MOTIVES OF RIGID ANALYTIC TUBES AND NEARBY MOTIVIC SHEAVES

by

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Abstract. — Let k be a field of characteristic zero, $R = k[\![t]\!]$ the ring of formal power series and $K = k(\!(t)\!)$ its fraction field. Let X be a finite type R-scheme with smooth generic fiber. Let $\mathscr{X}$ be the t-adic completion of X and $\mathscr{X}_\eta$ the generic fiber of $\mathscr{X}$. Let $Z \subset X_\sigma$ be a locally closed subset of the special fiber of X. In this article, we establish a relation between the rigid motive of $]Z[$ (the tube of Z in $\mathscr{X}_\eta$) and the restriction to Z of the nearby motivic sheaf associated with the R-scheme X. Our main result, Theorem 7.1, can be interpreted as a motivic analog of a theorem of Berkovich.

As an application, given a rational point $x \in X_\sigma$, we obtain an equality, in a suitable Grothendieck ring of motives, between the motivic Milnor fiber of Denef–Loeser at x and the class of the rigid motive of the analytic Milnor fiber of Nicaise–Sebag at x.

Résumé. — Soit k un corps de caractéristique nulle, $R = k[\![t]\!]$ l'anneau des séries formelles sur k et $K = k(\!(t)\!)$ son corps des fractions. Soit X un R schéma de type fini génériquement lisse. Soit $\mathscr{X}$ la complétion t-adique de X et $\mathscr{X}_\eta$ sa fibre générique. Soit $Z \subset X_\sigma$ un sous-ensemble localement fermé de X. Dans cet article, nous lions le motif rigide du tube $]Z[$ de Z dans $\mathscr{X}_\eta$ à la restriction à Z du faisceau cycles proches motivique associé au R-schéma X. Le théorème 7.1, qui est notre résultat principal, peut être interprété comme un analogue motivique d'un théorème de Berkovich.

Comme application, étant donné un point rationnel $x \in X_\sigma$, nous obtenons une égalité dans un anneau de Grothendieck de motifs adéquat entre la fibre de Milnor motivique de Denef–Loeser en x et la classe du motif rigide de la fibre de Milnor analytique de Nicaise–Sebag en x.

Contents

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1. Introduction

1.1. Let k be a field of characteristic zero, $R = k[[t]]$ be the ring of formal power series and $K = k((t))$ be its fraction field. Let Λ be a commutative ring (that we call the ring of coefficients). While the main body of the article is written in a greater generality, we restrict ourselves in the introduction to the categories of motives without transfers $\mathbf{DA}(k, \Lambda)$ and its rigid analytic version $\mathbf{RigDA}(K, \Lambda)$. These categories are related by triangulated functors

$$\mathbf{RigDA}(K, \Lambda) \xrightarrow{\mathfrak{R}} \mathbf{QUDA}(k, \Lambda) \xrightarrow{1^*} \mathbf{DA}(k, \Lambda),$$

where $\mathbf{QUDA}(k, \Lambda)$ is the full triangulated subcategory of $\mathbf{DA}(\mathbf{G}_{m,k}, \Lambda)$ whose objects are the quasi-unipotent motives; the functor $\mathfrak{R}$ is an equivalence of categories (see [**6**, Scholie 1.3.26]) and 1^* is the pullback functor along the unit section. For a quick recollection on motives and rigid motives, the reader is referred to §3.

1.2. Let X be a finite type R-scheme and denote by $f \colon X \to \mathsf{Spec}(R)$ its structural morphism. We denote by X_η and X_σ the generic and special fibers of X.

By [**3**, Chapitre 3] (see also [**6**, §A.1]), one has the nearby motivic sheaf $\Psi_f(\Lambda_{X_\eta})$ associated with f; this is an object of $\mathbf{DA}(X_\sigma, \Lambda)$. It realizes to the classical complexes of nearby cycles by [**4**, Théorème 4.9] (for the Betti realization and when X is the base-change of a finite type $k[t]$-scheme) and [**5**, Théorème 10.11] (for the ℓ-adic realization).

Consider the t-adic completion $\hat{f} : \mathscr{X} \to \mathsf{Spf}(R)$ of f and denote by $\mathscr{X}_\eta$ the generic fiber of $\mathscr{X}$. The rigid analytic variety $\mathscr{X}_\eta$ is an open analytic subvariety of the analytification X_η^{an} of the algebraic generic fiber X_η (e.g., see [**12**, (0.3.5)]). Given a locally closed subset $Z \subset X_\sigma$ (endowed with its reduced structure), denote by $]Z[$ its tube; this is an open rigid analytic subvariety of $\mathscr{X}_\eta$.

Assume that the rigid analytic variety $\mathscr{X}_\eta$ is smooth over K; this is the case for instance if the scheme X_η is smooth over K. Let $\mathrm{M}^\vee_{rig}(]Z[)$ be the cohomological motive of $]Z[$; this is an object of $\mathbf{RigDA}(K, \Lambda)$. The main theorem of this article is the following (see Theorem 7.1 for a more general statement):

Theorem. — *Denote by $z : Z \hookrightarrow X_\sigma$ the inclusion. Then, there is a canonical isomorphism*

$$1^* \circ \mathfrak{R}(\mathrm{M}^\vee_{rig}(]Z[)) \simeq (f_\sigma)_* z_* z^* \Psi_f(\Lambda_{X_\eta}) \tag{1}$$

in the category of motives $\mathbf{DA}(k, \Lambda)$.

Taking $Z = X_\sigma$, one gets that the cohomological motive $\mathrm{M}^\vee_{rig}(\mathscr{X}_\eta)$ is related to the nearby motivic sheaf by a canonical isomorphism

$$1^* \circ \mathfrak{R}(\mathrm{M}^\vee_{rig}(\mathscr{X}_\eta)) \simeq (f_\sigma)_* \Psi_f(\Lambda_{X_\eta})$$

in $\mathbf{DA}(k, \Lambda)$. In fact, we first prove this particular case of our main theorem (see Theorem 4.11 and Corollary 4.12) and then use it, with other ingredients, to derive the general case.

As a by-product of this work, we show that the rigid motives of tubes are compact (see proposition 5.9), and we extend to stable homotopy the computation of nearby motivic sheaf obtained previously by Ayoub in the context of étale motives (see theorem 6.1).

1.3. Our main theorem is a motivic analog of a theorem of Berkovich that we explain now. Let $\overline{K}$ be the completion of an algebraic closure of the valued field K and let $\overline{k}$ be its residue field. Set $\overline{Z} = Z \times_k \overline{k}$ and $]\overline{Z}[=]Z[\hat{\times}_K \overline{K}$. In [**10, 11**], Berkovich constructed canonical isomorphisms of étale hypercohomology groups

$$\mathrm{H}^i_{\text{ét}}(]\overline{Z}[, \mathbf{Q}_\ell) \simeq \mathbb{H}^i_{\text{ét}}(\overline{Z}, R\Psi_{\hat{f}}(\mathbf{Q}_{\ell,\mathscr{X}_\eta})|_{\overline{Z}}) \simeq \mathbb{H}^i_{\text{ét}}(\overline{Z}, R\Psi_f(\mathbf{Q}_{\ell,X_\eta})|_{\overline{Z}}). \tag{2}$$

(Here the tube $]Z[$ has to be considered as a Berkovich space in order to take its non-archimedean étale cohomology [**9**].) The first isomorphism is shown in [**11**, Corollary 3.5]; the second one follows from [**10**, Corollary 5.3].

We expect that the isomorphism (1) realizes to the composition of the isomorphisms in (2). However, we do not make any attempt to check this in this article. It is worth noting that Berkovich's theorem holds over general non-archimedean fields whereas, for the very statement of our theorem, we need to assume that K has equal characteristic zero. Indeed, this is required for [**6**, Scholie 1.3.26] which ensures the existence of the equivalence $\mathfrak{R}$.

1.4. Let $x \in X_\sigma$ be a rational point. In [**16**, Définition 4.2.1], Denef and Loeser have introduced the *motivic Milnor fiber* $\psi_{f,x} \in \mathscr{M}_k$ as the limit of the motivic zeta function associated with f ; in [**34**], Nicaise and Sebag have defined the analytic Milnor fiber at x to be $\mathscr{F}_x =]x[$. The present work and [**25**] show that (stable) motivic homotopy is a natural framework to relate and study these different notions of Milnor fiber. A particular case of our main theorem (see Theorem 8.8) gives an isomorphism of motives

$$1^* \circ \mathfrak{R}(\mathrm{M}^\vee_{rig}(\mathscr{F}_x)) \simeq x^* \Psi_f(\Lambda_{X_\eta}).$$

Theorem 6.1 shows that [**25**, Theorem 1.2] remains valid in a more general setting, and we deduce the following formula in the Grothendieck group of constructible motives

$$[1^* \circ \mathfrak{R}(\mathrm{M}^{\vee}_{rig}(\mathscr{F}_x))] = \chi_{k,\mathrm{c}}(\psi_{f,x}). \tag{3}$$

Here, we denote by $\chi_{k,\mathrm{c}} : \mathscr{M}_k \to \mathrm{K}_0(\mathbf{DA}_{\mathsf{ct}}(k,\Lambda))$ the motivic Euler characteristic [**25**, Lemma 2.1].

The formula (3) expresses the fact that the motivic Milnor fiber of Denef–Loeser, at least as a class in the Grothendieck ring of constructible motives, is determined by the rigid motive of the analytic Milnor fiber. A formula of a similar nature, comparing the motivic Milnor fiber of Denef–Loeser to the analytic Milnor fiber, appears in [**24**, Corollary 8.4.2]. (See Remarks 8.14 and 8.15 for an attempt to relate the two formulas.)

Notations, conventions

1.5. Although this is not really necessary, all schemes, formal schemes and rigid varieties will be assumed to be *separated*. Schemes and formal schemes will be also assumed to be *quasi-compact*.

When there is no risk of confusion, a scheme S will be identified with its maximal reduced subscheme that we denote by S_{red}. Also, a locally closed subset of a scheme will be automatically endowed with its reduced subscheme structure. The same applies for rigid analytic varieties.

We fix a ground field k of characteristic zero and an indeterminate t. We set $\mathbf{A}^1_k = \mathsf{Spec}(k[t])$ and $\mathbf{G}_{m,k} = \mathsf{Spec}(k[t,t^{-1}])$. We also set $R = k[[t]]$ and $K = k((t))$. Up to isomorphism, K is the unique non-archimedean field with discrete valuation ring and having k as residue field.

Unless otherwise stated, formal R-schemes will always be t-*adic*. We denote by $R\{T_1,\ldots,T_n\}$ the t-adic ring of *strictly convergent power series*. If $\mathscr{X}$ is a separated formal R-scheme topologically of finite type, we denote by $\mathscr{X}_\sigma$ its *special fiber*, that is a finite type k-scheme, and by $\mathscr{X}_\eta$ its *generic fiber* (in the sense of Raynaud), that is a quasi-compact rigid analytic variety over K. If $\mathscr{X} = \mathsf{Spf}(A)$ is affine, then $\mathscr{X}_\eta = \mathsf{Spm}(A[1/t])$ and $\mathscr{X}_\sigma = \mathsf{Spec}(A/(t))$.

As in [**6**], we denote by $-\hat{\otimes}-$ the completed tensor product, and by $-\hat{\times}-$ the fiber product in the category of rigid analytic varieties or the category of formal schemes. Following the notation of [**6**, §1.1.2], we denote by $\mathbf{B}^1_K = \mathsf{Spm}(K\{T\})$ the *unit ball* and, for X a rigid analytic K-variety, we set $\mathbf{B}^1_X = \mathbf{B}^1_K \hat{\times}_K X$. More generally, given a rigid analytic K-variety X, $f \in \mathcal{O}(X)^\times$ and $p \in \mathbf{N} \smallsetminus \{0\}$, we denote by $\mathbf{B}^1_X(o, |f|^{1/p})$ the relative ball over X with radius $|f(x)|^{1/p}$ at $x \in X$. If $X = \mathsf{Spm}(A)$ is affinoid and $f \in A^\circ$ (i.e., $|f|_\infty \leq 1$), then $\mathbf{B}^1_X(o,|f|^{1/p}) = \mathsf{Spm}(A\{T,U\}/(fU - T^p))$. Moreover, for f, $g \in \mathcal{O}(X)^\times$ and p, $q \in \mathbf{N} \smallsetminus \{0\}$ such that $|g(x)|^{1/q} \leq |f(x)|^{1/p}$ for every $x \in X$, we denote by $\mathbf{Cr}_X(o, |g|^{1/q}, |f|^{1/p})$ the *relative annulus* (aka., *relative corona*) with small radius $|g(x)|^{1/q}$ and big radius $f(x)^{1/p}$ at $x \in X$. If $X = \mathsf{Spm}(A)$ is affinoid and $f,\, g \in A^\circ$, then $\mathbf{Cr}_X(o, |g|^{1/q}, |f|^{1/p}) = \mathsf{Spm}(A_K\{T, U, W\}/(fU - T^p, T^q W - g))$. Finally, if $f = g$ and $p = q$, we denote the corresponding annulus by $\partial\mathbf{B}^1_X(o, |f|^{1/p})$; this is the *boundary* of the relative ball $\mathbf{B}^1_X(o, |f|^{1/p})$.

If $\mathscr{X}$ is a formal R-scheme, and $Z \subset \mathscr{X}_\sigma$ a locally closed subset (endowed with its reduced structure), we recall that its tube $]Z[$ is defined, set-theoretically, as the inverse image of Z by the specialization map $\mathrm{sp} : \mathscr{X}_\eta \to \mathscr{X}_\sigma$. If $\mathscr{X} = \mathsf{Spf}(A)$ is affine and $Z = (\cap_{i=1}^r V(f_i)) \cap (\cap_{j=1}^s D(g_j))$ in $\mathsf{Spec}(A/(t))$ where $f_i,\, g_j \in A$, this is the set of $x \in \mathscr{X}_\eta$ such that $|f_i(x)| < 1$, for all $1 \le i \le r$, and $|g_j(x)| = 1$, for at least one $1 \le j \le s$ (see, e.g., [**27**, §2.2]).

1.6. Let X be a finite type R-scheme and $f\colon X \to \mathsf{Spec}(R)$ be its structural morphism. We form the usual commutative diagram with cartesian squares

$$\begin{tikzcd} X_\eta \arrow[r, "j"] \arrow[d, "f_\eta"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \arrow[dr, phantom, "\square"] & X_\sigma \arrow[l, "i"'] \arrow[d, "f_\sigma"] \\ \eta = \mathsf{Spec}(K) \arrow[r, "j"] & \mathsf{Spec}(R) & \mathsf{Spec}(k) =: \sigma, \arrow[l, "i"'] \end{tikzcd}$$

where i is the inclusion of the special point of $\mathsf{Spec}(R)$ and j is the inclusion of its generic point.

1.7. We fix a ring of coefficients Λ. (The main examples we are interested in are $\mathbf{Z}$ and $\mathbf{Q}$.) More generally, we fix a category of coefficients $\mathfrak{M}$ in the sense of [**6**, Définition 1.2.31]. The reader may assume, without a real loss of generality, that $\mathfrak{M}$ is the category $\mathbf{Compl}(\Lambda)$ of complexes of Λ-modules or the category $\mathbf{Spect}^{\Sigma}_{S^1}(\Delta^{\mathrm{op}}\mathcal{S}et_\bullet)$ of symmetric S^1-spectra.

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2. Formal schemes and semi-stability

In this section we recall some basic facts concerning formal schemes, completions and rigid analytic varieties. We also make precise the definition of semi-stability used in this article. (For details on formal schemes, rigid analytic varieties, see, for example, [**20**, §10], [**1, 35, 13**] or [**6**, §1.1].)

2.1. Formal completion. — Let X be a finite type R-scheme and $f\colon X \to \mathsf{Spec}(R)$ its structural morphism. By the *completion of f* we mean the morphism of formal schemes $\hat{f}\colon \mathscr{X} \to \mathsf{Spf}(R)$ obtained from f by taking the t-adic completion. By construction, the formal R-scheme $\mathscr{X}$ is topologically of finite type.

Locally, one has the following description: if X is given as the spectrum of a finitely generated R-algebra $A = R[T_1, \ldots, T_n]/I$, then $\mathscr{X} = \mathsf{Spf}(R\{T_1, \ldots, T_n\}/I)$.

Lemma 2.1. — *Let X be a finite type R-scheme and $f\colon X \to \mathsf{Spec}(R)$ its structural morphism. We have the following properties:*

1. *if f is flat, so is the morphism of formal schemes $\hat{f}$;*
2. *if X is regular, so is the formal scheme $\mathscr{X}$;*
3. *if X_η is smooth, so is the generic fiber $\mathscr{X}_\eta$.*

Proof. — Everything in the statement is standard and well-known. For the sake of completeness we give some indications and references.

To prove the first two statements, note that, for every $x \in X_\sigma$, the canonical morphism of local rings $\mathcal{O}_{X,x} \to \mathcal{O}_{\mathscr{X},x}$ induces, by $\mathfrak{m}_x$-completion, an isomorphism of complete local rings

$$\widehat{\mathcal{O}_{X,x}} \to \widehat{\mathcal{O}_{\mathscr{X},x}}.$$

This is said, the first statement follows directly from [**31**, Theorem 22.4]. Similarly, the second statement follows directly from [**28**, Proposition **4**.2.26].

The last statement is clear since $\mathscr{X}_\eta$ is isomorphic to an open analytic subvariety of $(X_\eta)^{an}$. □

An important construction in formal geometry is that of *admissible blow-ups* (see for example [**14**, §2] or [**1**, §3.1]). In the following statement, we compare properties of blow-ups in algebraic and formal settings with respect to completion.

Lemma 2.2. — *Let X be a finite type R-scheme and $f\colon X \to \mathsf{Spec}(R)$ its structural morphism. Let $h\colon X' \to X$ be a blow-up with center a closed subscheme Z such that $Z_{\mathrm{red}} \subset (X_\sigma)_{\mathrm{red}}$. Denote $\hat{f} \circ \hat{h} : \mathscr{X}' \to \mathsf{Spf}(R)$ the completion of $f \circ h$ and $\hat{h} : \mathscr{X}' \to \mathscr{X}$ the induced morphism of formal R-schemes. We have the following properties:*

1. *if f is flat, so is the morphism $f \circ h : X' \to \mathsf{Spec}(R)$;*
2. *the morphism $\hat{h}$ is canonically isomorphic to the admissible blow-up of $\mathscr{X}$ with center Z;*
3. *the morphism $\hat{h}_\eta : \mathscr{X}'_\eta \to \mathscr{X}_\eta$ is an isomorphism;*
4. *if $T \subset X_\sigma$ is a locally closed subset and $T' = h^{-1}(T)$, then $\hat{h}_\eta$ induces an isomorphism $]T'[\simeq]T[$ on the tubes of T and T'.*

Proof. — Everything in the statement is standard and well-known. For the sake of completeness we give some indications and references.

The first statement follows from [**28**, Proposition **4**.3.9]. The second statement is a direct consequence of the definition of blow-ups for formal schemes. The third statement follows from [**14**, Lemma 2.2]. For the last statement, see, e.g., [**27**, Corollary 2.2.7]. □

2.2. Semi-stable reduction. — Remember that our base field k has characteristic zero. We will use the following terminology.

Definition 2.3. — A topologically finite type formal R-scheme $\mathscr{X}$ (resp. a finite type R-scheme X) is called *semi-stable* if it is flat over R and satisfies the following condition. For every $x \in \mathscr{X}_\sigma$ (resp. $x \in X_\sigma$), there exists a regular open formal subscheme $\mathscr{U} \subset \mathscr{X}$ (resp. a regular open subscheme $U \subset X$) containing x and elements $u, t_1, \ldots, t_n \in \mathcal{O}(\mathscr{U})$ (resp. $\in \mathcal{O}(U)$) verifying the following properties:

1. u is invertible and there are integers $a_1, \ldots, a_n \in \mathbf{N} \smallsetminus \{0\}$ such that $t = ut_1^{a_1} \cdots t_n^{a_n}$;
2. for every non empty subset $I \subset \{1, \ldots, n\}$, the subscheme $D_I \subset \mathscr{U}_\sigma$ (resp. $D_I \subset U_\sigma$) defined by the equations $t_i = 0$, for $i \in I$, is smooth over k, has codimension $\#(I) - 1$ in $\mathscr{U}_\sigma$ (resp. U_σ) and contains x.

We say that a semi-stable formal R-scheme (resp. R-scheme) is *strictly semi-stable*, if its special fiber is a reduced k-scheme (i.e., the integers a_i are always equal to 1).

Remark 2.4. — We warn the reader that our notion of semi-stability differs from the classical one. Classically, a semi-stable (formal) R-scheme is étale locally strictly semi-stable in the sense of Definition 2.3. Note also that our definition coincides with the definition of global semi-stable reduction of [**6**, Définition 1.1.57] and [**3**, Définition 3.3.33].

Proposition 2.5. — *Let X be a finite type R-scheme and $f : X \to \mathsf{Spec}(R)$ its structural morphism.*

1. *X is semi-stable if and only if its t-adic completion $\mathscr{X}$ is semi-stable.*
2. *If X is regular and $(X_\sigma)_{\mathrm{red}}$ is a simple normal crossing divisor in X, then X is semi-stable.*
3. *Conversely, if X is semi-stable, there exists a neighborhood of X_σ in X which is regular and in which $(X_\sigma)_{\mathrm{red}}$ is a simple normal crossing divisor.*

Proof. — Everything in the statement is standard and well-known. We only explain the second assertion.

Let $x \in X_\sigma$ and let $U \subset X$ be an affine neighborhood of x such that each component of the divisor $(U_\sigma)_{\mathrm{red}} = (X_\sigma)_{\mathrm{red}} \cap U$ is principal, i.e., defined by a single equation. Shrinking U, we may assume furthermore that all the components of U_σ contain x.

Let $D_1, \ldots, D_n$ be the irreducible components of $(U_\sigma)_{\mathrm{red}}$ and, for $1 \leqslant i \leqslant n$, let $t_i \in \mathcal{O}(U)$ be a generator of the ideal defining D_i. If a_i is the multiplicity of D_i in U_σ, then $t_1^{a_1} \cdots t_n^{a_n}$ is a generator of the ideal defining U_σ. This ideal is also generated by t (and more precisely by the image of t by the morphism $R \to \mathcal{O}(U)$). Therefore, there should be an invertible element $u \in \mathcal{O}(U)^\times$ such that $t = ut_1^{a_1} \cdots t_n^{a_n}$. □

Remark 2.6. — We will use Proposition 2.5 in the following way. Let X be a finite type R-scheme and $f : X \to \mathsf{Spec}(R)$ its structural morphism. Assume that f is flat and that the rigid variety $\mathscr{X}_\eta$ is smooth. Then X_σ admits an open neighborhood

$U \subset X$ such that U_η is smooth. Furthermore, by resolution of singularities, one can find a morphism $h : X' \to U$ satisfying the following properties:

- h is a blow-up of U with center a closed subscheme Z such that $Z_{\mathrm{red}} \subset (X_\sigma)_{\mathrm{red}}$;
- X' is regular and $(X'_\sigma)_{\mathrm{red}}$ is a simple normal crossing divisor.

It follows that $\mathscr{X}'_\eta \simeq \mathscr{X}_\eta$ and $\mathscr{X}'$ is a formal R-scheme with semi-stable reduction. Moreover, the morphism $\hat{h} : \mathscr{X}' \to \mathscr{X}$ is an admissible blow-up.

Example 2.7. — We recall here the definition of standard semi-stable (formal) R-schemes. For later use, we give actually a more general construction.

