C] + \max_{1 \le j \le J} c_j \sum_{k=1}^{J} (\alpha_k \lambda_{kj}/\alpha_j) I[(k,j) \in C] \Big\}, \end{aligned} \tag{4.3}$$

using the natural upper bounds on the rates at which an individual can make C-transitions in the forward and reversed processes.

As before, the immediate bound (4.3) is of little use when considering the flow into a state j_0, because, if $C = \{(k, j_0) : 1 \le k \le J\}$, the second of the maxima can be expected to be large. However, an ergodic argument can again be used to replace the sum of the two maxima by twice the smaller of them, provided that

$$\varepsilon_C^k(s) \le \tilde{\varepsilon}_C^k(s) \text{and} \qquad \eta_C^j(s) \le \tilde{\eta}_C^j(s), \tag{4.4}$$

where $\tilde{\varepsilon}$ and $\tilde{\eta}$ relate to typical individuals with given state at time 0, rather than to the 'extra' one. A condition sufficient to ensure (4.4) is that, for each j, the differences $\phi_j(r+1) - \phi_j(r)$ decrease with r in $r \ge 0$. This condition is certainly satisfied when $\phi_j(r) = c_j I[r > 0]$, leading to a bound

$$d_{TV}\left(\mathcal{L}(M_C^t), \mathrm{Po}\left(\nu^t \times \rho_C\right)\right) \le t^2 \rho(C) \max_{1 \le k \le J} c_k \lambda_{kj_0}, \tag{4.5}$$

to be compared with (4.1) and (4.2). Although none of the three bounds is always worse than the others, it is of interest to note that

$$\sum_{k=1}^J c_k \alpha_k \lambda_{kj_0}^2 \quad \text{and} \quad \rho(C) \max_{1 \le k \le J} c_k \lambda_{kj_0} = \Big(\max_{1 \le k \le J} c_k \lambda_{kj_0}\Big)\Big(\sum_{k=1}^J \alpha_k \lambda_{kj_0}\Big)$$

are often of comparable size, in which case (4.5) is of order the square of (4.2), and is hence very much better.

5. Proofs

Proof of Example 1, (a-c). Solving (2.1), we have $\alpha_1 = \nu_1$ and $\alpha_2 = \nu_1/(1-p) < 1$, and it thus follows that b_1 and b_2 are both finite, and that $\rho_{12} = \nu_1$ and $\rho_{22} = \nu_1 p/(1-p)$. Then simple calculations in the forward and reversed customer chains give

$$\eta_C^1 = 0, \quad \eta_C^2 = 1/(1-p), \quad \varepsilon_C^2 = p/(1-p) \quad \text{and thus} \quad \theta_C^{12} = p/(1-p) \text{ and } \theta_C^{22} = (1+p)/(1-p).$$

Hence

$$\sum_{(j,k)\in C} \rho_{jk}\theta_C^{jk} = \frac{2\nu_1 p}{(1-p)^2} \quad \text{and} \quad \frac{1}{\rho(C)} \sum_{(j,k)\in C} \rho_{jk}\theta_C^{jk} = 2p/(1-p),$$

and the upper bounds now follow from Theorem 2.

It now remains to consider the lower bound in part (a). The time $t > 0$ is fixed, and $h < t$ is to be specified at the end. Let M_1 and M_2 denote (simple) point processes and form the (simple) point process $\mathcal{M}$ on $[0,t]$ by defining for $s \le t$

$$\mathcal{M}_s = \int_0^s I[M_2((u+h)\wedge t) - M_2(u) \ge 1]\, dM_1(u), \tag{5.1}$$

so that, ignoring boundary effects, $\mathcal{M}$ counts the points of M_1 which are followed within h by a point of M_2. Now

$$\int_0^t (1 - \mathcal{M}_{s-})\, d\mathcal{M}_s \le I[\mathcal{M}_t \ge 1] \le \mathcal{M}_t, \tag{5.2}$$

which follows easily because the integrand of the left side is 1 at the first jump of $\mathcal{M}$ and ≤ 0 at each jump thereafter.

