THE MOTIVIC NEARBY CYCLES AND THE CONSERVATION CONJECTURE

JOSEPH AYOUB

To Jacob Murre for his 75th birthday

Contents

1. Introduction

Let X be a noetherian scheme. Following Morel and Voevodsky (see [24], [25], [28], [33] and [37]), one can associate to X the motivic stable homotopy category $\mathbf{SH}(X)$. Objects of $\mathbf{SH}(X)$ are T-spectra of simplicial sheaves on the smooth Nisnevich site $(\mathsf{Sm}/X)_{\mathrm{Nis}}$, where T is the pointed quotient sheaf $\mathbb{A}^1_X/\mathbb{G}_{\mathrm{m}X}$. As in topology, $\mathbf{SH}(X)$ is triangulated in a natural way. There is also a tensor product $-\otimes_X-$ and an "internal hom": $\underline{\mathsf{Hom}}_X$ on $\mathbf{SH}(X)$ (see [20] and [33]). Given a morphism $f: X \longrightarrow Y$ of noetherian schemes, there is a pair of adjoint functors (f^*, f_*) between $\mathbf{SH}(X)$ and $\mathbf{SH}(Y)$. When f is quasi-projective, one can extend the pair (f^*, f_*) to a quadruple $(f^*, f_*, f_!, f^!)$ (see [3] and [8]). In particular we have for $\mathbf{SH}(-)$ the full package of the Grothendieck six operators. It is then natural to ask if we have also the seventh one, that is, if we have a vanishing cycle formalism (analogous to the one in the étale case, developed in [9] and [10]).

In the third chapter of our PhD thesis [3], we have constructed a vanishing cycles formalism for motives. The goal of this paper is to give a detailed account of that construction, to put it in a historical perspective and to discuss some applications and conjectures. In some sense, it is complementary to [3] as it gives a quick introduction to the theory with emphasis on motivations rather than a systematic treatment. The reader will not find all the details here: some proofs will be omitted or quickly sketched, some results will be stated with some additional assumptions (indeed we will be mainly interested in motives with rational coefficients over characteristic zero schemes). For the full details of the theory, one should consult [3].

Let us mention also that M. Spitzweck has a theory of limiting motives which is closely related to our motivic vanishing cycles formalism. For more information, see [35].

The paper is organized as follows. First we recall the classical pictures: the étale and the Hodge cases. Although this is not achieved here, these classical constructions should be in a precise sense realizations of our motivic construction. In section 3 we introduce the notion of a specialization system which encodes some formal properties of the family of nearby cycles functors. We state without proofs some important theorems about specialization systems obtained in [3]. In section 4, we give our main construction and prove motivic analogues of some well-known classical results about nearby cycles functors: constructibility, commutation with tensor product and duality, etc. We also construct a monodromy operator on the unipotent part of the nearby cycles which is shown to be nilpotent. Finally, we propose a conservation conjecture which is weaker than the conservation of the classical realizations but strong enough to imply the Schur finiteness of constructible motives[1].

In the literature, the functors Ψ_f have two names: they are called "nearby cycles functors" or "vanishing cycles functors". Here we choose to call them the nearby cycles functors. The properties of these functors form what we call the vanishing cycles formalism (as in [9] and [10]).

2. The classical pictures

We briefly recall the construction of the nearby cycles functors $\mathsf{R}\Psi_f$ in étale cohomology. We then explain a construction of Rapoport and Zink which was the starting point of our definition of Ψ_f in the motivic context. After that we shall recall some facts about limits of variations of Hodge structures. A very nice exposition of these matters can be found in [15].

2.1. **The vanishing cycles formalism in étale cohomology.** Let us fix a prime number ℓ and a finite commutative ring Λ such that $\ell^\nu.\Lambda = 0$ for ν large enough. When dealing with étale cohomology, we shall always assume that ℓ is invertible on our schemes. For a reasonable scheme V, we denote by $\mathsf{D}^+(V, \Lambda)$ the derived category of bounded below complexes of étale sheaves on V with values in Λ-modules.

Let S be the spectrum of a strictly henselian DVR (discrete valuation ring). We denote by η the generic point of S and by s the closed point:

$$\eta \xrightarrow{j} S \xleftarrow{i} s \ .$$

We also fix a separable closure $\bar{\eta}$ of the point η. From the point of view of étale cohomology, the scheme S plays the role of a small disk so that η is a punctured small disk and $\bar{\eta}$ is a universal cover of that punctured disk. We will also need the normalization $\bar{S}$ of S in $\bar{\eta}$:

$$\bar{\eta} \xrightarrow{\bar{j}} \bar{S} \xleftarrow{\bar{i}} s \ .$$

[1] *Constructible motives* means *geometric motives* of [40]. They are also the compact objects in the sense Neeman [30] (see remark 3.3).

Now let $f: X \longrightarrow S$ be a finite type S-scheme. We consider the commutative diagram with cartesian squares

$$\begin{tikzcd} X_\eta \arrow[r, "j"] \arrow[d, "f_\eta"'] & X \arrow[d, "f"] & X_s \arrow[l, "i"'] \arrow[d, "f_s"] \\ \eta \arrow[r, "j"] & S & s \arrow[l, "i"'] \end{tikzcd} .$$

Following Grothendieck (see [10]), we look also at the diagram

$$\begin{tikzcd} X_{\bar\eta} \arrow[r, "\bar j"] \arrow[d, "f_{\bar\eta}"'] & \bar X \arrow[d, "\bar f"] & X_s \arrow[l, "\bar i"'] \arrow[d, "f_s"] \\ \bar\eta \arrow[r, "\bar j"] & \bar S & s \arrow[l, "\bar i"'] \end{tikzcd}$$

obtained in the same way by base-changing the morphism f. (This is what we will call the "Grothendieck trick"). We define then the triangulated functor:

$$\mathsf{R}\Psi_f : \mathsf{D}^+(X_\eta, \Lambda) \longrightarrow \mathsf{D}^+(X_s, \Lambda)$$

by the formula: $\mathsf{R}\Psi_f(A) = \bar{i}^*\mathsf{R}\bar{j}_*(A_{X_{\bar\eta}})$ for $A \in \mathsf{D}^+(X_\eta, \Lambda)$. By construction, the functor $\mathsf{R}\Psi_f$ comes with an action of the Galois group of $\bar\eta/\eta$, but we will not explicitly use this here. The basic properties of these functors concern the relation between $\mathsf{R}\Psi_g$ and $\mathsf{R}\Psi_{g\circ h}$ (see [9]):

Proposition 2.1 — *Let $g: Y \longrightarrow S$ be an S-scheme and suppose given an S-morphism $h: X \longrightarrow Y$ such that $f = g \circ h$. We form the commutative diagram*

$$\begin{tikzcd} X_\eta \arrow[r, "j"] \arrow[d, "h_\eta"'] & X \arrow[d, "h"] & X_s \arrow[l, "i"'] \arrow[d, "h_s"] \\ Y_\eta \arrow[r, "j"] & Y & Y_s . \arrow[l, "i"'] \end{tikzcd}$$

There exist natural transformations of functors

- $\alpha_h : h_s^*\mathsf{R}\Psi_g \longrightarrow \mathsf{R}\Psi_f h_\eta^*$,
- $\beta_h : \mathsf{R}\Psi_g \mathsf{R}h_{\eta*} \longrightarrow \mathsf{R}h_{s*}\mathsf{R}\Psi_f$.

Furthermore, α_h is an isomorphism when h is smooth and β_h is an isomorphism when h is proper.

The most important case, is maybe when $g = \mathrm{id}_S$ and $f = h$. Using the easy fact that $\mathsf{R}\Psi_{\mathrm{id}_S}\Lambda = \Lambda$, we get that:

- $\mathsf{R}\Psi_f\Lambda = \Lambda$ if f is smooth,
- $\mathsf{R}\Psi_{\mathrm{id}_S}\mathsf{R}f_{\eta*}\Lambda = \mathsf{R}f_{s*}\mathsf{R}\Psi_f\Lambda$ if f is proper.

The last formula can be rewritten in the following more expressive way: $\mathrm{H}^*_{\text{ét}}(X_{\bar\eta}, \Lambda) = \mathrm{H}^*_{\text{ét}}(X_s, \mathsf{R}\Psi_f\Lambda)$. In concrete terms, this means that for a proper S-scheme X, the étale cohomology of the constant sheaf on the generic geometric fiber $X_{\bar\eta}$ is isomorphic to the étale cohomology of the special fiber X_s with value in the complex of nearby cycles $\mathsf{R}\Psi_f\Lambda$. This is a very useful fact, because usually the scheme X_s is simpler than $X_{\bar\eta}$ and the complex $\mathsf{R}\Psi_f\Lambda$ can often be computed using local methods.

2.2. **The Rapoport-Zink construction.** We keep the notations of the previous paragraph. We now suppose that X is a semi-stable S-scheme i.e. locally for the étale topology X is isomorphic to the standard scheme $S[t_1, \ldots, t_n]/(t_1 \ldots t_r - \pi)$ where π is a uniformizer of S and $r \leq n$ are positive integers. In [32], Rapoport and Zink constructed an important model of the complex $\mathsf{R}\Psi_f(\Lambda)$. Their construction is based on the following two facts:

- There exists a canonical arrow $\theta : \Lambda_\eta \longrightarrow \Lambda_\eta(1)[1]$ in $\mathsf{D}^+(\eta, \Lambda)$ called the fundamental class with the property that the composition $\theta \circ \theta$ is zero,
- The morphism $\theta : i^*\mathsf{R}j_*\Lambda \longrightarrow i^*\mathsf{R}j_*\Lambda(1)[1]$ in $\mathsf{D}^+(X_s, \Lambda)$ has a representative on the level of complexes $\underline{\theta} : \mathcal{M}^\bullet \longrightarrow \mathcal{M}^\bullet(1)[1]$ such that the composition
$$\mathcal{M}^\bullet \longrightarrow \mathcal{M}^\bullet(1)[1] \longrightarrow \mathcal{M}^\bullet(2)[2]$$
is zero as a map of complexes.

Therefore we obtain a double complex
$$\mathcal{RZ}^{\bullet,\bullet} = [\cdots \to 0 \to \mathcal{M}^\bullet(1)[1] \to \mathcal{M}^\bullet(2)[2] \to \mathcal{M}^\bullet(3)[3] \to \cdots \to \mathcal{M}^\bullet(n)[n] \to \ldots]$$
where the complex $\mathcal{M}^\bullet(1)[1]$ is placed in degree zero. Furthermore, following Rapoport and Zink, we get a map $\mathsf{R}\Psi_f\Lambda \longrightarrow \mathsf{Tot}(\mathcal{RZ}^{\bullet,\bullet})$ which is an isomorphism in $\mathsf{D}^+(X_s, \Lambda)$ (see [32] for more details). Here $\mathsf{Tot}(-)$ means the simple complex associated to a double complex. In particular, Rapoport and Zink's result says that the nearby cycles complex $\mathsf{R}\Psi_f\Lambda$ can be constructed using two ingredients:

- The complex $i^*\mathsf{R}j_*\Lambda$,
- The fundamental class θ.

Our construction of the nearby cycles functor in the motivic context is inspired by this fact. Indeed, the above ingredients are motivic (see 4.1 for a definition of the motivic fundamental class). We will construct in paragraph 4.2 a motivic analogue of $\mathcal{RZ}^{\bullet,\bullet}$ based on these two motivic ingredients and then define the (unipotent) "motivic nearby cycles" to be the associated total motive. In fact, for technical reasons, we preferred to use a motivic analogue of the dual version of $\mathcal{RZ}^{\bullet,\bullet}$. By the dual of the Rapoport-Zink complex, we mean the bicomplex
$$\mathcal{Q}^{\bullet,\bullet} = [\cdots \to \mathcal{M}^\bullet(-n)[-n] \to \cdots \to \mathcal{M}^\bullet(-1)[-1] \to \mathcal{M}^\bullet \to 0 \to \ldots]$$
where the complex $\mathcal{M}^\bullet$ is placed in degree zero. It is true that by passing to the total complex, the double complex $\mathcal{Q}^{\bullet,\bullet}$ gives in the same way as $\mathcal{RZ}^{\bullet,\bullet}$ the nearby cycles complex.

2.3. **The limit of a variation of Hodge structures.** Let D be a small analytic disk, 0 a point of D and $D^\star = D - 0$. Let $f : X^\star \longrightarrow D^\star$ be an analytic family of smooth projective varieties. For $t \in D^\star$, we denote by X_t the fiber $f^{-1}(t)$ of f. For any integer q, the local system $\mathsf{R}^q f_*\mathbb{C} = (\mathsf{R}^q f_*\mathbb{Z}) \otimes \mathbb{C}$ on $D^\star$ with fibers $(\mathsf{R}^q f_*\mathbb{C})_t = \mathrm{H}^q(X_t, \mathbb{C})$ is the sheaf of horizontal sections of the Gauss-Manin connection ∇ on $\mathsf{R}^q f_*\Omega^\cdot_{X^\star/D^\star}$. The decreasing filtration F^k on the de Rham complex $\Omega^\cdot_{X^\star/D^\star}$ given by
$$F^k\Omega^\cdot_{X^\star/D^\star} = [0 \to \ldots 0 \to \Omega^k_{X^\star/D^\star} \to \cdots \to \Omega^n_{X^\star/D^\star}]$$
induces a filtration $F^k\mathsf{R}^q f_*\Omega^\cdot_{X^\star/D^\star}$ by locally free $\mathscr{O}_{D^\star}$-submodules on $\mathsf{R}^q f_*\Omega^\cdot_{X^\star/D^\star}$.

For any $t \in D^\star$, we get by applying the tensor product $- \otimes_{\mathscr{O}_{D^\star}} \mathbb{C}(t)$ a filtration F^k on $\mathrm{H}^q(X_t, \mathbb{C})$ which is the Hodge filtration. The data:

- The local system $\mathsf{R}^q f_* \mathbb{Z}$,
- The $\mathscr{O}_{D^\star}$-module $(\mathsf{R}^q f_* \mathbb{Z}) \otimes \mathscr{O}_{D^\star} = \mathsf{R}^q f_* \Omega^{\cdot}_{X^\star/D^\star}$ together with the Gauss-Manin connexion,
- The filtration F^k on $(\mathsf{R}^q f_* \mathbb{Z}) \otimes \mathscr{O}_{D^\star}$

satisfy the Griffiths transversality condition and are called a Variation of (pure) Hodge Structures.