Let X (resp. $\mathscr{X}$, $\mathcal{X}$) be an R-scheme (resp. a formal R-scheme, a rigid analytic variety over K). Let $\underline{a} = (a_1, \dots, a_n) \in (\mathbf{N}^\times)^n$, let $v \in \mathcal{O}(X)$ (resp. $v \in \mathcal{O}(\mathscr{X})$, $v \in \mathcal{O}(\mathcal{X})$). The *standard space of length n* associated with the triple $(X, v, \underline{a})$ (resp. $(\mathscr{X}, v, \underline{a})$, $(\mathcal{X}, v, \underline{a})$) is the R-scheme (resp. formal R-scheme, rigid analytic variety over K) given by:

$$\begin{aligned} \mathrm{St}^v_{X,\underline{a}} &= \mathsf{Spec}\, \mathcal{O}_X[T_1, \dots, T_n]/(T_1^{a_1} \cdots T_n^{a_n} - v) \\ (\text{resp. } \mathrm{St}^v_{\mathscr{X},\underline{a}} &= \mathsf{Spf}\, \mathcal{O}_{\mathscr{X}}\{T_1, \dots, T_n\}/(T_1^{a_1} \cdots T_n^{a_n} - v), \\ \mathrm{St}^v_{\mathcal{X},\underline{a}} &= \mathsf{Spm}\, \mathcal{O}_{\mathcal{X}}\{T_1, \dots, T_n\}/(T_1^{a_1} \cdots T_n^{a_n} - v)). \end{aligned}$$

If the R-scheme X is of finite type with t-adic completion $\mathscr{X}$, then $\mathrm{St}^v_{\mathscr{X},\underline{a}}$ is the t-adic completion of $\mathrm{St}^v_{X,\underline{a}}$. If the formal R-scheme $\mathscr{X}$ is of topologically of finite type, then $\mathrm{St}^v_{\mathscr{X}_\eta,\underline{a}}$ is the generic fiber of $\mathrm{St}^v_{\mathscr{X},\underline{a}}$.

If X (resp. $\mathscr{X}$) is a smooth R-scheme of finite type (resp. a smooth formal R-scheme topologically of finite), and if $v \in t\mathcal{O}_X(X)^\times$ (resp. $v \in t\mathcal{O}(\mathscr{X})^\times$), then the associated standard space $\mathrm{St}^v_{X,\underline{a}}$ (resp. $\mathrm{St}^v_{\mathscr{X},\underline{a}}$) is semi-stable. General semi-stable R-schemes (resp. formal R-schemes) are locally, for the Zariski topology, related to standard ones by [**3**, Proposition 3.3.39] (resp. [**6**, Proposition 1.1.62]).

Without necessarily assuming X (resp. $\mathscr{X}$) smooth over R, the subscheme $D_i \subset (\mathrm{St}^v_{X,\underline{a}})_\sigma$ (resp. $D_i \subset (\mathrm{St}^v_{\mathscr{X},\underline{a}})_\sigma$) defined by the equation $T_i = 0$ is called a *branch* of the standard scheme $\mathrm{St}^v_{X,\underline{a}}$ (resp. formal scheme $\mathrm{St}^v_{\mathscr{X},\underline{a}}$).

3. Motivic sheaves and rigid motives

In this section, we recall some elements of the theory of motives and rigid motives that are used in this article.

3.1. Recollections on motivic sheaves. — For a scheme S, we denote by $\mathbf{SH}_{\mathfrak{M}}(S)$ the category of motivic sheaves over S (for the Nisnevich topology and with coefficients in $\mathfrak{M}$). This category appears in [**3**, Définition 4.5.21] under the name $\mathbf{SH}^T_{\mathfrak{M}}(S)$, where T stands for a projective replacement of the presheaf

$$\frac{\mathbf{G}_{m,S} \otimes \mathbb{1}}{1_S \otimes \mathbb{1}}.$$

(The choice of T will not play any role in this article.) Also, for the construction of $\mathbf{SH}_{\mathfrak{M}}(S)$, one has to choose the Nisnevich topology (instead of the étale topology) at the beginning of [**3**, §4.5].

Example 3.1. —

1. When $\mathfrak{M} = \mathbf{Spect}^{\Sigma}_{S^1}(\Delta^{\mathrm{op}}\mathcal{S}et_{\bullet})$, it is customary to denote by $\mathbf{SH}(S)$ this category. This is the stable homotopy category of S-schemes of Morel–Voevodsky (see [**26, 32, 38**]).
2. When $\mathfrak{M} = \mathbf{Compl}(\Lambda)$, it is customary to denote by $\mathbf{DA}(S, \Lambda)$ this category. This is the Λ-linear counterpart of the stable homotopy category of S-schemes of Morel–Voevodsky.

Remark 3.2. — The theory developed in [**2, 3**] provides the categories $\mathbf{SH}_{\mathfrak{M}}(-)$ with the Grothendieck six operations and the formalism of vanishing cycles.

Actually, in *loc. cit.*, operations are only considered for quasi-projective morphisms as, by definition, a stable homotopic 2-functor is only assumed to be defined over quasi-projective schemes over a base S; however, $\mathbf{SH}_{\mathfrak{M}}(-)$ makes sense for any scheme and the operations f^*, f_* make sense for any morphism of schemes. The same holds true for the functors Ψ_f: their construction makes sense for any morphism of schemes $f : X \to \mathbf{A}^1_k$.

Definition 3.3. — Let $p : X \to S$ be a morphism of finite type k-schemes. We define the *cohomological motive* of the S-scheme X by [(1)]

$$\mathrm{M}^{\vee}_S(X) = p_* p^* \mathbb{1}_S = p_* \mathbb{1}_X.$$

(Here and later, $\mathbb{1}_S$ denotes the unit object of the monoidal category $\mathbf{SH}_{\mathfrak{M}}(S)$.) When p is smooth, we may also consider the *homological motive* $\mathrm{M}_S(X) = p_\sharp \mathbb{1}_X$, also given by the Tate spectrum $\mathrm{Sus}^0_T(X \otimes \mathbb{1})$. It is related to the *cohomological motive* by a canonical isomorphism $\mathrm{M}^{\vee}_S(X) \simeq \underline{\mathsf{Hom}}(\mathrm{M}_S(X), \mathbb{1}_S)$.

When the base scheme S is understood, we write simply $\mathrm{M}^{\vee}(X)$ and $\mathrm{M}(X)$ instead of $\mathrm{M}^{\vee}_S(X)$ and $\mathrm{M}_S(X)$.

It follows from [**2**, Scholie 2.2.34] that the motives introduced in Definition 3.3 are constructible motives, i.e., objects of $\mathbf{SH}_{\mathfrak{M},\mathsf{ct}}(S)$. The latter is defined as the smallest triangulated subcategory of $\mathbf{SH}_{\mathfrak{M}}(S)$ stable by direct factors, Tate twists and containing the homological motives of smooth quasi-projective S-schemes.

3.2. Nearby motivic sheaves. — Let X be a finite type R-scheme and denote by $f : X \to \mathsf{Spec}(R)$ its structural morphism. Using [**3**, §3.5] (see also [**6**, §A.1]), one has the *nearby motivic sheaf functor* $\Psi_{t\circ f} : \mathbf{SH}_{\mathfrak{M}}(X_\eta) \to \mathbf{SH}_{\mathfrak{M}}(X_\sigma)$ associated with the

1. In [**25**] the motive $\mathrm{M}^{\vee}_S(X)$ is denoted by $\mathrm{M}_S(X)$.

morphism $t \circ f : X \to \mathbf{A}^1_k$. (Of course, $t : \mathsf{Spec}(R) \to \mathbf{A}^1_k$ is the obvious morphism.) For convenience, we will (abusively) denote this functor by

$$\Psi_f : \mathbf{SH}_{\mathfrak{M}}(X_\eta) \to \mathbf{SH}_{\mathfrak{M}}(X_\sigma). \tag{4}$$

When X varies in the category of quasi-projective R-schemes, the functors (4) form a specialization system in the sense of [**3**, Définition 3.1.1]. Moreover all the results from [**3**, §3.5] apply to them.

The object $\Psi_f(\mathbb{1}_{X_\eta}) \in \mathbf{SH}_{\mathfrak{M}}(X_\sigma)$ will be called the *nearby motivic sheaf*[(2)] associated with the morphism f (or with the R-scheme X). For later use, we record the following result (see [**5**, Théorème 10.6]):

Proposition 3.4. — *Let X be a finite type R-scheme and denote by $f : X \to \mathsf{Spec}(R)$ its structural morphism. We assume that X is regular and that $D = (X_\sigma)_{\mathrm{red}}$ is a smooth k-scheme. We also assume that D is a principal divisor and we fix $g \in \mathcal{O}(X)$ a generator of its ideal of definition. Finally, we assume that there are $u \in \mathcal{O}(X)^\times$ and $m \in \mathbf{N}^\times$ such that $t = ug^m$. (In particular, the R-scheme X is semi-stable and X_σ is an irreducible divisor with multiplicity m.)*

Now, consider the finite étale cover

$$r_m : D_m = \mathsf{Spec}(\mathcal{O}_D[S]/(S^m - u_0)) \to D$$

where u_0 is the restriction of u to D. Then, for every object $M \in \mathbf{SH}_{\mathfrak{M}}(K)$, there is a canonical isomorphism

$$\Psi_f f_\eta^*(M) \simeq (r_m)_*(\Psi_{\mathrm{Id}}(e_m)^*_\eta M)|_{D_m}$$

where $e_m : \mathsf{Spec}(k[[t]]) \to \mathsf{Spec}(k[[t]])$ is the morphism given by $t \mapsto t^m$. In particular, taking M to be the unit object, one gets:

$$\Psi_f(\mathbb{1}_{X_\eta}) \simeq (r_m)_* \mathbb{1}_{D_m}.$$

Proof. — We only give a sketch of the proof since it is very similar to the proof of [**5**, Théorème 10.6].

We start by fixing some notations. Let $f_m : X_m = X \otimes_{R,e_m} R \to \mathsf{Spec}(R)$ be the base-change of f along e_m and let $e^X_m : X_m \to X$ be the projection to the first factor. By [**3**, Proposition 3.5.9] we have a natural isomorphism

$$\Psi_f \simeq \Psi_{f_m}(e^X_m)^*_\eta.$$

Now, let $\tilde{X}_m$ be the normalization of the scheme $X_m = \mathsf{Spec}(\mathcal{O}_X[T^m]/(T^m - t))$ and denote by $h_m : \tilde{X}_m \to X_m$ the canonical morphism. Using that $T^m = ug^m$ in $\mathcal{O}_X$, one gets that

$$\tilde{X}_m = \mathsf{Spec}(\mathcal{O}_X[S]/(S^m - u)).$$

2. This object was called *nearby motive* in [**25**].

In particular, the R-scheme $\tilde{X}_m$, with structure morphism $\tilde{f}_m = f_m \circ h_m$, is smooth with special fiber D_m. Using the second property of [**3**, Définition 3.1.1 (**SPE2**)], this shows that

$$\Psi_{\tilde{f}_m}(\tilde{f}_m)^*_\eta \simeq (\tilde{f}_m)^*_\sigma \Psi_{\mathrm{Id}}.$$

By putting these facts together, we obtain a sequence of isomorphisms

$$\begin{aligned}\Psi_f f^*_\eta(M) \simeq \Psi_{f_m}(e^X_m)^*_\eta f^*_\eta(M) \simeq \Psi_{f_m}(f_m)^*_\eta(e_m)^*_\eta(M) &\simeq (r_m)_*\Psi_{\tilde{f}_m}(\tilde{f}_m)^*_\eta(e_m)^*_\eta(M)\\ &\simeq (r_m)_*(\Psi_{\mathrm{Id}}(e_m)^*_\eta(M))|_{D_m}.\end{aligned}$$

The third isomorphism above uses the fact that h_m is finite, and hence projective, that $(h_m)_\eta$ is the identity and that $(h_m)_\sigma$ is equal to r_m up to nilradicals. □

Apart from Proposition 3.4, the computations with nearby cycles done in the present paper only require the defining properties of a specialization system (see [**3**, Définition 3.1.1]) and the formalism of the six operations of [**2, 3**], and especially the base-change theorem by a smooth morphism and the base-change theorem for a proper morphism [**2**, Corollaire 1.7.18].

3.3. Recollections on rigid motives. — In this subsection, we overview some constructions from [**6**] around the notion of *rigid motives*.

In [**6**] (see also [**7**, §2.2]), Ayoub developed a theory of motives in the context of rigid analytic geometry. In particular, one has a triangulated category of rigid motives $\mathbf{RigSH}_{\mathfrak{M}}(K)$. Its construction is parallel to the construction of the triangulated category of motives $\mathbf{SH}_{\mathfrak{M}}(K)$ except that smooth varieties are replaced with rigid analytic varieties and the affine line $\mathbf{A}^1_K = \mathsf{Spec}(K[T])$ is replaced with the unit ball $\mathbf{B}^1_K = \mathsf{Spm}(K\{T\})$. More precisely, one starts with the category $\mathbf{PSh}(\mathrm{SmRig}/K, \mathfrak{M})$ of presheaves on smooth rigid K-varieties with coefficients in $\mathfrak{M}$ endowed with its projective Nisnevich local model structure (see [**6**, Définition 1.2.8] for the definition of the Nisnevich topology in the rigid analytic context). A left Bousfield localisation with respect to the maps $\mathbf{B}^1_X \otimes A_{cst} \to X \otimes A_{cst}$, for $X \in \mathrm{SmRig}/K$ and $A \in \mathfrak{M}$, gives the projective $(\mathbf{B}^1, \mathrm{Nis})$-local model structure on $\mathbf{PSh}(\mathrm{SmRig}/K, \mathfrak{M})$ (see [**6**, Définition 1.3.2]). The category $\mathbf{RigSH}_{\mathfrak{M}}(K)$ is then the homotopy category of the category $\mathbf{Spt}^{\Sigma}_{T^{an}}(\mathbf{PSh}(\mathrm{SmRig}/K, \mathfrak{M}))$ of T^{an}-symmetric spectra endowed with its stable projective model structure obtained from the $(\mathbf{B}^1, \mathrm{Nis})$-local model structure. Here T^{an} is the image of T by the analytification functor. (See [**6**, Définition 1.3.19] and more generally [**6**, §1.3.1 and §1.3.3] for more details.)

Example 3.5. — Again, if $\mathfrak{M} = \mathbf{Spect}^{\Sigma}_{S^1}(\Delta^{\mathrm{op}}\mathcal{S}et_\bullet)$, this category is simply denoted by $\mathbf{RigSH}(K)$. If $\mathfrak{M} = \mathbf{Compl}(\Lambda)$, this category is denoted by $\mathbf{RigDA}(K, \Lambda)$.

Definition 3.6. — Let X be a smooth rigid variety over K. We denote by $\mathrm{M}_{rig}(X)$ the *homological motive* associated with X, i.e., the T^{an}-spectrum $\mathrm{Sus}^0_{T^{an}}(X \otimes \mathbb{1})$

considered as an object of $\mathbf{RigSH}_{\mathfrak{M}}(K)$. We will denote by $\mathrm{M}^{\vee}_{rig}(X)$ the *cohomological motive* associated with X given by the dual of $\mathrm{M}_{rig}(X)$. More precisely, we set

$$\mathrm{M}^{\vee}_{rig}(X) = \underline{\mathsf{Hom}}(\mathrm{M}_{rig}(X), \mathbb{1}_{\mathsf{Spm}(K)}).$$

(Compare with Definition 3.3.)

Recall that, given a K-scheme of finite type X, there is an associated rigid analytic K-variety X^{an}. The functor $X \mapsto X^{an}$ extends into a triangulated functor

$$\mathrm{Rig}^* : \mathbf{SH}_{\mathfrak{M}}(K) \to \mathbf{RigSH}_{\mathfrak{M}}(K)$$

such that $\mathrm{Rig}^*(\mathrm{M}(X)) = \mathrm{M}_{rig}(X^{an})$.

One of the main results of [**6**] gives an equivalence between the category of rigid motives over K and the category of quasi-unipotent (algebraic) motives over a torus over the residue field of K. More precisely, denote by $\mathbf{QUSH}_{\mathfrak{M}}(k)$ the triangulated subcategory of $\mathbf{SH}_{\mathfrak{M}}(\mathbf{G}_{m,k})$ closed under infinite direct sums and generated by the objects of the form $\mathrm{Sus}^p_T(Q^{gm}_r(X,g)\otimes \mathbb{1})$ where X is a smooth k-scheme, $g \in \mathcal{O}(X)^{\times}$, $r \in \mathbf{N}^{\times}$ and $Q^{gm}_r(X,g)$ is the smooth $\mathbf{G}_{m,k}$-scheme

$$Q^{gm}_r(X,g) := \mathsf{Spec}(\mathcal{O}_X[T,T^{-1},V]/(V^r - gT)) \to \mathsf{Spec}(k[T,T^{-1}]) = \mathbf{G}_{m,k}.$$

(See [**6**, Notation 1.3.24].) Then, the composition of the three functors

$$\mathbf{QUSH}_{\mathfrak{M}}(k) \hookrightarrow \mathbf{SH}_{\mathfrak{M}}(\mathbf{G}_{m,k}) \xrightarrow{t^*} \mathbf{SH}_{\mathfrak{M}}(K) \xrightarrow{\mathrm{Rig}^*} \mathbf{RigSH}_{\mathfrak{M}}(K)$$

is an equivalence of categories (see [**6**, Scholie 1.3.26]).

We fix a quasi-inverse to the above composition

$$\mathfrak{R} : \mathbf{RigSH}_{\mathfrak{M}}(K) \xrightarrow{\sim} \mathbf{QUSH}_{\mathfrak{M}}(k).$$

We will be interested in the composite functor

$$1^* \circ \mathfrak{R} : \mathbf{RigSH}_{\mathfrak{M}}(K) \to \mathbf{SH}_{\mathfrak{M}}(k)$$

where $1 : \mathsf{Spec}(k) \to \mathbf{G}_{m,k}$ is the unit section.

4. Rigid motives of generic fibers of formal schemes

The goal of this section is to establish Theorem 4.11, wich is the particular case $Z = X_\sigma$ of our main theorem. Theorem 4.11 will be obtained as a formal consequence of Theorem 4.1. We warn the reader that the main ingredients for proving Theorem 4.1 are already contained in [**6**]. More precisely, the proof depends ultimately on the description of the $(\mathbf{B}^1, \mathrm{Nis})$-localisation given in [**6**, §1.3.4, Théorèmes 1.3.37 et 1.3.38] and an important part of the argument consists in recalling these results.

4.1. Statement of preliminary results. — We start by introducing some notations. Let A be a smooth affinoid K-algebra. Consider the commutative diagram with cartesian squares

$$\begin{tikzcd}
\mathsf{Spec}(A) \arrow[r, "j"] \arrow[d, "f_\eta"] \arrow[dr, phantom, "\square"] & \mathsf{Spec}(A^\circ) \arrow[d, "f"] \arrow[dr, phantom, "\square"] & \mathsf{Spec}(\tilde{A}) \arrow[l, "i"'] \arrow[d, "f_\sigma"] \\
\mathsf{Spec}(K) \arrow[r, "j"] & \mathsf{Spec}(R) & \mathsf{Spec}(k). \arrow[l, "i"']
\end{tikzcd}$$

Here, as usual, $A^\circ = \{a \in A;\ |a|_\infty \leqslant 1\}$, $A^{\circ\circ} = \{a \in A;\ |a|_\infty < 1\}$ and $\tilde{A} = A^\circ/A^{\circ\circ}$, where $|\cdot|_\infty$ is the infinity norm (aka., spectral norm) on A. (Compare this with §1.6.)

Theorem 4.1. — *Let M be an object of $\mathbf{SH}_{\mathfrak{M}}(K)$. Then, there is a canonical isomorphism in $\mathbf{SH}_{\mathfrak{M}}(k)$:*

$$1^* \circ \mathfrak{R}(\underline{\mathsf{Hom}}(\mathrm{M}_{rig}(\mathsf{Spm}(A)), \mathrm{Rig}^*(M))) \simeq (f_\sigma)_* \Psi_f f_\eta^*(M). \tag{5}$$

Taking M to be the unit object of $\mathbf{SH}_{\mathfrak{M}}(K)$, one gets the following:

Corollary 4.2. — *There is a canonical isomorphism in $\mathbf{SH}_{\mathfrak{M}}(k)$:*

$$1^* \circ \mathfrak{R}(\mathrm{M}^\vee_{rig}(\mathsf{Spm}(A))) \simeq (f_\sigma)_* \Psi_f(\mathbb{1}_{\mathsf{Spec}(A)}).$$

Remark 4.3. — The statement of Theorem 4.1 makes use of the generalization of the theory of nearby motivic sheaves explained in [**6**, Appendice 1.A]. See also [**5**, §10].

Remark 4.4. — The statement of Theorem 4.1 can be made functorial as follows. Let $(\mathsf{Spm}(\mathcal{A}), \mathcal{I})$ be a diagram of smooth K-affinoids. This means that $\mathcal{I}$ is a small category and $\mathcal{A}$ is a contravariant functor from $\mathcal{I}$ to the category of smooth affinoid K-algebras. Consider the following commutative diagram of diagrams of schemes

$$\begin{tikzcd}
(\mathsf{Spec}(\mathcal{A}), \mathcal{I}) \arrow[r] \arrow[dd, "(f_\eta, p_\mathcal{I})"] \arrow[ddr, phantom, "\square"] & (\mathsf{Spec}(\mathcal{A}^\circ), \mathcal{I}) \arrow[dd, "(f, p_\mathcal{I})"] \arrow[ddr, phantom, "\square"] & (\mathsf{Spec}(\tilde{\mathcal{A}}), \mathcal{I}) \arrow[l] \arrow[dd, "(f_\sigma, p_\mathcal{I})"] \arrow[dr, "f_\sigma"] & \\
 & & & (\mathsf{Spec}(k), \mathcal{I}). \arrow[dl, "p_\mathcal{I}"] \\
\mathsf{Spec}(K) \arrow[r] & \mathsf{Spec}(R) & \mathsf{Spec}(k). \arrow[l] &
\end{tikzcd}$$

Then, there is a canonical isomorphism in $\mathbf{SH}_{\mathfrak{M}}(k, \mathcal{I})$:

$$1^* \circ \mathfrak{R}(\underline{\mathsf{Hom}}(\mathrm{M}_{rig}(\mathsf{Spm}(\mathcal{A})), \mathrm{Rig}^*(M))) \simeq (f_\sigma)_* \Psi_{(f, p_\mathcal{I})}(f_\eta, p_\mathcal{I})^*(M). \tag{6}$$

The proof is an easy adaptation of the proof for a single smooth K-affinoid. We leave the details to the reader.

Finally, we warn the reader that the "$\underline{\mathsf{Hom}}$" in (6) is not an "internal hom" in the category of $\mathbf{RigSH}_{\mathfrak{M}}(K, \mathcal{I})$. It is rather an "external hom" in the sense of [**19**, §3] going from $\mathbf{RigSH}_{\mathfrak{M}}(K, \mathcal{I}^{\mathrm{op}})$ to $\mathbf{RigSH}_{\mathfrak{M}}(K, \mathcal{I})$. More precisely,

$$\underline{\mathsf{Hom}}(\mathrm{M}_{rig}(\mathsf{Spm}(\mathcal{A})), \mathrm{Rig}^*(M))$$

is the diagram of rigid motives given, for $i \in \mathcal{I}$, by $\underline{\mathsf{Hom}}(\mathrm{M}_{rig}(\mathsf{Spm}(\mathcal{A}(i))), \mathrm{Rig}^*(M))$.

To prove Theorem 4.1, we first need to establish a variant where Ψ_f is replaced by the specialization system $\chi_f = i^* j_*$. (Recall that, for a base scheme S, Sm/S denotes the category of smooth S-schemes.)

