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The uniqueness of Atkinson and Reuter's epidemic waves

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Atkinson and Reuter (1) consider travelling wave solutions for the deterministic epidemic, with or without removals, spreading along the line. In the case where there are no removals, they reformulate the problem in terms of the solutions $X(\cdot)$ to the integral equation

$$X(u) = -c^{-1}\int_{-\infty}^{\infty}\{1-e^{X(u+v)}\}Q(v)\,dv \equiv TX(u), \tag{1}$$

which satisfy $X(-\infty) = -\infty$, $X(+\infty) = 0$, $X(u) < 0$ for $u \in (-\infty, \infty)$, where

$$Q(v) \equiv \int_{-\infty}^{v} dP(u)$$

is the left hand tail of the contact distribution, and where $c > 0$ is the velocity of the wave corresponding to X. They show that no solution is possible unless

$$F(\lambda) \equiv \int_{-\infty}^{\infty} Q(w)\,e^{-\lambda w}dw$$

converges for $\lambda > 0$ sufficiently small, and that any solution X must satisfy

$$X(u) = O(|u|) \quad \text{as} \quad u \to -\infty; \quad X(u) = O(e^{-Cu}) \quad \text{as} \quad u \to \infty, \tag{2}$$

for some $C > 0$. They then prove the following existence theorems.

THEOREM 1. *If $F(\lambda) = c$ has a positive root $\lambda = \alpha$, and if $F(\lambda)$ exists and is $< c$ in some interval $\alpha < \lambda < \alpha+\delta$, then equation* (1) *has a solution.*

THEOREM 2. *If $F(\lambda)$ is finite for $0 < \lambda \leqslant A$ and attains a minimum c_0 at $\lambda = \alpha_0 < A$, then there is no solution to equation* (1) *with $0 < c < c_0$.*

Their results for the case with removals are similar, F being replaced by

$$G(\lambda) \equiv F(\lambda) - b/\lambda,$$

where $b, 0 < b < 1$, is the relative removal rate, and equation (1) by

$$cX(u) = -\int_{-\infty}^{\infty}\{1-e^{X(u+v)}\}\left\{Q(v) - \int_0^{\infty} Q(v-s)\,\delta e^{-\delta s}ds\right\}dv,$$

where $\delta = b/c$. Brown and Carr (2) have shown that, under the conditions of Theorem 2, Theorem 1 also implies the existence of a solution with the critical velocity $c = c_0$.

In this paper, it is shown that the Atkinson–Reuter solution constructed in Theorem 1 is in fact unique. Only the case without removals is described: the argument for

the case with removals is exactly similar. Let $X(\cdot)$ be any solution to (1), and for each $\lambda > 0$, define

$$W_\lambda(t) = -X(t)\,e^{\lambda t} > 0;$$

$$g_\lambda(t) = c^{-1}\int_{-\infty}^{\infty} Q(v)\,e^{\lambda t}[1+X(t+v)-e^{X(t+v)}]\,dv;$$

suppose, as usual, that $F(\alpha) = c$, and that $F(\lambda) < c$ for λ in some interval $(\alpha, \alpha+\delta)$.

LEMMA. *Let λ be $\leqslant \alpha$. Then if, for some $\beta > \lambda/2$, $F(2\beta) < F(\lambda)$ and $W_\beta(t) \leqslant k_\beta$ uniformly in t for some k_β, it follows that $\lim_{t\to\infty} W_\lambda(t)$ exists and is finite.*

Proof. Note first, in preparation, that equation (1) can be rewritten as

$$W_\lambda(t) = \theta_\lambda \int_{-\infty}^{\infty} W_\lambda(t+v)\,dF_\lambda(v) + g_\lambda(t), \tag{3}$$

where $\theta_\lambda = c^{-1}F(\lambda) \geqslant 1$ and $dF_\lambda(v) = (c\theta_\lambda)^{-1}Q(v)\,e^{-\lambda v}dv$: and that, by hypothesis, and because $1+x-e^x \geqslant -x^2/2$ if $x < 0$,

$$\begin{aligned} 0 > g_\lambda(t) &\geqslant -\frac{1}{2c}k_\beta^2\,e^{(\lambda-2\beta)t}\int_{-\infty}^{\infty} Q(v)\,e^{-2\beta v}dv \\ &= -K_\beta\,e^{(\lambda-2\beta)t}, \quad \text{say.} \end{aligned} \tag{4}$$

