The uniqueness of Atkinson and Reuter’s epidemic waves

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Atkinson and Reuter (1) consider travelling wave solutions for the deterministic epidemic, with or without removals, spreading along the line. In the case where there are no removals, they reformulate the problem in terms of the solutions $X(u)$ to the integral equation

$$X(u) = -c^{-1} \int_{-\infty}^{\infty} \{1 - e^{X(u-x)}\} Q(v) dv = TX(u), \quad (1)$$

which satisfy $X(-\infty) = -\infty$, $X(+\infty) = 0$, $X(u) < 0$ for $u \in (-\infty, \infty)$, where

$$Q(v) = \int_{-\infty}^{v} dP(u)$$

is the left hand tail of the contact distribution, and where $c > 0$ is the velocity of the wave corresponding to $X$. They show that no solution is possible unless

$$F(\lambda) = \int_{-\infty}^{\infty} Q(v) e^{-\lambda v} dv$$

converges for $\lambda > 0$ sufficiently small, and that any solution $X$ must satisfy

$$X(u) = O(|u|) \quad \text{as} \quad u \to -\infty; \quad X(u) = O(e^{-\alpha u}) \quad \text{as} \quad u \to \infty, \quad (2)$$

for some $C > 0$. They then prove the following existence theorems.

**Theorem 1.** If $F(\lambda) = c$ has a positive root $\lambda = \alpha$, and if $F(\lambda)$ exists and is $< c$ in some interval $\alpha < \lambda < \alpha + \delta$, then equation (1) has a solution.

**Theorem 2.** If $F(\lambda)$ is finite for $0 < \lambda < A$ and attains a minimum $c_0$ at $\lambda = \alpha_0 < A$, then there is no solution to equation (1) with $0 < c < c_0$.

Their results for the case with removals are similar, $F$ being replaced by

$$G(\lambda) = F(\lambda) - b/\lambda,$$

where $b$, $0 < b < 1$, is the relative removal rate, and equation (1) by

$$cX(u) = -\int_{-\infty}^{\infty} \{1 - e^{X(u-x)}\} \left( Q(v) - \int_{0}^{\infty} Q(v-s) e^{-\lambda s} ds \right) dv,$$

where $\delta = b/c$. Brown and Corr (2) have shown that, under the conditions of Theorem 2, Theorem 1 also implies the existence of a solution with the critical velocity $c = c_0$.

In this paper, it is shown that the Atkinson–Reuter solution constructed in Theorem 1 is in fact unique. Only the case without removals is described: the argument for
the case with removals is exactly similar. Let \( X(t) \) be any solution to (1), and for each \( \lambda > 0 \), define

\[
W(\lambda) = -\frac{X(t)e^\lambda}{1 + e^\lambda};
\]

\[
g(\lambda) = C e^{\int_0^\lambda Q(v) e^{\nu[v]} dv};
\]

suppose, as usual, that \( P(x) = 1 \) and that \( P(\lambda) < e \) for \( \lambda \) in some interval \((a, a + \delta)\).

**Lemma.** Let \( \lambda \leq \alpha \). Then if, for some \( \beta > \lambda / 2 \), \( P(\beta) > P(\lambda) \) and \( W(\lambda) \leq k_1 \) uniformly in \( t \) for some \( k_1 \), it follows that \( \lim_{\lambda \to \infty} W(\lambda) \) exists and is finite.

**Proof.** Note first, in preparation, that equation (1) can be rewritten as

\[
W(\lambda) = \frac{1}{2} e^{\int_0^\lambda \frac{1}{2} e^{\nu[v]}} e^{\int_0^\lambda \frac{1}{2} e^{2\nu[v]}}
\]

where \( \theta = e^{-2P(\lambda)} \geq 1 \) and \( e^{\nu[v]} = (\varepsilon \theta^v) e^{\nu[v]} \). Then, by hypothesis, and because \( 1 + e^\lambda \geq 1 + e^{\lambda/2} \) if \( x < 0 \),

\[
0 \geq g(\lambda) \geq \frac{1}{2} e^{\int_0^\lambda \frac{1}{2} e^{2\nu[v]}}
\]

and so \( Y_n = W(\lambda) + \frac{1}{2} e^{\int_0^\lambda \frac{1}{2} e^{2\nu[v]}} \) is supermartingale. Now

\[
E[Y_n + 1] \geq E[\sum_{i=1}^{n} g_i(S_i)]
\]

and, from (4),

\[
Eg_1(S) \geq -K_1(1 - e^{-2\nu})
\]

with \( g_1(S) = 1 - e^{-2\nu} \).

Given

\[
0 \geq EY_n \geq E\left[\sum_{i=1}^{n} g_i(S_i)\right],
\]

uniformly in \( n \geq 1 \). Hence, by the semi-martingale convergence theorem, \( Y_n \) converges a.s. to a finite limit as \( n \to \infty \). Since \( \sum_{i=1}^{n} g_i(S_i) \) is a.s. decreasing, and its expectation is bounded below by (5), it follows that it converges a.s. to a finite limit, and hence \( W(\lambda) \) also converges a.s. to a finite limit.

Now \( EY_n \geq \frac{1}{2} e^{\int_0^\lambda \frac{1}{2} e^{2\nu[v]}} \), giving

\[
0 \geq EY_n \geq \frac{1}{2} e^{\int_0^\lambda \frac{1}{2} e^{2\nu[v]}}
\]

uniformly in \( n \geq 1 \). Hence, by the semi-martingale convergence theorem, \( X(t) \) converges a.s. to a finite limit as \( n \to \infty \). Since \( \sum_{i=1}^{n} g_i(S_i) \) is a.s. decreasing, and its expectation is bounded below by (5), it follows that it converges a.s. to a finite limit, and hence \( W(\lambda) \) also converges a.s. to a finite limit.