Theorem 4.5. — *Let M be an object of $\mathbf{SH}_{\mathfrak{M}}(K)$. Then, there is a canonical isomorphism in $\mathbf{SH}_{\mathfrak{M}}(k)$:*

$$q_* \circ \mathfrak{R}(\underline{\mathsf{Hom}}(\mathrm{M}_{rig}(\mathsf{Spm}(A)), \mathrm{Rig}^*(M))) \simeq (f_\sigma)_* \chi_f f_\eta^*(M).$$

4.2. Proof of Theorem 4.5. — Before we state our first lemma, we need to recall some notations from [**6**]. Given a k-variety X, we denote by $Q^{rig}(X)$ the generic fiber of the t-adic completion of the R-scheme $X \otimes_k R$. Note that, if X is the spectrum of a k-algebra E, then $Q^{rig}(X) = \mathsf{Spm}(E[[t]][t^{-1}])$. This gives a functor

$$Q^{rig} : \mathrm{Sm}/k \to \mathrm{SmRig}/K$$

which is continuous for the Nisnevich topology. (As in [**6**], SmRig/K denotes the category of smooth rigid analytic varieties over K.)

Using standard constructions, the functor Q^{rig} induces a pair of adjoint functors

$$((Q^{rig})^*, Q^{rig}_*) : \mathbf{SH}_{\mathfrak{M}}(k) \to \mathbf{RigSH}_{\mathfrak{M}}(K).$$

The functor $(Q^{rig})^*$ takes the homological motive of a smooth k-scheme X to the homological motive of the rigid analytic variety $Q^{rig}(X)$.

We will be mainly interested in the functor Q^{rig}_*. We have the following result which is a variant of [**7**, Théorème 2.24]. However, the proof here is much easier as everything is derived.

Lemma 4.6. — *There is a canonical invertible natural transformation of functors from $\mathbf{RigSH}_{\mathfrak{M}}(K)$ to $\mathbf{SH}_{\mathfrak{M}}(k)$*

$$q_* \circ \mathfrak{R} \simeq Q^{rig}_*.$$

Proof. — Recall that $\mathfrak{R}$ is a quasi-inverse to the following composition

$$\mathfrak{F} : \mathbf{QUSH}_{\mathfrak{M}}(k) \hookrightarrow \mathbf{SH}_{\mathfrak{M}}(\mathbf{G}_{m,k}) \xrightarrow{t^*} \mathbf{SH}_{\mathfrak{M}}(K) \xrightarrow{\mathrm{Rig}^*} \mathbf{RigSH}_{\mathfrak{M}}(K) \tag{7}$$

which is an equivalence of categories by [**6**, Scholie 1.3.26]. Therefore, to prove the lemma, it is enough to construct an isomorphism

$$(Q^{rig})^* \simeq \mathfrak{F} \circ q^*.$$

Now, let $Q^{an} : \mathrm{Sm}/k \to \mathrm{SmRig}/K$ be the functor that takes a k-variety X to the rigid analytic variety $(X \otimes_k K)^{an}$. It induces a functor

$$(Q^{an})^* : \mathbf{SH}_{\mathfrak{M}}(k) \to \mathbf{RigSH}_{\mathfrak{M}}(K)$$

which is nothing but $\mathfrak{F} \circ q^*$. On the other hand, there is a natural transformation $Q^{rig} \to Q^{an}$. It induces a natural transformation $(Q^{rig})^* \to (Q^{an})^*$ which is an isomorphism by [**6**, Théorème 1.3.11]. □

Therefore, to prove Theorem 4.5, it is enough to establish the following proposition.

Proposition 4.7. — *Keep the notation as for Theorem 4.5. There is a canonical isomorphism*

$$Q^{rig}_* \underline{\mathsf{Hom}}(\mathrm{M}_{rig}(\mathsf{Spm}(A)), \mathrm{Rig}^*(M)) \simeq (f_\sigma)_* \chi_f f^*_\eta(M).$$

Remark 4.8. — The proof of this proposition uses similar ideas and techniques as those exposed in [**6**, §1.3.4] and especially in the proof of [**6**, Scholie 1.3.26]. The reader who finds our proof below a bit sketchy is advised to read [**6**, §1.3.4] where he can find enough material to complement the arguments.

To prove Proposition 4.7, we need to recall the construction of the $(\mathbf{B}^1, \mathrm{Nis})$-localization of the T^{an}-spectrum $\mathrm{Rig}^*(M)$ given in [**6**, §1.3.4, Théorèmes 1.3.37 et 1.3.38]. We start by recalling the necessary notation. Let

$$\mathscr{D} : \mathrm{SmAfnd}/K \to \mathrm{Sch}/R$$

be the functor from the category SmAfnd/K of smooth K-affinoids to the category Sch/R of R-schemes (not necessarily of finite type) that takes a K-affinoid X to the R-scheme

$$\mathscr{D}(X) = \mathsf{Spec}(\mathcal{O}(X)^\circ).$$

We will think about $\mathscr{D}$ as a diagram of R-schemes. There are two other related diagrams $\mathscr{D}_\eta$ and $\mathscr{D}_\sigma$ defined on SmAfnd/K, and with values in Sch/K and Sch/k respectively. These are given by

$$\mathscr{D}_\eta(X) = \mathsf{Spec}(\mathcal{O}(X)) \qquad \text{and} \qquad \mathscr{D}_\sigma(X) = \mathsf{Spec}(\mathcal{O}(X)\tilde{\ }).$$

Thus, we have a diagram of diagrams of schemes (see [**6**, (1.86)]):

$$\begin{tikzcd}
\mathscr{D}_\eta \arrow[r, "j"] \arrow[d, "\mathrm{u}_\eta"] \arrow[dr, phantom, "\square"] & \mathscr{D} \arrow[d, "\mathrm{u}"] \arrow[dr, phantom, "\square"] & \mathscr{D}_\sigma \arrow[l, "i"'] \arrow[d, "\mathrm{u}_\sigma"] \\
\mathsf{Spec}(K) \arrow[r, "j"] & \mathsf{Spec}(R) & \mathsf{Spec}(k). \arrow[l, "i"']
\end{tikzcd}$$

There is an obvious diagonal functor

$$\mathrm{diag} : \mathrm{SmAfnd}/K \to \mathrm{Sm}/\mathscr{D}.$$

(For the definition of "Sm/a diagram of schemes", see the beginning of [**3**, §4.5.1].) It takes an object $\mathsf{Spm}(B)$ of SmAfnd/K to the couple

$$(\mathsf{Spm}(B), \mathrm{Id}_{\mathsf{Spec}(B^\circ)}).$$

Composing with diag yields a functor

$$\mathrm{diag}^* : \mathbf{PreShv}(\mathrm{Sm}/\mathscr{D}, \mathfrak{M}) \to \mathbf{PreShv}(\mathrm{SmAfnd}/K, \mathfrak{M}).$$

This functor extends to T-spectra and can be derived into a functor

$$\mathrm{Rdiag}^* : \mathbf{SH}_{\mathfrak{M}}(\mathscr{D}) \to \mathbf{Ho}(\mathbf{Spect}^{\Sigma}_{\mathrm{diag}^*(T)}(\mathbf{PreShv}(\mathrm{SmAfnd}/K, \mathfrak{M}))).$$

(In fact, it is shown in [**6**, §1.3.4] that $\mathrm{diag}^*(T)$ is weakly equivalent to T^{an}.) With these notations, we can state [**6**, Théorèmes 1.3.37 et 1.3.38] as follows:

Theorem 4.9. — *Let M be an object of $\mathbf{SH}_{\mathfrak{M}}(K)$. Then the symmetric* diag^*T*-spectrum*

$$\mathrm{diag}^* i_* i^* j_* \mathrm{u}_\eta^* M$$

is a stably $(\mathbf{B}^1, \mathrm{Nis})$*-local object of*

$$\mathbf{Ho}(\mathbf{Spect}^{\Sigma}_{\mathrm{diag}^*(T)}(\mathbf{PreShv}(\mathrm{SmAfnd}/K, \mathfrak{M}))).$$

Moreover, there is a canonical $(\mathbf{B}^1, \mathrm{Nis})$*-equivalence*

$$\mathrm{r}_* \mathrm{Rig}^*(M) \to \mathrm{diag}^* i_* i^* j_* \mathrm{u}_\eta^* M.$$

In the statement of Theorem 4.9, $\mathrm{r} : \mathrm{SmAfnd}/K \hookrightarrow \mathrm{SmRig}/K$ is the inclusion of the subcategory of smooth affinoid varieties over K and r_* is the functor induced by composition with r. Similarly, we denote by $\mathrm{r} : \mathrm{SmAf}/k \hookrightarrow \mathrm{Sm}/k$ the inclusion of the subcategory of smooth affine k-schemes and r_* the functor induced by composition with r. (Below, we use implicitly that the functors r induce equivalences of Nisnevich sites, and thus Quillen equivalences with respect to the $(\mathbf{B}^1, \mathrm{Nis})$ and $(\mathbf{A}^1, \mathrm{Nis})$-local structures.)

Using Theorem 4.9 and going back to the construction of the different functors, we obtain canonical isomorphisms

$$\begin{aligned}\mathrm{r}_* Q^{rig}_* \underline{\mathsf{Hom}}(\mathrm{M}_{rig}(\mathsf{Spm}(A)), \mathrm{Rig}^*(M)) &= Q^{rig}_* \underline{\mathsf{Hom}}(\mathrm{M}_{rig}(\mathsf{Spm}(A)), \mathrm{r}_* \mathrm{Rig}^*(M)) \\ &\simeq Q^{rig}_* \underline{\mathsf{Hom}}(\mathrm{M}_{rig}(\mathsf{Spm}(A)), \mathrm{diag}^* i_* i^* j_* \mathrm{u}_\eta^* M) = \delta_A^* i_* i^* j_* \mathrm{u}_\eta^* M\end{aligned}$$

in $\mathbf{Ho}_{\mathbf{A}^1-\mathrm{Nis}}(\mathbf{Spect}^{\Sigma}_T(\mathbf{PreShv}(\mathrm{SmAf}/k, \mathfrak{M}))) \simeq \mathbf{SH}_{\mathfrak{M}}(k)$. The second and third Q^{rig} above stand for the functor $Q^{rig} : \mathrm{SmAf}/k \to \mathrm{SmAfnd}/K$; the functor

$$\delta_A : \mathrm{SmAf}/k \to \mathrm{Sm}/\mathscr{D}$$

takes a smooth affine scheme $U = \mathsf{Spec}(E)$ to the couple

$$\Big(\mathsf{Spm}(A) \hat{\times}_K Q^{rig}(U) = \mathsf{Spm}(A \hat{\otimes}_K E[[t]][t^{-1}])\ ,\ \mathrm{Id}_{\mathsf{Spec}(A^\circ \hat{\otimes}_R E[[t]])}\Big)\,;$$

and δ_A^* is the functor induced by composition with δ_A.

Consider now the diagram of schemes $\mathscr{F}_A : \mathrm{SmAf}/k \to \mathrm{Sch}/R$ that takes a smooth affine k-scheme $\mathsf{Spec}(E)$ to $\mathsf{Spec}(A^\circ \hat{\otimes}_R E[[t]])$. Similarly, let $\mathscr{F}_{A,\eta} : \mathrm{SmAf}/k \to \mathrm{Sch}/K$ and $\mathscr{F}_{A,\sigma} : \mathrm{SmAf}/k \to \mathrm{Sch}/k$ be the diagrams of schemes that takes $\mathsf{Spec}(E)$ to $\mathsf{Spec}(A \hat{\otimes}_K E[[t]])$ and $\mathsf{Spec}(\tilde{A} \otimes_k E)$ respectively. One has a commutative diagram of diagrams of schemes:

$$\begin{tikzcd}
\mathscr{F}_{A,\eta} \arrow[r, "j"] \arrow[d, "\mathfrak{f}_\eta"] \arrow[dr, phantom, "\square"] & \mathscr{F}_A \arrow[d, "\mathfrak{f}"] \arrow[dr, phantom, "\square"] & \mathscr{F}_{A,\sigma} \arrow[l, "i"'] \arrow[d, "\mathfrak{f}_\sigma"] \\
\mathsf{Spec}(K) \arrow[r, "j"] & \mathsf{Spec}(R) & \mathsf{Spec}(k). \arrow[l, "i"']
\end{tikzcd}$$

Moreover, there is an obvious morphism of diagrams of schemes $\mathscr{F}_A \to \mathscr{D}$ induced by the functor on the indexing categories $\mathrm{SmAf}/k \to \mathrm{SmAfnd}/K$ that takes $\mathsf{Spec}(E)$ to $\mathsf{Spm}(A \hat{\otimes}_K E[[t]][t^{-1}])$.

Let $\mathrm{diag}_A : \mathrm{SmAf}/k \to \mathrm{Sm}/\mathscr{F}_A$ be the diagonal functor given by $\mathrm{diag}_A(\mathsf{Spec}(E)) = (\mathsf{Spec}(E), \mathrm{Id}_{\mathsf{Spec}(A^\circ \hat{\otimes}_R E[[t]])})$. Using the following commutative triangle

$$\begin{tikzcd}
\mathrm{SmAf}/k \arrow[r, "\mathrm{diag}_A"] \arrow[dr, "\delta_A"'] & \mathrm{Sm}/\mathscr{F}_A \arrow[d] \\
& \mathrm{Sm}/\mathscr{D},
\end{tikzcd}$$

we get canonical isomorphisms

$$\delta_A^* i_* i^* j_* \mathfrak{u}_\eta^* M \simeq \mathrm{diag}_A^* i_* i^* j_* \mathfrak{f}_\eta^* M \simeq \mathrm{diag}_{A,\sigma}^* i^* j_* \mathfrak{f}_\eta^* M$$

where $\mathrm{diag}_{A,\sigma}$ is the diagonal functor that takes $\mathsf{Spec}(E)$ to $(\mathsf{Spec}(E), \mathrm{Id}_{\mathsf{Spec}(\tilde{A} \otimes_k E)})$.

Finally, one has a commutative diagram of diagrams of schemes:

$$\begin{tikzcd}
\mathscr{F}_{A,\eta} \arrow[r, "j"] \arrow[d, "\mathfrak{a}_\eta"] \arrow[dr, phantom, "\square"] & \mathscr{F}_A \arrow[d, "\mathfrak{a}"] \arrow[dr, phantom, "\square"] & \mathscr{F}_{A,\sigma} \arrow[l, "i"'] \arrow[d, "\mathfrak{a}_\sigma"] \\
\mathsf{Spec}(A) \arrow[r, "j"] & \mathsf{Spec}(A^\circ) & \mathsf{Spec}(\tilde{A}), \arrow[l, "i"']
\end{tikzcd}$$

with regular vertical maps. By [**6**, Corollaire 1.A.4], this gives a canonical isomorphism

$$i^* j_* \mathfrak{f}_\eta^* M \simeq \mathfrak{a}_\sigma^* i^* j_* M|_{\mathsf{Spec}(A)}.$$

Now, it is obvious that $\mathrm{diag}_{A,\sigma}^* \circ \mathfrak{a}_\sigma^* = (f_\sigma)_*$. This finishes the proof of Proposition 4.7 and hence of Theorem 4.5.

4.3. Proof of Theorem 4.1. — We have to recall the definition of the nearby motivic sheaf functor. Let Δ be the category of finite ordinals $\underline{n} = \{0 < 1 < \cdots < n\}$, for $n \in \mathbf{N}$, and $\mathbf{N}^\times = \mathbf{N} \smallsetminus \{0\}$ ordered by the opposite of the division relation. In

[**3**, Définition 3.5.3], Ayoub introduced a diagram of k-schemes $(\mathscr{R}, \Delta \times \mathbf{N}^{\times})$ with a morphism

$$(\theta^{\mathscr{R}}, p_{\Delta\times\mathbf{N}^{\times}}) : (\mathscr{R}, \Delta \times \mathbf{N}^{\times}) \to \mathbf{G}_{m,k}.$$

Let $(\theta^{\mathscr{R}}_f, p_{\Delta\times\mathbf{N}^{\times}}) : (\mathscr{R}_f, p_{\Delta\times\mathbf{N}^{\times}}) \to \mathsf{Spec}(A)$ be the morphism of diagrams obtained by base-change along the morphism $\mathsf{Spec}(A) \to \mathbf{G}_{m,k}$ (given by the composition of $f : \mathsf{Spec}(A) \to \mathsf{Spec}(K)$ and $t : \mathsf{Spec}(K) \to \mathbf{G}_{m,k}$). The nearby motivic sheaf functor is then given by

$$\begin{aligned}\Psi_f(-) &= (p_{\Delta\times\mathbf{N}^{\times}})_\sharp \circ \chi_{f,p_{\Delta\times\mathbf{N}^{\times}}} \circ (\theta^{\mathscr{R}}_f)_* \circ (\theta^{\mathscr{R}}_f)^* \circ (p_{\Delta\times\mathbf{N}^{\times}})^*(-)\\ &\simeq \chi_f \circ (p_{\Delta\times\mathbf{N}^{\times}})_\sharp \circ (\theta^{\mathscr{R}}_f)_* \circ (\theta^{\mathscr{R}}_f)^* \circ (p_{\Delta\times\mathbf{N}^{\times}})^*(-).\end{aligned}$$

The isomorphism above is a consequence of the fact that inverse and direct images commute with homotopy colimits in the case of $\mathbf{SH}_{\mathfrak{M}}(-)$. Moreover, after composing with f_η^*, one has further isomorphisms as follows:

$$\begin{aligned}\Psi_f f_\eta^*(-) &\simeq \chi_f \circ (p_{\Delta\times\mathbf{N}^{\times}})_\sharp \circ (\theta^{\mathscr{R}}_f)_* \circ (\theta^{\mathscr{R}}_f)^* \circ (p_{\Delta\times\mathbf{N}^{\times}})^* \circ f_\eta^*(-)\\ &\simeq \chi_f \circ f_\eta^* \circ (p_{\Delta\times\mathbf{N}^{\times}})_\sharp \circ (\theta^{\mathscr{R}}_t)_* \circ (\theta^{\mathscr{R}}_t)^* \circ (p_{\Delta\times\mathbf{N}^{\times}})^*(-)\\ &\simeq \chi_f \circ f_\eta^*((-) \otimes t^*\mathcal{U})\end{aligned}$$

where $\mathcal{U} = (p_{\Delta\times\mathbf{N}^{\times}})_\sharp(\theta^{\mathscr{R}})_*\mathbb{1}_{(\mathscr{R},\Delta\times\mathbf{N}^{\times})}$ and $t : \mathsf{Spec}(K) \to \mathbf{G}_{m,k}$. Applying Theorem 4.5 with $M \otimes t^*\mathcal{U}$ instead of M, we get an isomorphism

$$(f_\sigma)_*\Psi_f f_\eta^* M \simeq q_* \circ \mathfrak{R}(\underline{\mathsf{Hom}}(\mathrm{M}_{rig}(\mathsf{Spm}(A)), \mathrm{Rig}^*(M \otimes t^*\mathcal{U})).$$

Therefore, it is enough to show that

$$q_* \circ \mathfrak{R}(\underline{\mathsf{Hom}}(\mathrm{M}_{rig}(\mathsf{Spm}(A)), \mathrm{Rig}^*(M \otimes t^*\mathcal{U}))) \simeq 1^* \circ \mathfrak{R}(\underline{\mathsf{Hom}}(\mathrm{M}_{rig}(\mathsf{Spm}(A)), \mathrm{Rig}^*(M))).$$

Let us recall the following lemma that is a consequence of results in [**6**]:

Lemma 4.10. — *Every compact object of* $\mathbf{RigSH}_{\mathfrak{M}}(K)$ *is strongly dualizable.*

Proof. — By [**6**, Théorème 1.3.22] and [**2**, Proposition 2.1.24], it is enough to show that, for every smooth k-scheme X, every $p \in \mathbf{N}$, $r \in \mathbf{N} \smallsetminus \{0\}$ and every $g \in \mathcal{O}(X)^{\times}$, the objects $\mathrm{Sus}^p_{T^{an}}(Q^{rig}_r(X, g) \otimes \mathbb{1})$ are strongly dualizable (see [**6**, Notation 1.3.10]). By [**6**, Lemma 1.3.12], the map

$$\mathrm{Sus}^p_{T^{an}}(Q^{rig}_r(X, g) \otimes \mathbb{1}) \to \mathrm{Sus}^p_{T^{an}}(Q^{an}_r(X, g) \otimes \mathbb{1}) = \mathrm{Rig}^*(\mathrm{Sus}^p_T(Q^{geo}_r(X, g) \otimes \mathbb{1}))$$

is an isomorphism in $\mathbf{RigSH}_{\mathfrak{M}}(K)$. As the functor Rig^* is symmetric monoidal and unitary, it suffices to check that $\mathrm{Sus}^p_T(Q^{geo}_r(X, g) \otimes \mathbb{1})$ is strongly dualizable in $\mathbf{SH}_{\mathfrak{M}}(K)$. This follows from [**36**] (see also [**6**, Lemme 1.3.29]). □

By [**6**, Proposition 1.2.34], $\mathrm{M}_{rig}(\mathsf{Spm}(A))$ is a compact object in $\mathbf{RigSH}_{\mathfrak{M}}(K)$, hence strongly dualizable by Lemma 4.10. Therefore, using that Rig^* is monoidal, one has a canonical isomorphism

$$\underline{\mathsf{Hom}}(\mathrm{M}_{rig}(\mathsf{Spm}(A)), \mathrm{Rig}^*(M \otimes t^*\mathcal{U})) \simeq \underline{\mathsf{Hom}}(\mathrm{M}_{rig}(\mathsf{Spm}(A)), \mathrm{Rig}^*(M)) \otimes \mathrm{Rig}^* t^* \mathcal{U}.$$

Now, $\mathcal{U}$ is an object of $\mathbf{QUSH}_{\mathfrak{M}}(k)$ (see [**6**, Définition 1.3.25]). Therefore, we can write

$$\mathrm{Rig}^* t^* \mathcal{U} = \mathfrak{F}(\mathcal{U}).$$

By putting these facts together, we are left to show that

$$\begin{aligned} q_* \circ \mathfrak{R}(\underline{\mathsf{Hom}}(\mathrm{M}_{rig}(\mathsf{Spm}(A)), \mathrm{Rig}^*(M)) \otimes \mathfrak{F}(\mathcal{U})) \\ \simeq 1^* \circ \mathfrak{R}(\underline{\mathsf{Hom}}(\mathrm{M}_{rig}(\mathsf{Spm}(A)), \mathrm{Rig}^*(M))). \end{aligned}$$

Now, as $\mathfrak{R}$ is a monoidal equivalence of categories, one has a projection formula:

$$\mathfrak{R}((-) \otimes \mathfrak{F}(\mathcal{U})) \simeq \mathfrak{R}(-) \otimes \mathcal{U}.$$

At the end, we are left to construct an invertible natural transformation

$$q_*(- \otimes \mathcal{U}) \simeq 1^*(-)$$

between functors from $\mathbf{QUSH}_{\mathfrak{M}}(k)$ to $\mathbf{SH}_{\mathfrak{M}}(k)$. In [**6**, (1.112)], an isomorphism of functors

$$(p_{\Delta \times \mathbf{N}^\times})_\sharp q_* \left((p_{\Delta \times \mathbf{N}^\times})^*(-) \otimes (\theta^{\mathscr{R}})_* \mathbb{1}_{(\mathscr{R}, \Delta \times \mathbf{N}^\times)}\right) =: \Psi^{qu} \to 1^*(-)$$

is constructed. Using that $(p_{\Delta \times \mathbf{N}^\times})_\sharp q_* \simeq q_*(p_{\Delta \times \mathbf{N}^\times})_\sharp$ and projection formula, it is easy to see that Ψ^{qu} is canonically isomorphic to $q_*(- \otimes \mathcal{U})$. This finishes the proof of Theorem 4.1.

4.4. A particular case of the main theorem. — Here we prove the case $Z = X_\sigma$ of our main theorem. This is done using the functorial version of Theorem 4.1 (see Remark 4.4).