Let M_1 be a Poisson process of rate $\rho_{12} = \nu_1$ and M_2 be an independent Poisson process of rate $\rho_{22} = \nu_1 p/(1-p)$. Let $\mathcal{G}_s = \sigma(M_1(u), u \leq s) \vee \sigma(M_2)$. Then the point process $\mathcal{M}$ is adapted to the history $\mathcal{G}$, and the conditional intensity of $\mathcal{M}$ at time $s \leq t$ is $\nu_1 J_s$, where $J_s = I[M_2((s+h)\wedge t) - M_2(s) \geq 1]$. Using (5.1) and (5.2), we have

$$\begin{aligned} P(\mathcal{M}_t \geq 1) &\leq E(\mathcal{M}_t) \\ &= E\Big(\int_0^{t-h} I[M_2(s+h) - M_2(s) \geq 1]\nu_1 ds\Big) + E\Big(\int_{t-h}^{t} I[M_2(t) - M_2(s) \geq 1]\nu_1 ds\Big) \\ &= \int_0^{t-h} (1 - e^{\frac{-p\nu_1 h}{1-p}})\nu_1 ds + \int_{t-h}^{t} (1 - e^{\frac{-p\nu_1 (t-s)}{1-p}})\nu_1 ds \\ &\leq \frac{\nu_1}{1-p} hp\nu_1 (t - \frac{h}{2}), \end{aligned} \tag{5.3}$$

the middle step following from an interchange of expectation and Lebesgue integration.

We now let M_1 be the process of transitions from queue 1 to queue 2, and let M_2 be the process of feedback transitions from queue 2 to queue 2. The lower bound in (5.2) may be further bounded:

$$\begin{aligned} E(\int_0^t (1 - \mathcal{M}_{s-}) d\mathcal{M}_s) \;\geq\; & E(\mathcal{M}_t) - E\Big(\int_h^t \mathcal{M}_{(s-h)-} d\mathcal{M}_s\Big) \\ & - E\Big(\int_h^t (M_1(s-) - M_1((s-h)-)) d\mathcal{M}_s\Big) - E\Big(\int_0^h \mathcal{M}_{s-} d\mathcal{M}_s\Big), \end{aligned} \tag{5.4}$$

since each of the points of $\mathcal{M}$ is a point of M_1 .

Taking optional projections and using the Markov property of the system,

$$E(\mathcal{M}_t) \;=\; E\Big(\sum_n \int_0^t P(M_2((s+h)\wedge t) - M_2(s) \geq 1 | N_s = n) I[N_s = n] \; dM_1(s)\Big)$$

where the sum is over $n = (n_1, n_2)$ such that both elements are non-negative. The conditional probability in the integral is at least the probability that queue 2 completes at least one service in $(s, (s+h)\wedge t)$, and the first such service is fed back to queue 2. This latter probability, for $n_2 > 0$ and $0 \leq s \leq t-h$, is equal to $p(1 - e^{-h})$ and for $n_2 > 0$ and $t - h \leq s \leq t$ is equal to $p(1 - e^{-(t-s)})$. At the points of M_1 we have $N_2 \geq 1$ so

$$\begin{aligned} E(\mathcal{M}_t) \;\geq\; & E\Big(\int_0^{t-h} p(1 - e^{-h}) \; dM_1(s) + \int_{t-h}^{t} p(1 - e^{-(t-s)}) \; dM_1(s)\Big) \\ =\; & (1 - e^{-h}) p\nu_1 (t-h) \; + p\nu_1 (h - (1 - e^{-h})) \geq (1 - \frac{h}{2}) hp\nu_1 (t - \frac{h}{2}) \end{aligned} \tag{5.5}$$

the second step following on integrating with respect to the mean measure of M_1 . On the other hand, using the fact that if X is a nonnegative integer valued random variable the conditional probability of $X \geq 1$ is bounded above by the conditional expectation of X ,

$$E(\mathcal{M}_t) \;\leq\; hpE(M_1(t)) \;=\; hp\nu_1 t. \tag{5.6}$$

Looking next at the second term in (5.4), and using (5.6), we have

$$
\begin{aligned}
E\Big(\int_h^t \mathcal{M}_{(s-h)-} d\mathcal{M}_s\Big) &\le E\Big(\int_h^t \mathcal{M}_{(s-h)-} hp\ dM_1(s)\Big) \\
&= hp \int_h^t E(\mathcal{M}_{(s-h)-} I[N_1(s) \ge 1])\ ds \qquad (5.7)\\
&= hp \int_h^t E(\mathcal{M}_{(s-h)-})\nu_1\ ds \ \le\ \frac{hp\nu_1 t}{2} hp\nu_1(t-\frac{h}{2}),
\end{aligned}
$$