Consider $Y_n = W_\lambda(S_n) + \sum_{j=0}^{n-1} g_\lambda(S_j)$, where, for some s_0, $S_n = s_0 + \sum_{r=1}^{n} X_r$, and $(X_r)_{r\geqslant 1}$ are independent and identically distributed with distribution $dF_\lambda(\cdot)$. Then, if $\mathscr{F}_n$ denotes the σ-field $\sigma(X_1, \ldots, X_n)$,

$$\begin{aligned} E[Y_{n+1}|\mathscr{F}_n] &= \int_{-\infty}^{\infty} W_\lambda(S_n+v)\,dF_\lambda(v) + \sum_{j=0}^{n} g_\lambda(S_j) \\ &= \theta_\lambda^{-1}W_\lambda(S_n) + \sum_{j=0}^{n-1} g_\lambda(S_j) + g_\lambda(S_n)(1-\theta_\lambda^{-1}) \\ &\leqslant W_\lambda(S_n) + \sum_{j=0}^{n-1} g_\lambda(S_j) = Y_n; \end{aligned}$$

and so $(Y_n, \mathscr{F}_n, n \geqslant 0)$ is a super-martingale. Now

$$0 \geqslant EY_n^- \geqslant E\left[\sum_{j=0}^{n-1} g_\lambda(S_j)\right],$$

and, from (4),

$$Eg_\lambda(S_j) \geqslant -K_\beta[(c\theta_\lambda)^{-1}F(2\beta)]^j\,e^{(\lambda-2\beta)s_0},$$

giving

$$0 \geqslant EY_n^- \geqslant -K_\beta\,e^{(\lambda-2\beta)s_0}F(\lambda)/(F(\lambda)-F(2\beta)) \tag{5}$$

uniformly in $n \geqslant 1$. Hence, by the semi-martingale convergence theorem, Y_n converges a.s. to a finite limit as $n\to\infty$. Since $\sum_{j=0}^{n} g_\lambda(S_j)$ is a.s. decreasing, and its expectation is bounded below by (5), it follows that it converges a.s. to a finite limit, and hence $W_\lambda(S_n)$ also converges a.s. to a finite limit.

Now $EX_j = -(c\theta_\lambda)^{-1}F'(\lambda) > 0$. So, taking $s_0 = 0$, it follows from Blackwell's renewal theorem that, for any $h > 0$,

$$\begin{aligned} P[\bigcup_{n\geqslant 1}\{S_n \in [t, t+h]\}](1+m(-h,h)) &\geqslant m(t, t+h) \\ &\to h/\mu \quad \text{as} \quad t\to\infty, \end{aligned}$$

where $m(a,b) = \sum_{j=1}^{\infty} P[a \leqslant S_j \leqslant b] < \infty$; and thus that

$$\liminf_{t\to\infty} P[\bigcup_{n\geqslant 1}\{S_n \in [t, t+h]\}] > 0.$$

Since choosing $s_0 \neq 0$ is equivalent to changing the origin, it follows in general that there exists a t_0 and an $\eta = \eta(h) > 0$ such that, for all $t \geqslant t_0 + s_0$,

$$P[\bigcup_{n\geqslant 1}\{S_n \in [t, t+h]\}] \geqslant \eta > 0.$$

Hence, if $J = \bigcup_{j\geqslant 1} I_j \subseteq R^+$ is any countable collection of disjoint intervals of length h, and if A_J denotes the event $\bigcup_{n\geqslant 1}\{S_n \in J\}$, $P[A_J|\mathscr{F}_m] \geqslant \eta$ for all $m \geqslant 0$. But, as $m\to\infty$, $P[A_J|\mathscr{F}_m]$ converges a.s. to the indicator of A_J, and so $P[A_J] = 1$, whatever the value of s_0. Hence also,

$$P[S_n \in J \text{ i.o.}] = 1. \tag{6}$$

It is impossible that $W_\lambda(t)\to\infty$ as $t\to\infty$, since this would contradict the convergence of $W_\lambda(S_n)$ to an a.s. finite limit. Hence there must exist a sequence $(t_j)_{j\geqslant 1}$ such that, as $j\to\infty$, $t_j\to\infty$ and $W_\lambda(t_j)$ converges to some finite limit w. For any such sequence, since $W'_\lambda(t) = \lambda W_\lambda(t) - X'(t)\,e^{\alpha t} \leqslant \lambda W_\lambda(t)$, it follows that, for all j sufficiently large, and for any given $\epsilon < 1/2$,

$$\begin{aligned} W_\lambda(t) &\geqslant w - 2\epsilon \quad \text{whenever} \quad t_j - \epsilon/\lambda w \leqslant t \leqslant t_j; \\ W_\lambda(t) &\leqslant w + 2\epsilon \quad \text{whenever} \quad t_j \leqslant t \leqslant t_j + \epsilon/\lambda w. \end{aligned}$$