Let us suppose for simplicity that f extends to a semi-stable proper analytic morphism: $X \longrightarrow D$. We denote by $\omega^{\cdot}_{X/D}$ the relative de Rham complex with logarithmic poles on $Y = X - X^\star$, that is,

$$\omega^1_{X/D} = \Omega^1_X(\log(Y))/\Omega^1_D(\log(0)).$$

We fix a uniformizer $t : D \to \mathbb{C}$, a universal cover $\bar{D}^\star \to D^\star$ and a logarithm $\log t$ on $\bar{D}^\star$. In [36], Steenbrink constructed an isomorphism $(\omega^{\cdot}_{X/D})_{|Y} \longrightarrow \mathsf{R}\Psi_f \mathbb{C}$ depending on these choices. From this, he deduced a mixed Hodge structure on $\mathrm{H}^q(Y, (\omega^{\cdot}_{X/D})_{|Y})$ which is by definition the limit of the above Variation of Hodge Structures.

2.4. **The analogy between the situations in étale cohomology and Hodge theory.** Let V be a smooth projective variety defined over a field k of characteristic zero. Suppose also given an algebraic closure $\bar{k}/k$ with Galois group G_k and an embedding $\sigma : k \subset \mathbb{C}$. In the étale case, the ℓ-adic cohomology of $V_{\bar{k}}$ is equipped with a structure of a continuous G_k-module. In the complex analytic case, the Betti cohomology of $V(\mathbb{C})$ is equipped with a Hodge structure.

Now let $f : X \longrightarrow C$ be a flat and proper family of smooth varieties over k parametrized by an open k-curve C. Then for any $\bar{k}$-point t of C, we have a continuous Galois module[2] $\mathrm{H}^q(X_t, \mathbb{Q}_\ell)$. These continuous Galois modules can be thought of as a "Variation of Galois Representations" parametrized by C which is the étale analogue of the Variation of Hodge structures $(\mathrm{H}^q(X_t(\mathbb{C}), \mathbb{Q}), F^k)$ that we discussed in the above paragraph.

Now let s be a point of the boundary of C and choose a uniformizer near s. As in the Hodge–theoretic case, the variation of Galois modules above has a "limit" on s which is a "mixed" Galois module given by the following data:

- A monodromy operator N which is nilpotent. This operator induces the monodromy filtration which turns out to be compatible with the weight filtration of Steenbrink's mixed Hodge structure on the limit cohomology (see [15]),
- The grading associated to the monodromy filtration is a continuous Galois module of "pure" type.

As in the analytic case, this limit is defined via the nearby cycles complex. Indeed, choose an extension of f to a projective scheme X' over $C' = C \cup \{s\}$. Let Y be the special fiber of X'. The choice of a uniformizer gives us a complex $\mathsf{R}\Psi_{X'/C'}\mathbb{Q}_\ell$ on Y. Then the "limit" of our "Variation of Galois representations" is given by $H^q(Y, \mathsf{R}\Psi_{X'/C'}\mathbb{Q}_\ell)$. The monodromy operator N is induced from the representation

[2]In general only an open subgroup of G_k acts on the cohomology, unless t factors trough a k-rational point.

on $\mathsf{R}\Psi_{X'/C'}\mathbb{Q}_\ell$ of the étale fundamental group of the punctured henselian neighbourhood of s in C.

3. Specialization systems

The goal of this section is to axiomatize some formal properties of the nearby cycles functors that we expect to hold in the motivic context. The result will be the notion of specialization systems. We then state some consequences of these axioms which play an important role in the theory. Before doing that we recall briefly the motivic categories we use.

3.1. **The motivic categories.** Let X be a noetherian scheme. In this paper we will use two triangulated categories associated to X:

(1) The motivic stable homotopy category $\mathbf{SH}(X)$ of Morel and Voevodsky,
(2) The stable category of mixed motives $\mathbf{DM}(X)$ of Voevodsky.

These categories are respectively obtained by taking the *homotopy category* (in the sense of Quillen [31]) associated to the two *model categories* of $T = (\mathbb{A}^1_X/\mathbb{G}_{\mathrm{m}X})$-spectra:

(1) The category $\mathbf{Spect}^T_s(X)$ of T-spectra of simplicial sheaves on the smooth Nisnevich site $(\mathsf{Sm}/X)_{\mathrm{Nis}}$,
(2) The category $\mathbf{Spect}^T_{\mathrm{tr}}(X)$ of T-spectra of complexes of sheaves with transfers on the smooth Nisnevich site $(\mathsf{Sm}/X)_{\mathrm{Nis}}$.

Recall that a T-spectrum E is a sequence of objects $(E_n)_{n\in\mathbb{N}}$ connected by maps of the form $E_n \longrightarrow \underline{\mathsf{Hom}}(T, E_{n+1})$. We sometimes denote by $\mathbf{Spect}^T(X)$ one of the two categories $\mathbf{Spect}^T_s(X)$ or $\mathbf{Spect}^T_{\mathrm{tr}}(X)$. We do not intend to give the detailed construction of these model categories as this has already been done in several places (cf. [5], [20], [24], [25], [28], [33], [37]). For the reader's convenience, we however give some indications. We focus mainly on the class of weak equivalences; indeed this is enough to define the homotopy category which is obtained by formally inverting the arrows in this class. The weak equivalences in these two categories of T-spectra are called the stable $\mathbb{A}^1$-weak equivalences and are defined in the three steps. We restrict ourself to the case of simplicial sheaves; the case of complexes of sheaves with transfers is completely analogous.

Step 1. We first define simplicial weak equivalences for simplicial sheaves. A map $A_\bullet \longrightarrow B_\bullet$ of simplicial sheaves on $(\mathsf{Sm}/X)_{\mathrm{Nis}}$ is a simplicial weak equivalence if for any smooth X-scheme U and any point $u \in U$, the map of simplicial sets[3] $A_\bullet(\mathsf{Spec}(\mathscr{O}^h_{U,u})) \longrightarrow B_\bullet(\mathsf{Spec}(\mathscr{O}^h_{U,u}))$ is a weak equivalence (i.e. induces isomorphisms on the set of connected components and on the homotopy groups).

Step 2. Next we perform a Bousfield localization of the simplicial model structure on simplicial sheaves in order to invert the projections $\mathbb{A}^1_U \longrightarrow U$ for smooth X-schemes U (see [13] for a general existence theorem on localizations and [28] for this particular case). The model structure thus obtained is the $\mathbb{A}^1$-model structure on simplicial sheaves over $(\mathsf{Sm}/X)_{\mathrm{Nis}}$. We denote $\mathbf{Ho}_{\mathbb{A}^1}(X)$ the associated homotopy category.

[3]This map of simplicial sets is the stalk of $A_\bullet \longrightarrow B_\bullet$ at the point $u \in U$ with respect to the Nisnevich topology.

Step 3. If A is a pointed simplicial sheaf and $E = (E_n)_n$ is a T-spectrum of simplicial sheaves we define the stable cohomology groups of A with values in E to be the colimit: $\mathsf{Colim}_n \hom_{\mathbf{Ho}_{\mathbb{A}^1}(X)}(T^{\wedge n} \wedge A, E_n)$. We then say that a morphism of spectra $(E_n)_n \longrightarrow (E'_n)_n$ is a stable $\mathbb{A}^1$-weak equivalence if it induces isomorphisms on cohomology groups for every simplicial sheaf A.

By inverting stable $\mathbb{A}^1$-weak equivalences in $\mathbf{Spect}^T_s(X)$ and $\mathbf{Spect}^T_{\mathrm{tr}}(X)$ we get respectively the categories $\mathbf{SH}(X)$ and $\mathbf{DM}(X)$. Let U be a smooth X-scheme. We can associate to U the pointed simplicial sheaf U_+ which is simplicially constant, represented by $U \coprod X$ and pointed by the trivial map $X \longrightarrow U \coprod X$. Then, we can associate to U_+ its infinite T-suspension $\Sigma^\infty_T(U_+)$ given in level n by $T^{\wedge n} \wedge U_+$. This provides a covariant functor $M : \mathsf{Sm}/X \longrightarrow \mathbf{SH}(X)$ which associates to U its motive $M(U)$. Similarly we can associate to U the complex $\mathbb{Z}_{\mathrm{tr}}(U)$, concentrated in degree zero, and then take its infinite suspension given in level n by $\mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^n \times U)/\mathbb{Z}_{\mathrm{tr}}((\mathbb{A}^n - 0) \times U) \simeq T^{\overset{\mathrm{L}}{\otimes} n} \overset{\mathrm{L}}{\otimes} U$. This also gives a covariant functor $M : \mathsf{Sm}/X \longrightarrow \mathbf{DM}(X)$. The images in $\mathbf{SH}(X)$ and $\mathbf{DM}(X)$ of the identity X-scheme are respectively denoted by $\mathbb{I}_X$ and $\mathbb{Z}_X$. When there is no confusion we will drop the index X.

Remark 3.1 — Sometimes it is useful to stop in the middle of the above construction and consider the homotopy category $\mathbf{Ho}_{\mathbb{A}^1}(X)$ of *step 2*. The abelian version with transfers of $\mathbf{Ho}_{\mathbb{A}^1}(X)$ is the category $\mathbf{DM}_{\mathrm{eff}}(X)$ which is used at the end of the paper. This is the category of effective motives whose objects are complexes of Nisnevich sheaves with transfers and morphisms obtained by inverting $\mathbb{A}^1$-weak equivalences.

Remark 3.2 — One can also consider the categories $\mathbf{SH}_{\mathbb{Q}}(X)$ and $\mathbf{DM}_{\mathbb{Q}}(X)$ obtained from $\mathbf{SH}(X)$ and $\mathbf{DM}(X)$ by killing torsion objects (using a Verdier localization) or equivalently by repeating the above three steps using simplicial sheaves and complexes of sheaves with transfers of $\mathbb{Q}$-vector spaces (instead of sets and abelian groups). It is important to note that the categories $\mathbf{SH}_{\mathbb{Q}}(X)$ and $\mathbf{DM}_{\mathbb{Q}}(X)$ are essentially the same at least for X a field. Indeed, an unpublished result of Morel (see however the announcement [27]) claims that $\mathbf{SH}_{\mathbb{Q}}(k)$ decomposes into $\mathbf{DM}_{\mathbb{Q}}(k) \oplus ?(k)$ with $?(k)$ a "small part" equivalent to the zero category unless the field k is formally real (i.e., if (-1) is not a sum of squares in k).

Remark 3.3 — The triangulated categories $\mathbf{SH}(X)$ and $\mathbf{DM}(X)$ have infinite direct sums. It is then possible to speak about compact motives. A motive M is compact if the functor $\hom(M, -)$ commutes with infinite direct sums (see [30]). If U is a smooth X-scheme, then its motive $M(U)$ (in $\mathbf{SH}(X)$ or $\mathbf{DM}(X)$) is known to be compact (see for example [33]). Therefore, the triangulated categories with infinite sums $\mathbf{SH}(X)$ and $\mathbf{DM}(X)$ are compactly generated in the sense of [30]. We shall denote $\mathbf{SH}^{ct}(X)$ and $\mathbf{DM}^{ct}(X)$ the triangulated subcategories of $\mathbf{SH}(X)$ and $\mathbf{DM}(X)$ whose objects are the compact ones. The letters *ct* stand for constructible and we shall call them the categories of constructible motives (by analogy with the notion of constructible sheaves in étale cohomology considered in [2]).

The elementary functorial operators f^*, f_* and $f_\#$ of the categories $\mathbf{SH}(-)$ and $\mathbf{DM}(-)$ are defined by deriving the usual operators f^*, f_* and $f_\#$ on the level of sheaves. For $\mathbf{Ho}_{\mathbb{A}^1}(-)$, the details can be found in [28]. It is possible to extend these operators to spectra (see [34]). For $\mathbf{DM}(-)$ one can follow the same construction. Details will appear in [6]. The tensor product is obtained by using the category of symmetric spectra. The details for $\mathbf{SH}(-)$ can be found in [20]. For $\mathbf{DM}(-)$ this will be included in [6]. Using the elementary functorial operators: f^*, $f_\#$, f_* and $\otimes$, it is possible to fully develop the Grothendieck formalism of the six operators (see chapters I and II of [3]). For example, assuming resolution of singularities one can prove that all the Grothendieck operators preserve constructible motives.

Except for the monodromy triangle, the formalism of motivic vanishing cycles can be developed equally using the categories $\mathbf{SH}(-)$ or $\mathbf{DM}(-)$. In fact, one can more generally work in the context of a *stable homotopical 2-functor*. See [3] for a definition of this notion and for the construction of the functors Ψ in this abstract setting.

3.2. **Definitions and examples.** Let B be a base scheme. We fix a diagram

$$\eta \xrightarrow{j} B \xleftarrow{i} s$$

with j (resp. i) an open (resp. closed) immersion. We do not suppose that B is the spectrum of a DVR or that s is the complement of η. Every time we are given a B-scheme $f : X \longrightarrow B$, we form the commutative diagram with cartesian squares

$$\begin{tikzcd} X_\eta \arrow[r, "j"] \arrow[d, "f_\eta"'] & X \arrow[d, "f"] & X_s \arrow[l, "i"'] \arrow[d, "f_s"] \\ \eta \arrow[r, "j"] & B & s. \arrow[l, "i"'] \end{tikzcd}$$

We recall the following definition from [3], chapter III:

Definition 3.4 — *A* specialization system *sp over (B, j, i) is given by the following data:*

(1) *For a B-scheme $f : X \longrightarrow B$, a triangulated functor:*

$$sp_f : \mathbf{SH}(X_\eta) \longrightarrow \mathbf{SH}(X_s)$$

(2) *For a morphism $g : Y \longrightarrow X$ a natural transformation of functors:*

$$\alpha_g : g_s^* sp_f \longrightarrow sp_{f\circ g} g_\eta^*.$$

These data should satisfy the following three axioms:

- *The natural transformations $\alpha_?$ are compatible with the composition of morphisms. More precisely, given a third morphism $h : Z \longrightarrow Y$, the diagram*

$$\begin{tikzcd} (g\circ h)_s^* sp_f \arrow[rr] \arrow[d, "\sim"'] & & sp_{fgh}(g\circ h)_\eta^* \arrow[d, "\sim"] \\ h_s^* g_s^* sp_f \arrow[r] & h_s^* sp_{fg} g_\eta^* \arrow[r] & sp_{fgh} h_\eta^* g_\eta^* \end{tikzcd}$$

  is commutative,
- *The natural transformation α_g is an isomorphism when g is smooth,*

- *If we define the natural transformation* $\beta_g : sp_f g_{\eta *} \longrightarrow g_{s*} sp_{f \circ g}$ *by the composition*

$$sp_f g_{\eta *} \longrightarrow g_{s*} g_s^* sp_f g_{\eta *} \xrightarrow{\alpha_g} g_{s*} sp_{fg} g_\eta^* g_{\eta *} \longrightarrow g_{s*} sp_{fg}$$

then β_g *is an isomorphism when* g *is projective.*

Remark 3.5 — A morphism $sp \longrightarrow sp'$ of specialization systems is a collection of natural transformations $sp_f \longrightarrow sp'_f$, one for every B-scheme f, commuting with the α_g, i.e., such that the squares

$$\begin{tikzcd} g_s^* sp_f \arrow[r] \arrow[d] & sp_{fg} g_\eta^* \arrow[d] \\ g_s sp'_f \arrow[r] & sp'_{fg} f_\eta^* \end{tikzcd}$$

are commutative.

