RAPID SOLUTION OF THE WAVE EQUATION IN UNBOUNDED DOMAINS: ABRIDGED VERSION

L. Banjai[†,*], **S. Sauter**[‡,*]

[*]Mathematical Institute, University of Zürich, Switzerland

Email: [†]lehelb@math.unizh.ch, [‡] stas@math.unizh.ch

Abstract

We propose and analyze a new fast method for the numerical solution of time-domain boundary integral formulations of the wave equation. Discretization in time is achieved by Lubich's convolution quadrature method and in space by a Galerkin boundary element method. We show that the arising block Toeplitz system is after a small perturbation equivalent to a a decoupled system of discretized Helmholtz equations. Each of these systems can efficiently be solved by a fast data-sparse method (e.g. FMM, panel clustering). Further savings can be achieved by noticing that in some cases the solutions of many of the Helmholtz problems can be replaced by zero. Finally the proposed method is inherently parallel.

We prove that the excellent stability and optimal convergence of the convolution quadrature are inherited by the new method. These results thereby pave the way to the efficient solution using fast data-sparse techniques.

1 Introduction

Boundary value problems governed by the wave equation arise in many physical applications such as electromagnetic wave propagation or the computation of transient acoustic waves. Since such problems are typically formulated in unbounded domains, the method of integral equations is an elegant tool to transform this partial differential equation to an integral equation on the bounded surface of the scatterer.

In this paper we represent the solution of the wave equation as a single layer potential thereby arriving at a boundary integral formulation of the problem. For discretisation we employ the convolution quadrature method in time [5], [6] and a Galerkin boundary element method in space.

The coefficient matrix in the arising system of linear equations is a block-triangular Toeplitz matrix consisting of N blocks of dimension $M \times M$, where N denotes the number of time steps and M is the number of spatial degrees of freedom. Due to the non-localness of the arising boundary integral operators, the $M \times M$ matrix blocks are densely populated.

In the literature, there exist (at least) two alternatives to solve this system efficiently. In [3], an FFT-technique is employed which makes use of the Toeplitz structure of the system matrix and the computational complexity is reduced to $\mathcal{O}\left(M^2 N \log^2 N\right)$, while the storage complexity stays at $\mathcal{O}\left(M^2 N\right)$. In [2], [4] the $M \times M$ block matrices are approximated by data sparse representations based on a cutoff and panel-clustering strategy. This leads to a significant reduction of the storage complexity while the computational complexity is reduced compared to the naive approach (cost: $\mathcal{O}\left(N^2 M^2\right)$) but increased compared to the FFT approach.

In this paper, we will propose a third approach which combines the advantages of the FFT-technique with the data sparse approximation: The block Toeplitz system is transformed to a decoupled system of discretized Helmholtz problems which can then be efficiently solved by fast data sparse approximations. Thereby we reduce *both* the storage and complexity estimates to $\mathcal{O}(MN \log^a N)$, for a small $a \geq 1$. A full version of the paper [1], where all the results stated here are proved, is in preparation.

2 Integral formulation of the wave equation

Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain with boundary Γ. We consider the homogeneous wave equation

$$\partial_t^2 u = \Delta u \qquad \text{in } \Omega \times (0,T) \tag{1 a}$$

$$u(\cdot,0) = \partial_t u(\cdot,0) = 0 \qquad \text{in } \Omega \tag{1 b}$$

$$u = g \quad \text{on } \Gamma \times (0,T) \tag{1 c}$$

on a time interval $(0,T)$ for some $T > 0$. For its solution, we employ an ansatz as a *single layer potential*

$$u(x,t) = \int_0^t \int_\Gamma k(x-y, t-\tau)\phi(y,\tau) d\Gamma_y d\tau \tag{2}$$

for $(x,t) \in \Omega \times (0,T)$ and where $k(z,t)$ is the fundamental solution of the wave equation,

$$k(z,t) = \frac{\delta(t - \|z\|)}{4\pi\|z\|}, \tag{3}$$

$\delta(t)$ being the Dirac delta distribution. The ansatz (2) satisfies the homogeneous equation (1 a) and the initial conditions (1 b). The extension $x \to \Gamma$ is continuous and hence, the unknown density ϕ in (2) is determined via the boundary condition (1 c), $u(x,t) = g(x,t)$. This results in the boundary integral equation for ϕ,

$$\int_0^t \int_\Gamma k(x-y, t-\tau)\phi(y,\tau) d\Gamma_y d\tau = g(x,t) \qquad (4)$$

for all $(x,t) \in \Gamma \times (0,T)$.