Let X be a finite type R-scheme and let $f : X \to \mathsf{Spec}(R)$ be its structural morphism. Assume that X_η is smooth over K and consider the t-adic completion $\mathscr{X}$ of X.

Theorem 4.11. — *Let M be an object of $\mathbf{SH}_{\mathfrak{M}}(K)$. Then, there is a canonical isomorphism in $\mathbf{SH}_{\mathfrak{M}}(k)$:*

$$1^* \circ \mathfrak{R}(\underline{\mathsf{Hom}}(\mathrm{M}_{rig}(\mathscr{X}_\eta), \mathrm{Rig}^*(M))) \simeq (f_\sigma)_* \Psi_f f_\eta^*(M). \tag{8}$$

When $X = \mathsf{Spec}(R)$ (and $f = \mathrm{Id}$), the above theorem simply states that $1^* \circ \mathfrak{R} \circ \mathrm{Rig}^*$ is isomorphic to the nearby motive functor Ψ_{Id}, which we already know by [**6**, Scholie 1.3.26(2)]. Thus, in some sense, Theorem 4.11 can be considered as a generalization of [**6**, Scholie 1.3.26(2)].

Taking M to be the unit object of $\mathbf{SH}_{\mathfrak{M}}(K)$ in Theorem 4.11, one gets the following:

Corollary 4.12. — *There is a canonical isomorphism in* $\mathbf{SH}_{\mathfrak{M}}(k)$*:*

$$1^* \circ \mathfrak{R}(\mathrm{M}^{\vee}_{rig}(\mathscr{X}_\eta)) \simeq (f_\sigma)_* \Psi_f(\mathbb{1}_{X_\eta}).$$

Proof. — As we already said, the proof relies on the functorial version of Theorem 4.1 described in Remark 4.4.

Let $(U_i)_{i\in I}$ be a finite covering of X by open affine subschemes. Let $\mathcal{P}^*(I)$ be the set of non-empty subsets of I ordered by reverse inclusion. We have a diagram of schemes $(U, \mathcal{P}^*(I))$ that takes $J \in \mathcal{P}^*(I)$ to $U_J = \cap_{j\in J} U_j$.

Let $(u, p) : (U, \mathcal{P}^*(I)) \to X$ be the canonical morphism. (We wrote p instead of $p_{\mathcal{P}^*(I)}$ to ease the notation.) Using Zariski descent and the second property in [**3**, Définition 3.2.1 (**SPE2**)], we see that the canonical maps

$$\Psi_f f_\eta^*(M) \to (u_\sigma, p)_*(u_\sigma, p)^* \Psi_f f_\eta^*(M) \to (u_\sigma, p)_* \Psi_{(f\circ u,p)}(f_\eta \circ u_\eta, p)^*(M)$$

are isomorphisms in $\mathbf{SH}_{\mathfrak{M}}(X_\sigma)$. Applying $(f_\sigma)_*$, we get a canonical isomorphism

$$(f_\sigma)_* \Psi_f f_\eta^*(M) \simeq p_*((f\circ u)_\sigma)_* \Psi_{(f\circ u,p)}((f\circ u)_\eta, p)^*(M)$$

in $\mathbf{SH}_{\mathfrak{M}}(k)$.

Now, consider the diagram of formal schemes $(\mathscr{U}, \mathcal{P}^*(I))$ obtained as the completion of U. As every $\mathscr{U}_J$ is affine, one can also form the diagram of schemes $(V, \mathcal{P}^*(I))$ where $V_J = \mathsf{Spec}(\mathcal{O}(\mathscr{U}_J))$. Now, one has a regular morphism of diagrams of R-schemes

$$r : (V, \mathcal{P}^*(I)) \to (U, \mathcal{P}^*(I))$$

inducing the identity between the special fibers. It follows from [**6**, Proposition 1.A.6] that

$$\Psi_{(f\circ u,p)}((f\circ u)_\eta, p)^*(M) \simeq \Psi_{(f\circ u\circ r,p)}((f\circ u\circ r)_\eta, p)^*(M)$$

in $\mathbf{SH}_{\mathfrak{M}}(U_\sigma, \mathcal{P}^*(I))$. On the other hand, the functorial version of Theorem 4.1 (see Remark 4.4) provides an isomorphism

$$((f\circ u)_\sigma)_* \Psi_{(f\circ u\circ r,p)}((f\circ u\circ r)_\eta, p)^*(M) \simeq 1^* \circ \mathfrak{R}(\underline{\mathsf{Hom}}(\mathrm{M}_{rig}(\mathscr{U}_\eta), \mathrm{Rig}^*(M))).$$

We therefore have an isomorphism

$$(f_\sigma)_* \Psi_f f_\eta^*(M) \simeq p_* \circ 1^* \circ \mathfrak{R}(\underline{\mathsf{Hom}}(\mathrm{M}_{rig}(\mathscr{U}_\eta), \mathrm{Rig}^*(M)))$$

and it remains to check that

$$p_* \circ 1^* \circ \mathfrak{R}(\underline{\mathsf{Hom}}(\mathrm{M}_{rig}(\mathscr{U}_\eta), \mathrm{Rig}^*(M))) \simeq 1^* \circ \mathfrak{R}(\underline{\mathsf{Hom}}(\mathrm{M}_{rig}(\mathscr{X}_\eta), \mathrm{Rig}^*(M))).$$

Using [**8**, Proposition 1.15], we get an isomorphism

$$p_* \circ 1^* \circ \mathfrak{R} \simeq 1^* \circ \mathfrak{R} \circ p_*.$$

Therefore, it is enough to check that one has an isomorphism

$$p_* \underline{\mathsf{Hom}}(\mathrm{M}_{rig}(\mathscr{U}_\eta), \mathrm{Rig}^*(M)) \simeq \underline{\mathsf{Hom}}(\mathrm{M}_{rig}(\mathscr{X}_\eta), \mathrm{Rig}^*(M))$$

in $\mathbf{RigSH}_{\mathfrak{M}}(K)$. Now the left hand side is canonically isomorphic to

$$\underline{\mathsf{Hom}}(p_\sharp \mathrm{M}_{rig}(\mathscr{U}_\eta), \mathrm{Rig}^*(M)).$$

Thus, we are left to check that $p_\sharp \mathrm{M}_{rig}(\mathscr{U}_\eta) \to \mathrm{M}_{rig}(\mathscr{X}_\eta)$ is an isomorphism. This follows by Zariski descent. □

5. Rigid motives of tubes in a semi-stable situation

The goal of this section is to prove some preparatory results about rigid motives of tubes in a semi-stable situation. A striking consequence of these results is that the rigid motive of a tube (in a quasi-compact rigid analytic variety) is always a compact motive.

5.1. Tubes in rigid analytic geometry. — Let $\mathscr{X}$ be a formal R-scheme topologically of finite type. Let $Z \subset \mathscr{X}_\sigma$ be a locally closed subset. The *tube* of Z, denoted by $]Z[$, is the inverse image of Z under the *specialization map* $\mathrm{sp}\colon \mathscr{X}_\eta \to \mathscr{X}_\sigma$. This is an admissible open rigid analytic subvariety of $\mathscr{X}_\eta$, which is not quasi-compact in general.

If $U \subset \mathscr{X}_\sigma$ is an open subset and $\mathscr{U} \subset \mathscr{X}$ is the formal open subscheme such that $\mathscr{U}_\sigma = U$, then $]U[= \mathscr{U}_\eta$; in this case, the tube is quasi-compact. For more details concerning tubes, see, for example, [**12**] or [**27**, §2.1.2].

5.2. Statement of the results. — Assume that $\mathscr{X}$ is a semi-stable formal R-scheme. Let us denote by $(D_i)_{i\in I}$ the irreducible components of $(\mathscr{X}_\sigma)_{\mathrm{red}}$. Given a subset $J \subset I$, denote by D_J and $D(J)$ the reduced closed subschemes of $\mathscr{X}_\sigma$ given by

$$D_J = \cap_{i\in J} D_i \qquad \text{and} \qquad D(J) = \cup_{i\in J} D_i$$

with the convention that $D_\emptyset = (\mathscr{X}_\sigma)_{\mathrm{red}}$ and $D(\emptyset) = \emptyset$.

Fix a subset $J \subset I$ and let Z be a closed subscheme of $D(J)$. For $I' \subset I \smallsetminus J$, we set

$$Z^\circ_{I'} = Z \smallsetminus D(I').$$

When $I' = I \smallsetminus J$, we simply write Z° for $Z^\circ_{I\smallsetminus J}$.

Theorem 5.1. — *Keep the notation as before. Assume that Z is a union of closed subsets of the form $D_{J'}$, for some $\emptyset \neq J' \subset J$. Then, for $I' \subset I'' \subset I \smallsetminus J$, the inclusion $]Z^\circ_{I''}[\hookrightarrow]Z^\circ_{I'}[$ induces an isomorphism in $\mathbf{RigSH}_{\mathfrak{M}}(K)$:*

$$\mathrm{M}_{rig}(]Z^\circ_{I''}[) \simeq \mathrm{M}_{rig}(]Z^\circ_{I'}[).$$

At the end, we are only concerned with the following particular case.

Corollary 5.2. — *Keep the notation as before. The inclusion $]D(J)^\circ[\hookrightarrow]D(J)[$ induces an isomorphism in $\mathbf{RigSH}_{\mathfrak{M}}(K)$:*

$$\mathrm{M}_{rig}(]D(J)^\circ[) \simeq \mathrm{M}_{rig}(]D(J)[).$$

5.3. Reductions. — We start the proof of Theorem 5.1 by proving the following lemma.

Lemma 5.3. — *It is enough to prove Theorem 5.1 when $Z = D_J$, $J \neq \emptyset$, $I' = \emptyset$ and $\#(I'') = 1$.*

Proof. — Let us assume this particular case proven and suppose that Z, I' and I'' are as in the statement of Theorem 5.1. When $Z = \emptyset$, there is nothing to be proven; so we can assume that $Z \neq \emptyset$. (This forces that $J \neq \emptyset$.) We can write

$$Z = D_{J_1} \cup \cdots \cup D_{J_n}$$

for some integer $n \geqslant 1$, with $\emptyset \neq J_i \subset J$ for $1 \leqslant i \leqslant n$. We argue by induction on the integer n.

First, let us assume that $n = 1$. This means that $Z = D_{J_1}$ for some $J_1 \subset J$. As I' and I'' are also subsets of $I \smallsetminus J_1$, we may actually assume that $J_1 = J$. Also, by an easy induction we may assume that $\#(I'' \smallsetminus I') = 1$.

Now, consider the open formal subscheme $\mathscr{X}' \subset \mathscr{X}$ given by $\mathscr{X} \smallsetminus D(I')$. Then $\mathscr{X}'$ is a semi-stable formal R-scheme and $(\mathscr{X}'_\sigma)_{\mathrm{red}} = \cup_{i \in I \smallsetminus I'} D'_i$ with $D'_i = D_i \smallsetminus D(I')$. Moreover, letting $Z' = Z \cap \mathscr{X}'_\sigma$, one has (with the notations of §5.2):

$$Z'^{\circ}_{\emptyset} = Z^{\circ}_{I'} \qquad \text{and} \qquad Z'^{\circ}_{I'' \smallsetminus I'} = Z^{\circ}_{I''}.$$

Therefore, the map $\mathrm{M}_{rig}(]Z^{\circ}_{I''}[) \to \mathrm{M}_{rig}(]Z^{\circ}_{I'}[)$ identifies with

$$\mathrm{M}_{rig}(]Z'^{\circ}_{I'' \smallsetminus I'}[) \to \mathrm{M}_{rig}(]Z'^{\circ}_{\emptyset}[)$$

which is an isomorphism by the assumption of the lemma.

Next, assume that $n \geqslant 2$. We may then write $Z = Z_1 \cup Z_2$ where

$$Z_1 = D_{J_1} \cup \cdots \cup D_{J_{n-1}} \quad \text{and} \quad Z_2 = D_{J_n}.$$

Set $W = Z_1 \cap Z_2$. We therefore have admissible open coverings:

$$]Z^{\circ}_{I'}[\,=\,](Z_1)^{\circ}_{I'}[\,\cup\,](Z_2)^{\circ}_{I'}[\qquad \text{and} \qquad]Z^{\circ}_{I''}[\,=\,](Z_1)^{\circ}_{I''}[\,\cup\,](Z_2)^{\circ}_{I''}[.$$

Moreover, we have:

$$](Z_1)^{\circ}_{I'}[\,\cap\,](Z_2)^{\circ}_{I'}[\,=\,]W^{\circ}_{I'}[\qquad \text{and} \qquad](Z_1)^{\circ}_{I''}[\,\cap\,](Z_2)^{\circ}_{I''}[\,=\,]W^{\circ}_{I''}[.$$

Using Mayer–Vietoris distinguished triangles, we are left to treat the cases of Z_1, Z_2 and W. These cases follow by induction. □

We prove a further reduction.

Lemma 5.4. — *It is enough to prove Theorem 5.1 when $\#(J) = 1$ (and hence Z is an irreducible component of $\mathscr{X}_\sigma$), $I' = \emptyset$ and $\#(I'') = 1$.*

Proof. — By the previous lemma, we may assume that $Z = D_J$ (for J non-empty and not necessarily a singleton) $I' = \emptyset$ and $\#(I'') = 1$. Let $h : \mathscr{X}' \to \mathscr{X}$ be the admissible blow-up of $\mathscr{X}$ at Z and $E \subset \mathscr{X}'_\sigma$ its exceptional divisor. Then the morphism $\mathrm{M}_{rig}(]Z \smallsetminus D(I'')[) \to \mathrm{M}_{rig}(]Z[)$ identifies with

$$\mathrm{M}_{rig}(]E \smallsetminus h^{-1}(D(I''))[) \to \mathrm{M}_{rig}(]E[).$$

But, if $I'' = \{i\}$, then $h^{-1}(D_i)$ is simply the strict transform of D_i and hence is an irreducible divisor of $\mathscr{X}'_\sigma$. This enables us to conclude. □

By Lemmas 5.3 and 5.4, we may assume that $I = \{1, \ldots, n\}$, $J = \{1\}$, $I' = \emptyset$ and $I'' = \{2\}$. We are thus left to show that

$$\mathrm{M}_{rig}(]D_1 \smallsetminus D_2[) = \mathrm{M}_{rig}(]D_1 \smallsetminus D_{1,2}[) \to \mathrm{M}_{rig}(]D_1[)$$

is an isomorphism in $\mathbf{RigSH}_{\mathfrak{M}}(K)$. (Recall that $D_{1,2} = D_1 \cap D_2$.) From now on, we argue by induction on the integer n. We use this to obtain the following reduction.

Lemma 5.5. — *To prove Theorem 5.1, it is enough to show that*

$$\mathrm{M}_{rig}(]D_1 \smallsetminus D_I[) \to \mathrm{M}_{rig}(]D_1[) \tag{9}$$

is an isomorphism in $\mathbf{RigSH}_{\mathfrak{M}}(K)$.

Proof. — Assume that (9) is an isomorphism. Thus, by the previous discussion, we are left to check that

$$\mathrm{M}_{rig}(]D_1 \smallsetminus D_{1,2}[) \to \mathrm{M}_{rig}(]D_1 \smallsetminus D_I[)$$

is an isomorphism. Note that $(\mathscr{X} \smallsetminus D_i)_{1 \leqslant i \leqslant n}$ is an open covering of the formal scheme $\mathscr{X} \smallsetminus D_I$. This induces admissible open coverings

$$(]D_1 \smallsetminus (D_{1,2} \cup D_i)[)_{2 \leqslant i \leqslant n} \qquad \text{and} \qquad (]D_1 \smallsetminus D_i[)_{2 \leqslant i \leqslant n}$$

of $]D_1 \smallsetminus D_{1,2}[$ and $]D_1 \smallsetminus D_I[$ respectively, where $D_{1,2} = D_1 \cap D_2$. Hence, thanks to Mayer–Vietoris distinguished triangles, it is enough to show that, for every integer i, $2 \leqslant i \leqslant n$, the morphism

$$\mathrm{M}_{rig}(]D_1 \smallsetminus (D_{1,2} \cup D_i)[) \to \mathrm{M}_{rig}(]D_1 \smallsetminus D_i[)$$

is invertible in $\mathbf{RigSH}_{\mathfrak{M}}(K)$. As the special fiber of $\mathscr{X} \smallsetminus D_i$ has $n-1$ irreducible components, we may use induction to conclude when $i \geqslant 3$. □

Before we give our final reduction, we note the following fact (where $\mathscr{X}$ is not necessarily the semi-stable formal R-scheme of Theorem 5.1).

Lemma 5.6. — *Let $\mathscr{X}$ be a formal R-scheme topologically of finite type and assume that $\mathscr{X}_\eta$ is smooth. Let $\mathrm{e} : \mathscr{X}' \to \mathscr{X}$ be an étale morphism of formal R-schemes. Let H and Z be closed subschemes of the special fiber $\mathscr{X}_\sigma$. Assume that the induced morphism $\mathrm{e}^{-1}(Z) \to Z$ is an isomorphism. Then, the following assertions are equivalent:*

1. *the morphism* $\mathrm{M}_{rig}(]H \smallsetminus Z[) \to \mathrm{M}_{rig}(]H[)$ *is an isomorphism;*

2. the morphism $\mathrm{M}_{rig}(]e^{-1}(H \smallsetminus Z)[) \to \mathrm{M}_{rig}(]e^{-1}(H)[)$ *is an isomorphism.*

Proof. — Let $\mathscr{U} = \mathscr{X} \smallsetminus Z$ and $\mathscr{U}' = \mathscr{X}' \smallsetminus e^{-1}(Z)$. Consider the commutative cube of rigid analytic varieties over K:

$$\begin{tikzcd}
]e^{-1}(H \smallsetminus Z)[\arrow[rr] \arrow[dr] \arrow[dd] & &]e^{-1}(H)[\arrow[dr] \arrow[dd, dotted] & \\
& \mathscr{U}'_\eta \arrow[rr] \arrow[dd] & & \mathscr{X}'_\eta \arrow[dd, "e_\eta"] \\
]H \smallsetminus Z[\arrow[rr, dotted] \arrow[dr] & &]H[\arrow[dr, dotted, "v"] & \\
& \mathscr{U}_\eta \arrow[rr] & & \mathscr{X}_\eta .
\end{tikzcd}$$

All the faces of this cube are cartesian squares, and the frontal face is, by [**6**, Proposition 1.2.23], a distinguished Nisnevich square of quasi-compact rigid analytic varieties over K (in the sense of [**6**, Définition 1.2.20]).

One has a morphism of distinguished triangles in $\mathbf{RigSH}_{\mathfrak{M}}(\mathscr{X}_\eta)$:

$$\begin{tikzcd}
\mathrm{M}_{\mathscr{X}_\eta, rig}(\mathscr{U}'_\eta) \arrow[r] \arrow[d] & \mathrm{M}_{\mathscr{X}_\eta, rig}(\mathscr{X}'_\eta) \arrow[r] \arrow[d] & \mathrm{M}_{\mathscr{X}_\eta, rig}(\mathscr{X}'_\eta / \mathscr{U}'_\eta) \arrow[r, "+1"] \arrow[d, "\sim"] & {} \\
\mathrm{M}_{\mathscr{X}_\eta, rig}(\mathscr{U}_\eta) \arrow[r] & \mathrm{M}_{\mathscr{X}_\eta, rig}(\mathscr{X}_\eta) \arrow[r] & \mathrm{M}_{\mathscr{X}_\eta, rig}(\mathscr{X}_\eta / \mathscr{U}_\eta) \arrow[r, "+1"] & {}
\end{tikzcd}$$

where the third vertical arrow is an isomorphism thanks to [**6**, Corollaire 1.2.27].

Denote $q :]H[\to \mathsf{Spm}(K)$ the structural morphism. Applying the functor $q_\sharp v^*$, and using [**6**, Lemme 1.4.32], we get a morphism of distinguished triangles in $\mathbf{RigSH}_{\mathfrak{M}}(K)$:

$$\begin{tikzcd}
\mathrm{M}_{rig}\left(]e^{-1}(H \smallsetminus Z)[\right) \arrow[r] \arrow[d] & \mathrm{M}_{rig}\left(]e^{-1}(H)[\right) \arrow[r] \arrow[d] & \mathrm{M}_{rig}\left(]e^{-1}(H)[\,/\,]e^{-1}(H \smallsetminus Z)[\right) \arrow[r, "+1"] \arrow[d, "\sim"] & {} \\
\mathrm{M}_{rig}\left(]H \smallsetminus Z[\right) \arrow[r] & \mathrm{M}_{rig}\left(]H[\right) \arrow[r] & \mathrm{M}_{rig}\left(]H[\,/\,]H \smallsetminus Z[\right) \arrow[r, "+1"] & {} .
\end{tikzcd}$$

That concludes the proof. □

Now using Lemma 5.6 and [**6**, Proposition 1.1.62], which relates general semi-stable formal R-schemes to standard semi-stable formal R-schemes (as in Example 2.7), we obtain the following final reduction.

Lemma 5.7. — *To prove Theorem 5.1, we may assume that* $\mathscr{X} = \mathrm{St}^{v}_{\mathscr{Y}, \underline{a}}$ *where* $\mathscr{Y}$ *is a smooth formal* R*-scheme,* $v \in t\mathcal{O}(\mathscr{Y})^\times$ *and* $\underline{a} = (a_1, \dots, a_n) \in (\mathbf{N}^\times)^n$*. Moreover, it is enough to show, in this case, that*

$$\mathrm{M}_{rig}(]D_1 \smallsetminus D_I[) \to \mathrm{M}_{rig}(]D_1[)$$

is an isomorphism in $\mathbf{RigSH}_{\mathfrak{M}}(K)$*. (Recall that* $I = \{1, \dots, n\}$*.)*

Proof. — Note that the morphism $\mathrm{M}_{rig}(]D_1 \smallsetminus D_I[) \to \mathrm{M}_{rig}(]D_1[)$ is a direct summand of the corresponding morphism for the formal R-scheme $\mathscr{X}\{T, T^{-1}\}$. Using [**6**, Proposition 1.1.62] and Mayer–Vietoris distinguished triangles, we may therefore assume that there exists an étale morphism of formal R-schemes

$$e : \mathscr{X} \to \mathrm{St}^{Ut}_{\mathsf{Spf}(R\{U,U^{-1}\}),\underline{a}}\{S_1, \ldots, S_r\},$$

where $\underline{a} = (a_1, \ldots, a_n) \in (\mathbf{N}^{\times})^n$ and $U, S_1, \ldots, S_r$ are independent variables. We denote by $\mathscr{S}$ the target of the morphism e; recall (from Example 2.7) that this formal R-scheme is given by

$$\mathscr{S} = \mathsf{Spf}\left(R\{U, U^{-1}, T_1, \ldots, T_n, S_1, \ldots, S_r\}/(T_1^{a_1} \cdots T_n^{a_n} - Ut)\right).$$

The formal R-scheme $\mathscr{S}$ is semi-stable and the irreducible components of $\mathscr{S}_\sigma$ are defined by the equations $T_i = 0$, for $1 \leqslant i \leqslant n$. We denote by C their intersection, i.e., the subscheme of $\mathscr{S}_\sigma$ given by the ideal $(T_1, \ldots, T_n)$. Clearly, we have $C = \mathsf{Spec}(k[U, U^{-1}, S_1, \ldots, S_r])$.