Remark 3.6 — Let us keep the notations of the Definition 3.4. It is possible to construct from $\alpha_?$ two natural transformations (see chapter III of [3])

$$sp_{f\circ g} g_\eta^! \longrightarrow g_s^! sp_f \quad \text{and} \quad g_{s!} sp_{f \circ g} \longrightarrow sp_f g_{\eta !} \,.$$

These natural transformations are important for the study of the action of the duality operators on the motivic nearby cycles functors in paragraph 4.5. However, we will not need them for the rest of the paper.

Remark 3.7 — The above definition makes sense for any stable homotopical 2-functor from the category of schemes to the 2-category of triangulated categories (see chapter I of [3]). In particular, one can speak about specialization systems in $\mathbf{DM}(-)$, $\mathbf{SH}_{\mathbb{Q}}(-)$ and of course in $\mathsf{D}^+(-, \Lambda)$. For example, the family of nearby cycles functors $\Psi = (\Psi_f)_{f \in \mathsf{Fl}(\mathsf{Sch})}$ of the paragraph 2.1 is in a natural way a specialization system in $\mathsf{D}^+(-, \Lambda)$ with base (S, j, i).

Example 3.8 — It is easy to produce examples of specialization systems. The most simple (but still very interesting) example is what we call in chapter III of [3] the canonical specialization system χ. It is defined by $\chi_f(A) = i^* j_*(A)$.

Example 3.9 — Given a specialization system sp and an object $E \in \mathbf{SH}(\eta)$, we can define a new specialization system by the formula: $sp'_f(-) = sp_f(- \otimes f_\eta^* E)$. In the same way, given an object F of $\mathbf{SH}(s)$, we define a third specialization system by the formula: $sp''_f(-) = sp_f(-) \otimes f_s^* F$.

3.3. **The basic results.** We state here some (non-trivial) results that follow from the axioms of Definition 3.4. For the proofs (which are too long to be included here) the reader can consult chapter III of [3]. For simplicity, we shall stick to the case where B is an affine, smooth and geometrically irreducible curve over a field k of *characteristic zero*, s a closed point of B and η a non-empty open subscheme of $B - s$ or the generic point of B.

We fix a section $\pi \in \Gamma(B, \mathscr{O}_B)$ which we suppose to have a zero of order one on s and to be invertible on η. We then define for $n \in \mathbb{N}$, two simple B-schemes:

- $B_n = B[t]/(t^n - \pi)$ and $e_n : B_n \longrightarrow B$ the obvious morphism,

- $B'_n = B[t,u,u^{-1}]/(t^n - u.\pi)$ and $e'_n : B'_n \longrightarrow B$ the obvious morphism.

Recall that the unit objects of $\mathbf{SH}(X)$ and $\mathbf{DM}(X)$ were respectively denoted by $\mathbb{I} = \mathbb{I}_X$ and $\mathbb{Z} = \mathbb{Z}_X$. We shall also denote by $\mathbb{Q} = \mathbb{Q}_X$ the unit object of $\mathbf{DM}_{\mathbb{Q}}(X)$. The proofs of the following three theorems are in [3], chapter III.

Theorem 3.10 — ***1-*** *Let sp be a specialization system over (B,j,i) for* $\mathbf{SH}$ *(resp. for* $\mathbf{DM}$*). Suppose that for all $n \in \mathbb{N}$, the objects:*

- $sp_{e_n}(\mathbb{I}) \in \mathsf{Ob}(\mathbf{SH}((B_n)_s))$ *(resp.* $sp_{e_n}(\mathbb{Z}) \in \mathsf{Ob}(\mathbf{DM}((B_n)_s))$*),*
- $sp_{e'_n}(\mathbb{I}) \in \mathsf{Ob}(\mathbf{SH}((B'_n)_s))$ *(resp.* $sp_{e'_n}(\mathbb{Z}) \in \mathsf{Ob}(\mathbf{SH}((B'_n)_s))$*),*

are constructible (see remark 3.3). Then for any B-scheme $f : X \longrightarrow B$, and any constructible object A of $\mathbf{SH}(X_\eta)$ *(resp.* $\mathbf{DM}(X_\eta)$*), the object $sp_f(A)$ is constructible.*

2- *Let sp be a specialization system over (B,j,i) for* $\mathbf{DM}_{\mathbb{Q}}(-)$*. Suppose that for all $n \in \mathbb{N}$, the objects $sp_{e_n}(\mathbb{Q}) \in \mathbf{DM}_{\mathbb{Q}}(s)$ are constructible. Then for any B-scheme $f : X \longrightarrow B$, and any constructible object $A \in \mathbf{DM}_{\mathbb{Q}}(X_\eta)$, the object $sp_f(A)$ is constructible.*

The following result will play an important role:

Theorem 3.11 — ***1-*** *Let $sp \longrightarrow sp'$ be a morphism between two specialization systems over (B,j,i) for* $\mathbf{SH}$ *(resp.* $\mathbf{DM}$*). Suppose that for every $n \in \mathbb{N}$, the induced morphisms:*

- $sp_{e_n}(\mathbb{I}) \longrightarrow sp'_{e_n}(\mathbb{I})$ *(resp.* $sp_{e_n}(\mathbb{Z}) \longrightarrow sp'_{e_n}(\mathbb{Z})$ *),*
- $sp_{e'_n}(\mathbb{I}) \longrightarrow sp'_{e'_n}(\mathbb{I})$ *(resp.* $sp_{e'_n}(\mathbb{Z}) \longrightarrow sp'_{e'_n}(\mathbb{Z})$ *),*

are isomorphisms.

Then for any B-scheme $f : X \longrightarrow B$, and any constructible object A of $\mathbf{SH}(X_\eta)$ *(resp. of* $\mathbf{DM}(X_\eta)$*) the morphism*

$$sp_f(A) \longrightarrow sp'_f(A)$$

is an isomorphism. When sp_f and sp'_f both commute with infinite sums, the constructibility condition on A can be dropped.

2- *If we are working in* $\mathbf{DM}_{\mathbb{Q}}(-)$ *the same conclusions hold under the following weaker condition: For every $n \in \mathbb{N}$ the morphisms $sp_{e_n}(\mathbb{Q}) \longrightarrow sp'_{e_n}(\mathbb{Q})$ are isomorphisms.*

Remark 3.12 — In part *2* of Theorems 3.8 and 3.9, we cannot replace $\mathbf{DM}_{\mathbb{Q}}$ by $\mathbf{SH}_{\mathbb{Q}}$. Indeed, we use in an essential way the fact that the stable homotopical 2-functor $\mathbf{DM}_{\mathbb{Q}}$ is separated (like "separated" for presheaves) (see chapter II of [3]), that is, the functor e^* is conservative for a finite surjective morphism e. This property for $\mathbf{DM}_{\mathbb{Q}}$ is easily proved by reducing to a finite field extension and using transfers. It fails for $\mathbf{SH}_{\mathbb{Q}}$ already for the morphism $\mathsf{Spec}(\mathbb{C}) \longrightarrow \mathsf{Spec}(\mathbb{R})$. However, using Morel's result [27], one sees that $\mathbf{SH}_{\mathbb{Q}}$ is separated when restricted to the category of schemes on which (-1) is a sum of squares.

The previous two theorems are deduced using resolution of singularities from the following result:

Theorem 3.13 — *Let sp be a specialization system over B. Let $f: X \longrightarrow B$ be a B-scheme. Suppose that X is regular, X_s is a reduced normal crossing divisor in X and fix a smooth branch $D \subset X_s$. We denote by D^0 the smooth locus of f contained in D, i.e., D^0 is the complement in X_s of the union of all the branches that meet D properly. Let us denote by u the closed immersion $D \subset X_s$ and v the open immersion $D^0 \subset D$. The obvious morphism $id \longrightarrow v_* v^*$ induces an isomorphism: $[u^* sp_f f_\eta^*] \longrightarrow v_* v^* [u^* sp_f f_\eta^*]$. Furthermore, if p is the projection of D^0 over s then $v^*[u^* sp_f f_\eta^*] \simeq p^* sp_{id_B}$.*

Remark 3.14 — The previous theorem says that in good situations, the knowledge of $sp_{\mathrm{id}_B}\mathbb{I}$ suffices to determine (up to extension problems) the motive $sp_f\mathbb{I}$.

Remark 3.15 — If we work in $\mathbf{DM}_{\mathbb{Q}}(-)$ and over a field of characteristic zero, then we can drop the condition that X_s is reduced in Theorem 3.11 and still have an isomorphism: $[u^* sp_f f_\eta^*] \longrightarrow v_* v^* [u^* sp_f f_\eta^*]$. However it is no longer true that $v^*[u^* sp_f f_\eta^*] \simeq p^* sp_{\mathrm{id}_B}$, unless the branch D is of multiplicity one.

Example 3.16 — To help the reader understand the content of Theorem 3.11, we use it to make a computation in a familiar situation:

- We shall work with étale cohomology, that is in the stable homotopical 2-functor $\mathsf{D}^+(-, \Lambda)$, and with the nearby cycles specialization system $\mathsf{R}\Psi$.
- We take $f: X \longrightarrow S$ to be a semi-stable curve (not necessarily proper) over a henselian discrete valuation ring S. We suppose that X_s has two branches D_1 and D_2 that meet in a point $C = D_1 \cap D_2$.

We will compute the cohomology sheaves of the complex of nearby cycles $\mathsf{R}\Psi_f\Lambda$. We have the following commutative diagram:

$$\begin{tikzcd} & D_1 \arrow[dr, "u_1"] & \\ C \arrow[ur, "c_1"] \arrow[rr, "c"] \arrow[dr, "c_2"'] & & X_s \\ & D_2. \arrow[ur, "u_2"'] & \end{tikzcd}$$

For $i \in \{1, 2\}$ we denote $v_i: D_i - C \longrightarrow D_i$ the inclusion of the smooth locus of f in D_i. By Theorem 3.11 the restriction $u_i^* \mathsf{R}\Psi_f\Lambda$ of $\mathsf{R}\Psi_f\Lambda$ to D_i is given by $\mathsf{R}v_{i*}\Lambda$. As v_i is the complement of a closed point in a smooth curve over a field we know that $\mathsf{R}^p v_{i*}\Lambda = 0$ for $p \notin \{0, 1\}$, $\mathsf{R}^0 v_{i*}\Lambda = \Lambda_{D_i}$ and $\mathsf{R}^1 v_{i*}\Lambda = c_{i*}\Lambda(-1)$. This immediately gives that $\mathsf{R}^p\Psi_f\Lambda = 0$ for $p \notin \{0, 1\}$, $\mathsf{R}^0\Psi_f\Lambda = \Lambda_{X_s}$ and $\mathsf{R}^1\Psi_f\Lambda = c_*\Lambda(-1)$.

4. Constructing the vanishing cycles formalism

The goal of this section is to construct in the motivic context a specialization system (in the sense of 3.4) that behaves as much as possible like the nearby cycles functors in étale cohomology. We begin by explaining why the definition of $\mathsf{R}\Psi_f$ given in paragraph 2.1 does not give the right functors in the motivic context. Let S be as in 2.1. For an S-scheme $f: X \longrightarrow S$ consider the functor

$$\Phi_f : \mathbf{DM}(X_\eta) \longrightarrow \mathbf{DM}(X_s)$$

defined by the formula $\Phi_f(A) = \bar{i}^*\bar{j}_* A_{|X_{\bar{\eta}}}$. It is easy to check that Φ is indeed a specialization system over S. There is at least one problem with this definition: we have $\Phi_{\mathrm{id}}(\mathbb{Z}) \neq \mathbb{Z}$ (this means for example that Φ_{id} cannot be monoidal). Indeed, let k be an algebraically closed field of characteristic zero and suppose that S is the henselization of the affine line over $\mathbb{A}^1_k = \mathsf{Spec}(k[T])$ in its zero section. In this case, $\bar{S}$ is the limit of $S_n \longrightarrow S$ where $S_n = S[T^{1/n}]$. To compute $\Phi_{\mathrm{id}_S}\mathbb{Z}$, we consider the diagrams:

$$\begin{tikzcd} \eta_n \arrow[r, "j_n"] \arrow[d, "(e_n)_\eta"'] & S_n \arrow[d, "e_n"] & s \arrow[l, "i_n"'] \arrow[d, equal] \\ \eta \arrow[r, "j"] & S & s. \arrow[l, "i"'] \end{tikzcd}$$

By definition, $\Phi_{\mathrm{id}_B}\mathbb{Z}$ is the colimit over $n \in \mathbb{N}^\times$ of $i_n^* j_{n*}\mathbb{Z}$. By an easy computation, we have that $i_n^* j_{n*}\mathbb{I} = \mathbb{Z} \oplus \mathbb{Z}(-1)[-1]$ and for n dividing m the morphism $i_n^* j_{n*}\mathbb{Z} \longrightarrow i_m^* j_{m*}\mathbb{Z}$ is given by the matrix:

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{m}{n} \end{pmatrix} : \mathbb{Z} \oplus \mathbb{Z}(-1)[-1] \longrightarrow \mathbb{Z} \oplus \mathbb{Z}(-1)[-1] .$$

Because we are working with integral coefficients, it follows that $\Phi_{\mathrm{id}_S}\mathbb{Z}$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Q}(-1)[-1]$. This problem disappears in étale cohomology, where the colimit of the diagram $(- \times \frac{m}{n} : \Lambda \longmapsto \Lambda)_{n \text{ divides } m}$ is zero because Λ is torsion.

4.1. **The idea of the construction.** We will construct the specialization system Ψ out of the canonical specialization system χ of example 3.8. It is possible to make the definition over an arbitrary base of dimension one, but unfortunately our main results are known to hold only over an equi-characteristic zero base. This is because Theorems 3.8 and 3.9 are not true (as they are stated here) when the special point s is of positive characteristic [4] even if one assumes resolution of singularities. For some details on more general situations, the reader can consult [3], chapter III. In this paper we consider only the base

$$\begin{tikzcd} \eta = \mathbb{G}_{\mathrm{m}} \arrow[r, "j"] \arrow[dr] & B = \mathbb{A}^1_k \arrow[d] & s = 0 \arrow[l, "i"'] \arrow[dl, equal] \\ & k & \end{tikzcd}$$

where k is a field of characteristic zero (not necessarily algebraically closed) and s is the zero section of the affine line. Note that whenever we have a smooth affine curve C, a closed point $x \in C$ and a function $\pi_C \in \Gamma(C, \mathscr{O}_C)$ invertible on $C - x$ with $\pi_C(x) = 0$, we get by restriction a specialization system over C induced by the map $\pi_C : C \longrightarrow \mathbb{A}^1_k$. So the real restriction is not to work over $\mathbb{A}^1_k$ but to work over an equi-characteristic zero base.