3 Time and space discretisation

For the time discretisation, we employ the convolution quadrature approach which has been developed by Lubich in [5], [6]

We split the time interval $[0,T]$ into $N+1$ time steps of equal length $\Delta t = T/N$ and compute an approximate solution at the discrete time levels $t_n = n\Delta t$. The continuous convolution operator is replaced by a discrete convolution operator, and the resulting semi-discrete problem is given by

$$\sum_{j=0}^{n} \int_\Gamma \omega_{n-j}^{\Delta t}(\|x-y\|)\phi_{\Delta t,j}(y) d\Gamma_y = g(x, t_n), \qquad (5)$$

for all $n = 1, \ldots, N, \quad x \in \Gamma$.

If the time discretisation is related to a multistep method defined by its generating polynomial $\gamma(\zeta)$ (see [5]) the kernel functions $\omega_n^{\Delta t}(d)$ are implicitly defined by

$$\hat{k}\left(d, \frac{\gamma(\zeta)}{\Delta t}\right) = \sum_{n=0}^{\infty} \omega_n^{\Delta t}(d)\zeta^n.$$

Here, $\hat{k}$ denotes the Laplace transform of the original kernel $\hat{k}(d,s) = \frac{e^{-sd}}{4\pi d}$. For the rest of this paper we consider the BDF2 multistep method defined by $\gamma(\zeta) = \frac{1}{2}(\zeta-1)(\zeta-3)$.

For the space discretisation, we employ a standard Galerkin boundary element method. The space of piecewise constant, discontinuous functions is denoted by $S_{-1,0}$ and the space of continuous, piecewise linear functions by $S_{0,1}$ The general notation is S for the boundary element space and $(b_m)_{m=1}^M$ for the basis. The mesh width is denoted by h. For the space-time discrete solution at time t_n we employ the ansatz

$$\phi_{\Delta t,h,n}(y) = \sum_{m=1}^{M} \phi_{n,m} b_m(y)\,, \qquad (6)$$

where $(\phi_{n,m})_{m=1}^M \in \mathbb{R}^M$ are the nodal values of the discrete solution at time step t_n.

We impose the integral equation not pointwise but in a weak form: Find $\phi_{\Delta t,h,n} \in S$ of the form (6) such that

$$\sum_{j=0}^{n}\sum_{m=1}^{M} \phi_{j,m} \int_\Gamma \int_\Gamma \omega_{n-j}^{\Delta t}(x-y) b_m(y) b_k(x) d\Gamma_y d\Gamma_x = \int_\Gamma g(x,t_n) b_k(x) d\Gamma_x \qquad (7)$$

for all $1 \leq k \leq M$ and $n = 1, \ldots, N$. This can be written as a linear system

$$\sum_{j=0}^{n} \mathbf{A}_{n-j}\phi_{j,\star} = \mathbf{g}_{n,\star}\,, \qquad n = 0, \ldots, N\,, \qquad (8)$$

with

$$(\mathbf{A}_n)_{k,m} := \int_\Gamma \int_\Gamma \omega_n^{\Delta t}(x-y) b_m(y) b_k(x) d\Gamma_y d\Gamma_x\,,$$

and

$$\mathbf{g}_{n,\star} = \left(\int_\Gamma g(x,t_n) b_k(x) d\Gamma_x\right)_{k=1}^{M}.$$

4 Transformation to a decoupled system of Helmholtz problems

Let us define $\omega_n^{\Delta t} := 0$ for $n = -N, -N+1, \ldots, -1$. With this definition we can extend the sum in (7) to $j = 0, 1, \ldots, N$ to obtain

$$\sum_{j=0}^{N}\sum_{m=1}^{M} \phi_{j,m} \int_\Gamma \int_\Gamma \omega_{n-j}^{\Delta t}(x-y) b_m(y) b_k(x) d\Gamma_y d\Gamma_x = \int_\Gamma g_{\Delta t,n}(x) b_k(x) d\Gamma_x. \qquad (9)$$