After reordering the irreducible components of $\mathscr{X}_\sigma$, we may assume that $D_i \subset \mathscr{X}_\sigma$ is given by the equation $T_i \circ e = 0$. The morphism e induces an étale morphism $e_0 : D_I \to C$. In fact, one has a cartesian square of formal R-schemes:

$$\begin{tikzcd} D_I \arrow[r] \arrow[d, "e_0"'] \arrow[dr, phantom, "\square"] & \mathscr{X} \arrow[d, "e"] \\ C \arrow[r] & \mathscr{S}. \end{tikzcd}$$

As in [**6**, Notation 1.2.35], we denote by $Q^{for}(C)$ the formal R-scheme given by the t-adic completion of the R-scheme $C \otimes_k R$. Since the morphism e_0 is étale, by Lemma 2.1 and [**15**, Lemma 1.2], the morphism of formal R-schemes

$$Q^{for}(e_0) : Q^{for}(D_I) \to Q^{for}(C) = \mathsf{Spf}(R\{U, U^{-1}, S_1, \ldots, S_r\})$$

is also étale and induces an étale morphism of standard schemes

$$e' : \mathscr{X}' = \mathrm{St}^{Ut}_{Q^{for}(D_I),\underline{a}} \to \mathscr{S} = \mathrm{St}^{Ut}_{\mathsf{Spf}(R\{U,U^{-1},S_1,\ldots,S_r\}),\underline{a}}.$$

Moreover, by construction, one has a cartesian square of formal R-schemes:

$$\begin{tikzcd} D_I \arrow[r] \arrow[d, "e_0"'] \arrow[dr, phantom, "\square"] & \mathscr{X}' \arrow[d, "e'"] \\ C \arrow[r] & \mathscr{S}. \end{tikzcd}$$

Now, consider the fiber product $\mathscr{X} \hat{\times}_{\mathscr{S}} \mathscr{X}'$. By construction, one has

$$(\mathscr{X} \hat{\times}_{\mathscr{S}} \mathscr{X}') \times_{\mathscr{S}} C \simeq D_I \times_C D_I.$$

As $e_0 : D_I \to C$ is étale, the diagonal embedding $D_I \hookrightarrow D_I \times_C D_I$ is an open and closed immersion and hence induces a decomposition $D_I \times_C D_I \simeq D_I \sqcup F$. We set

$$\mathscr{X}'' = (\mathscr{X} \hat{\times}_{\mathscr{S}} \mathscr{X}') \smallsetminus F.$$

By construction, one has étale morphisms

$$f : \mathscr{X}'' \to \mathscr{X} \qquad \text{and} \qquad f' : \mathscr{X}'' \to \mathscr{X}'$$

inducing isomorphisms $f^{-1}(D_I) \xrightarrow{\sim} D_I$ and $f'^{-1}(D_I) \xrightarrow{\sim} D_I$. Therefore, we can apply Lemma 5.6 twice:

- for $\mathscr{X}'' \to \mathscr{X}$ with $H = D_1$ and $Z = D_I$, and
- for $\mathscr{X}'' \to \mathscr{X}'$ with $H \subset \mathscr{X}'_\sigma$ given by the equation $T_1 = 0$ and $Z = D_I$.

This shows that to prove the property stated in Lemma 5.5 for $\mathscr{X}$, it is enough to prove it for $\mathscr{X}'$. As the latter is a standard semi-stable formal R-scheme, we are done. □

5.4. The case of a standard semi-stable formal R-scheme. — Here, we finish the proof of Theorem 5.1 by showing the property stated in Lemma 5.7. This property is obtained as a consequence of the following statement which is slightly more general than what is needed. Indeed, we are only concerned with the case where $\mathscr{Y}$ is smooth over R and $v \in t\mathcal{O}(\mathscr{Y})^\times$. However, the extra generality in the following statement gives a flexibility that we use in its proof.

Proposition 5.8. — *Let $\mathscr{Y}$ be a formal R-scheme topologically of finite type with smooth generic fiber. Let $v \in \sqrt{t\mathcal{O}(\mathscr{Y})}$ dividing a power of t. Let $\underline{a} = (a_1, \ldots, a_n)$ be an n-tuple of strictly positive integers. Let $\mathscr{X} = \mathrm{St}^v_{\mathscr{Y},\underline{a}}$ be the associated standard formal scheme. Let D be a branch of $\mathscr{X}_\sigma$ and D° the complement in D of the union of the remaining branches. Then the canonical morphism*

$$\mathrm{M}_{rig}(]D^\circ[) \to \mathrm{M}_{rig}(]D[)$$

is an isomorphism in $\mathbf{RigSH}_{\mathfrak{M}}(K)$.

We refer to subsection 1.5, or originally to [**6**, §1.1.2, Exemple 1.1.14, Exemple 1.1.15], for a definition of the notion of relative annulus which plays an important role in the proof of proposition 5.8.

Proof. — The condition that v divides a power of t ensures that $\mathscr{X}_\eta$ is a smooth rigid analytic variety over K. Therefore, the statement of the proposition makes sense.

We may assume that $D = D_1$, i.e., the branch of $\mathscr{X}_\sigma$ defined by the equation $T_1 = 0$ (see Example 2.7). When $n = 1$, there is nothing to prove. Thus, we may assume that $n \geqslant 2$. We split the proof in three parts.

Step 1. The case $n = 2$. — In this case, we have:

$$\mathscr{X} = \mathsf{Spf}\,\frac{\mathcal{O}_{\mathscr{Y}}\{T_1, T_2\}}{(T_1^{a_1}T_2^{a_2} - v)} \simeq \mathsf{Spf}\,\frac{\mathcal{O}_{\mathscr{Y}}\{w, T_1, T_2\}}{(w^e - v, T_1^{a_1/e}T_2^{a_2/e} - w)}$$

with e the greatest common divisor of a_1 and a_2. Replacing $\mathscr{Y}$ by $\mathsf{Spf}(\mathscr{O}_{\mathscr{Y}}\{w\}/w^e - v)$ and v by w, we may assume that a_1 and a_2 are coprime.

We fix a Bézout relation

$$a_1 d_1 + a_2 d_2 = -1$$

where $d_1 \geqslant 0$ and $d_2 < 0$ are relative integers. The equation $T_1^{a_1} T_2^{a_2} = v$ in $\mathcal{O}(\mathscr{X}_\eta)$ can be written as

$$(T_1^{-d_2} T_2^{d_1})^{a_2} = T_1 v^{d_1}.$$

This shows in particular that

$$|T_1^{-d_2} T_2^{d_1}|_\infty \leqslant |v|_\infty^{d_1/a_1} \leqslant 1.$$

(Here, $|\cdot|_\infty$ is the infinity norm computed on $\mathscr{X}_\eta$.) Using this, we may construct an isomorphism of rigid analytic varieties over $\mathscr{Y}_\eta$:

$$\mathscr{X}_\eta \xrightarrow{\sim} \mathsf{Spm}\, \frac{\mathcal{O}_{\mathscr{Y}_\eta}\{T, U, V\}}{(T^{a_1} U - v^{-d_2}, v^{d_1} V - T^{a_2})}$$

given, on the structural sheaves of functions, by $T \mapsto T_1^{-d_2} T_2^{d_1}$, $U \mapsto T_2$ and $V \mapsto T_1$. Compositing this isomorphism with the obvious open immersion

$$\mathsf{Spm}\, \frac{\mathcal{O}_{\mathscr{Y}_\eta}\{T, U, V\}}{(T^{a_1} U - v^{-d_2}, v^{d_1} V - T^{a_2})} \hookrightarrow \mathsf{Spm}(\mathcal{O}_{\mathscr{Y}_\eta}\{T\}),$$

yields an open immersion

$$j : \mathscr{X}_\eta \hookrightarrow \mathbf{B}^1_{\mathscr{Y}_\eta}$$

which identifies $\mathscr{X}_\eta$ with the relative annulus (aka., relative corona)

$$\mathbf{Cr}_{\mathscr{Y}_\eta}(o, |v|^{-d_2/a_1}, |v|^{d_1/a_2})$$

inside the relative ball $\mathbf{B}^1_{\mathscr{Y}_\eta}$. (Here, we are using the notation as in [**6**, Exemple 1.1.14].)

Now, by definition, $]D_1[= \{x \in \mathscr{X}_\eta\,;\ |T_1(x)|_\infty < 1\}$. Using that $T^{a_2} = v^{d_1} T_1$, we get an identification

$$]D_1[= \bigcup_{R \to 1^-} \mathbf{Cr}_{\mathscr{Y}_\eta}(o, |v|^{-d_2/a_1}, R \cdot |v|^{d_1/a_2}).$$

On the other hand, we have $]D_1^\circ[= \{x \in \mathscr{X}_\eta\,;\ |T_2(x)|_\infty = 1\}$. Using that $T^{a_1} T_2 = v^{-d_2}$, we get an identification

$$]D_1^\circ[= \partial \mathbf{B}^1_{\mathscr{Y}_\eta}(o, |v|^{-d_2/a_1}).$$

Thus, it is enough to show that the inclusion

$$\partial \mathbf{B}^1_{\mathscr{Y}_\eta}(o, |v|^{-d_2/a_1}) \hookrightarrow \mathbf{Cr}_{\mathscr{Y}_\eta}(o, |v|^{-d_2/a_1}, R|v|^{d_1/a_2})$$

induces an isomorphism in $\mathbf{RigSH}_{\mathfrak{M}}(K)$ for R close enough to 1. This is done in [**6**, Proposition 1.3.4].

Step 2. The case where $a_2 = \cdots = a_n = d$. — Here we treat the case of the standard scheme

$$\mathscr{X} = \mathrm{St}^v_{\mathscr{Y},(a_1,\underline{d}_{n-1})} = \mathsf{Spf}\,\frac{\mathcal{O}_{\mathscr{Y}}\{T_1,\ldots,T_n\}}{(T_1^{a_1}T_2^d\cdots T_n^d - v)}$$

and its branch D_1 defined by the equation $T_1 = 0$. (Above, $\underline{d}_{n-1}$ denotes the constant $(n-1)$-tuple with value $d \in \mathbf{N}^{\times}$.)

We argue by induction on the integer n. By the previous step, we may assume that $n \geqslant 3$. Consider the standard formal R-scheme

$$\mathscr{Z} = \mathrm{St}^v_{\mathscr{Y}\{T_n\},(a_1,\underline{d}_{n-2})} = \mathsf{Spf}\,\frac{\mathcal{O}_{\mathscr{Y}}\{T_n\}\{T_1,\ldots,T_{n-1}\}}{(T_1^{a_1}T_2^d\cdots T_{n-1}^d - v)}$$

and its admissible blow-up $\mathscr{Z}'$ at the ideal (T_{n-1},T_n). The formal R-scheme $\mathscr{Z}'$ has an open covering given by the following two open formal subschemes:

$$\mathsf{Spf}(\mathcal{O}_{\mathscr{Z}}\{S_{n-1}\}/(T_{n-1}S_{n-1} - T_n)) = \mathsf{Spf}\,\frac{\mathcal{O}_{\mathscr{Y}}\{S_{n-1}\}\{T_1,\ldots,T_{n-1}\}}{(T_1^{a_1}T_2^d\cdots T_{n-1}^d - v)} \simeq \mathscr{Z}$$

and

$$\mathsf{Spf}(\mathcal{O}_{\mathscr{Z}}\{S_n\}/(T_nS_n - T_{n-1})) = \mathsf{Spf}\,\frac{\mathcal{O}_{\mathscr{Y}}\{T_1,\ldots,T_{n-2},S_n,T_n\}}{(T_1^{a_1}T_2^d\cdots T_{n-2}^dS_n^dT_n^d - v)} \simeq \mathscr{X}.$$

Their intersection is given by

$$\mathscr{W} = \mathsf{Spf}\,\frac{\mathcal{O}_{\mathscr{Y}}\{S_{n-1},S_{n-1}^{-1}\}\{T_1,\ldots,T_{n-1}\}}{(T_1^{a_1}T_2^d\cdots T_{n-1}^d - v)}.$$

Let's denote by $]D_1[_{\mathscr{Z}_\eta}$ (resp. $]D_1[_{\mathscr{X}_\eta}$, etc) the tube, taken in $\mathscr{Z}_\eta$ (resp. $\mathscr{X}_\eta$, etc), of the branch D_1 defined by the equation $T_1 = 0$. We use similar notations with D_1° instead of D_1. We then have

$$]D_1[_{\mathscr{Z}'_\eta} =]D_1[_{\mathscr{Z}_\eta} \cup]D_1[_{\mathscr{X}_\eta} \qquad \text{and} \qquad]D_1[_{\mathscr{W}_\eta} =]D_1[_{\mathscr{Z}_\eta} \cap]D_1[_{\mathscr{X}_\eta},$$

and similarly

$$]D_1^\circ[_{\mathscr{Z}'_\eta} =]D_1^\circ[_{\mathscr{Z}_\eta} \cup]D_1^\circ[_{\mathscr{X}_\eta} \qquad \text{and} \qquad]D_1^\circ[_{\mathscr{W}_\eta} =]D_1^\circ[_{\mathscr{Z}_\eta} \cap]D_1^\circ[_{\mathscr{X}_\eta}.$$

Now, by the induction hypothesis, the conclusion of the proposition holds for the standard formal schemes $\mathscr{Z}$ and $\mathscr{W}$ and their branches D_1. On the other hand, the blow-up morphism $\mathscr{Z}' \to \mathscr{Z}$ induces isomorphism $]D_1[_{\mathscr{Z}'_\eta} \simeq]D_1[_{\mathscr{Z}_\eta}$ and $]D_1^\circ[_{\mathscr{Z}'_\eta} \simeq]D_1^\circ[_{\mathscr{Z}_\eta}$. Using Mayer–Vietoris distinguished triangles, the conclusion of the proposition follows now for $\mathscr{X}$ and its branch D_1.

Step 3. The general case. — We will use the same trick as in the proof of [**6**, Lemme 1.2.38]. Namely, we blow-up intersections of two components to increase the multiplicities and reduce the general case to the one treated in Step 2. We will argue by induction on the n-tuple $\underline{a}$.

By the previous step, we may assume that $(a_2,\ldots,a_n)$ is not constant. Let $i,j \in \{2,\ldots,n\}$ such that $a_i \neq a_j$. We may assume that $a_i > a_j$. Let $\underline{b} = (b_1,\ldots,b_n)$ be the n-tuple given by $b_r = a_r$ for $r \neq i$ and $b_i = a_i - a_j$. Also, let $\underline{a}'$ be the n-tuple given by $a'_r = a_r$ for $r \notin \{i, j\}$, $a'_i = a_i - a_j$ and $a'_j = a_i$.

Consider the standard formal R-scheme

$$\mathscr{Z} = \mathrm{St}^v_{\mathscr{Y},\underline{b}} = \mathsf{Spf}\, \frac{\mathcal{O}_{\mathscr{Y}}\{T_1, \dots, T_n\}}{(T_1^{b_1} \cdots T_n^{b_n} - v)}$$

and its admissible blow-up $\mathscr{Z}'$ at the ideal (T_i, T_j). The formal R-scheme $\mathscr{Z}'$ has an open covering given by the following two open formal subschemes:

$$\mathsf{Spf}(\mathcal{O}_{\mathscr{Z}}\{S_j\}/(T_iS_j - T_j)) \simeq \mathrm{St}^v_{\mathscr{Y},\underline{a}} \quad \text{and} \quad \mathsf{Spf}(\mathcal{O}_{\mathscr{Z}}\{S_i\}/(T_jS_i - T_i)) \simeq \mathrm{St}^v_{\mathscr{Y},\underline{a}'}.$$

We identify $\mathscr{X}$ with the first open formal subscheme and we denote by $\mathscr{V}$ the second one. The intersection $\mathscr{W} = \mathscr{X} \cap \mathscr{V}$ is given by

$$\mathsf{Spf}\, \frac{\mathcal{O}_{\mathscr{Y}}\{S_j, S_j^{-1}\}\{T_1, \dots, T_{j-1}, T_{j+1}, \dots, T_n\}}{(T_1^{a_1} \cdots T_{j-1}^{a_{j-1}} T_{j+1}^{a_{j+1}} \cdots T_n^{a_n} - vS_j^{-a_j})}.$$

Hence, $\mathscr{W}$ is a standard formal R-scheme of length $n-1$.

Now, using the same notation as in Step 2, we have

$$]D_1[_{\mathscr{Z}'_\eta} =]D_1[_{\mathscr{X}_\eta} \cup]D_1[_{\mathscr{V}_\eta} \qquad \text{and} \qquad]D_1[_{\mathscr{W}_\eta} =]D_1[_{\mathscr{X}_\eta} \cap]D_1[_{\mathscr{V}_\eta},$$

and similarly

$$]D_1^\circ[_{\mathscr{Z}'_\eta} =]D_1^\circ[_{\mathscr{X}_\eta} \cup]D_1^\circ[_{\mathscr{V}_\eta} \qquad \text{and} \qquad]D_1^\circ[_{\mathscr{W}_\eta} =]D_1^\circ[_{\mathscr{X}_\eta} \cap]D_1^\circ[_{\mathscr{V}_\eta}.$$

Moreover, the blow-up morphism $\mathscr{Z}' \to \mathscr{Z}$ induces isomorphisms $]D_1[_{\mathscr{Z}'_\eta} \simeq]D_1[_{\mathscr{Z}_\eta}$ and $]D_1^\circ[_{\mathscr{Z}'_\eta} \simeq]D_1^\circ[_{\mathscr{Z}_\eta}$. Using Mayer–Vietoris distinguished triangles and induction, one gets that

$$\mathrm{M}_{rig}(]D_1^\circ[_{\mathscr{X}_\eta}) \oplus \mathrm{M}_{rig}(]D_1^\circ[_{\mathscr{V}_\eta}) \to \mathrm{M}_{rig}(]D_1[_{\mathscr{X}_\eta}) \oplus \mathrm{M}_{rig}(]D_1[_{\mathscr{V}_\eta})$$

is an isomorphism. This finishes the proof. $\square$

We finish this subsection by indicating how to deduce the property stated in Lemma 5.7 from Proposition 5.8.

Let $\mathscr{Y}$ be a smooth formal R-scheme, $v \in t\mathcal{O}(\mathscr{Y})^\times$ and $\underline{a} = (a_1, \dots, a_n) \in (\mathbf{N}^\times)^n$. Let $\mathscr{X} = \mathrm{St}^v_{\mathscr{Y},\underline{a}}$ be the associated standard formal R-scheme. Let D_I, with $I = \{1, \dots, n\}$, be the intersection of all branches in $\mathscr{X}$ and let $D = D_1$ be the branch given by the equation $T_1 = 0$. We need to show that $]D \smallsetminus D_I[\hookrightarrow]D[$ induces and isomorphism in $\mathbf{RigSH}_{\mathfrak{M}}(K)$.

By Proposition 5.8, one has an isomorphism in $\mathbf{RigSH}_{\mathfrak{M}}(K)$:

$$\mathrm{M}_{rig}(]D^\circ[) \xrightarrow{\sim} \mathrm{M}_{rig}(]D[). \tag{10}$$

On the other hand, for every $2 \leqslant i \leqslant n$, the formal R-scheme $\mathscr{X} \smallsetminus D_i$ is isomorphic to standard formal R-scheme of length $n-1$. Applying Proposition 5.8 to it and its branch $D_1 \smallsetminus D_i$, yields an isomorphism in $\mathbf{RigSH}_{\mathfrak{M}}(K)$:

$$\mathrm{M}_{rig}(]D^\circ[) \xrightarrow{\sim} \mathrm{M}_{rig}(]D \smallsetminus D_i[). \tag{11}$$

Using induction and Mayer–Vietoris distinguished triangles, the isomorphisms (11) can be "glued" to produce an isomorphism in $\mathbf{RigSH}_{\mathfrak{M}}(K)$:

$$\mathrm{M}_{rig}(]D^{\circ}[) \xrightarrow{\sim} \mathrm{M}_{rig}(]D \smallsetminus \cap_{i=2}^{n} D_i[) = \mathrm{M}_{rig}(]D \smallsetminus D_I[).$$

Combining this with the isomorphism (10) gives the required isomorphism. This finishes the proof of Theorem 5.1.

5.5. A consequence on motives of tube. — We finish this section with the following application.

Proposition 5.9. — *Let $\mathscr{X}$ be a formal R-scheme topologically of finite type and let $Z \subset \mathscr{X}_{\sigma}$ be a locally closed subset. Assume that $\mathscr{X}_{\eta}$ is smooth over K. Then, the rigid motive* $\mathrm{M}_{rig}(]Z[)$ *is a compact object of* $\mathbf{RigSH}_{\mathfrak{M}}(K)$.

Proof. — By resolution of singularities, we may find an admissible blow-up $e : \mathscr{X}' \to \mathscr{X}$ with $\mathscr{X}'$ a semi-stable formal R-scheme and such that $Z' = e^{-1}(Z)$ is a union of irreducible components of $(\mathscr{X}'_{\sigma})_{\mathrm{red}}$. As e induces an isomorphism of rigid analytic varieties $]Z'[\simeq]Z[$, we may assume from the beginning that $\mathscr{X}$ is a semi-stable formal R-scheme and Z is a union of irreducible components of $(\mathscr{X}_{\sigma})_{\mathrm{red}}$.

Denote $(D_i)_{i\in I}$ the irreducible components of $(\mathscr{X}_{\sigma})_{\mathrm{red}}$ and let $J \subset I$ be the subset such that $Z = D(J) = \cup_{j\in J} D_j$. By Corollary 5.2, the obvious inclusion $D(J)^{\circ} \hookrightarrow D(J)$ induces an isomorphism in $\mathbf{RigSH}_{\mathfrak{M}}(K)$:

$$\mathrm{M}_{rig}(]D(J)^{\circ}[) \xrightarrow{\sim} \mathrm{M}_{rig}(]D(J)[).$$

Now, $D(J)^{\circ} = (\mathscr{X}_{\sigma})_{\mathrm{red}} \smallsetminus D(I \smallsetminus J)$ is an open subset of $(\mathscr{X}_{\sigma})_{\mathrm{red}}$ and hence its tube $]D(J)^{\circ}[$ is quasi-compact. Therefore, the rigid motive of $]D(J)^{\circ}[$ is a compact object by [**6**, Corollaire 1.3.21]. This finishes the proof. □

6. Nearby motivic sheaves in a semi-stable situation

The goal of this section is to prove Theorem 6.1 that is the analog of Theorem 5.1 for nearby motivic sheaves. The proofs of both theorems share some similarities but differ at a crucial point, namely, at the treatment of the case of a standard space of length 2. For Theorem 6.1, this case will be treated using Theorem 4.1.

6.1. Statement of the results. — Let X be a semi-stable R-scheme. We denote by $(D_i)_{i\in I}$ the irreducible components of $(X_{\sigma})_{\mathrm{red}}$. Given a subset $J \subset I$, denote by D_J and $D(J)$ the reduced closed subschemes of X_{σ} given by

$$D_J = \cap_{i\in J} D_i \qquad \text{and} \qquad D(J) = \cup_{i\in J} D_i$$

with the convention that $D_{\emptyset} = (X_{\sigma})_{\mathrm{red}}$ and $D(\emptyset) = \emptyset$.

Fix a subset $J \subset I$ and let Z be a closed subscheme of $D(J)$. For $I' \subset I \smallsetminus J$, we set

$$Z^{\circ}_{I'} = Z \smallsetminus D(I')$$

and denote by $v_{Z,I'} : Z^{\circ}_{I'} \hookrightarrow Z$ the obvious inclusion. When $I' = I \smallsetminus J$, we simply write Z° and v_Z instead of $Z^{\circ}_{I \smallsetminus J}$ and $v_{Z,I\smallsetminus J}$.

Theorem 6.1. — *Keep the notation as before. Assume that Z is a union of closed subschemes of the form $D_{J'}$, for some $\emptyset \neq J' \subset J$. Let M be an object of $\mathbf{SH}_{\mathfrak{M}}(K)$. Then, for $I' \subset I \smallsetminus J$, the canonical morphism*

$$(\Psi_f f^*_\eta(M))|_Z \to (v_{Z,I'})_*(v_{Z,I'})^*(\Psi_f f^*_\eta(M))|_Z \tag{12}$$

is an isomorphism in $\mathbf{SH}_{\mathfrak{M}}(Z)$.

Later, we only need the following particular case of Theorem 6.1.