In the rest of the section, we will denote by π the variable so that $\mathbb{A}^1_k = \mathsf{Spec}(k[\pi])$. We shall also use the notations in section 3. Theorem 3.9 shows that a specialization system is to a large extent determined by its values at $\mathbb{I}_\eta \in \mathbf{SH}(\eta)$. Thus our main objective will be to find a specialization system Υ over B such that: $\Upsilon_{\mathrm{id}_B}\mathbb{I} = \mathbb{I}$. We

[4]Part *2* of Theorems 3.8 and 3.9 is valid over an arbitrary base of dimension 1 if in the condition we replace b_n by any quasi-finite extension of B (see [3]).

then modify Υ by a variant of the "Grothendieck trick" to get the *nearby cycles specialization system* Ψ.

In order to obtain Υ from the canonical specialization system $\chi = i^*j_*$ of Example 3.8 one has to kill in $\chi_{\mathrm{id}_B}\mathbb{I} = \mathbb{I} \oplus \mathbb{I}(-1)[-1]$ the component $\mathbb{I}(-1)[-1]$. A way to do this is to look for an object $P \in \mathbf{SH}(\eta)$ such that $\chi_{\mathrm{id}}P = \mathbb{I}$ and then take the specialization system $\chi(- \otimes P)$ as in Example 3.9. In $\mathbf{DM}_{\mathbb{Q}}$ there is a natural candidate for such P given by the motivic "logarithmic sheaf" over $\mathbb{G}\mathrm{m}$ (see § 4.6). With integral coefficients, we do not know of any natural object $P \in \mathbf{SH}(\mathbb{G}\mathrm{m})$ with this property, but there is a diagram $\mathcal{A}^\vee$ of motives in $\mathbf{SH}(\mathbb{G}\mathrm{m}_k)$ such that the homotopy colimit of $\chi_{\mathrm{id}_B}(\mathcal{A}^\vee)$ is indeed $\mathbb{I}$. In fact, $\mathcal{A}^\vee$ can be taken to be the "simplicial motive" obtained from the "cosimplicial motive" $\mathcal{A}$ of the next paragraph by applying $\underline{\mathsf{Hom}}(-, \mathbb{I})$ component-wise. The simplicial motive $\mathcal{A}^\vee$ gives the motivic analogue of (the dual of) the Rapoport-Zink bicomplex $\mathcal{Q}^{\bullet,\bullet}$ in étale cohomology (see remark 4.6). So we will define the unipotent part Υ of the nearby cycles to be the "homotopy colimit" of the simplicial specialization system $\chi(- \otimes \mathcal{A}^\vee)$.

4.2. **The cosimplicial motive $\mathcal{A}$ and the construction of Υ.** We mentioned that our construction is inspired by the Rapoport-Zink bicomplex $\mathcal{Q}^{\bullet,\bullet}$. We also pointed out that this bicomplex is built from $i^*\mathsf{R}j_*\Lambda$ and the fundamental class morphism $\theta : \Lambda \longrightarrow \Lambda(1)[1]$. The motivic analogue of $i^*\mathsf{R}j_*\Lambda$ is of course $i^*j_*\mathbb{I}$. We describe the motivic fundamental class in Definition 4.1 below. Recall that given an X-scheme U and a section $s : X \longrightarrow U$, we denote by (U, s) the X-scheme pointed by s. The motive $M(U, s)$ of a pointed X-scheme (U, s) is the cofiber of $M(X) \xrightarrow{M(s)} M(U)$. Moreover, we have a canonical decomposition $M(U) = \mathbb{I}_X \oplus M(U, s)$. For example, the motive of $(\mathbb{G}\mathrm{m}_X, 1)$ is by definition $\mathbb{I}_X(1)[1]$ and $M(\mathbb{G}\mathrm{m}_X) = \mathbb{I}_X \oplus \mathbb{I}_X(1)[1]$.

Definition 4.1 — *The* motivic fundamental class $\theta : \mathbb{I} \longrightarrow \mathbb{I}(1)[1]$ *is the morphism in* $\mathbf{SH}(\mathbb{G}m)$ *defined by the diagram*

$$\begin{tikzcd}
\mathbb{I}_{\mathbb{G}m} \arrow[rrr, "\theta"] \arrow[d, equal] & & & \mathbb{I}_{\mathbb{G}m}(1)[1] \arrow[d, equal] \\
M[\mathbb{G}m \xmapsto{id} \mathbb{G}m] \arrow[r, "\Delta"] & M[\mathbb{G}m \times \mathbb{G}m \xmapsto{pr_1} \mathbb{G}m] \arrow[rr] & & M[\mathbb{G}m \times (\mathbb{G}m, 1) \xmapsto{pr_1} \mathbb{G}m]
\end{tikzcd}$$

where M is the "associated motive" functor, $[X \xmapsto{f} \mathbb{G}\mathrm{m}_k]$ denotes a $\mathbb{G}m$-scheme X, pr_1 is the projection to the first factor and Δ is the diagonal immersion.

In $\mathbf{DM}(\mathbb{G}\mathrm{m})$, one can equivalently define θ as an element of the motivic cohomology group $\mathrm{H}^{1,1}(\mathbb{G}\mathrm{m}) = \Gamma(\mathbb{G}\mathrm{m}, \mathscr{O}^\times)$ because of the identification

$$\hom_{\mathbf{DM}(\mathbb{G}\mathrm{m}_k)}(\mathbb{Z}, \mathbb{Z}(1)[1]) = \hom_{\mathbf{DM}(k)}(\mathbb{G}\mathrm{m}, \mathbb{Z}(1)[1]).$$

It corresponds then to the class of the variable $\pi \in k[\pi, \pi^{-1}]$. It is an easy exercise to check that the étale realization of θ gives indeed the classical fundamental class. One difference with the classical situation is that $\theta \circ \theta$ is non zero even in $\mathbf{DM}$. In fact $\theta \circ \theta$ corresponds in the Milnor K-theory group $K_2^M(k[T, T^{-1}])$ to the symbol $\{T, T\} = \{T, -1\}$ which is 2-torsion. Of course one can kill 2-torsion, and try to find a representative $\underline{\theta}$ of θ such that $\underline{\theta}^2$ is zero in the model category. We shall do something different. Note first the following lemma:

Lemma 4.2 — *Let $\mathcal{C}$ be a category having direct products. Consider a diagram in $\mathcal{C}$:*

$$A \xrightarrow{f} B \xleftarrow{f'} A' .$$

There exists a cosimplicial object $(A\tilde{\times}_B A')^\bullet$ in $\mathcal{C}$ such that for $n \in \mathbb{N}$, we have:

- $(A\tilde{\times}_B A')^n = A \times B \times \cdots \times B \times A' = A \times B^n \times A'$,
- $d^0(a, b_1, \ldots, b_n, a') = (a, f(a), b_1, \ldots, b_n, a')$,
- $d^{n+1}(a, b_1, \ldots, b_n, a') = (a, b_1, \ldots, b_n, f'(a'), a')$,
- *For* $1 \leq i \leq n$, $d^i(a, b_1, \ldots, b_n, a') = (a, b_1, \ldots, b_i, b_i, \ldots, b_n, a')$,
- *For* $1 \leq i \leq n-1$, $s^i(a, b_1, \ldots, b_n, a') = (a, b_1, \ldots, b_i, b_{i+2}, \ldots, b_n, a')$

where a, a' and the b_i are respectively elements of $\hom(X, A)$, $\hom(X, A')$ *and* $\hom(X, B)$ *for a fixed object X of $\mathcal{C}$. Moreover, if f is an isomorphism then the obvious morphism $(A\tilde{\times}_B A') \longrightarrow A'$ is a cosimplicial cohomotopy equivalence*[5]*, where A' is the constant cosimplicial object with value A'.*

We apply Lemma 4.2 to the following diagram in the category $\mathsf{Sm}/\mathbb{G}_m$ of smooth $\mathbb{G}_m$-schemes:

$$[\mathbb{G}_m \xmapsto{\mathrm{id}} \mathbb{G}_m] \xrightarrow{\Delta} [\mathbb{G}_m \times \mathbb{G}_m \xmapsto{pr_1} \mathbb{G}_m] \xleftarrow{(x,1)} [\mathbb{G}_m \xmapsto{\mathrm{id}} \mathbb{G}_m].$$

We denote by $\mathcal{A}^\bullet$ the cosimplicial $\mathbb{G}_m$-scheme thus obtained. We will usually look at $\mathcal{A}^\bullet$ as a cosimplicial object in the model category of T-spectra over $\mathbb{G}_m$: $\mathbf{Spect}^T_s(\mathbb{G}_m)$ or $\mathbf{Spect}^T_{\mathrm{tr}}(\mathbb{G}_m)$. We claim (and show in remark 4.6) that for a semi-stable B-scheme f the simplicial object $\chi_f \underline{\mathsf{Hom}}(\mathcal{A}^\bullet, \mathbb{I})$ is the motivic analogue of the double-complex $\mathcal{Q}^{\bullet,\bullet}$ of the paragraph 2.2. This motivates the following definition:

Definition 4.3 — *Let $f : X \longrightarrow \mathbb{A}^1_k$ be a morphism of schemes. Let E be an object of $\mathbf{Spect}^T_s(X_\eta)$. We put*

$$\Upsilon_f(E) = \mathsf{Tot}\left[\mathsf{L}^{\mathbb{A}^1} i^* \mathsf{R}^{\mathbb{A}^1} j_* \underline{\mathsf{Hom}}(f_\eta^* \mathcal{A}^\bullet, (E)_{\mathbb{A}^1-Fib})\right]$$

where $(E)_{\mathbb{A}^1-Fib}$ is a functorial fibrant replacement and $\mathsf{L}^{\mathbb{A}^1} i^*$ and $\mathsf{R}^{\mathbb{A}^1} j_*$ are the left and right derived functors of i^* and j_* on the level of T-spectra. (Everything being with respect to the stable $\mathbb{A}^1$-model structure.) This functor sends stable $\mathbb{A}^1$-weak equivalences to stable $\mathbb{A}^1$-weak equivalences and induces a triangulated functor $\Upsilon_f : \mathbf{SH}(X_\eta) \longrightarrow \mathbf{SH}(X_s)$ which we call the motivic unipotent nearby cycles functor.

Remark 4.4 — Recall that the functor Tot associates to a simplicial object in a model category $\mathcal{M}$ its homotopy colimit. It is simply the left derived functor of the functor $\pi_0 : \mathbf{\Delta}^{\mathrm{op}}\mathcal{M} \longrightarrow \mathcal{M}$ that associates to a simplicial object $E_\bullet$ the equalizer of the two first cofaces: $E_1 \rightrightarrows E_0$. In our case the Tot functor has a simple description. Indeed, a simplicial object in the category of simplicial sheaves is simply a bisimplicial sheaf and its homotopy colimit is given by the restriction to

[5]By a cosimplicial cohomotopy equivalence we mean that if we view this cosimplicial morphism as a simplicial morphism between simplicial objects with values in the category $\mathcal{C}^{\mathrm{op}}$, then it is a simplicial homotopy equivalence. Note that the notion of a simplicial homotopy equivalence is combinatorial and makes sense for any category.

the diagonal $\mathbf{\Delta} \longrightarrow \mathbf{\Delta} \times \mathbf{\Delta}$. Similarly, the homotopy colimit of a simplicial object in the category of complexes of sheaves is given by the total complex associated to the double complex obtained by taking the alternating sum of the cofaces.

Remark 4.5 — A better way to define the functors Υ_f is to use the categories $\mathbf{SH}(-, \mathbf{\Delta})$ which are obtained as the homotopy categories of the model categories $\mathbf{\Delta}^{\mathrm{op}}\mathbf{Spect}_s^T(-)$. One can take for example the Reedy model structure induced from the stable $\mathbb{A}^1$-model structure on $\mathbf{Spect}_s^T(-)$ (or another one depending on the functor we want to derive). Our functor Υ_f is then the following composition of triangulated functors:

$$\mathbf{SH}(X_\eta) \xrightarrow{\underline{\mathsf{Hom}}(\mathcal{A}^\bullet,-)} \mathbf{SH}(X_\eta, \mathbf{\Delta}) \xrightarrow{j_*} \mathbf{SH}(X, \mathbf{\Delta}) \xrightarrow{i^*} \mathbf{SH}(X_s, \mathbf{\Delta}) \xrightarrow{\mathsf{Tot}} \mathbf{SH}(X_s).$$

Even better, one can use the notion of algebraic derivator to define Υ_f using only basic operators of the form a^*, a_* and $a_\#$. This is the point of view we use in [3].

Remark 4.6 — Let us explain the relation between our definition and the Rapoport-Zink bicomplex $\mathcal{Q}^{\bullet,\bullet}$. We will work with Nisnevich sheaves with transfers over $\mathsf{Sm}/\mathbb{G}\mathrm{m}$. Let $\mathrm{N}(\mathcal{A})$ be the normalized complex of sheaves with transfers associated to the cosimplicial sheaf $\mathbb{Z}_{\mathrm{tr}}(\mathcal{A}^\bullet)$. The complex $\mathrm{N}(\mathcal{A})$ is concentrated in (homological) negative degrees and is given by

$$\mathrm{N}(\mathcal{A})_{-n} = \mathsf{Ker}(\oplus_{i=1}^{n-1} s^i : \ \mathbb{Z}_{\mathrm{tr}}(\mathcal{A}^n) \longrightarrow \oplus_{i=1}^{n-1}\mathbb{Z}_{\mathrm{tr}}(\mathcal{A}^{n-1})\ ) \qquad \text{for } n \geq 0.$$

Recall that $\mathcal{A}^n = [(\mathbb{G}\mathrm{m})^{n+1} \xmapsto{pr_1} \mathbb{G}\mathrm{m}]$ and s^i is given by the projection that forgets the $(i{+}1)$–st coordinate. Because of the decomposition $\mathbb{Z}_{\mathrm{tr}}(\mathbb{G}\mathrm{m}_X) = \mathbb{Z}_{\mathrm{tr}}(X) \oplus \mathbb{Z}_{\mathrm{tr}}(\mathbb{G}\mathrm{m}_X, 1)$ it follows that $\mathrm{N}(\mathcal{A})_{-n}$ is isomorphic to $\mathbb{Z}_{\mathrm{tr}}[\mathbb{G}\mathrm{m} \times (\mathbb{G}\mathrm{m}, 1)^{\wedge n} \longmapsto \mathbb{G}\mathrm{m}]$. In particular, viewed as a complex of objects in $\mathbf{DM}(\mathbb{G}\mathrm{m})$, the complex $\mathrm{N}(\mathcal{A})$ looks like:

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{I}_{\mathbb{G}\mathrm{m}} \longrightarrow \mathbb{I}_{\mathbb{G}\mathrm{m}}(1)[1] \longrightarrow \cdots \longrightarrow \mathbb{I}_{\mathbb{G}\mathrm{m}}(n)[n] \longrightarrow \cdots.$$

It is easy to check that the first non-zero differential $\mathbb{I}_{\mathbb{G}\mathrm{m}} \longrightarrow \mathbb{I}_{\mathbb{G}\mathrm{m}}(1)[1]$ is given by the motivic fundamental class θ of Definition 4.1. One can prove that the n-th differential $\mathbb{I}_{\mathbb{G}\mathrm{m}}(n-1)[n-1] \longrightarrow \mathbb{I}_{\mathbb{G}\mathrm{m}}(n)[n]$ is always given by $\theta + \epsilon$ where, ϵ is zero in étale cohomology. It is now clear that when we apply $\underline{\mathsf{Hom}}(-, \mathbb{I})$ component-wise and then the functor χ_f we get a motivic analogue of $\mathcal{Q}^{\bullet,\bullet}$.