The next step in our method is to replace each $\omega_n^{\Delta t}$ by the trapezoid rule approximation of the Cauchy integral representation. The details and an error estimate are given in the following proposition. For clearer exposition we introduce some extra notation. Let $\zeta_{N+1} := \exp(-2\pi i/(N+1))$ and $\kappa_l := i\gamma(\lambda\zeta_{N+1}^l)/\Delta t$. Then we notice that

$$\hat{k}(\|x-y\|, \gamma(\lambda\zeta_{N+1}^l)/\Delta t) = G_{\kappa_l}(\|x-y\|),$$

where $G_\kappa(\cdot)$ is the fundamental solution of the Helmholtz operator $\Delta + \kappa^2\cdot$:

$$G_\kappa(d) = \frac{e^{i\kappa d}}{4\pi d}.$$

Proposition 4.1 *Let $N \in \mathbb{N}$, $d > 0$, $\Delta t = T/N$, and $\lambda < e^{-\Delta t}$ be given. There exists a constant $C > 0$ independent of the parameters such that, for $-N \leq j \leq N$,*

$$|\omega_j^{\Delta t}(d) - \frac{\lambda^{-j}}{N+1}\sum_{l=0}^{N} G_{\kappa_l}(d)\zeta_{N+1}^{-lj}| \leq Ce^{2T}\frac{\lambda^{N+1}}{d}.$$

Substituting the above approximation in to (9) we obtain a new system

$$\frac{\lambda^n}{N+1}\sum_{l=0}^{N}\left(\hat{A}_l\,\hat{\phi}_{l,\star}\right)_k \zeta_{N+1}^{-nl} = \int_\Gamma g(x,t_n)b_k(x)d\Gamma_x,$$

where

$$\hat{\phi}_{l,\star} := \sum_{j=0}^{N}\lambda^j\tilde{\phi}_{j,\star}\zeta_{N+1}^{lj}$$

and

$$(\hat{A}_l)_{k,m} = \int_\Gamma\int_\Gamma G_{\kappa_l}(\|x-y\|)b_m(y)b_k(x)d\Gamma_y d\Gamma_x.$$

Note that the unknowns (in the time domain) of the above systems are denoted by $\tilde{\phi}_{j,m}$. The approximation $\tilde{\phi}_{\Delta,h,n} \in S$ at time t_n is then given analogously to (6) Now, notice that if we take the Fourier transform of both sides we obtain $N+1$ decoupled problems:

$$\hat{A}_l\hat{\phi}_{l,\star} = \int_\Gamma \hat{g}_{\Delta t,l}(x)b_\star(x)d\Gamma_x, \qquad (10)$$

where

$$\hat{g}_{\Delta t,l}(x) = \sum_{n=0}^{N}\lambda^n g_(x,t_n)\zeta_{N+1}^{ln}.$$

Therefore we have indeed reduced the problem of solving numerically the wave equation to a system of Helmholtz problems with wavenumbers κ_l, $l = 0,1,\ldots N$.

The stability and convergence of the new method can be derived using the techniques developed in [6] and [2].

Theorem 4.2 *Let g be sufficiently compatible and smooth (see [6]) and let $S = S_{m-1,m}$ for $m \in \{0,1\}$. Then if $\lambda^{N+1} \leq Ch\Delta t^8$, the solution $\tilde{\phi}_{\Delta t,h,n}$ exists and satisfies the error estimate*

$$\|\tilde{\phi}_{\Delta t,h,n} - \phi(\cdot,t_n)\|_{H^{-1/2}(\Gamma)} \leq C_g(\Delta t^2 + h^{m+3/2}),$$

where C depends on T and C_g depends on the right-hand side g.

5 Fast solution of the decoupled systems

To solve the N dense systems from the previous section, fast methods (e.g. FMM, panel clustering) designed for high-frequency Helmholtz problems (see e.g. [7]), can be applied. In these methods dense sub-blocks of the matrix are replaced by data-sparse (e.g. low rank) matrix approximations. It is then possible to solve each dense $M \times M$ system (10) in $\mathcal{O}(M\log^a M)$, for some small $a \geq 0$, time and storage complexity. The conditions on the error introduced so that the stability and optimal convergence is preserved are developed in [1].

To reduce the costs even further we notice that if $\partial_t^r g(x,0) = \partial_t^r g(x,T) = 0$, for $r = 0,1,\ldots R$ (corresponding to a smooth, time limited incoming signal) only $\mathcal{O}(N^{3/R})$ systems need to be solved (see [1]). In fact if $\partial_t^r g(x,0) = \partial_t^r g(x,T) = 0$, for all $r \geq 0$ then we expect to only have to solve $O(\log N)$ systems. For supporting numerical evidence see [1].

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