Corollary 6.2. — *Keep the notation as before. Let M be an object of $\mathbf{SH}_{\mathfrak{M}}(K)$. The canonical morphism*

$$(\Psi_f f^*_\eta(M))|_{D(J)} \to (v_{D(J)})_*(v_{D(J)})^*(\Psi_f f^*_\eta(M))|_{D(J)}$$

is an isomorphism in $\mathbf{SH}_{\mathfrak{M}}(D(J))$.

The scheme $D(J)$ being a union of irreducible components of the special fiber, it is rather natural, so as to prove the corollary, to try to use the Mayer–Vietoris triangles associated with this closed covering. However Corollary 6.2 is not the right statement to do so. This is exactly where actually proving Theorem 6.1 instead becomes handy.

Remark 6.3. — Note that Theorem 6.1 is a generalization of [**3**, Théorème 3.3.44], inspired by [**8**, Proposition 1.20]. Also, at least for the stable homotopical 2-functor $\mathbf{SH}_{\mathfrak{M}}(-)$ and the specialization system Ψ, it shows that the hypothesis of $\mathbf{Q}$-linearity and separatedness are not needed for the conclusion of [**3**, Théorème 3.3.44]. This answers affirmatively the question raised in [**3**, Remarque 3.3.26], at least for $\mathbf{SH}_{\mathfrak{M}}(-)$ and Ψ, and over some special bases.

6.2. Reductions. — We start with the following simple reduction.

Lemma 6.4. — *If the conclusion of Theorem 6.1 holds for $I \smallsetminus J$, then it holds for every $I' \subset I \smallsetminus J$.*

Proof. — Let $u : Z^{\circ} = Z^{\circ}_{I\smallsetminus J} \hookrightarrow Z^{\circ}_{I'}$ be the obvious open immersion. Then, $v_Z = v_{Z,I\smallsetminus J} = v_{Z,I'} \circ u$. We are assuming that there is an isomorphism

$$(\Psi_f(M|_{X_\eta}))|_Z \simeq (v_Z)_*(v_Z)^*(\Psi_f(M|_{X_\eta}))|_Z.$$

Therefore, to show that the canonical morphism

$$(\Psi_f(M|_{X_\eta}))|_Z \to (v_{Z,I'})_*(v_{Z,I'})^*(\Psi_f(M|_{X_\eta}))|_Z$$

is invertible, it is enough to show that the natural transformation

$$(v_Z)_* \to (v_{Z,I'})_*(v_{Z,I'})^*(v_Z)_*$$

is invertible. This is obvious since $v_Z = v_{Z,I'} \circ u$ and the counit $(v_{Z,I'})^*(v_{Z,I'})_* \to \mathrm{Id}$ is invertible. □

Lemma 6.5. — *It is enough to prove Theorem 6.1 when* $\#(J) = 1$ *(and hence* Z *is an irreducible component of* X_σ*) and* $I' = I \smallsetminus J$.

Proof. — We assume that the case $\#(J) = 1$ and $I' = I \smallsetminus J$ is settled and we explain how to prove the general case of Theorem 6.1. This will be done in two steps. We first deal with an intersection of components using a blow-up as in [**25**].

Step1. Assume $Z = D_J$ *and* $I' = I \smallsetminus J$. — We will prove the assertion by induction on the cardinal of J. The case $\#(J) = 1$ being settled by assumption, we may assume $\#(J) \geqslant 2$. Consider $h : Y \to X$ the blow-up of X with center Z and let E be its exceptional divisor. The reduced special fiber $(Y_\sigma)_{\mathrm{red}}$ of the R-scheme Y is again a simple normal crossings divisor in Y, whose irreducible components are the closed subscheme E and the strict transforms of the D_i's, for $i \in I$ (e.g., see [**28**, Lemma 8.1.2]). In accordance with the notation in §6.1, we denote by E° the open subscheme of E defined as the complement in E of all the strict transforms of the D_i's. We have the following commutative diagram

$$\begin{tikzcd} E^\circ \arrow[r, "v"] \arrow[d, "q"] & E \arrow[r, "e"] \arrow[d, "p"] \arrow[dr, phantom, "\square"] & Y_\sigma \arrow[d, "h_\sigma"] \\ D_J^\circ \arrow[r, "v_{D_J}"] & D_J \arrow[r, "z"] & X_\sigma, \end{tikzcd} \tag{13}$$

with a cartesian square on the right (but not on the left).

By our assumption (applied to the R-scheme Y and the component E), the canonical morphism

$$e^*\Psi_{f\circ h}(M|_{Y_\eta}) \to v_*v^*e^*\Psi_{f\circ h}(M|_{Y_\eta}) \tag{14}$$

is an isomorphism in $\mathbf{SH}_{\mathfrak{M}}(E)$. By applying the third property of [**3**, Définition 3.1.1 (**SPE2**)] to the projective morphism h, we see that the morphism

$$\Psi_f(M|_{X_\eta}) \to (h_\sigma)_*\Psi_{f\circ h}(M|_{Y_\eta}) \tag{15}$$

is an isomorphism in $\mathbf{SH}_{\mathfrak{M}}(X_\sigma)$. Using these isomorphisms and the base-change for projective morphism [**2**, Corollaire 1.7.18] applied to the cartesian square (i.e., the right square) in (13), we obtain the following chain of canonical isomorphisms:

$$\begin{aligned} z^*\Psi_f(M|_{X_\eta}) \simeq z^*(h_\sigma)_*\Psi_{f\circ h}(M|_{Y_\eta}) \simeq p_*e^*\Psi_{f\circ h}(M|_{Y_\eta}) \simeq p_*v_*v^*e^*\Psi_{f\circ h}(M|_{Y_\eta}) \\ \simeq (v_{D_J})_*q_*v^*e^*\Psi_{f\circ h}(M|_{Y_\eta}). \end{aligned}$$

Therefore, to show our claim, it is enough to check that the canonical morphism

$$(v_{D_J})_*M \to (v_{D_J})_*(v_{D_J})^*(v_{D_J})_*M$$

is invertible for $M = q_* v^* e^* \Psi_{f \circ h}(M|_{Y_\eta})$. But, this is obviously true for any $M \in \mathbf{SH}_{\mathfrak{M}}(D_J^\circ)$.

Step 2. End of the proof. — We consider now the general case. If $Z = \emptyset$, there is nothing to be proven. Hence, we may assume that $Z \neq \emptyset$ (which forces that $J \neq \emptyset$).

The closed subscheme Z is then of the form $Z = D_{J_1} \cup \cdots \cup D_{J_n}$ for some integer $n \geqslant 1$ where $\emptyset \neq J_i \subset J$ for $1 \leqslant i \leqslant n$. For $n = 1$, the result follows from the first step and Lemma 6.4. Let us prove the result by induction on n. If $n \geqslant 2$, we may then write $Z = Z_1 \cup Z_2$ where

$$Z_1 = D_{J_1} \cup \cdots \cup D_{J_{n-1}} \quad \text{and} \quad Z_2 = D_{J_n}.$$

Let $i_1 : Z_1 \hookrightarrow Z$, $i_2 : Z_2 \hookrightarrow Z$ be the obvious inclusions and denote by $i : W \hookrightarrow Z$ the inclusion of the intersection $W = Z_1 \cap Z_2$. Using the Mayer–Vietoris distinguished triangle, associated with the closed covering $Z = Z_1 \cup Z_2$, we obtain a morphism of distinguished triangles:

$$\begin{tikzcd}
(\Psi_f(M|_{X_\eta}))|_Z \arrow[r, "(12)"] \arrow[d] & (v_{Z,I'})_*(v_{Z,I'})^*(\Psi_f(M|_{X_\eta}))|_Z \arrow[d] \\
\begin{matrix} (i_1)_*(i_1)^*(\Psi_f(M|_{X_\eta}))|_Z \\ \oplus \\ (i_2)_*(i_2)^*(\Psi_f(M|_{X_\eta}))|_Z \end{matrix} \arrow[r] \arrow[d] & \begin{matrix} (v_{Z,I'})_*(v_{Z,I'})^*(i_1)_*(i_1)^*(\Psi_f(M|_{X_\eta}))|_Z \\ \oplus \\ (v_{Z,I'})_*(v_{Z,I'})^*(i_2)_*(i_2)^*(\Psi_f(M|_{X_\eta}))|_Z \end{matrix} \arrow[d] \\
i_*i^*(\Psi_f(M|_{X_\eta}))|_Z \arrow[r] \arrow[d, "+1"] & (v_{Z,I'})_*(v_{Z,I'})^*i_*i^*(\Psi_f(M|_{X_\eta}))|_Z. \arrow[d, "+1"] \\
{} & {}
\end{tikzcd}$$

Note that W is also a union of $n-1$ subschemes of the form $D_{J'}$ for some $\emptyset \neq J' \subset J$. Therefore one sees that (12) is an isomorphism by induction on n using the following remark. □

Remark 6.6. — Let $Z' \subset Z$ be a closed subscheme and assume that the canonical morphism

$$(\Psi_f(M|_{X_\eta}))|_{Z'} \to (v_{Z',I'})^*(v_{Z',I'})_*(\Psi_f(M|_{X_\eta}))|_{Z'} \tag{16}$$

is an isomorphism in $\mathbf{SH}_{\mathfrak{M}}(Z')$. Then the canonical morphism

$$i_*i^*(\Psi_f(M|_{X_\eta}))|_Z \to (v_{Z,I'})_*(v_{Z,I'})^*i_*i^*(\Psi_f(M|_{X_\eta}))|_Z \tag{17}$$

is also an isomorphism in $\mathbf{SH}_{\mathfrak{M}}(Z)$ with $i : Z' \hookrightarrow Z$ is the obvious inclusion. This follows immediately using base-change for projective morphisms (in fact closed immersions) applied to the cartesian square

$$\begin{tikzcd} (Z')^{\circ}_{I'} \arrow[r, "v_{Z',I'}"] \arrow[d, "i^{\circ}_{I'}"'] \arrow[dr, phantom, "\square"] & Z' \arrow[d, "i"] \\ Z^{\circ}_{I'} \arrow[r, "v_{Z,I'}"] & Z. \end{tikzcd} \tag{18}$$

Now using [**3**, Proposition 3.3.39] that relates semi-stable R-schemes to standard semi-stable R-schemes (as in Example 2.7), we obtain the following further reduction.

Lemma 6.7. — *To prove Theorem 6.1, we may assume that X is the standard semi-stable R-scheme*

$$\mathrm{St}^{Ut}_{R[U,U^{-1}],\underline{a}} = \mathsf{Spec}\,\frac{R[U,U^{-1},T_1,\dots,T_n]}{(T_1^{a_1}\cdots T_n^{a_n}-Ut)}$$

where $\underline{a}=(a_1,\dots,a_n)\in(\mathbf{N}^{\times})^n$. Moreover, in this case, it is enough to show that

$$\Psi_f(M|_{X_\eta})|_{D_1} \to (v_{D_1})_*(v_{D_1})^*\Psi_f(M|_{X_\eta})|_{D_1} \tag{19}$$

is an isomorphism in $\mathbf{SH}_{\mathfrak{M}}(D_1)$.

Proof. — The problem is local for the Zariski topology and we may replace X by the R-scheme $X[T,T^{-1}]$. Using [**3**, Proposition 3.3.39], we can assume that there exists a smooth morphism of R-schemes

$$h : X \to S = \mathrm{St}^{Ut}_{R[U,U^{-1}],\underline{a}},$$

for some $\underline{a}=(a_1,\dots,a_n)\in(\mathbf{N}^{\times})^n$. Using base-change by a smooth morphism and the second property of [**3**, Définition 3.1.1 (**SPE2**)], one sees easily that the morphism

$$\Psi_f(M|_{X_\eta})|_{D_1} \to (v_{D_1})_*(v_{D_1})^*\Psi_f(M|_{X_\eta})|_{D_1}$$

identifies with the inverse image along h_σ of the corresponding morphism for the R-scheme $\mathrm{St}^{Ut}_{R[U,U^{-1}],\underline{a}}$. This finishes the proof. □

Our final reduction is the following.

Lemma 6.8. — *To prove Theorem 6.1 it is enough to show the case $n=2$ of the property stated in Lemma 6.7. More precisely, it suffices to show that* (19) *is an isomorphism for the standard semi-stable R-scheme of length* 2*:*

$$\mathrm{St}^{Ut}_{R[U,U^{-1}],a_1,a_2} = \mathsf{Spec}\,\frac{R[U,U^{-1},T_1,T_2]}{(T_1^{a_1}T_2^{a_2}-Ut)}$$

where $a_1,a_2\in\mathbf{N}^{\times}$.

Proof. — We need to prove the property stated in Lemma 6.7 assuming that it holds for $n=2$. We argue by induction on $n\geqslant 3$. We split the proof in two steps. (These steps correspond to Step 2 and 3 of the proof of Proposition 5.8.)

Step 1. The case where $a_2 = \cdots = a_n = d$. — Using the same method as in the proof of [**3**, Théorème 3.3.10], we will treat in this step the case of the standard semi-stable R-scheme

$$X = \mathrm{St}^{Ut}_{R[U,U^{-1}],(a_1,\underline{d}_{n-1})} = \mathsf{Spec}\,\frac{R[U,U^{-1},T_1,\ldots,T_n]}{(T_1^{a_1}T_2^d\cdots T_n^d - Ut)}.$$

(Above, $\underline{d}_{n-1}$ denotes the constant $(n-1)$-tuple with value $d \in \mathbf{N}^\times$.) Recall that D_1 is the branch defined by the equation $T_1 = 0$. We denote by $f : X \to \mathsf{Spec}(R)$ the structural morphism.

As $n \geqslant 3$, we may consider the standard semi-stable R-scheme

$$Z = \mathrm{St}^{Ut}_{R[U,U^{-1},T_n],(a_1,\underline{d}_{n-2})} = \mathsf{Spec}\,\frac{R[U,U^{-1},T_1,\ldots,T_n]}{(T_1^{a_1}T_2^d\cdots T_{n-1}^d - Ut)}$$

and its admissible blow-up Z' at the ideal (T_{n-1},T_n). The R-scheme Z' has an open covering given by the following two open subschemes:

$$\mathsf{Spec}(\mathcal{O}_Z\{S_{n-1}\}/(T_{n-1}S_{n-1} - T_n)) = \mathsf{Spec}\,\frac{R[U,U^{-1},S_{n-1},T_1,\ldots,T_{n-1}]}{(T_1^{a_1}T_2^d\cdots T_{n-1}^d - Ut)} \simeq Z$$

and

$$\mathsf{Spec}(\mathcal{O}_Z\{S_n\}/(T_nS_n - T_{n-1})) = \mathsf{Spec}\,\frac{R[U,U^{-1},T_1,\ldots,T_{n-2},S_n,T_n]}{(T_1^{a_1}T_2^d\cdots T_{n-2}^dS_n^dT_n^d - v)} \simeq X.$$

In particular, one has an open immersion $X \hookrightarrow Z'$.

Let $E_1' \subset (Z'_\sigma)_{\mathrm{red}}$ be the irreducible component defined by the equation $T_1 = 0$ and let $E_1'^{\circ}$ be the complement in E_1' of the union of the remaining irreducible components. Denote $K_{Z'}$ the cone of the morphism

$$(\Psi_{g'}(M|_{Z'_\eta}))|_{E_1'} \to (v_{E_1'})_*(v_{E_1'})^*(\Psi_{g'}(M|_{Z'_\eta}))|_{E_1'}$$

(where $g' : Z' \to \mathsf{Spec}(R)$ is the structural morphism). Also let K_X be the similar cone where g', Z' and E_1' are replaced by f, X and D_1

We need to prove that $K_X = 0$. As K_X is isomorphic to the restriction of $K_{Z'}$ to the open subset $D_1 \subset E_1'$, it is enough to show that $K_{Z'} = 0$.

Let C be the intersection of all branches in X, i.e., the closed subset of X_σ defined by the ideal $(T_1,\ldots,T_n)$. Denote also by C its image along the inclusion $X \hookrightarrow Z'$. This is also a closed subset of Z'_σ. Moreover, $Z' \smallsetminus C$ can be covered by standard semi-stable R-schemes of length at most $n-1$. This shows that $(K_{Z'})|_{E_1'\smallsetminus C} = 0$, i.e., $K_Z \in \mathbf{SH}_{\mathfrak{M}}(E_1')$ is supported on C.

Now, let $h : Z' \to Z$ be the blow-up morphism. We have a commutative diagram with cartesian squares:

$$\begin{tikzcd}
E_1'^{\circ} \arrow[r, "v_{E_1'}"] \arrow[d, "h_1^{\circ}"] \arrow[dr, phantom, "\square"] & E_1' \arrow[r] \arrow[d, "h_1"] \arrow[dr, phantom, "\square"] & Z'_\sigma \arrow[r, "i"] \arrow[d, "h_\sigma"] \arrow[dr, phantom, "\square"] & Z' \arrow[d, "h"] \\
E_1^{\circ} \arrow[r, "v_{E_1}"] & E_1 \arrow[r] & Z_\sigma \arrow[r, "i"] & Z.
\end{tikzcd} \tag{20}$$

(Again, E_1 is the irreducible component of Z_σ defined by the equation $T_1 = 0$ and E_1° is complement in E_1 of the union of the remaining irreducible components.) It is easy to see that h_σ induces an isomorphism $C \simeq h_\sigma(C)$. Therefore, as $K_{Z'}$ is supported over C, it is enough to show that $(h_1)_* K_{Z'} = 0$. Equivalently, we will show that

$$(h_1)_*(\Psi_{g'}(M|_{Z'_\eta}))|_{E'_1} \to (h_1)_*(v_{E'_1})_*(v_{E'_1})^*(\Psi_{g'}(M|_{Z'_\eta}))|_{E'_1}$$

is an isomorphism. Using base-change for projective morphisms [**2**, Corollaire 1.7.18] and the third property of [**3**, Définition 3.1.1 (**SPE2**)], one easily sees that the above morphism identifies with

$$(\Psi_g(M|_{Z_\eta}))|_{E_1} \to (v_{E_1})_*(v_{E_1})^*(\Psi_g(M|_{Z_\eta}))|_{E_1}.$$

As Z is a standard semi-stable R-scheme with $n-1$ branches, we may use induction to conclude.

Step 2. The general case. — The argument below is based on a trick used in the proofs of [**3**, Théorèmes 3.3.4 et 3.3.6]. It consists of blowing-up intersections of two components to increase the multiplicities and reduce the general case to the one treated in Step 1. We will argue by induction on $|\underline{a}| = a_1 + \cdots + a_n$.

By the previous step, we may assume that $(a_2, \ldots, a_n)$ is not constant. Let $i, j \in \{2, \ldots, n\}$ such that $a_i \neq a_j$. We may assume that $a_i > a_j$. Let $\underline{b} = (b_1, \ldots, b_n)$ be the n-tuple given by $b_r = a_r$ for $r \neq i$ and $b_i = a_i - a_j$. Also, let $\underline{a}'$ be the n-tuple given by $a'_r = a_r$ for $r \notin \{i, j\}$, $a'_i = a_i - a_j$ and $a'_j = a_i$.

Consider the standard semi-stable R-scheme

$$Z = \mathrm{St}^{Ut}_{R[U,U^{-1}],\underline{b}} = \mathsf{Spec}\,\frac{R[U, U^{-1}, T_1, \ldots, T_n]}{(T_1^{b_1} \cdots T_n^{b_n} - Ut)}.$$

As $|\underline{b}| < |\underline{a}|$, we may assume by induction that the result is known for Z. Let Z' be the blow-up of Z at the ideal (T_i, T_j). The R-scheme Z' has an open covering given by the following two open formal subschemes:

$$\begin{aligned} \mathsf{Spec}(\mathcal{O}_Z\{S_j\}/(T_iS_j - T_j)) \quad &\simeq \quad \mathrm{St}^{Ut}_{R[U,U^{-1}],\underline{a}} \\ \text{and} \quad \mathsf{Spec}(\mathcal{O}_Z\{S_i\}/(T_jS_i - T_i)) \quad &\simeq \quad \mathrm{St}^{Ut}_{R[U,U^{-1}],\underline{a}'}. \end{aligned}$$

We identify X with the first open subscheme and we denote by V the second one.

Let E'_1, E'°_1 and $K_{Z'}$ be as in Step 1. Again, the restriction of $K_{Z'}$ to D_1 (viewed as an open subscheme of E'_1 thanks to the inclusion $X \hookrightarrow Z'$) is isomorphic to K_X. Therefore, it is enough to show that $K_{Z'} = 0$.

Let $C_X \subset (X_\sigma)_{\mathrm{red}}$ (resp. $C_V \subset (V_\sigma)_{\mathrm{red}}$) be the intersection of the n irreducible components of $(X_\sigma)_{\mathrm{red}}$ (resp. of $(V_\sigma)_{\mathrm{red}}$). Then the map $X \sqcup V \to Z'$ identifies $C = C_X \sqcup C_V$ with a closed subset of Z'_σ. Moreover, $Z \smallsetminus C$ can be covered by standard semi-stable R-schemes of length at most $n-1$. Therefore, by induction on n, one gets that $(K_{Z'})|_{E'_1 \smallsetminus C} = 0$, i.e., $K_{Z'}$ is supported at C.

Now, the blow-up morphism $h : Z' \to Z$ induces isomorphisms $C_X \simeq h_\sigma(C)$ and $C_V \simeq h_\sigma(C)$. Therefore, it is enough to prove that $(h_1)_* K_{Z'} = 0$, with $h_1 : E_1' \to E_1$ the morphism induced by h. Finally, note that one also has a commutative diagram with cartesian squares as in (20). Using this, one can conclude exactly as we did in the last part of Step 1. □

6.3. The case of a standard semi-stable R-scheme of length 2. — In this subsection we finish the proof of Theorem 6.1 by showing the property stated in Lemma 6.8. We start with the following key observation.

Lemma 6.9. — *Let Y be a finite type R-scheme with smooth generic fiber. Let $v \in \sqrt{t\mathcal{O}(Y)}$ dividing a power of t. Let $X = \mathrm{St}^v_{Y,a_1,a_2}$ be the associated standard R-scheme of length 2. Let $f : X \to \mathsf{Spec}(R)$ and $q_1 : D_1^\circ \to \mathsf{Spec}(k)$ be the structural morphisms. Then the canonical morphism*

$$(f_\sigma)_* \Psi_f(M|_{X_\eta}) \to (q_1)_*(\Psi_f(M|_{X_\eta})|_{D_1^\circ})$$

is an isomorphism in $\mathbf{SH}_{\mathfrak{M}}(k)$. (As usual, D_1 is the branch given by the equation $T_1 = 0$ and $D_1^\circ = D_1 \smallsetminus D_2$ where D_2 is the branch given by the equation $T_2 = 0$.)

Proof. — The proof of this lemma makes use of Theorem 4.11.

As in Step 1 of the proof of Proposition 5.8, we may assume that a_1 and a_2 are coprime. (This will be needed later in the proof.) Let $\mathscr{X}$ be the t-adic completion of X and $\mathscr{U}$ the t-adic completion of $X \smallsetminus D_2$. Then $\mathscr{U}$ is an open formal subscheme of $\mathscr{X}$, and $\mathscr{X}_\sigma = X_\sigma$ and $\mathscr{U}_\sigma = D_1^\circ$. Using Theorem 4.11, the morphism we are interested in can be written as

$$1^* \circ \mathfrak{R}(\underline{\mathsf{Hom}}(\mathrm{M}_{rig}(\mathscr{X}_\eta), \mathrm{Rig}^*(M))) \to 1^* \circ \mathfrak{R}(\underline{\mathsf{Hom}}(\mathrm{M}_{rig}(\mathscr{U}_\eta), \mathrm{Rig}^*(M))).$$

Therefore, it suffices to show that

$$\mathrm{M}_{rig}(\mathscr{U}_\eta) \to \mathrm{M}_{rig}(\mathscr{X}_\eta)$$

is an isomorphism in $\mathbf{RigSH}_{\mathfrak{M}}(K)$.