Hence, by (6), and since $W_\lambda(S_n)$ converges a.s. it follows that $W_\lambda(S_n)\to w$ a.s. This analysis shows that $W_\lambda(t)$ converges to w as $t\to\infty$, since otherwise, from the continuity of W_λ, it would be possible to choose a sequence (t_j) converging to some other finite w', and argue also that $W_\lambda(S_n)\to w'$ a.s.

THEOREM. *Under the hypotheses of Theorem 1, there is a unique solution to equation (1) such that $X(u) < 0$ for all $u \in (-\infty, \infty)$ and $X(0) = -1$.*

Remark. The condition $X(0) = -1$ is an arbitrary normalization, needed because the translates $X^t(\cdot) = X(\cdot - t)$ also satisfy (1).

Proof. From the estimates (2), if $X(\cdot)$ is a solution to (1), $X(u) = O(e^{-Cu})$ as $u\to\infty$, for some $C > 0$. The lemma then implies that $X(u) = O(e^{-\lambda u})$ and

$$W_\lambda(u)\to w_\lambda < \infty \quad \text{as} \quad u\to\infty,$$

for any $\lambda \leqslant \alpha$ such that $\lambda < 2C$. Replacing C by λ, and repeating the argument as many times as is necessary, it follows that $W_\alpha(t)\to w < \infty$ for some $w \geqslant 0$: and since $\theta_\alpha = 1$ and $g_\lambda(t) < 0$, it is necessary that $w > 0$, as otherwise W_α would reach its maximum at some $-\infty < t_0 < \infty$, which is inconsistent with (3). Similarly, to the translate X^t corresponds W_α^t with $W_\alpha^t(u) = X(u-t)\,e^{\alpha u}\to w\,e^{\alpha t}$ as $u\to\infty$. Note also that, from (2), $W_\alpha(t)\to 0$ as $t\to-\infty$.

So, given any two solutions X_1^* and X_2 of (1), compare X_2 with X_1, where X_1 is the translate of X_1^* for which

$$\lim_{u\to\infty} X_1(u)\,e^{\alpha u} = \lim_{u\to\infty} X_2(u)\,e^{\alpha u}.$$

Let $W_1(u) = X_1(u)\,e^{\alpha u}$, $W_2(u) = X_2(u)\,e^{\alpha u}$, and consider the quantity

$$\|W_1 - W_2\| \equiv \sup_{-\infty<u<\infty} |W_1(u) - W_2(u)|:$$

since W_1 and W_2 are continuous, and have equal finite limits at $-\infty$ and $+\infty$,

$$\|W_1 - W_2\| < \infty \quad \text{and} \quad \|W_1 - W_2\| = |W_1(t_0) - W_2(t_0)|$$

for some $-\infty < t_0 < \infty$. Now, since X_1 and X_2 are solutions of (1),

$$|W_1(u) - W_2(u)| = |TX_1(u) - TX_2(u)|\,e^{\alpha u},$$

and so, for any u for which $|W_1(u) - W_2(u)| > 0$,

$$\begin{aligned} 0 < |W_1(u) - W_2(u)| &\leqslant c^{-1}\int_{-\infty}^{\infty} Q(v)\,|e^{X_1(u+v)} - e^{X_2(u+v)}|\,e^{\alpha u}\,dv \\ &< c^{-1}\int_{-\infty}^{\infty} Q(v)\,e^{-\alpha v}e^{\alpha(u+v)}\,|X_1(u+v) - X_2(u+v)|\,dv \qquad (7) \\ &\leqslant c^{-1}F(\alpha)\,\|W_1 - W_2\| = \|W_1 - W_2\|. \end{aligned}$$

Hence, by considering $u = t_0$, $\|W_1 - W_2\| > 0$ is impossible, and it follows that

$$X_1(u) = X_2(u)$$

for all u. This proves the theorem.

The theorem does not cover the case $c = c_0$. By considering equation (3), it can be shown that any solution of (1) corresponding to velocity c_0 has a translate X for which $X(t) \sim -t\,e^{-\alpha_0 t}$, but this is not a strong enough restriction on X for the subsequent argument to be applied.

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