Remark 4.7 — Markus Spitzweck gave us a topological interpretation of the functor Υ_f which gives yet another motivation for our definition. His interpretation is as follows. One can look at the cosimplicial object $\mathcal{A}^\bullet$ as the space of paths in $\mathbb{G}\mathrm{m}$ with end-point equal to 1. This means that $\mathcal{A}^\bullet$ is in a sense the universal cover of $\mathbb{G}\mathrm{m}$. When taking $\underline{\mathsf{Hom}}(\mathcal{A}^\bullet, E)$, we are looking at the sections over the universal cover of $\mathbb{G}\mathrm{m}$ with values in E. Finally, when applying i^*j_*, we are taking the restriction of these sections to the "boundary" of the universal cover. This picture is of course very similar to the classical one we have in the analytic case.

Proposition 4.8 — *The family* $(\Upsilon_?)$ *extends naturally to a specialization system over* $\mathbb{A}^1_k$. *It is called the* unipotent nearby cycles specialization system.

Proof. We have to define the natural transformations $\alpha_?$ and prove that the axioms of Definition 3.4 hold. Suppose given a morphism of $\mathbb{A}^1_k$-schemes

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[dr] & X \arrow[d, "f"] \\ & \mathbb{A}^1_k. \end{tikzcd}$$

We define a natural transformation $\alpha_g : \; g_s^*\Upsilon_f \longrightarrow \Upsilon_{f\circ g} g_\eta^*$ by taking the composition

$$\begin{tikzcd} g_s^*\mathsf{Tot}\; i^*j_*\underline{\mathsf{Hom}}(f_\eta^*\mathcal{A}^\bullet, -) & \\ \mathsf{Tot}\; g_s^*i^*j_*\underline{\mathsf{Hom}}(f_\eta^*\mathcal{A}^\bullet, -) \arrow[u, "\sim"'] \arrow[r] & \mathsf{Tot}\; i^*j_*g_\eta^*\underline{\mathsf{Hom}}(f_\eta^*\mathcal{A}^\bullet, -) \arrow[d] \\ & \mathrm{Tot}\; i^*j_*\underline{\mathsf{Hom}}(g_\eta^*f_\eta^*\mathcal{A}^\bullet, g_\eta^*(-)). \end{tikzcd}$$

It is easy to check that these $\alpha_?$ are compatible with composition (see the third chapter of [3] for details). Furthermore, α_g is an isomorphism when g is smooth by the "base change theorem by a smooth morphism" and the formula $g_\eta^*\underline{\mathsf{Hom}}(-,-) = \underline{\mathsf{Hom}}(g_\eta^*(-), g_\eta^*(-))$. We still need to check that β_g is an isomorphism for g projective. It is easy to see that β_g is given by the composition

$$\begin{tikzcd} \mathsf{Tot}\; i^*j_*\underline{\mathsf{Hom}}(f_\eta^*\mathcal{A}^\bullet, g_{\eta *}(-)) \arrow[d, "\sim"] & \\ \mathsf{Tot}\; i^*j_*g_{\eta *}\underline{\mathsf{Hom}}(g_\eta^*f_\eta^*\mathcal{A}^\bullet, -) \arrow[r] & \mathsf{Tot}\; g_{s*}i^*j_*\underline{\mathsf{Hom}}(g_\eta^*f_\eta^*\mathcal{A}^\bullet, -) \arrow[d] \\ & g_{s*}\mathrm{Tot}\; i^*j_*\underline{\mathsf{Hom}}(g_\eta^*f_\eta^*\mathcal{A}^\bullet, -). \end{tikzcd}$$

The first map is an adjunction formula and is always invertible. The second is an isomorphism when g is projective due to the "base change theorem by a projective morphism" (proved in chapter I of [3]). The last morphism is also an isomorphism when g is projective because then $g_{s*} = g_{s!}$ (see also the first chapter of [3]) and the operation $g_{s!}$ commutes with colimits. $\square$

Let us denote a the natural morphism $\mathcal{A}^\bullet \longrightarrow \mathbb{I}$. This morphism induces a natural transformation $a : \; i^*j_* = \chi_f \longrightarrow \Upsilon_f$ which is a morphism of specialization systems. We have the following normalization, which is the main reason for our definition:

Proposition 4.9 — *The composition:* $\mathbb{I}_s \longrightarrow i^*j_*\mathbb{I}_{\mathbb{G}m} \longrightarrow \Upsilon_{id}\mathbb{I}_{\mathbb{G}m}$ *is an isomorphism.*

Proof. Recall the commutative diagram

$$\begin{tikzcd} \mathbb{G}_{\mathrm{m}} \arrow[r, "j"] \arrow[dr, "q"'] & \mathbb{A}^1_k \arrow[d, "p"] & s \arrow[l, "i"'] \arrow[dl, equal] \\ & k. & \end{tikzcd}$$

We define a natural transformation: $q_* \longrightarrow i^* j_*$ by the following composition:

$$q_* \xrightarrow{\sim} p_* j_* \longrightarrow p_* i_* i^* j_* \xrightarrow{\sim} i^* j_*.$$

Note that this natural transformation is an isomorphism when applied to q^*. In particular, the maps $q_* \mathbb{I}(m) \longrightarrow i^* j_* \mathbb{I}(m)$ are isomorphisms for every $m \in \mathbb{Z}$. This implies that the natural map of simplicial objects

$$q_* \underline{\mathsf{Hom}}(\mathcal{A}^\bullet, \mathbb{I}) \longrightarrow i^* j_* \underline{\mathsf{Hom}}(\mathcal{A}^\bullet, \mathbb{I})$$

is an isomorphism.

To prove the proposition, we only need to show that the composition

$$\mathbb{I} \longrightarrow q_* \mathbb{I} \longrightarrow \mathsf{Tot}\ q_* \underline{\mathsf{Hom}}(\mathcal{A}^\bullet, \mathbb{I})$$

is invertible. By the adjunction formula we have an identification: $\underline{\mathsf{Hom}}(q_\# \mathcal{A}^\bullet, \mathbb{I}) \simeq q_* \underline{\mathsf{Hom}}(\mathcal{A}^\bullet, \mathbb{I})$. It is then sufficient to check that the morphism of simplicial objects $\mathbb{I} \longrightarrow \underline{\mathsf{Hom}}(q_\# \mathcal{A}^\bullet, \mathbb{I})$ (where $\mathbb{I}$ is considered as a constant simplicial motive) is a simplicial homotopy equivalence. The latter is induced from a map of cosimplicial objects $q_\# \mathcal{A}^\bullet \longrightarrow \mathbb{I}$ which we check to be a cosimplicial cohomotopy equivalence.

The cosimplicial motive $q_\# \mathcal{A}^\bullet$ is the one associated to the cosimplicial k-scheme obtained by forgetting in $\mathcal{A}^\bullet$ the structure of $\mathbb{G}_{\mathrm{m}}$-scheme. An easy computation shows that this cosimplicial scheme is obtained using Lemma 4.2 from the diagram of k-schemes

$$\mathbb{G}_{\mathrm{m}} \xrightarrow{\mathrm{id}} \mathbb{G}_{\mathrm{m}} \xleftarrow{1} k\ .$$

Furthermore the map $q_\# \mathcal{A}^\bullet \longrightarrow \mathbb{I}$ is induced via the projection to the second factor of $\mathbb{G}_{\mathrm{m}} \tilde{\times}_{\mathbb{G}_{\mathrm{m}}} k$. By the last assertion of the Lemma 4.2, this is indeed a cosimplicial cohomotopy equivalence. □

4.3. **The construction of Ψ.** Now we come to the construction of the nearby cycles functors. For this we introduce the morphisms $e_n : \mathbb{A}^1_k \longrightarrow \mathbb{A}^1_k$ which are given by elevation to the n-th power. Note that these morphisms are isomorphic to the $e_n : B_n \longrightarrow B$ we introduce in paragraph 3.3 when $B = \mathbb{A}^1_k$. Given a morphism $f : X \longrightarrow \mathbb{A}^1_k$, we form the cartesian square

$$\begin{tikzcd} X^n \arrow[r, "e_n"] \arrow[d, "f^n"'] & X \arrow[d, "f"] \\ \mathbb{A}^1_k \arrow[r, "e_n"] & \mathbb{A}^1_k. \end{tikzcd}$$

Lemma 4.10 — *For any non zero positive integer n there is a natural transformation*

$$\mu_n : \Upsilon_f \longrightarrow \Upsilon_{f^n}(e_n)^*_\eta.$$

Moreover, if d is another non zero positive integer, we have: $(f^n)^d = f^{nd}$, $e_{nd} = e_n \circ e_d$ and μ_{nd} is given by the composition

$$\Upsilon_f \xrightarrow{\mu_n} \Upsilon_{f^n}(e_n)^*_\eta \xrightarrow{\mu_d} \Upsilon_{(f^n)^d}(e_d)^*_\eta(e_n)^*_\eta \simeq \Upsilon_{f^{nd}}(e_{nd})^*_\eta.$$

Proof. There is an obvious transformation: $\Upsilon_f \longrightarrow (e_n)_{s*}\Upsilon_{e_n\circ f^n}(e_n)^*_\eta$ given by the composition:

$$\Upsilon_f \longrightarrow (e_n)_{s*}(e_n)^*_s\Upsilon_f \xrightarrow{\alpha_{e_n}} (e_n)_{s*}\Upsilon_{f\circ e_n}(e_n)^*_\eta = (e_n)_{s*}\Upsilon_{e_n\circ f^n}(e_n)^*_\eta.$$

To define μ_n we need to specify a transformation $\nu_n : (e_n)_{s*}\Upsilon_{e_n\circ f^n} \longrightarrow \Upsilon_{f^n}$. As $(e_n)_s : (X^n)_s \longrightarrow X_s$ induces an isomorphism on the associated reduced schemes, the functor $(e_n)_{s*}$ is an equivalence of categories. Moreover, the two functors $(e_n)_{s*}\chi_{e_n\circ f^n}$ and χ_{f^n} are naturally isomorphic. To obtain our ν_n, it is then sufficient to define a map of cosimplicial objects: $(f^n)^*_\eta\mathcal{A}^\bullet \longrightarrow (e_n\circ f^n)^*_\eta\mathcal{A}^\bullet$.

First note that $(e_n)^*_\eta\mathcal{A}^\bullet$ is the cosimplicial scheme obtained by Lemma 4.2 from the diagram

$$[\mathbb{G}\mathrm{m} \xmapsto{\mathrm{id}} \mathbb{G}\mathrm{m}] \xrightarrow{(x,x^n)} [\mathbb{G}\mathrm{m}\times\mathbb{G}\mathrm{m} \xmapsto{pr_1} \mathbb{G}\mathrm{m}] \xleftarrow{(x,1)} [\mathbb{G}\mathrm{m} \xmapsto{\mathrm{id}} \mathbb{G}\mathrm{m}].$$

The commutative diagram of $\mathbb{G}\mathrm{m}$-schemes

$$\begin{tikzcd}
{[\mathbb{G}\mathrm{m} \xmapsto{\mathrm{id}} \mathbb{G}\mathrm{m}]} \arrow[r, "\Delta"] \arrow[d, equal] & {[\mathbb{G}\mathrm{m}\times\mathbb{G}\mathrm{m} \xmapsto{pr_1} \mathbb{G}\mathrm{m}]} \arrow[d, "{(x,y^n)}"] & {[\mathbb{G}\mathrm{m} \xmapsto{\mathrm{id}} \mathbb{G}\mathrm{m}]} \arrow[l, "\Delta"'] \arrow[d, equal] \\
{[\mathbb{G}\mathrm{m} \xmapsto{\mathrm{id}} \mathbb{G}\mathrm{m}]} \arrow[r, "{(x,x^n)}"] & {[\mathbb{G}\mathrm{m}\times\mathbb{G}\mathrm{m} \xmapsto{pr_1} \mathbb{G}\mathrm{m}]} & {[\mathbb{G}\mathrm{m} \xmapsto{\mathrm{id}} \mathbb{G}\mathrm{m}]} \arrow[l, "{(x,1)}"']
\end{tikzcd}$$

induces a map of cosimplicial schemes $\mathcal{A}^\bullet \longrightarrow (e_n)^*_\eta\mathcal{A}^\bullet$. This gives for any $\mathbb{A}^1_k$-scheme f a map

$$(f^n)^*_\eta(\mathcal{A}^\bullet) \longrightarrow (f^n)^*_\eta(e_n)^*_\eta(\mathcal{A}^\bullet) \simeq (e_n)^*_\eta f^*_\eta(\mathcal{A}^\bullet) \simeq (f^n\circ e_n)^*_\eta(\mathcal{A}^\bullet).$$

The last assertion is an easy verification which we leave to the reader. □

Definition 4.11 — *We define the* (total) motivic nearby cycles functor

$$\Psi_f : \mathbf{SH}(X_\eta) \longrightarrow \mathbf{SH}(X_s)$$

by the formula: $\Psi_f = \mathsf{HoColim}_{n\in\mathbb{N}^\times}\ \Upsilon_{f^n}(e_n)^*_\eta$.

Remark 4.12 — Because the homotopy colimit is not functorial in a triangulated category, one needs to work more to get a well–defined triangulated functor. A way to do this is to define categories $\mathbf{SH}(-,\mathbb{N}^\times)$ corresponding to $\mathbb{N}^\times$-diagrams of spectra. Then extend the functor Υ_f to a more elaborate one that goes from $\mathbf{SH}(X_\eta)$ to $\mathbf{SH}(X_s,\mathbb{N}^\times)$ and associates to A the full diagram $(\Upsilon_{f^n}(e_n)^*_\eta A)_n$. Finally apply the colimit functor $\mathbf{SH}(X_s, \mathbb{N}^\times) \longrightarrow \mathbf{SH}(X_s)$. For more details, the reader can consult the third chapter of [3].

Proposition 4.13 — *The family* $(\Psi_?)$ *extends naturally to a specialization system over* $\mathbb{A}^1_k$*. It is called the* (total) nearby cycles specialization system.