Let $\mathscr{Y}$ be the t-adic completion of the R-scheme Y. Then $\mathscr{X}$ is the standard semi-stable formal R-scheme $\mathrm{St}^v_{\mathscr{Y},a_1,a_2}$. Now, the rigid analytic varieties $\mathscr{U}_\eta$ and $\mathscr{X}_\eta$ were identified in Step 1 of the proof of Proposition 5.8 with the following relative annulus and boundary of relative ball:

$$\mathbf{Cr}_{\mathscr{Y}_\eta}(o, |v|^{-d_2/a_1}, |v|^{d_1/a_2}) \qquad \text{and} \qquad \partial\mathbf{B}^1_{\mathscr{Y}_\eta}(o, |v|^{-d_2/a_1}).$$

Thus, it is enough to show that the inclusion

$$\partial\mathbf{B}^1_{\mathscr{Y}_\eta}(o, |v|^{-d_2/a_1}) \hookrightarrow \mathbf{Cr}_{\mathscr{Y}_\eta}(o, |v|^{-d_2/a_1}, |v|^{d_1/a_2})$$

induces an isomorphism in $\mathbf{RigSH}_{\mathfrak{M}}(K)$. This is done in [**6**, Proposition 1.3.4]. □

From Lemma 6.9, we deduce the following variant of what is needed.

Corollary 6.10. — *Let $a_1, a_2 \in \mathbf{N}^\times$ and $v \in R$ a uniformizing element (i.e., $v \in tR^\times$). Let $X = \mathrm{St}^v_{R,a_1,a_2}$ and denote by $f : X \to \mathsf{Spec}(R)$ the structural morphism. Then, the morphism*

$$(\Psi_f(M|_{X_\eta}))|_{D_1} \to (v_{D_1})_*(v_{D_1})^*(\Psi_f(M|_{X_\eta}))|_{D_1}$$

is an isomorphism.

Proof. — We split the proof in two steps.

Step 1.— For $i \in \{1, 2\}$, denote by $z_{D_i} : D_i \hookrightarrow X_\sigma$ the obvious inclusion. Consider the following morphism of distinguished triangles in $\mathbf{SH}_{\mathfrak{M}}(X_\sigma)$:

$$\begin{tikzcd}
N \arrow[r] \arrow[d] & \Psi_f(M|_{X_\eta}) \arrow[r] \arrow[d, equal] & (z_{D_1})_*(z_{D_1})^*\Psi_f(M|_{X_\eta}) \arrow[r, "+1"] \arrow[d] & {} \\
N' \arrow[r] & \Psi_f(M|_{X_\eta}) \arrow[r] & (z_{D_1})_*(v_{D_1})_*(v_{D_1})^*(z_{D_1})^*\Psi_f(M|_{X_\eta}) \arrow[r, "+1"] & {}
\end{tikzcd}$$

where the objects N and N' are defined (up to isomorphism) as the homotopy fibers (aka., shifted cone) of the horizontal arrows in the middle.

It is enough to show that $N \to N'$ is an isomorphism. Let $C = D_1 \cap D_2$. The third vertical arrow in the previous diagram is an isomorphism after restriction to $X_\sigma \smallsetminus C$. Thus, it is also the case for $N \to N'$. In other words, $\mathrm{Cone}(N \to N')$ is supported over C. As $C \simeq \mathsf{Spec}(k)$, we see that it suffices to show that

$$(f_\sigma)_*(N) \to (f_\sigma)_*(N')$$

is an isomorphism in $\mathbf{SH}_{\mathfrak{M}}(k)$. Now, by Lemma 6.9, we have $(f_\sigma)_*(N') = 0$. Hence, to finish the proof, we are left to show that $(f_\sigma)_*(N) = 0$. This will be done in the second step.

Step 2.— Using the localization triangle associated with the closed subset $D_1 \subset X_\sigma$ and its complement D_2°, one gets that:

$$N \simeq (z_{D_2})_*(v_{D_2})_!(\Psi_f(M|_{X_\eta}))|_{D_2^\circ}$$

where $(-)|_{D_2^\circ} = (z_{D_2} \circ v_{D_2})^*$. Therefore, one has:

$$(f_\sigma)_*(N) \simeq (p_2)_*(v_{D_2})_!(\Psi_f(M|_{X_\eta}))|_{D_2^\circ}$$

with $p_2 = f_\sigma \circ z_{D_2} : D_2 \to \mathsf{Spec}(k)$ the structural morphism.

Now, $\Psi_f(M|_{X_\eta})|_{D_2^\circ}$ can be computed explicitly using Proposition 3.4. To state the result, we need some notations. Assume that $v = ut$, with $u \in R^\times$. Note that $D_2 = \mathsf{Spec}(k[T_1])$ and $D_2^\circ = \mathsf{Spec}(k[T_1, T_1^\times])$. Consider the following finite étale cover of D_2°:

$$r_2^\circ : E_2^\circ = \mathsf{Spec}(k[T_1, T_1^{-1}][S]/(S^{a_2} - u_0 T_1^{-a_1})) \to D_2^\circ = \mathsf{Spec}(k[T_1, T_1^{-1}])$$

(where u_0 is the residue class of u). With these notations, one has

$$\Psi_f(M|_{X_\eta})|_{D_2^\circ} \simeq (r_2^\circ)_*(A|_{E_2^\circ})$$

with $A = \Psi_{\mathrm{Id}}((e_{a_2})^*_\eta M)$ where, for $m \in \mathbf{N}^\times$, $e_m : \mathsf{Spec}(k[[t]]) \to \mathsf{Spec}(k[[t]])$ is given by $t \mapsto t^m$.

Now, let E_2 be the normal finite D_2-scheme extending E_2°. If e is the greatest common divisor of a_1 and a_2, $a_1 d_1 + a_2 d_2 = e$ a Bézout relation, and $l = k[w]/(w^e - u_0)$, then

$$E_2 \simeq \mathsf{Spec}(l[T_1, S']/(S'^{a_2/e} - w^{-d_1} T_1)) \simeq \mathbf{A}^1_l.$$

(The first isomorphism above is induced by the substitution $S' = S^{-d_1} T_1^{d_2}$.) We have a cartesian square

$$\begin{tikzcd} E_2^\circ \simeq \mathbf{G}_{m,l} \arrow[r, "v_{E_2}"] \arrow[d, "r_2^\circ"] & E_2 \simeq \mathbf{A}^1_l \arrow[d, "r_2"] \\ D_2^\circ \arrow[r, "v_{D_2}"] & D_2. \end{tikzcd}$$

Since r_2°, r_2 are finite, we have $(r_2^\circ)_! = (r_2^\circ)_*$ and $(r_2)_! = (r_2)_*$ by [**2**, Théorème 1.7.17]. This gives canonical isomorphisms

$$(v_{D_2})_! \Psi_f(M|_{X_\eta})|_{D_2^\circ} \simeq (v_{D_2})_!(r_2^\circ)_*(A|_{E_2^\circ}) \simeq (r_2)_*(v_{E_2})_!(A|_{E_2^\circ}).$$

Therefore, to finish the proof it remains to show that $p_* j_! q^* \simeq 0$ where $j : \mathbf{G}_{m,l} \hookrightarrow \mathbf{A}^1_l$, $p : \mathbf{A}^1_l \to \mathsf{Spec}(l)$ and $q : \mathbf{G}_{m,l} \to \mathsf{Spec}(l)$ are the obvious morphisms. This is an easy exercise. Indeed, by localization, one has a distinguished 2-triangle

$$p_* j_! j^* p^* \to p_* p^* \to p_* i_* i^* p^* \xrightarrow{+1}$$

where $i : \mathsf{Spec}(l) \to \mathbf{A}^1_l$ is the zero section. Now, clearly, $p_* j_! j^* p^* \simeq p_* j_! q^*$ and $p_* i_* i^* p^* \simeq \mathrm{Id}$ as $p \circ i = \mathrm{Id}_{\mathsf{Spec}(l)}$. Also, we have $p_* p^* \simeq \mathrm{Id}$ by homotopy invariance. This finishes the proof. $\square$

We are now ready to prove the following statement, and thus complete the proof of Theorem 6.1 (see Lemma 6.8).

Proposition 6.11. — *Let $a_1, a_2 \in \mathbf{N}^\times$ and let*

$$X = \mathrm{St}^{Ut}_{R[U,U^{-1}],a_1,a_2} = \mathsf{Spec}\,\frac{R[U, U^{-1}, T_1, T_2]}{(T_1^{a_1} T_2^{a_2} - Ut)}.$$

Denote $f : X \to \mathsf{Spec}(R)$ the structural morphism. Then, the morphism

$$(\Psi_f(M|_{X_\eta}))|_{D_1} \to (v_{D_1})_*(v_{D_1})^*(\Psi_f(M|_{X_\eta}))|_{D_1} \tag{21}$$

is an isomorphism.

Proof. — We start as in the proof of Corollary 6.10 from which we keep the notations. As there, we must show that $N \to N'$ is an isomorphism. The difficulty we need to overcome here is caused by the fact that $C = D_1 \cap D_2$, on which $L = \mathrm{Cone}(N \to N')$ is supported, is now a 1-dimensional scheme (isomorphic to $\mathsf{Spec}(k[U, U^{-1}])$). Therefore, it is no longer sufficient to check that $(f_\sigma)_*(L) = 0$.

However, it would suffice to check that $(f_\sigma)_*(L) = 0$ if we knew that L was supported on a 0-dimensional closed subset of C. This is what we will prove in Step 1 below. In Step 2, we complete the proof by checking that $(f_\sigma)_*(L) = 0$ using the same method as in the proof of Corollary 6.10.

Before starting with Step 1, we note that we may assume that M is compact, i.e., $M \in \mathbf{SH}_{\mathfrak{M},\mathsf{ct}}(K)$. Indeed, all the operations in (21) commute with infinite sums and are triangulated. As $\mathbf{SH}_{\mathfrak{M}}(K)$ is a compactly generated triangulated category with infinite sums (see [**3**, Théorème 4.5.67]), we may indeed assume that M is compact. The compactness of M will be useful in Step 1.

Step 1. M is supported on a 0-dimensional subset of C. — As M is assumed to be compact, it follows from [**2**, Scholie 2.2.34] and [**3**, Théorème 3.5.14] that L is a compact object of $\mathbf{SH}_{\mathfrak{M}}(X_\sigma)$.

Let $\eta_C \simeq \mathsf{Spec}(k(U))$ the generic point of C. We also denote by η_C its inclusion in X_σ. As L is compact and supported in C, [**6**, Corollaire 1.A.3] shows that L is supported on a 0-dimensional closed subset of C if and only if $(\eta_C)^*(L) = 0$.

Now in order to prove that $(\eta_C)^*(L) = 0$, we introduce some notations. Let $\tilde{k} = k(U)$, $\tilde{R} = \tilde{k}[[t]]$ and $\tilde{K} = \tilde{R}[t^{-1}]$. There is a morphism of R-scheme

$$s : Y = \mathrm{St}^{Ut}_{\tilde{R},a_1,a_2} \to X = \mathrm{St}^{Ut}_{R[U,U^{-1}],a_1,a_2}$$

which is regular. Indeed, we have $Y = X \otimes_{k[[t]][U,U^{-1}]} k(U)[[t]]$ and $k(U)[[t]]$ is a regular $k[[t]][U, U^{-1}]$-algebra. Let $g : Y \to \mathsf{Spec}(R)$ be the structural morphism. Using [**6**, Corollaire 1.A.4] and the definition of the nearby motivic sheaf functors, we deduce that the canonical morphism

$$(s_\sigma)^* \Psi_f(M|_{X_\eta}) \to \Psi_g(M|_{Y_\eta})$$

is an isomorphism. Also, note that $\Psi_g(M|_{Y_\eta}) = \Psi_{\tilde{g}}(M|_{Y_\eta})$ where $\tilde{g} : Y \to \mathsf{Spec}(\tilde{R})$, i.e., the nearby motivic sheaf for Y can be computed equally using its structure of an R-scheme or an $\tilde{R}$-scheme.

The morphism $s_\sigma : (Y_\sigma)_{\mathrm{red}} \to (X_\sigma)_{\mathrm{red}}$ is the pro-open immersion

$$\mathsf{Spec}(k(U)[T_1, T_2]/(T_1T_2)) \hookrightarrow \mathsf{Spec}(k[U, U^{-1}, T_1, T_2]/(T_1T_2)).$$

Let $E_1 \subset Y_\sigma$ be the irreducible component defined by the equation $T_1 = 0$. We have $E_1 = \mathsf{Spec}(k(U)[T_2])$ and the morphism $E_1 \to D_1$, induced by s_σ, is simply the pro-open immersion $\mathsf{Spec}(k(U)[T_2]) \hookrightarrow \mathsf{Spec}(k[U, U^{-1}, T_2])$.

The inverse image of (21) along the pro-open immersion $E_1 \hookrightarrow D_1$ identifies with the morphism

$$(\Psi_g(M|_{Y_\eta}))|_{E_1} \to (v_{E_1})_*(v_{E_1})^*(\Psi_g(M|_{Y_\eta}))|_{E_1}.$$

(Use [**6**, Corollaire 1.A.4].) The latter is an isomorphism by Corollary 6.10. Therefore, the inverse image of $N \to N'$ along the pro-open immersion $s_\sigma : Y_\sigma \to X_\sigma$ is an isomorphism. This shows that $(s_\sigma)^*(L) = 0$. Now, the inclusion of the point η_C in X_σ factors through s_σ. This gives that $(\eta_C)^*(L) = 0$ as claimed.

Step 2. End of the proof. — Thanks to Step 1, it remains to show that $(f_\sigma)_*(L) = 0$. This is equivalent to showing that

$$(f_\sigma)_*(N) \to (f_\sigma)_*(N')$$

is an isomorphism in $\mathbf{SH}_{\mathfrak{M}}(k)$. Now, by Lemma 6.9, we have $(f_\sigma)_*(N') = 0$. Hence, to finish the proof, we are left to show that $(f_\sigma)_*(N) = 0$.

The rest of the proof is identical to Step 2 of the proof of Corollary 6.10. As there, we have:

$$(f_\sigma)_*(N) \simeq (p_2)_*(v_{D_2})_!(\Psi_f(M|_{X_\eta}))|_{D_2^\circ}.$$

Here also, $\Psi_f(M|_{X_\eta})|_{D_2^\circ}$ can be computed explicitly using Proposition 3.4: Note that

$$D_2 = \mathsf{Spec}(k[U, U^{-1}, T_1]) \qquad \text{and} \qquad D_2^\circ = \mathsf{Spec}(k[U, U^{-1}, T_1, T_1^\times]).$$

Consider the following finite étale cover of D_2°:

$$r_2^\circ : E_2^\circ = \mathsf{Spec}(k[T_1, T_1^{-1}][S]/(S^{a_2} - UT_1^{-a_1})) \to D_2^\circ = \mathsf{Spec}(k[T_1, T_1^{-1}])$$

With these notations, we have

$$\Psi_f(M|_{X_\eta})|_{D_2^\circ} \simeq (r_2^\circ)_*(A|_{E_2^\circ})$$

with $A = \Psi_{\mathrm{Id}}((e_{a_2})_\eta^* M)$ where, for $m \in \mathbf{N}^\times$, $e_m : \mathsf{Spec}(k[[t]]) \to \mathsf{Spec}(k[[t]])$ is given by $t \mapsto t^m$.

Now, let E_2 be the normal finite D_2-scheme extending E_2°. If e is the greatest common divisor of a_1 and a_2, $a_1 d_1 + a_2 d_2 = e$ a Bézout relation, and $P = \mathsf{Spec}(k[w]/(w^e - U))$, then

$$E_2 \simeq \mathsf{Spec}(\mathcal{O}_P[T_1, S']/(S'^{a_2/e} - w^{-d_1} T_1)) \simeq \mathbf{A}^1_P.$$

(The first isomorphism above is induced by the substitution $S' = S^{-d_1} T_1^{d_2}$.) We have a cartesian square

$$\begin{tikzcd} E_2^\circ \simeq \mathbf{G}_{m,P} \arrow[r, "v_{E_2}"] \arrow[d, "r_2^\circ"] & E_2 \simeq \mathbf{A}^1_P \arrow[d, "r_2"] \\ D_2^\circ \arrow[r, "v_{D_2}"] & D_2. \end{tikzcd}$$

Since r_2°, r_2 are finite, we have $(r_2^\circ)_! = (r_2^\circ)_*$ and $(r_2)_! = (r_2)_*$ by [**2**, Théorème 1.7.17]. This gives isomorphisms

$$(v_{D_2})_! \Psi_f(M|_{X_\eta})|_{D_2^\circ} \simeq (v_{D_2})_!(r_2^\circ)_*(A|_{E_2^\circ}) \simeq (r_2)_*(v_{E_2})_!(A|_{E_2^\circ}).$$

We conclude using that $p_* j_! q^* \simeq 0$ for $j : \mathbf{G}_{m,P} \hookrightarrow \mathbf{A}^1_P$, $p : \mathbf{A}^1_P \to P$ and $q : \mathbf{G}_{m,P} \to P$ the obvious morphisms. $\square$

7. Nearby motivic sheaves and rigid motives of tubes

In this section, we prove the main result of this article (see Theorem 7.1) that extends Theorem 4.11 to motives of tubes of locally closed subsets of the special fiber.

7.1. Statement of the main theorem. — Let X be a finite type R-scheme and let $f : X \to \mathsf{Spec}(R)$ be its structural morphism. Assume that X_η is smooth over K and consider the t-adic completion $\mathscr{X}$ of X.

The following statement is our main theorem.

Theorem 7.1. — *Let $Z \subset X_\sigma$ be a locally closed subset and denote by $z : Z \hookrightarrow X_\sigma$ its inclusion. Consider the tube $]Z[$ of Z in $\mathscr{X}_\eta$. Let M be an object of $\mathbf{SH}_{\mathfrak{M}}(K)$. Then, there exists a canonical isomorphism in $\mathbf{SH}_{\mathfrak{M}}(k)$:*

$$1^* \circ \mathfrak{R}(\underline{\mathsf{Hom}}(\mathrm{M}_{rig}(]Z[), \mathrm{Rig}^*(M))) \simeq (f_\sigma \circ z)_*(\Psi_f(M|_{X_\eta})|_Z).$$

When $X = \mathsf{Spec}(R)$, $f = \mathrm{Id}$ and $Z = \mathsf{Spec}(k)$, the above theorem simply states that $1^* \circ \mathfrak{R} \circ \mathrm{Rig}^*$ is isomorphic to the nearby motive functor Ψ_{Id}, which we already know by [**6**, Scholie 1.3.26(2)]. Thus, in some sense, Theorem 7.1 can be considered as a generalization of [**6**, Scholie 1.3.26(2)].

Taking M to be the unit object of $\mathbf{SH}_{\mathfrak{M}}(K)$ in Theorem 7.1, one gets the following:

Corollary 7.2. — *With the notation of Theorem 7.1, there is a canonical isomorphism in $\mathbf{SH}_{\mathfrak{M}}(k)$:*

$$1^* \circ \mathfrak{R}(\mathrm{M}^\vee_{rig}(]Z[)) \simeq (f_\sigma \circ z)_*(\Psi_f(\mathbb{1}_{X_\eta}))|_Z.$$

7.2. The proof of Theorem 7.1. — The proof consists of using Corollary 5.2 and Corollary 6.2 to deduce Theorem 7.1 from its particular case obtained in Theorem 4.11. We split the proof in three steps.

Step 1. Reduction to the case where Z is closed. — Let $U \subset X$ be an open neighborhood of Z in which Z is closed. Let $f_U : U \to \mathsf{Spec}(R)$ be the structural morphism of U, $\mathscr{U}$ its t-adic completion and $z_U : Z \hookrightarrow U_\sigma$ the obvious inclusion. Clearly, the tube of Z in $\mathscr{X}_\eta$ is also the tube of Z in $\mathscr{U}_\eta$ (see [**27**, Proposition 2.2.2]). On the other hand, we have

$$(f_\sigma \circ z)_*(\Psi_f(M|_{X_\eta}))|_Z \simeq ((f_U)_\sigma \circ z_U)_*(\Psi_{f_U}(M|_{U_\eta}))|_Z.$$

Therefore, we may replace X by U and assume that Z is closed.

Step 2. Reduction to the case where X is semi-stable and Z is a subdivisor. — Let $h\colon X' \to X$ be a projective morphism such that X' is a semi-stable R-scheme, h_η is an isomorphism and $((h_\sigma)^{-1}(Z))_{\mathrm{red}}$ is a union of irreducible components of $(X'_\sigma)_{\mathrm{red}}$. (Such a morphism exists by Hironaka's resolution of singularities.)

By [**27**, Corollary 2.2.7], we have $]Z[\, = \,]h^{-1}(Z)[$ as admissible open rigid subvarieties of $\mathscr{X}_\eta$ which we identifies with $\mathscr{X}'_\eta$. On the other hand, using the third property of [**3**, Définition 3.1.1 (**SPE2**)] and the base-change theorem for projective morphisms [**2**, Corollaire 1.7.18], we have canonical isomorphisms

$$(f_\sigma)_* z_* z^* \Psi_f(M|_{X_\eta}) \simeq (f_\sigma)_* z_* z^* (h_\sigma)_* \Psi_{f\circ h}(M|_{X'_\eta}) \simeq (f_\sigma \circ h_\sigma)_* z'_* z'^* \Psi_{f\circ h}(M|_{X'_\eta})$$

where $z' : h^{-1}(Z) \hookrightarrow X'$ is the obvious inclusion. Therefore, it is enough to show that there is an isomorphism

$$1^* \circ \mathfrak{R}(\underline{\mathsf{Hom}}(\mathrm{M}_{rig}(]h^{-1}(Z)[), \mathrm{Rig}^*(M))) \simeq (f'_\sigma \circ z')_*(\Psi_{f\circ h}(M|_{X'_\eta})|_{h^{-1}(Z)}).$$

In other words, we may assume that the R-scheme X is semi-stable and that Z is a union of irreducible components of $(X_\sigma)_{\mathrm{red}}$.

Step 3. End of the proof. — Here, we assume that the R-scheme X is semi-stable and we denote by $(D_i)_{i\in I}$ the irreducible components of $(X_\sigma)_{\mathrm{red}}$. We also assume that $Z = D(J) = \cup_{j\in J} D_j$ for a subset $J \subset I$. Recall that $D(J)^\circ = D(J) \smallsetminus \cup_{i\in I\smallsetminus J} D_i$; this is an open subset of X_σ.

Now, by Corollary 5.2, we have a canonical isomorphism

$$1^* \circ \mathfrak{R}(\underline{\mathsf{Hom}}(\mathrm{M}_{rig}(]D(J)[), \mathrm{Rig}^*(M))) \xrightarrow{\sim} 1^* \circ \mathfrak{R}(\underline{\mathsf{Hom}}(\mathrm{M}_{rig}(]D(J)^\circ[), \mathrm{Rig}^*(M))).$$

On the other hand, by Corollary 6.2, we have canonical isomorphisms

$$\begin{aligned}(f_\sigma)_*(z_{D(J)})_*(z_{D(J)})^*\Psi_f(M|_{X_\eta}) &\xrightarrow{\sim} (f_\sigma)_*(z_{D(J)})_*(v_{D(J)})_*(v_{D(J)})^*(z_{D(J)})^*\Psi_f(M|_{X_\eta})\\ &\simeq (f_\sigma)_*(z_{D(J)^\circ})_*(z_{D(J)^\circ})^*\Psi_f(M|_{X_\eta})\end{aligned}$$

where $z_{D(J)} : D(J) \hookrightarrow X_\sigma$, $z_{D(J)^\circ} : D(J)^\circ \hookrightarrow X_\sigma$ and $v_{D(J)} : D(J)^\circ \hookrightarrow D(J)$ are the obvious inclusions. Therefore, it is enough to prove Theorem 7.1 for $D(J)^\circ$. As $D(J)^\circ$ is an open subset, we may apply Theorem 4.11 to the R-scheme $X \smallsetminus \cup_{j\in I\smallsetminus J} D_j$ to get the result.