because $\mathcal{M}_{(s-h)-}$ is independent of $I[N_1(s) \ge 1]$. To see this note that the random variable $\mathcal{M}_{(s-h)-}$ is measurable with respect to $\sigma(M_1(z), M_2(z), z \le s)$, and this σ-field is independent of $\sigma(N_1(s))$. To see this, note that the σ-fields can be defined from the reverse-time system in which s becomes 0. The reverse time system is also an open queueing network, but the point processes now feed queue number 1. Hence, the reverse-time point processes can be constructed from random variables which are independent of the state of the first queue at time 0.

Similarly,

$$
E\Big(\int_h^t \{M_1(s-) - M_1((s-h)-)\}\ d\mathcal{M}_s\Big) \ \le h^2 p\nu_1^2(t-h) \text{ and } E\Big(\int_0^h \ \mathcal{M}_{s-}\ d\mathcal{M}_s\Big) \ \le\ \frac{h^3 p\nu_1^2}{2}. \qquad (5.8)
$$

For the tandem queue, combining (5.5), (5.7) and (5.8) in (5.4) and using (5.2) gives

$$
P(\mathcal{M}_t \ge 1) \ \ge\ (1 - h(\frac{1}{2} + \nu_1(1+\frac{pt}{2}))hp\nu_1(t-\frac{h}{2}). \qquad (5.9)
$$

Taking the difference between the right side of (5.9) and (5.3) gives the difference of the probabilities of $\mathcal{M}_t \ge 1$ under the actual and Poisson approximation as at least

$$
(\delta - h\gamma)hp\nu_1(t-\frac{h}{2}), \qquad (5.10)
$$

where

$$
\delta = 1 - \frac{\nu_1}{1-p} \quad \text{and} \quad \gamma = \frac{1}{2} + \nu_1(1+\frac{pt}{2}) \le \frac{7}{4},
$$

this last from the upper bound on $\nu_1 t$.

We now take $h = \delta/(2\gamma)$, and note that

$$
h \le (1-\nu_1-p)/[(1-p)(1+2\nu_1)] \le 1/2\nu_1 \le t
$$

from the lower bound for t, so that (5.10) then becomes

$$
\frac{\delta^2\nu_1 p(t-\frac{h}{2})}{4\gamma} \ge \frac{\delta^2\nu_1 pt}{8\gamma} \ge \frac{\delta^2\nu_1 pt}{14}. \qquad (5.11)
$$

Thus the total variation distance of the two processes, which dominates the absolute difference between these two probabilities, is bounded below by the right hand side of (5.11), because this is nonnegative.

Lemma 2. *If* K *is a random variable taking values in* $\{1, 2, \ldots\}$ *such that* $P[K \geq j] \leq c^{j-1}$, $j \geq 2$, *for some* $c \leq 1/2$, *then*

$$\frac{E\{K(K-1)\}}{EK} \leq 2c/(1-c).$$

Proof. Let $g_j = \sum_{l \geq j} P[K \geq l]$, $j \geq 2$; then

$$0 \leq g_j \leq \sum_{l \geq j} c^{l-1} = c^{j-1}/(1-c) \tag{5.12}$$

by hypothesis, and

$$\frac{E\{K(K-1)\}}{EK} = \frac{2\sum_{j\geq 2} g_j}{1+g_2} = 2\Big[1 - \frac{(1-\sum_{j\geq 3} g_j)}{1+g_2}\Big].$$

This latter expression is clearly increasing in each g_j for $j \geq 3$, and in g_2 also if $c \leq 1/2$ for any choice of the g_j such that (5.12) holds, since then

$$\sum_{j\geq 3} g_j \leq c^2/(1-c)^2 \leq 1.$$

The maximum possible is thus obtained when equality holds for all j in (5.12), proving the lemma.

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