We have the following simple lemma:

Lemma 4.14 — ***1-*** *Suppose that the morphism* $f : X \longrightarrow \mathbb{A}^1_k$ *is smooth. Then the canonical morphism:* $\Upsilon_f \mathbb{I} \longrightarrow \Psi_f \mathbb{I}$ *is an isomorphism.*

2- *For every* n*, there exist a natural isomorphism* $\Psi_{f^n}(e_n)^*_\eta \xrightarrow{\sim} \Psi_f$ *making the triangle*

$$\begin{tikzcd} \Upsilon_{f^n}(e_n)^*_\eta \arrow[r] \arrow[dr] & \Psi_{f^n}(e_n)^*_\eta \arrow[d, "\sim"] \\ & \Psi_f \end{tikzcd}$$

commutative.

Proof. The first point is easy, and comes from the fact that the two objects are isomorphic to $\mathbb{I}$. The second point is left as an exercise. □

Our next step is the computation of $\Psi_{e_n}\mathbb{I}$ and $\Psi_{e'_n}\mathbb{I}$ (see the notations of paragraph 3.3):

Proposition 4.15 — *For every* $n \in \mathbb{N}^\times$ *the canonical morphisms*

$$\Upsilon_{(e_n)^m}\mathbb{I} \longrightarrow \Psi_{(e_n)}\mathbb{I} \qquad \textit{and} \qquad \Upsilon_{(e'_n)^m}\mathbb{I} \longrightarrow \Psi_{e'_n}\mathbb{I}$$

are isomorphisms when m *is divisible by* n*.*

Proof. In both cases, the proofs are exactly the same in the two cases and are based on the fact that the normalizations of $(e_n)^m$ and $(e'_n)^m$ are smooth $\mathbb{A}^1_k$-schemes. Indeed let us denote $B = \mathbb{A}^1$, B_n and B'_n as in paragraph 3.3. Then we have:

- $(B_n)^m = B_n \times_B B_m = \mathsf{Spec}(k[\pi][t_1]/(t_1^n - \pi)[t_2]/(t_2^m - \pi))$. When m is divisible by n, the normalization Q^m_n of $(B_n)^m$ is $\mathsf{Spec}(k[\pi][v]/(v^n - 1)[t^2]/(t_2^m - \pi))$ with $v = t_1/t_2^{\frac{m}{n}}$. In particular Q^m_n is étale over B_m.
- $(B'_n)^m = B'_n \times_B B_m = k[\pi][t_1, u, u^{-1}]/(t_1^n - u.\pi)[t_2][t_2^m - \pi]$. When m is divisible by n its normalization Q'^m_n is $k[\pi][w, w^{-1}][t_2]/(t_2^m - \pi)$ with $w^n = u$. In particular Q'^m_n is smooth over B_m.

Consider now the morphisms $t_m : Q^m_n \longrightarrow (B_n)^m$ and $t'_m : Q'^m_n \longrightarrow (B'_n)^m$. They are both finite, and induce isomorphisms on the generic fibers. We have the following commutative diagram:

$$\begin{tikzcd} (t_m)_{s*}\Upsilon_{(e_n)^m \circ t_m}\mathbb{I} \arrow[r, "(a)"] \arrow[d, "\beta_{t_m}"'] & (t_m)_{s*}\Psi_{(e_n)^m \circ t_m}\mathbb{I} \arrow[d, "\beta_{t_m}"] & \\ \Upsilon_{(e_n)^m}\mathbb{I} \arrow[r] & \Psi_{(e_n)^m}\mathbb{I} \arrow[r, "(b)"] & \Psi_{e_n}. \end{tikzcd} \tag{1}$$

The two vertical arrows are the transformations β_{t_m} of Definition 3.4 modulo the identification $(t_m)_\eta = \mathrm{id}$. As t_m is a finite map, these two arrows are invertible.

By Lemma 4.14, we know that the horizontal arrows labeled (a) and (b) are also invertible. Thus we are done with the first case. The second case is settled using exactly the same argument. $\square$

The proof of Proposition 4.15 gives a more precise statement. It computes exactly the motives $\Psi_{e_n}\mathbb{I}$ and $\Psi_{e'_n}\mathbb{I}$. Indeed in the diagram (1) we have $\Upsilon_{(e_n)^m \circ t_m}\mathbb{I} = \mathbb{I}$. It follows that:

- $\Psi_{e_n}\mathbb{I} = (t_n)_{s*}\mathbb{I}$,
- $\Psi_{e'_n}\mathbb{I} = (t'_n)_{s*}\mathbb{I}$.

These are Artin 0-motives and they are constructible. Theorem 3.8 then applies to give us the following:

Theorem 4.16 — *For any quasi-projective f, the functor Ψ_f takes constructible motives of $\mathbf{SH}(X_\eta)$ to constructible motives of $\mathbf{SH}(X_s)$.*

Later on, we will need the following result:

Theorem 4.17 — *Let $f : X \longrightarrow \mathbb{A}^1_k$ be a finite type morphism. For any constructible object $A \in \mathbf{SH}(X_\eta)$ there exists an integer N such that the natural morphism: $\Upsilon_{f^m}(e_m)^*_\eta A \longrightarrow \Psi_f(A)$ is an isomorphism for non zero m divisible by N.*

Proof. Note that for every non-zero n, the family of functors $(\Upsilon_{f^n}(e_n)_\eta)_f$ is a specialization system over $\mathbb{A}^1$ and the obvious morphisms $\Upsilon_{f^n}(e_n)^*_\eta \longrightarrow \Psi_f$ give a morphism of specialization systems. The conclusion of the theorem follows from Proposition 4.15 and a refined version of Theorem 3.9. Indeed, suppose that in 3.9 we only knew that $sp_{e_n}\mathbb{I} \longrightarrow sp'_{e_n}\mathbb{I}$ and $sp_{e'_n}\mathbb{I} \longrightarrow sp'_{e'_n}\mathbb{I}$ are invertible for n dividing a fixed number N. Then it is still possible to conclude that $sp_f(A) \longrightarrow sp'_f(A)$ is invertible for A "coming" from varieties with semi-stable reduction over B_N. For more details, see the third chapter of [3]. $\square$

4.4. **Pseudo-monoidal structure.** We continue our study of the functors Ψ_f by constructing a pseudo-tensor structure on them. We denote $\mathcal{A}^\bullet \otimes \mathcal{A}^\bullet$ the bicosimplicial $\mathbb{G}_m$-scheme (or its associated motive) obtained by taking fiber products over $\mathbb{G}_{m_k}$. We will denote $(\mathcal{A} \otimes \mathcal{A})^\bullet$ the cosimplicial object obtained from $\mathcal{A}^\bullet \otimes \mathcal{A}^\bullet$ by restricting to the diagonal $\mathbf{\Delta} \longrightarrow \mathbf{\Delta} \times \mathbf{\Delta}$. We have the following lemma:

Lemma 4.18 — *The cosimplicial scheme $(\mathcal{A} \otimes \mathcal{A})^\bullet$ is the one obtained from Lemma 4.2 applied to the following diagram in $\mathsf{Sm}/\mathbb{G}_m$:*

$$[\mathbb{G}_m \xmapsto{id} \mathbb{G}_m] \xrightarrow{\Delta_3} [\mathbb{G}_m \times \mathbb{G}_m \times \mathbb{G}_m \xmapsto{pr_1} \mathbb{G}_m] \xleftarrow{(x,1,1)} [\mathbb{G}_m \xmapsto{id} \mathbb{G}_m]$$

where Δ_3 is the diagonal embedding.

Let $m : \mathcal{A}^\bullet \longrightarrow (\mathcal{A} \otimes \mathcal{A})^\bullet$ be the morphism of cosimplicial objects induced by the commutative diagram

$$\begin{tikzcd}
{[\mathbb{G}\mathrm{m} \xmapsto{\mathrm{id}} \mathbb{G}\mathrm{m}]} \arrow[r, "\Delta"] \arrow[d, equal] & {[\mathbb{G}\mathrm{m} \times \mathbb{G}\mathrm{m} \xmapsto{pr_1} \mathbb{G}\mathrm{m}]} \arrow[d, "{(x,y,y)}"] & {[\mathbb{G}\mathrm{m} \xmapsto{\mathrm{id}} \mathbb{G}\mathrm{m}]} \arrow[l, "{(x,1)}"'] \arrow[d, equal] \\
{[\mathbb{G}\mathrm{m} \xmapsto{\mathrm{id}} \mathbb{G}\mathrm{m}]} \arrow[r, "\Delta_3"] & {[\mathbb{G}\mathrm{m} \times \mathbb{G}\mathrm{m} \times \mathbb{G}\mathrm{m} \xmapsto{pr_1} \mathbb{G}\mathrm{m}]} & {[\mathbb{G}\mathrm{m} \xmapsto{\mathrm{id}} \mathbb{G}\mathrm{m}].} \arrow[l, "{(x,1,1)}"']
\end{tikzcd}$$

We make the following definition:

Definition 4.19 — *Let $f : X \longrightarrow \mathbb{A}^1_k$ be a morphism. We define a bi-natural transformation $m : \Upsilon_f(-) \otimes \Upsilon_f(-') \longrightarrow \Upsilon_f(- \otimes -')$ by the composition*

$$\begin{tikzcd}
\mathsf{Tot}\ i^*j_*\underline{\mathsf{Hom}}(\mathcal{A}^\bullet, -) \otimes \mathsf{Tot}\ i^*j_*\underline{\mathsf{Hom}}(\mathcal{A}^\bullet, -') \arrow[d] \\
\mathsf{Tot}\ [i^*j_*\underline{\mathsf{Hom}}(\mathcal{A}^\bullet, -) \otimes i^*j_*\underline{\mathsf{Hom}}(\mathcal{A}^\bullet, -')] \arrow[d] \\
\mathsf{Tot}\ i^*j_*[\underline{\mathsf{Hom}}(\mathcal{A}^\bullet, -) \otimes \underline{\mathsf{Hom}}(\mathcal{A}^\bullet, -')] \arrow[d, equal] \\
\mathsf{Tot}\ i^*j_*\underline{\mathsf{Hom}}(\mathcal{A}^\bullet \otimes \mathcal{A}^\bullet, - \otimes -') \arrow[d, "(a)"', "\sim"] \\
\mathsf{Tot}\ i^*j_*\underline{\mathsf{Hom}}((\mathcal{A} \otimes \mathcal{A})^\bullet, - \otimes -') \arrow[d, "{\underline{\mathsf{Hom}}(m,-)}"] \\
\mathsf{Tot}\ i^*j_*\underline{\mathsf{Hom}}(\mathcal{A}^\bullet, - \otimes -')
\end{tikzcd}$$

where the arrow labelled (a) is the identification of the homotopy colimit of a bisimplicial object with the homotopy colimit of its diagonal.

One checks (as in chapter III of [3]) that:

Proposition 4.20 — *The bi-natural transformation m of the above definition makes Υ_f into a pseudo-monoidal functor. Moreover the natural transformation $\chi_f = i^*j_* \longrightarrow \Upsilon_f$ is compatible with the pseudo-monoidal structures.*

Note that the above proposition defines a "χ-module" structure on Υ in the sense that there exists a binatural transformation

$$m' : \chi_f(-) \otimes \Upsilon_f(-') \longrightarrow \Upsilon_f(- \otimes -')$$

which is nothing but the composition of the canonical morphism $\chi \longrightarrow \Upsilon$ with the morphism of the definition. It is easy to check that m' is given by the following composition:

$$\begin{tikzcd}
i^*j_*- \otimes \mathsf{Tot}\ i^*j_*\underline{\mathsf{Hom}}(\mathcal{A}^\bullet, -') \arrow[d] & \\
\mathsf{Tot}\ i^*j_*- \otimes i^*j_*\underline{\mathsf{Hom}}(\mathcal{A}^\bullet, -') \arrow[r] & \mathsf{Tot}\ i^*j_*(- \otimes \underline{\mathsf{Hom}}(\mathcal{A}^\bullet, -')) \arrow[d] \\
& \mathsf{Tot}\ i^*j_*\underline{\mathsf{Hom}}(\mathcal{A}^\bullet, - \otimes -').
\end{tikzcd}$$

Corollary 4.21 — *For any object A in $\mathbf{SH}(X_\eta)$ the composition*

$$\Upsilon_f(A) = \Upsilon_f(A) \otimes \mathbb{I} \longrightarrow \Upsilon_f(A) \otimes \Upsilon_f\mathbb{I} \longrightarrow \Upsilon_f(A)$$

is the identity.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\Upsilon_f(A) \arrow[r, equal] \arrow[d, equal] & \Upsilon_f(A) \otimes \mathbb{I} \arrow[r] & \Upsilon_f(A) \otimes \Upsilon_f\mathbb{I} \arrow[r] & \Upsilon_f(A) \arrow[d, equal] \\
\Upsilon_f(A) \arrow[r] & \Upsilon_f(A) \otimes f_s^*\chi_{\mathrm{id}}\mathbb{I} \arrow[r] \arrow[u, "(a)"] & \Upsilon_f(A) \otimes \chi_f\mathbb{I} \arrow[r] \arrow[u] & \Upsilon_f(A)
\end{tikzcd}$$

where the arrow labelled (a) is the one induced from the canonical splitting: $\chi_{\mathrm{id}}\mathbb{I} \to \Upsilon_{\mathrm{id}}\mathbb{I} = \mathbb{I}$. So we need only to check that the composition of the bottom sequence is the identity. For this we can use the description of the "χ-module" structure given above. Going back to the definition of Υ, we see that it suffices to check that the composition

$$\chi_f B \longrightarrow \chi_f B \otimes f_s^* i^* j_* \mathbb{I} \longrightarrow \chi_f B \otimes \chi_f \mathbb{I} \longrightarrow \chi_f B$$

is the identity for $B \in \mathbf{SH}(X_\eta)$. This is an easy exercise. □

In order to extend the pseudo-monoidal structure from Υ to Ψ we use the following lemma:

Lemma 4.22 — *With the notations of paragraph 4.3, we have a commutative diagram of binatural transformations*

$$\begin{tikzcd}
\Upsilon_f(-) \otimes \Upsilon_f(-') \arrow[rr] \arrow[d] & & \Upsilon_{f^n}(e_n)^*_\eta(-) \otimes \Upsilon_{f^n}(e_n)^*_\eta(-') \arrow[d] \\
\Upsilon_f(- \otimes -') \arrow[r] & \Upsilon_{f^n}(e_n)^*_\eta(- \otimes -') \arrow[r, "\sim"] & \Upsilon_{f^n}(e_n)^*_\eta(-) \otimes (e_n)^*_\eta(-').
\end{tikzcd}$$

Proof. Going back to the definitions we see that we must check the commutativity of the corresponding diagram of cosimplicial objects

$$\begin{tikzcd}
(\mathcal{A} \otimes \mathcal{A})^\bullet \arrow[r] & \mathrm{diag}[(e_n)^*_\eta\mathcal{A}^\bullet \otimes (e_n)^*_\eta\mathcal{A}^\bullet] \arrow[r] & (e_n)^*_\eta(\mathcal{A} \otimes \mathcal{A})^\bullet) \\
\mathcal{A}^\bullet \arrow[rr] \arrow[u] & & (e_n)^*_\eta\mathcal{A}^\bullet \arrow[u]
\end{tikzcd}$$

This diagram is obviously commutative. □

Lemma 4.22 allows us to define a bi-natural transformation

$$\Psi_f(-) \otimes \Psi_f(-') \longrightarrow \Psi_f(- \otimes -')$$

by taking the colimit of the bi-natural transformations

$$\Upsilon_{f^n}(e_n)^*_\eta(-) \otimes \Upsilon_{f^n}(e_n)^*_\eta(-') \longrightarrow \Upsilon_{f^n}(e_n)^*_\eta(- \otimes -').$$

We have:

Theorem 4.23 — *For every $f : X \longrightarrow \mathbb{A}^1_k$, the functor Ψ_f is naturally a pseudo-monoidal functor. Furthermore, the morphisms*

$$\chi_f \longrightarrow \Upsilon_f \longrightarrow \Psi_f$$

are natural transformations of pseudo-monoidal functors.