8. Applications and remarks

In this section, we use Theorem 7.1 and [**25**, Theorem 5.1] to establish a link between the motivic Milnor fiber introduced by Denef–Loeser [**16**, Définition 4.2.1] and the rigid motive of the analytic Milnor fiber introduced by Nicaise–Sebag [**34**].

8.1. Two definitions. — Let X be a finite type R-scheme and denote by $f : X \to \mathsf{Spec}(R)$ its structural morphism. Assume that X_η is smooth.

Remark 8.1. — Although it is unnecessary, the reader may want to assume throughout this section that the morphism $f : X \to \mathsf{Spec}(R)$ is the base-change by $\mathsf{Spec}(R) \to \mathbf{A}^1_k$ of a morphism $\tilde{f} : \tilde{X} \to \mathbf{A}^1_k$ with $\tilde{X}$ a smooth k-scheme of finite type; this assumption is sometimes necessary to quote results from the existing literature, word for word.

For the reader who wants to keep the degree of generality that was adopted so far in this article, we mention that the rationality of the motivic zeta function for finite type R-schemes with smooth generic fiber has been verified in [**37**] and [**34**, Corollary 7.7].

Definition 8.2. — Let $x \in X_\sigma(k)$ be a rational point. Following Nicaise–Sebag [**34**], we define the *analytic Milnor fiber* of f at x to be the tube $]x[\subset \mathscr{X}_\eta$ of the closed point x. This is a rigid analytic variety over K which is denoted by $\mathscr{F}_x$.(3)

Given a base-scheme S, let $\mathrm{K}_0(\mathrm{Var}_S)$ be the Grothendieck group of S-schemes. This group is the quotient of the free abelian group on isomorphism classes of quasi-projective S-schemes by the scissor relation $[Y] = [Y \smallsetminus Z] + [Z]$ (where Y is a quasi-projective S-scheme and $Z \subset Y$ is a closed subscheme). Fiber product over S endows $\mathrm{K}_0(\mathrm{Var}_S)$ with a ring structure. One sets

$$\mathscr{M}_S = \mathrm{K}_0(\mathrm{Var}_S)[\mathbf{L}^{-1}]$$

where $\mathbf{L} = [\mathbf{A}^1_S]$.

Going back to our setting, one has by Denef–Loeser [**16**] the *motivic zeta function* associated with the R-scheme X (or, more precisely, to the morphism $\tilde{f} : \tilde{X} \to \mathbf{A}^1_k$):

$$Z_f(T) = \sum_{n \geqslant 1} Z^1_n T^n \quad \in \quad \mathscr{M}_{X_\sigma}[[T]],$$

with $Z^1_n = \mathbf{L}^{-nd}[\{\phi \in \mathscr{L}_n(X), f \circ \phi = t^n + O(t^{n+1})\}] \in \mathscr{M}_{X_\sigma}$ where $\mathscr{L}_n(X)$ is the n-jets space of X and d the dimension of X (that we may assume constant). For $x \in X_\sigma(k)$, one gets by applying the natural ring homomorphism $x^* : \mathscr{M}_{X_\sigma} \to \mathscr{M}_k$, $[Y] \mapsto [Y \times_{X_\sigma} x]$, the *local motivic zeta function* at x denoted by $Z_{f,x}(T)$.

By Denef–Loeser [**16**], one knows that $Z_f(T)$ is a rational function and that the limit

$$\psi_f = -\Big(\lim_{T \to \infty} Z_f(T)\Big)$$

exists in $\mathscr{M}_{X_\sigma}$.

Definition 8.3. — For $x \in X_\sigma(k)$, the image of ψ_f by $x^* : \mathscr{M}_{X_\sigma} \to \mathscr{M}_k$ is called the *motivic Milnor fiber* at x and is denoted by $\psi_{f,x}$.

Remark 8.4. — Thanks to a motivic analog of the Thom–Sebastiani formula established by Guibert, Loeser and Merle in [**21**], Lunts and Schnürer explain in [**30**] how the motivic vanishing cycles of Denef–Loeser give rise to a motivic measure on $\mathrm{K}_0(\mathrm{Var}_{\mathbf{A}^1_k})$. In the last part of *loc. cit.*, they also compare their construction to another motivic measure of categorical nature, based on the associated category of matrix factorizations. Since we strongly believe that a Thom–Sebastiani formula exists in the world of motives, analogs of these measures should exist at the level of the

3. In [**34**], the analytic Milnor fiber is considered as a Berkovich space. In this article, we prefer to consider it as a rigid analytic variety in the sense of Tate.

corresponding Grothendieck ring of motives. The comparison of the categorical and non-archimedean points of vue should illuminate each other. We thank the referee for having pointed out this reference to us.

8.2. Recollection from Ivorra–Sebag [25]. — Here we recall the main results of [**25**] and explain how to obtain variants which are more suitable for our purposes. Roughly speaking, we claim that everything in [**25**] still hold when $\mathbf{DA}^{\text{ét}}(-,\mathbf{Q})$, the category of étale motivic sheaves with rational coefficients, is replaced by $\mathbf{SH}_{\mathfrak{M}}(-)$. This is rendered possible primarily thanks to Theorem 6.1 showing that the conclusion of [**3**, Théorème 3.3.44] holds for $\mathbf{SH}_{\mathfrak{M}}(-)$ even though the latter is not $\mathbf{Q}$-linear nor separated (cf. Remark 6.3).

First, note the following:

Lemma 8.5 **([25], Lemma 2.1)**. — *Let S be a base-scheme. Then, there exists a ring homomorphism*

$$\chi_{S,c}:\mathscr{M}_S\to \mathrm{K}_0(\mathbf{SH}_{\mathfrak{M},\mathsf{ct}}(S)),$$

which is uniquely determined by the formula

$$\chi_{S,c}([Y])=[\mathrm{M}_{S,c}(Y)]$$

where Y is a quasi-projective S-scheme and $\mathrm{M}_{S,c}(Y)$ is its motive with compact support defined to be $(p_Y)_!(p_Y)^*\mathbb{1}_S$ with $p_Y:Y\to S$ the structural morphism.

Proof. — The proof given in [**25**] extends word for word to the case of $\mathbf{SH}_{\mathfrak{M}}(-)$.[4] Note that $(p_Y)_!(p_Y)^*\mathbb{1}_S$ is a compact object of $\mathbf{SH}_{\mathfrak{M}}(S)$ by [**3**, Scholie 2.2.34 **B**]. □

Theorem 8.6 **([25], Theorem 3.1)**. — *Let X be a semi-stable R-scheme and recall the notations from §6.1. For $\emptyset\neq J\subset I$, let $\rho_J:\tilde{D}_J^\circ\to D_J^\circ$ be the étale finite cover defined as in* [**25**, §3.1.3]*. Then, one has the formula*

$$[\Psi_f(\mathbb{1}_{X_\eta})]=\sum_{\emptyset\neq J\subset I}(-1)^{|J|-1}\left[\mathrm{M}_{X_\sigma,c}(\tilde{D}_J^\circ\times_k \mathbf{G}_{m,k}^{|J|-1})\right]$$

in $\mathrm{K}_0(\mathbf{SH}_{\mathfrak{M},\mathsf{ct}}(X_\sigma))$.

Proof. — The proof given [**25**, §4] extends with very few modifications: there are only two points where new ingredients are needed. More precisely, in the proof of [**25**, Proposition 4.4], the reference to [**3**, Théorème 3.3.44] is no longer sufficient for $\mathbf{SH}_{\mathfrak{M}}(-)$ which is not $\mathbf{Q}$-linear nor separated. Happily, we now can use Theorem 6.1 to overcome this difficulty. Also, the reference to [**5**, Théorème 10.6] needs to be changed: one can use Proposition 3.4 instead.

The rest of the proof, i.e., [**25**, Lemmas 4.1, 4.2 and 4.3], [**25**, Proposition 4.5] and [**25**, §4.3], extend with no modification. Note also that the extension of the

4. We warn the reader that there is a misprint in the proof of [**25**, Lemma 2.1]: the image of $\mathbf{L}$ by $\chi_{S,c}$ is $[\mathbb{1}_S(-1)]$ instead of $[\mathbb{1}_S(1)]$.

argument in [**25**, §4.3] (which is based on Verdier duality and its compatibility with the nearby motivic sheaf functors) is indeed possible because we are working over a field of characteristic zero [(5)].

Finally, note that due to the lack of orientability in $\mathbf{SH}_{\mathfrak{M}}(-)$, Thom equivalences are not always trivial, i.e., if $\mathcal{M}$ is a locally free $\mathcal{O}_S$-module of rank r on a scheme S, then $\mathrm{Th}(\mathcal{M})(-)$ can be different from the Tate twist $(-)(r)[2r]$. However, these two equivalences induce the same action in the Grothendieck ring of motives.

Indeed, recall that by definition, the Tate twist $(-)(r)[2r]$ is defined as the Thom equivalence associated with the free $\mathcal{O}_S$-module $\mathcal{O}_S^r$ of rank r. (see [**2**, §1.5.3]). Let $U_1, \dots, U_n$ be a covering of S by open subschemes such that $\mathcal{M}|_{U_i}$ is isomorphic to $\mathcal{O}_{U_i}^r$. Denote by $u_I : U_I \hookrightarrow S$ the open immersion of $U_I := \cap_{i\in I} U_i$. Using Mayer–Vietoris triangles and [**3**, Proposition 1.5.2], one gets, for $A \in \mathbf{SH}_{\mathfrak{M},\mathsf{ct}}(S)$, the equalities

$$\begin{aligned}
[\mathrm{Th}(\mathcal{M})(A)] &= \sum_{\emptyset\neq I\subseteq\{1,\dots,n\}} (-1)^{|I|-1}[(u_I)_*(u_I)^*\mathrm{Th}(\mathcal{M})(A)] \\
&= \sum_{\emptyset\neq I\subseteq\{1,\dots,n\}} (-1)^{|I|-1}[(u_I)_*\mathrm{Th}(\mathcal{M}|_{U_I})((u_I)^*A)] \\
&= \sum_{\emptyset\neq I\subseteq\{1,\dots,n\}} (-1)^{|I|-1}[(u_I)_*\mathrm{Th}(\mathcal{O}_S^r|_{U_I})((u_I)^*A)] \\
&= \sum_{\emptyset\neq I\subseteq\{1,\dots,n\}} (-1)^{|I|-1}[(u_I)_*(u_I)^*\mathrm{Th}(\mathcal{O}_S^r)(A)] = [A(r)[2r]]
\end{aligned}$$

in $\mathrm{K}_0(\mathbf{SH}_{\mathfrak{M},\mathsf{ct}}(S))$. □

As in [**25**] one gets the following statement as a consequence of Theorem 8.6 and known formulas for ψ_f in a semi-stable situation.

Corollary 8.7 **([25], Theorem 5.1)**. — *Let X be a finite type R-scheme with smooth generic fiber and denote by $f : X \to \mathsf{Spec}(R)$ its structural morphism. We have the equality*

$$[\Psi_f(\mathbb{1}_{X_\eta})] = \chi_{X_\sigma,c}(\psi_f)$$

in $\mathrm{K}_0(\mathbf{SH}_{\mathfrak{M},\mathsf{ct}}(X_\sigma))$. *Also, for every $x \in X_\sigma(k)$, we have the equality*

$$[x^*\Psi_f(\mathbb{1}_{X_\eta})] = \chi_{k,c}(\psi_{f,x})$$

in $\mathrm{K}_0(\mathbf{SH}_{\mathfrak{M},\mathsf{ct}}(k))$.

5. For Verdier duality and its compatibility with nearby cycles, see [**3**, Théorème 3.5.20]. Note that all results of [**3**, §3.5.3] hold in characteristic zero.

8.3. An application. — We are now ready to give our application. Let X be a finite type R-scheme with smooth generic fiber and denote by $f : X \to \mathsf{Spec}(R)$ its structural morphism. Also, fix a rational point $x \in X_\sigma(k)$.

Theorem 8.8. — *There is a canonical isomorphism in* $\mathbf{SH}_{\mathfrak{M}}(k)$*:*

$$1^* \circ \mathfrak{R}(\mathrm{M}^\vee_{rig}(\mathscr{F}_x)) \simeq x^*\Psi(\mathbb{1}_{X_\eta}).$$

Proof. — This is a particular case of Theorem 7.1. □

Corollary 8.9. — *The following equality holds in* $\mathrm{K}_0(\mathbf{SH}_{\mathfrak{M},\mathsf{ct}}(k))$*:*

$$[1^* \circ \mathfrak{R}(\mathrm{M}^\vee_{rig}(\mathscr{F}_x))] = \chi_{k,c}(\psi_{f,x}). \tag{22}$$

Proof. — This result follows directly from Theorem 8.8 and Corollary 8.7. □

Remark 8.10. — Corollary 8.9 shows that the motivic Milnor fiber of Denef–Loeser, viewed as a class in $\mathrm{K}_0(\mathbf{SH}_{\mathfrak{M},\mathsf{ct}}(k))$ via the morphism $\chi_{k,c}$, depends only on the rigid motive of the analytic Milnor fiber.

8.4. Some remarks. — We gather here some remarks that a reader familiar with the literature on "motivic integration" might find useful.

Let X be a finite type R-scheme with smooth generic fiber and denote by $f : X \to \mathsf{Spec}(R)$ its structural morphism.

Remark 8.11. — Theorem 7.1 and Corollary 8.7 give a positive answer to the question asked in [**34**, page 163]. Indeed, by [**34**, Theorem 9.13], the motivic volume $S(\mathscr{X}_\eta; \widehat{K}^s)$ is equal to ψ_f up to a twist by a power of $\mathbf{L}$.

Remark 8.12. — Assume that k contains all roots of unity. The trace formula of Denef–Loeser [**17**, Theorem 1.1] links the Lefschetz numbers of the monodromy action on the Milnor fiber with the Euler characteristic of the coefficients of the local zeta function. In [**34**, Theorem 5.4] and [**33**, Theorem 6.4] this trace formula has been extended in different directions. In particular, given a locally closed subset $Z \subset X_\sigma$, one has for all $d \in \mathbf{N}^\times$

$$\mathrm{Tr}(\varphi^d \mid \mathrm{H}_{\text{ét}}(\overline{]Z[}, \mathbf{Q}_\ell)) = \chi_{\ell,c}(S_Z(\mathscr{X}_d)). \tag{23}$$

In this formula, φ is a topological generator of the Galois group $\hat{\mu}$ of the extension $\cup_{d\in\mathbf{N}^\times} k(\!(t^{1/d})\!)$ of $K = k(\!(t)\!)$, $S_Z(\mathscr{X}_d)$ is the motivic Serre invariant with support in Z associated with the $t^{1/d}$-adic completion of $X_d = X \otimes_{k[\![t]\!]} k[\![t^{1/d}]\!]$, and $\chi_{\ell,c} : \mathscr{M}_k \to \mathbf{Z}$ is the ℓ-adic Euler characteristic with compact supports.

Using corollary 7.2, one can formulate this trace formula in a more motivic way. Indeed, the group $\hat{\mu}$ acts by natural transformations on the functor $1^* : \mathbf{QUSH}_{\mathfrak{M}}(k) \to \mathbf{SH}_{\mathfrak{M}}(k)$. In particular, one has an action of $\hat{\mu}$ on $1^* \circ \mathfrak{R}(\mathrm{M}^\vee_{rig}(]Z[))$. Moreover, after semi-simplification, the action of $\hat{\mu}$ on the étale realization of $1^* \circ \mathfrak{R}(\mathrm{M}^\vee_{rig}(]Z[))$ agrees with its action on $\mathrm{H}_{\text{ét}}(\overline{]Z[}, \mathbf{Q}_\ell)$. In particular, the left hand side of (23) can be written as $\mathrm{Tr}(\varphi^d \mid 1^* \circ \mathfrak{R}(\mathrm{M}^{\vee}_{rig}(]Z[)))$. (Note that the object $1^* \circ \mathfrak{R}(\mathrm{M}^{\vee}_{rig}(]Z[))$ is strongly dualizable thanks to Lemma 4.10 and Proposition 5.9. Hence, the trace of an endomorphism of this object makes sense.) Therefore, we may reformulate the trace formula as follows:

$$\mathrm{Tr}(\varphi^d \mid 1^* \circ \mathfrak{R}(\mathrm{M}^{\vee}_{rig}(]Z[))) = \chi_{\ell,c}(S_Z(\mathscr{X}_d)).$$

This shows that the monodromy zeta function of A'Campo only depends on the motive of the analytic Milnor fiber.

Remark 8.13. — Keep the notation as in Remarks 8.11 and 8.12. As explained in §8.1, a motivic zeta function $Z_f(T) = \sum_{n\geq 1} Z^1_n T^n \in \mathscr{M}_{X_\sigma}[[T]]$ is associated with a flat morphism $f\colon X \to \mathbf{A}^1_k$ of k-varieties such that X is smooth. As proved by Denef and Loeser in [**16**], this zeta function gives rise to motivic nearby cycles at the level of the Grothendieck rings of varieties by taking a limit as T goes to ∞ (assuming that the characteristic of k is zero). By Corollary 8.7 or [**25**, Theorem 5.1], one knows that these motivic nearby cycles can be compared with the motivic nearby sheaves of [**3**], and, in this way, can be directly linked to the classical sheaves of nearby cycles.

It would be very interesting to provide a categorical interpretation of the motivic zeta function and the limiting process, as T goes to ∞, in the world of motives. For instance, by [**34**], one knows that the coefficients of the motivic zeta function can be realized as the motivic integrals of well-chosen gauge forms on the generic fiber $\mathscr{X}_\eta$ of the t-adic completion $\mathscr{X}$ of X. (The relevant theory of motivic integration has been introduced in [**29**].) Also, the n-th coefficient of $Z_f(T)$ coincides with the motivic Serre invariant $S(\mathscr{X}_n)$ in the quotient of $\mathscr{M}_{X_\sigma}$ by the class of $\mathbf{G}_{m,X_\sigma}$. On the other hand, in [**18**], Drinfeld conjectures the existence of a "refined" theory of motivic integration which takes values in the derived category $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Spec}(k), \mathbf{Z}_\ell)$ of constructible ℓ-adic sheaves on $\mathrm{Spec}(k)$. The basic idea in *loc. cit.* is to consider an alternative version of integrals of top-degree differential forms on rigid analytic spaces. All these considerations suggest that these various theories are connected and that it would be interesting to develop further relations between the motivic zeta function of Denef–Loeser and the theory of motives (rigid or classical), as it has been already emphasized in [**16**, Remarks, page 12].

Remark 8.14. — In [**22, 23**], Hrushovski and Kazhdan introduced Grothendieck rings associated with the theory $\mathrm{ACVF}(0,0)$ of algebraically closed valued fields of equi-characteristic zero. From *loc. cit.* and [**24**], one has the following ring homomorphisms:

$$\mathrm{K}_0(\mathrm{volVF}_K) \xrightarrow{\tilde{\Theta}\circ\Upsilon\circ\int} \mathrm{K}_0^{\hat{\mu}}(\mathrm{Var}_k)[[\mathbf{A}^1_k]^{-1}] \xrightarrow{\mathrm{f}} \mathrm{K}_0(\mathrm{Var}_k)[[\mathbf{A}^1_k]^{-1}] = \mathscr{M}_k.$$

The group $\mathrm{K}_0(\mathrm{volVF}_K)$ is the Grothendieck group of definable subsets of VF^n over K with volume form. The group $\mathrm{K}_0^{\hat{\mu}}(\mathrm{Var}_k)$ is the Grothendieck ring of k-schemes endowed with a continuous action of the profinite group $\hat{\mu} = \lim_{n\in\mathbf{N}^\times} \mu_n(k)$ (k is assumed to have all roots of unity). The morphism f is induced by the forgetful functor. For the definitions of the morphisms $\tilde{\Theta}$, Υ and $\int$, we refer the reader to [**24**].

The analytic Milnor fiber $\mathscr{F}_x$ is a definable subset in $\mathrm{ACVF}(0,0)$ and hence admits a class $[\mathscr{F}_x]$ in the ring $\mathrm{K}_0(\mathrm{volVF}_K)$. With the previous notation, [**24**, Corollary 8.4.2] gives the following formula:

$$\psi_{f,x} = \mathrm{f} \circ \tilde{\Theta} \circ \Upsilon \circ \int ([\mathscr{F}_x]) \,. \tag{24}$$

In the same spirit as Corollary 8.9, this formula shows that the motivic Milnor fiber of Denef–Loeser depends only on the class of the analytic Milnor fiber in $\mathrm{K}_0(\mathrm{volVF}_K)$.

Remark 8.15. — We keep the notation as in the previous remark. Combining the formula (24) with the formula (22) of Corollary 8.9 gives an equality in $\mathrm{K}_0(\mathbf{SH}_{\mathfrak{M},\mathsf{ct}}(k))$:

$$\chi_{k,c} \circ \mathrm{f} \circ \tilde{\Theta} \circ \Upsilon \circ \int ([\mathscr{F}_x]) = [1^* \circ \mathfrak{R}(\mathrm{M}^{\vee}_{rig}(\mathscr{F}_x))].$$

It is therefore tempting to speculate the existence of a morphism of rings (6)

$$\dagger : \mathrm{K}_0(\mathrm{volVF}_K) \to \mathrm{K}_0(\mathbf{SH}_{\mathfrak{M},\mathsf{ct}}(k))$$

sending the class $[V] \in \mathrm{K}_0(\mathrm{volVF}_K)$ of a definable smooth rigid analytic variety V to the class $[\mathrm{M}^{\vee}_{rig}(V)] \in \mathrm{K}_0(\mathbf{SH}_{\mathfrak{M},\mathsf{ct}}(k))$ of its associated cohomological rigid motive $\mathrm{M}^{\vee}_{rig}(V)$. Moreover, there should exist a morphism of rings

$$\chi^{\hat{\mu}}_{k,c} : \mathrm{K}^{\hat{\mu}}_0(\mathrm{Var}_k)[[\mathbf{A}^1_k]^{-1}] \to \mathrm{K}_0(\mathbf{QUSH}_{\mathfrak{M},\mathsf{ct}}(k)),$$

analogous to the morphism $\chi_{k,c}$ obtained in Lemma 8.5 (see also [**25**, Lemma 2.1]), which makes the following diagram commutative

$$\begin{tikzcd}
\mathrm{K}_0(\mathrm{volVF}_K) \arrow[r, "\tilde{\theta}\circ\Upsilon\circ\int"] \arrow[d, dotted, "\dagger"] & \mathrm{K}^{\hat{\mu}}_0(\mathrm{Var}_k)[[\mathbf{A}^1_k]^{-1}] \arrow[r, "\mathrm{f}"] \arrow[d, "\chi^{\hat{\mu}}_{k,c}"] & \mathrm{K}_0(\mathrm{Var}_k)[[\mathbf{A}^1_k]^{-1}] \arrow[d, "\chi_{k,c}"] \\
\mathrm{K}_0(\mathbf{RigSH}_{\mathfrak{M},\mathsf{ct}}(K)) \arrow[r, "\mathfrak{R}", "\sim"'] & \mathrm{K}_0(\mathbf{QUSH}_{\mathfrak{M},\mathsf{ct}}(k)) \arrow[r, "1^*"] & \mathrm{K}_0(\mathbf{SH}_{\mathfrak{M},\mathsf{ct}}(k)).
\end{tikzcd}$$

If such a morphism † exists, our formula (22) would then follows from the formula of Hrushovski–Loeser [**24**, Corollary 8.4.2].

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6. To the best of our knowledge, there is no result in that direction.

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