Finally, noting that \( X(t) \to X(t^+) \) as \( t \to \infty \), we deduce that "The uniqueness of Atkinson and Reuter's wave equation" where \( m(a, b) = \int_{a}^{b} P(x) e^{-\theta x} dx \); and thus that

\[
\lim_{\lambda \to \infty} W(\lambda) \geq \frac{1}{2} e^{\int_0^\lambda \frac{1}{2} e^{2\nu[v]}}
\]

Since choosing \( \epsilon_0 \geq 0 \) is equivalent to changing the origin, it follows in general that there exists a \( \epsilon_0 \) and an \( \eta = \eta(\epsilon_0) > 0 \) such that, for all \( \epsilon > \epsilon_0 \),

\[
P(\epsilon) \geq \frac{1}{2} e^{\int_0^\lambda \frac{1}{2} e^{2\nu[v]}} \eta \geq 0.
\]

Hence, if \( I = \bigcup I_j \) is any countable collection of disjoint intervals of length \( h \), and if \( \epsilon \) denotes the event \( \bigcup_{\epsilon > \epsilon_0} P(A_j), P(A_j) \geq \eta \) for all \( m > 0 \). But, as \( m \to \infty \),

\[
P(\epsilon) \geq \frac{1}{2} e^{\int_0^\lambda \frac{1}{2} e^{2\nu[v]}} \eta \geq 0.
\]

Hence, \( \epsilon \) is a solution to (1), \( X(\epsilon) = \epsilon^{-\theta} e^{-\lambda} \). This analysis shows that \( W(\lambda) \geq \frac{1}{2} e^{\int_0^\lambda \frac{1}{2} e^{2\nu[v]}} \eta \), for all \( \lambda \), and for all \( \epsilon > \epsilon_0 \).

Finally, \( W(\lambda) \to \infty \) as \( \lambda \to \infty \), since, otherwise, from the continuity of \( W(\lambda) \), \( W(\lambda) \) converges a.s. to some finite limit \( \eta \). For any such sequence, since \( W(\lambda) = \lambda W(\lambda - \epsilon) \), \( \lambda W(\lambda - \epsilon) \to \lambda W(\lambda) \), it follows, for all \( \epsilon > \epsilon_0 \), that for all \( \epsilon > \epsilon_0 \),

\[
W(\lambda) = \frac{1}{2} e^{\int_0^\lambda \frac{1}{2} e^{2\nu[v]}} \eta \geq 0.
\]

Hence, \( \epsilon \) is a solution to (1), \( X(\epsilon) = \epsilon^{-\theta} e^{-\lambda} \). This analysis shows that \( W(\lambda) \geq \frac{1}{2} e^{\int_0^\lambda \frac{1}{2} e^{2\nu[v]}} \eta \), for all \( \lambda \), and for all \( \epsilon > \epsilon_0 \).

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Finally, \( W(\lambda) \to \infty \) as \( \lambda \to \infty \), since, otherwise, from the continuity of \( W(\lambda) \), \( W(\lambda) \) converges a.s. to some finite limit \( \eta \). For any such sequence, since \( W(\lambda) = \lambda W(\lambda - \epsilon) \), \( \lambda W(\lambda - \epsilon) \to \lambda W(\lambda) \), it follows, for all \( \epsilon > \epsilon_0 \), that for all \( \epsilon > \epsilon_0 \),

\[
W(\lambda) = \frac{1}{2} e^{\int_0^\lambda \frac{1}{2} e^{2\nu[v]}} \eta \geq 0.
\]
Let \( W_1(u) = X_1(u) e^{tu} \), \( W_2(u) = X_2(u) e^{tu} \), and consider the quantity

\[
\|W_1 - W_2\| = \sup_{-\infty < u < \infty} |W_1(u) - W_2(u)|.
\]

since \( W_1 \) and \( W_2 \) are continuous, and have equal finite limits at \(-\infty \) and \(+\infty\),

\[
\|W_1 - W_2\| < \infty \quad \text{and} \quad \|W_1(u) - W_2(t_0)\| = \|W_1(u) - W_2(t_0)\|
\]

for some \(-\infty < t_0 < \infty\). Now, since \( X_1 \) and \( X_2 \) are solutions of (1),

\[
|W_1(u) - W_2(u)| = |TX_1(u) - TX_2(u)| e^{tu},
\]

and so, for any \( u \) for which \( |W_1(u) - W_2(u)| > 0 \),

\[
0 < |W_1(u) - W_2(u)| \leq c^{-1} \int_{-\infty}^{\infty} Q(y) e^{\tau y} |X_1(u + y) - X_2(u + y)| e^{\tau y} \, dy
\]

\[
\leq c^{-1} \int_{-\infty}^{\infty} Q(y) e^{-\tau y} |X_1(u + y) - X_2(u + y)| e^{\tau y} \, dy
\]

(7)

\[
\leq c^{-1} P(u) \|W_1 - W_2\| = \|W_1 - W_2\|.
\]

Hence, by considering \( u = t_0 \), \( \|W_1 - W_2\| > 0 \) is impossible, and it follows that

\[
X_1(u) = X_2(u)
\]

for all \( u \). This proves the theorem.

The theorem does not cover the case \( c = c_0 \). By considering equation (3), it can be shown that any solution of (1) corresponding to velocity \( c_0 \) has a translate \( X \) for which \( X(t) \sim -t e^{-\alpha t} \), but this is not a strong enough restriction on \( X \) for the subsequent argument to be applied.

I would like to thank Professor Reuter for his helpful suggestions as to the presentation of the argument.

REFERENCES