We have the following important result:

Theorem 4.24 — *Let F be an object of $\mathbf{SH}(\eta)$. Then for any $f : X \longrightarrow \mathbb{A}^1_k$ and any object A of $\mathbf{SH}(X_\eta)$, the composition:*

$$\Psi_f(A) \otimes f_s^*\Psi_{id}(F) \longrightarrow \Psi_f(A) \otimes \Psi_f f_\eta^* F \longrightarrow \Psi_f(A \otimes f_\eta^* F)$$

is an isomorphism. In particular, Ψ_{id} is a monoidal functor.

Proof. We will apply Theorem 3.9 to a well chosen morphism between two specialization systems. These specialization systems are (see Example 3.9):

(1) $\Psi^{(a)}$, given by the formula: $\Psi^{(a)}_f(A) = \Psi_f(A) \otimes f_s^*\Psi_{\mathrm{id}} F$,
(2) $\Psi^{(b)}$, given by the formula: $\Psi^{(b)}_f(A) = \Psi_f(A \otimes f_\eta^* F)$.

One sees immediately that the composition in the statement of the theorem defines a morphism of specialization systems: $\Psi^{(a)} \longrightarrow \Psi^{(b)}$. Note also that $\Psi^{(a)}_f$ and $\Psi^{(b)}_f$ both commute with infinite sums. So by Theorem 3.9, we only need to consider the two special cases:

- $f = e_n$ and $A = \mathbb{I}$,
- $f = e'_n$ and $A = \mathbb{I}$.

The proofs in these two cases are similar to the proof of Proposition 4.15. We will concentrate on the first case and use the notations in the proof of 4.15. Recall that we denoted $t_m : Q^m_n \longrightarrow (B_n)^m$ the normalization of $(B_n)^m$. We can suppose that F is of finite type. By Theorem 4.17 we can choose a sufficiently divisible m such that:

- $\Psi_{e_n}(\mathbb{I}) \otimes (e_n)^*_s\Psi_{\mathrm{id}}(F) \simeq \Upsilon_{(e_n)^m}(\mathbb{I}) \otimes ((e_n)^m)^*_s\Upsilon_{\mathrm{id}}(e_m)^*_\eta(F)$
$\simeq (t_m)_{s*}\Upsilon_{(e_n)^m\circ t_m}(\mathbb{I}) \otimes ((e_n)^m)^*_s\Upsilon_{\mathrm{id}}(e_m)^*_\eta(F)$,
- $\Psi_{e_n}(\mathbb{I}) \otimes \Psi_{e_n}(e_n)^*_\eta F \simeq \Upsilon_{(e_n)^m}(\mathbb{I}) \otimes \Upsilon_{(e_n)^m}(e_m)^*_\eta((e_n)^m)^*_\eta F$
$\simeq (t_m)_{s*}\Upsilon_{(e_n)^m\circ t_m}(\mathbb{I}) \otimes (t_m)_{s*}\Upsilon_{(e_n)^m\circ t_m}(e_m)^*_\eta((e_n)^m)^*_\eta F$
- $\Psi_{e_n}(\mathbb{I} \otimes (e_n)^*_\eta F) \simeq \Upsilon_{(e_n)^m}(e_m)^*_\eta(\mathbb{I} \otimes ((e_n)^m)^*_\eta F)$
$\simeq (t_m)_{s*}\Upsilon_{(e_n)^m\circ t_m}(e_m)^*_\eta(\mathbb{I} \otimes ((e_n)^m)^*_\eta F)$.

Denoting by f the smooth morphism $(e_n)^m \circ t_m$, we end up with the following problem: is the composition

$$\Upsilon_f \mathbb{I} \otimes f_s^* \Upsilon_{\mathrm{id}} F \longrightarrow \Upsilon_f \mathbb{I} \otimes \Upsilon_f f_\eta^* F \longrightarrow \Upsilon_f(\mathbb{I} \otimes f_\eta^* F)$$

invertible? This is indeed the case by Corollary 4.21. □

4.5. **Compatibility with duality.** It is a well-known fact that in étale cohomology the nearby cycles functors commute with duality (see for example [15]). We extend this result to the motivic context. We first specify our duality functors.

Definition 4.25 — *Let $f : X \longrightarrow \mathbb{A}_k^1$. We define two duality functors D_η and D_s on $\mathbf{SH}(X_\eta)$ and $\mathbf{SH}(X_s)$ by:*

(1) $\mathrm{D}_\eta(-) = \underline{\mathsf{Hom}}(-, f_\eta^! \mathbb{I})$,
(2) $\mathrm{D}_s(-) = \underline{\mathsf{Hom}}(-, f_s^! \mathbb{I})$.

(The "extraordinary inverse image" operation $(-)^!$ is constructed in the first chapter of [3]*.)*

Remark 4.26 — Note that D_η differs by a Tate twist and a double suspension from the usual duality functor on $\mathbf{SH}(X_\eta)$. Indeed we used $f_\eta^! \mathbb{I}$ instead of the dualising motive $(q \circ f_\eta)^! \mathbb{I}$ (where q is the projection of $\mathbb{G}_m$ to k).

We define for any $f : X \longrightarrow \mathbb{A}_k^1$ a natural transformation $\delta_f : \Psi_f \mathrm{D}_\eta \longrightarrow \mathrm{D}_s \Psi_f$ in the following way:

(1) First note that for an object $A \in \mathbf{SH}(X_\eta)$ there is a natural pairing
$$A \otimes \mathrm{D}_\eta(A) \longrightarrow f_\eta^! \mathbb{I}.$$
(2) We define a pairing $\Psi_f(A) \otimes \Psi_f \mathrm{D}_\eta(A) \longrightarrow f_s^! \mathbb{I}$ by the following composition:
$$\Psi_f(A) \otimes \Psi_f \mathrm{D}_\eta(A) \longrightarrow \Psi_f(A \otimes \mathrm{D}_\eta A) \longrightarrow \Psi_f f_\eta^! \mathbb{I} \longrightarrow f_s^! \Psi_f \mathbb{I} = f_s^! \mathbb{I}. \qquad (2)$$
(3) Using adjunction, we get from the above pairing the desired natural morphism $\delta_f : \Psi_f \mathrm{D}_\eta(A) \longrightarrow \mathrm{D}_s \Psi_f(A)$.

Theorem 4.27 — *When A is constructible in $\mathbf{SH}(X_\eta)$, the morphism*

$$\delta_f : \Psi_f D_\eta(A) \longrightarrow D_s \Psi_f(A)$$

is an isomorphism.

Proof. Once again the proof is based on Theorem 3.9. First note that when A is constructible, $\mathrm{D}_\eta \mathrm{D}_\eta(A) = A$ (by [3], chapter II). Thus we only need to prove that the natural transformation $\delta'_f : \Psi_f \longrightarrow \mathrm{D}_s \Psi_f \mathrm{D}_\eta$ is an isomorphism when evaluated on constructible objects. Note that δ'_f is nothing but the second adjoint deduced from the pairing (2). Now we have two specialization systems: Ψ and $\mathrm{D}_s \Psi \mathrm{D}_\eta$ and a morphism $\delta'_?$ between them. Using Theorem 3.9 we only need to check the theorem when $f = e_n$ or e'_n and $A = \mathbb{I}$. This is done using the same method as in the proof of Proposition 4.15. For more details, the reader can consult the third chapter of [3]. □

4.6. **The monodromy operator.** In this section, we construct the monodromy operator on the unipotent part of the nearby cycles functors Υ_f. In order to do this we will work in $\mathbf{DM}_{\mathbb{Q}}(-)$. Note that one extends our definition of the specialization systems Υ and Ψ from $\mathbf{SH}(-)$ to $\mathbf{DM}(-)$ by using the same definitions. The main result of this paragraph is:

Theorem 4.28 — *Let $f: X \longrightarrow \mathbb{A}^1_k$ be an $\mathbb{A}^1$-scheme. There exists a natural (in A) distinguished triangle in the triangulated category of motives with rational coefficients $\mathbf{DM}_{\mathbb{Q}}(X_s)$:*

$$\Upsilon_f(A)(-1)[-1] \longrightarrow \chi_f(A) \longrightarrow \Upsilon_f(A) \xrightarrow{N} \Upsilon_f(A)(-1).$$

We call N the monodromy operator*. Moreover, when A is of finite type, this operator is nilpotent.*

Our strategy is as follows: we introduce a new specialization system log on $\mathbf{DM}_{\mathbb{Q}}(-)$ for which we can easily construct a monodromy sequence. Then we construct a morphism of specialization system $\mathsf{log} \longrightarrow \Upsilon$ and prove that it is an isomorphism.

The specialization system log is defined using a pro-motive $\mathcal{L}og$ called the logarithmic pro-motive. To define $\mathcal{L}og$ we first need the motivic Kummer torsor:

Definition 4.29 — *The* motivic Kummer torsor *is the object $\mathcal{K}$ of $\mathbf{DM}(\mathbb{G}_m)$ defined up to a unique isomorphism by the distinguished triangle*

$$\mathbb{Z}_{\mathbb{G}m}(1) \longrightarrow \mathcal{K} \longrightarrow \mathbb{Z}_{\mathbb{G}m}(0) \xrightarrow{\theta} \mathbb{Z}_{\mathbb{G}m}(1)[1]$$

where θ is the motivic fundamental class of Definition 4.1.

Remark 4.30 — The uniqueness of the motivic Kummer torsor follows from the vanishing of the group $\hom_{\mathbf{DM}(\mathbb{G}\mathrm{m})}(\mathbb{Z}(0) \oplus \mathbb{Z}(1)[1], \mathbb{Z}(1)) = \mathrm{H}^{1,0}(k) \oplus \mathrm{H}^{0,-1}(k)$ (see [40]). Indeed, let a be an automorphism of the above triangle which is the identity on $\mathbb{Z}(1)$ and $\mathbb{Z}(0)$. To prove that a is the identity, we look at $\mathrm{id} - a$. This gives a morphism of distinguished triangles

$$\begin{tikzcd}
\mathbb{Z}(1) \arrow[r] \arrow[d, "0"] & \mathcal{K} \arrow[r] \arrow[d, "\epsilon"] & \mathbb{Z}(0) \arrow[r, "e"] \arrow[d, "0"] \arrow[dl, dotted, "u"] & \mathbb{Z}(1)[1] \arrow[d, "0"] \\
\mathbb{Z}(1) \arrow[r] & \mathcal{K} \arrow[r] & \mathbb{Z}(0) \arrow[r, "e"] & \mathbb{Z}(1)[1].
\end{tikzcd}$$

It is easy to see that ϵ factors through some morphism u. To show our claim, it suffices to prove that the group $\hom_{\mathbf{DM}(\mathbb{G}\mathrm{m})}(\mathbb{Z}(0), \mathcal{K})$ is zero. For this we look at the exact sequence of hom groups in $\mathbf{DM}(\mathbb{G}\mathrm{m})$:

$$\hom(\mathbb{Z}(0), \mathbb{Z}(1)) \longrightarrow \hom(\mathbb{Z}(0), \mathcal{K}) \longrightarrow \hom(\mathbb{Z}(0), \mathbb{Z}(0)) \longrightarrow \hom(\mathbb{Z}(0), \mathbb{Z}(1)[1]).$$

Because $\hom(\mathbb{Z}_{\mathbb{G}\mathrm{m}}(0), \mathbb{Z}_{\mathbb{G}\mathrm{m}}(1))$ is zero, we only need to show the injectivity of

$$\mathbb{Z} = \hom(\mathbb{Z}_{\mathbb{G}\mathrm{m}}(0), \mathbb{Z}_{\mathbb{G}\mathrm{m}}(0)) \longrightarrow \hom(\mathbb{Z}_{\mathbb{G}\mathrm{m}}(0), \mathbb{Z}_{\mathbb{G}\mathrm{m}}(1)[1]) = \Gamma(\mathbb{G}\mathrm{m}, \mathscr{O}^{\times}).$$

This morphism sends $1 \in \mathbb{Z}$ to the class of the variable $\pi \in k[\pi, \pi^{-1}]$. The desired injectivity follows from the fact that this element is non-torsion.

Remark 4.31 — When the base field is a number field, there is a way to think about $\mathcal{K}$ as an extension of Tate motives in some abelian sub-category of $\mathbf{DM}(\mathbb{G}\mathrm{m})$.

Indeed, the Beilinson-Soulé conjecture is known for $\mathbb{G}\mathrm{m}_k$ and all its points when k is a number field. It is then possible to define a motivic t-structure on the sub-category of Tate-motives over $\mathbb{G}\mathrm{m}_k$. We will use this (non-elementary) point of view to simplify the proofs of Lemmas 4.36 and 4.45. Note that these two lemmas admit elementary proofs that can be found in the third chapter of [3].

Definition 4.32 — *For $n \in \mathbb{N}$, we define the object $\mathcal{L}og^n$ of $\mathbf{DM}_{\mathbb{Q}}(\mathbb{G}m)$ by*

$$\mathcal{L}og^n = \mathrm{Sym}^n(\mathcal{K})$$

where Sym^n *is the symmetric n-th power. This object is called the n-th* logarithmic motive.

Remark 4.33 — The definition of the logarithmic motive $\mathcal{L}og^n$ only makes sense after inverting some denominators. Indeed, the projector Sym^n is given by

$$\frac{1}{|\Sigma_n|}\sum_{\sigma\in\Sigma_n}\sigma$$

where Σ_n is the n-th symmetric group.

Remark 4.34 — Logarithmic motives, or at least their realizations, are well–known objects in the study of Beilinson's conjectures and polylogarithms. Lemmas 4.35, 4.36 and 4.45 are surely well-known.

Lemma 4.35 — *Let n and m be integers. We have two canonical morphisms:*

- $\alpha_{n,n+m} : \mathcal{L}og^n(m) \longrightarrow \mathcal{L}og^{n+m}$
- $\beta_{n+m,m} : \mathcal{L}og^{n+m} \longrightarrow \mathcal{L}og^m$

Moreover, if l is a third integer, we have: $\alpha_{n+m,n+m+l}\circ\alpha_{n,n+m} = \alpha_{n,n+m+l}$ and $\beta_{m+l,l}\circ\beta_{n+m+l,m+l} = \beta_{n+m+l,l}$. We also have a commutative square

$$\begin{tikzcd}
\mathcal{L}og^{n+m}(l) \arrow[r] \arrow[d] & \mathcal{L}og^{n+m+l} \arrow[d] \\
\mathcal{L}og^{m}(l) \arrow[r] & \mathcal{L}og^{m+l}.
\end{tikzcd}$$

Proof. Consider the $\mathbb{Q}$-algebras $\mathbb{Q}[\Sigma_n]$ and $\mathbb{Q}[\Sigma_m]$ as sub-algebras of $\mathbb{Q}[\Sigma_{n+m}]$ corresponding to the partition $n+m$. In $\mathbb{Q}[\Sigma_{n+m}]$ we have three projectors: Sym^n, Sym^m and Sym^{n+m} with the relations:

$$\mathrm{Sym}^n.\mathrm{Sym}^m = \mathrm{Sym}^m.\mathrm{Sym}^n \qquad \mathrm{Sym}^{n+m} = \mathrm{Sym}^{n+m}.\mathrm{Sym}^n = \mathrm{Sym}^{n+m}.\mathrm{Sym}^m.$$

We then see immediately that $\mathrm{Sym}^{n+m}\mathcal{K}$ is canonically a direct factor of $\mathrm{Sym}^n\mathcal{K}\otimes\mathrm{Sym}^m\mathcal{K}$. On the other hand, we have natural morphisms:

- $\mathbb{Q}(m) \longrightarrow \mathrm{Sym}^m\mathcal{K}$,
- $\mathrm{Sym}^n\mathcal{K} \longrightarrow \mathbb{Q}(0)$.

We get the desired morphisms by taking the compositions:

- $\mathbb{Q}(m)\otimes\mathrm{Sym}^n\mathcal{K} \longrightarrow \mathrm{Sym}^m\mathcal{K}\otimes\mathrm{Sym}^n\mathcal{K} \longrightarrow \mathrm{Sym}^{n+m}\mathcal{K}$,
- $\mathrm{Sym}^{n+m}\mathcal{K} \longrightarrow \mathrm{Sym}^m\mathcal{K}\otimes\mathrm{Sym}^n\mathcal{K} \longrightarrow \mathrm{Sym}^m\mathcal{K}\otimes\mathbb{Q}(0)$.

We leave the verification of the two identities and the commutativity of the square to the reader (see [3], chapter III). □

Lemma 4.36 — *There is a canonical distinguished triangle*

$$\mathcal{L}og^n(m+1) \xrightarrow{\alpha} \mathcal{L}og^{n+m+1} \xrightarrow{\beta} \mathcal{L}og^m \longrightarrow \mathcal{L}og^n(m+1)[+1].$$

Proof. We have chosen to give a short and simple proof of Lemma 4.36 based on a non-elementary result, rather than a complicated and self-contained one (see [3] for an elementary proof). The non-elementary result we shall use is the existence of an abelian category $\mathbf{MTM}(\mathbb{G}_m)$ of mixed Tate motives over $\mathbb{G}_m$, which is the heart of a motivic t-structure on the sub-category of $\mathbf{DM}(\mathbb{G}_m)$ generated by $\mathbb{Q}(i)$ for $i \in \mathbb{Z}$. Of course, $\mathbf{MTM}(\mathbb{G}_m)$ is known to exist only when the base field is a number field. So we first construct our distinguished triangle when our base field is $\mathbb{Q}$ and then extend it to arbitrary field of characteristic zero by taking its pull-back.

Let us first prove that $\beta \circ \alpha$ is zero. It clearly suffices to prove that for any two subsets I and J of $E = \{1, \ldots, n+m+1\}$ of respective cardinality n and m, the composition

$$\mathcal{K}^{\otimes I} \otimes \mathbb{Q}(1)^{\otimes E-I} \longrightarrow \mathcal{K}^{\otimes E} \longrightarrow \mathcal{K}^{\otimes J} \otimes \mathbb{Q}(0)^{\otimes E-J}$$

is zero. But this is indeed the case, because $(E-I) \cap (E-J)$ is always nonempty. The next step will be to prove that the sequence:

$$0 \longrightarrow \mathcal{L}og^n(m+1) \xrightarrow{\alpha} \mathcal{L}og^{n+m+1} \xrightarrow{\beta} \mathcal{L}og^m \longrightarrow 0$$

is a short exact sequence in $\mathbf{MTM}(\mathbb{G}_m)$. This will imply our statement.

One can easily see that α is injective and β surjective. So we have to prove exactness at the middle term. For this, we use the fact that $\mathbf{MTM}(\mathbb{G}_m)$ is a neutral Tannakian $\mathbb{Q}$-linear category and all its non-zero objects have a strictly positive dimension (given by the trace of the identity). So to prove the exactness at the middle term we only need to show that

$$\dim(\mathcal{L}og^{n+m+1}) = \dim(\mathcal{L}og^n) + \dim(\mathcal{L}og^m).$$

But this is true because $\dim(\mathcal{L}og^l) = l+1$, which is an easy consequence of $\dim(\mathcal{K}) = 2$. □

By Lemma 4.35, the logarithmic motives define a pro-object in the category of mixed Tate motives over $\mathbb{G}_m$. This pro-object

$$(\mathcal{L}og^{n+1} \to \mathcal{L}og^n)_n$$

will be denoted by $\mathcal{L}og$. We will use this particular case of 4.36 to get our monodromy sequence:

Corollary 4.37 — *There is canonical pro-distinguished triangle*

$$\mathcal{L}og^{n-1}(1) \longrightarrow \mathcal{L}og^n \longrightarrow \mathbb{Q}(0) \longrightarrow \mathcal{L}og^{n-1}(1)[1]$$

in $\mathbf{DM}_{\mathbb{Q}}(\mathbb{G}_m)$.

We make the following definition:

Definition 4.38 — *Given an $\mathbb{A}^1_k$-scheme $f: X \longrightarrow \mathbb{A}^1_k$ and an object $A \in$ $\mathbf{SH}(X_\eta)$, we define*

$$\mathsf{log}_f(A) = \mathsf{Colim}_n \ \chi_f \underline{\mathsf{Hom}}(f_\eta^* \mathcal{L}og^n, A).$$

The arguments used in the proof of Proposition 4.8 show that this formula extends to a specialization system log.

Proposition 4.39 — *For any f, there is a natural distinguished triangle*

$$\mathsf{log}_f(A)(-1)[-1] \longrightarrow \chi_f(A) \longrightarrow \mathsf{log}_f(A) \xrightarrow{N} \mathsf{log}_f(A)(-1).$$

Proof. This is clear from Corollary 4.37. □

To obtain the first part of Theorem 4.28 from Proposition 4.39 we need to compare the two specialization systems Υ and log. We do this in three steps: *Step 1.* If $E^\bullet$ is a cosimplicial object in an additive category we will denote by $cE^\bullet$ the usual complex associated to it (by taking the alternating sum of the faces). Given a complex $K = K^\bullet$ in some additive category, we denote $K^{\leq n}$ the complex obtained by replacing the objects K^i by a zero object for all $i \geq n$. Given a smooth X-scheme U let us simply denote by X (and not $\mathbb{Q}_{\mathrm{tr}}(X)$) the Nisnevich sheaf of $\mathbb{Q}$-vector spaces with transfers represented by X. Consider the complexes of sheaves with transfers $c\mathcal{A}^{\leq n}$. We have canonical morphisms

$$c\mathcal{A}^{\leq n+1} \longrightarrow c\mathcal{A}^{\leq n}$$

that give a pro-object $(c\mathcal{A}^{\leq n})_{n \in \mathbb{N}}$.

First remark that $(c\mathcal{A}^{\leq 1})$ maps naturally to the motivic Kummer torsor. Indeed, we take the morphism induced by the morphisms of complexes

$$\begin{array}{ccccc}
c\mathcal{A}^{\leq 1} \simeq & [\mathrm{id}: \mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m}] & \xrightarrow{\Delta - (x,1)} & [pr_1 : \mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m}] \\
 & \| & & \downarrow \\
\mathcal{K} \simeq & [\mathrm{id}: \mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m}] & \xrightarrow{\theta} & [pr_1 : \mathbb{G}_\mathrm{m} \times (\mathbb{G}_\mathrm{m}, 1) \to \mathbb{G}_\mathrm{m}]
\end{array}$$

where the two horizontal arrows are the first and only non-zero differentials of $c\mathcal{A}^{\leq 1}$ and $\mathcal{K}$. This canonical morphism will be denoted by γ_1. By composing with the obvious morphisms $(c\mathcal{A}^{\leq n}) \longrightarrow (c\mathcal{A}^{\leq 1})$ we get for $n \geq 1$ canonical morphisms

$$\gamma_1 : (c\mathcal{A}^{\leq n}) \longrightarrow \mathcal{K}.$$

Passing to the symmetric powers, we get morphisms

$$\gamma_r : \mathrm{Sym}^r(c\mathcal{A}^{\leq n}) \longrightarrow \mathcal{L}og^r \ .$$

Using the obvious morphism: $(c\mathcal{A}^{\leq n}) \longrightarrow \mathbb{Q}(0)$ we can define in the same way as for $\mathcal{L}og$ a pro-object structure on $(\mathrm{Sym}^r(c\mathcal{A}^{\leq n}))_{r \in \mathbb{N}}$, and the family (γ_r) becomes a morphism of pro-objects for $n \geq 1$. Given a morphism of k-schemes $f: X \longrightarrow \mathbb{A}^1$, we get from (γ_r) a natural transformation

$$\mathsf{log}_f \longrightarrow \mathsf{Colim}_{n,r} \ i^* j_* \underline{\mathsf{Hom}}(f_\eta^* \mathrm{Sym}^r(c\mathcal{A}^{\leq n}), -). \tag{3}$$

Step 2. Let us denote by $(\mathcal{A}^{\otimes r})^\bullet$ the cosimplicial object obtained by taking self products of $\mathcal{A}$ in the category of cosimplicial $\mathbb{G}_{\mathrm{m}}$-schemes. There is an action of Σ_r on $(\mathcal{A}^{\otimes r})^\bullet$, so that the symmetric part $(\mathrm{Sym}^r\mathcal{A})^\bullet$ can be defined in the category of cosimplicial sheaves of $\mathbb{Q}$-vector spaces. Using the projection $\mathcal{A} \longrightarrow \mathbb{G}_{\mathrm{m}}$, we get a pro-object of cosimplicial sheaves $(\,(\mathrm{Sym}^{r+1}\mathcal{A})^\bullet \longrightarrow (\mathrm{Sym}^r\mathcal{A})^\bullet\,)_r$. As in the first step, we consider the complexes $c(\mathrm{Sym}^r\mathcal{A})^{\leq n}$ and $c(\mathcal{A}^{\otimes r})^{\leq n}$. We have an obvious Σ_r-equivariant morphism: $(c\mathcal{A}^{\leq n})^{\otimes r} \longrightarrow c(\mathcal{A}^{\otimes r})^{\leq n}$. Passing to the symmetric part, we get morphisms

$$\mathrm{Sym}^r(c\mathcal{A}^{\leq n}) \longrightarrow c(\mathrm{Sym}^r\mathcal{A})^{\leq n}$$

of $\mathbb{N}\times\mathbb{N}$-pro-objects. This induces for any $\mathbb{A}^1$-scheme $f : X \longrightarrow \mathbb{A}^1$ a natural transformation

$$\begin{array}{c} \mathsf{Colim}_{n,r}\ i^*j_*\underline{\mathsf{Hom}}(f_\eta^* c(\mathrm{Sym}^r\mathcal{A})^{\leq n}, -) \\ \big\downarrow \\ \mathsf{Colim}_{n,r}\ i^*j_*\underline{\mathsf{Hom}}(f_\eta^*\mathrm{Sym}^r(c\mathcal{A}^{\leq n}), -). \end{array} \tag{4}$$

Lemma 4.40 — *The natural transformation (4) is an isomorphism.*

Proof. It suffices to fix r and to prove that the natural transformation:

$$\begin{array}{c} \mathsf{Colim}_{n}\ i^*j_*\underline{\mathsf{Hom}}(f_\eta^* c(\mathrm{Sym}^r\mathcal{A})^{\leq n}, -) \\ \big\downarrow \\ \mathsf{Colim}_{n}\ i^*j_*\underline{\mathsf{Hom}}(f_\eta^*\mathrm{Sym}^r(c\mathcal{A}^{\leq n}), -) \end{array}$$

is invertible. This natural transformation is a direct factor of

$$\mathsf{Colim}_n\ i^*j_*\underline{\mathsf{Hom}}(f_\eta^* c(\mathcal{A}^{\otimes r})^{\leq n}, -) \longrightarrow \mathsf{Colim}_n\ i^*j_*\underline{\mathsf{Hom}}(f_\eta^*(c\mathcal{A}^{\leq n})^{\otimes r}, -).$$

Let us show that the latter is invertible. The left hand side is nothing but the total space of the simplicial space

$$i^*j_*\underline{\mathsf{Hom}}(f_\eta^*(\mathcal{A}^{\otimes r})^\bullet, -).$$

The right hand side is the total space of the r-simplicial space

$$i^*j_*\underline{\mathsf{Hom}}(f_\eta^*(\mathcal{A}^\bullet \otimes \cdots \otimes \mathcal{A}^\bullet), -)$$

and the morphism we are looking at is the one induced from the identification of $i^*j_*\underline{\mathsf{Hom}}(f_\eta^*(\mathcal{A}^{\otimes r})^\bullet, -)$ with the restriction of $i^*j_*\underline{\mathsf{Hom}}(f_\eta^*(\mathcal{A}^\bullet)^{\otimes r}, -)$ to the diagonal inclusion of categories $\mathbf{\Delta} \longrightarrow \mathbf{\Delta} \times \cdots \times \mathbf{\Delta}$. But it is a well–known fact that the total space of an r-simplicial object is quasi-isomorphic to the total space of its diagonal. □

Step 3. Using Lemma 4.40 and the two natural transformations (3) and (4), we get for any k-morphism $f : X \longrightarrow \mathbb{A}^1_k$ a natural transformation

$$\mathsf{log}_f \longrightarrow \mathsf{Colim}_{n,r}\ i^*j_*\underline{\mathsf{Hom}}(f_\eta^* c(\mathrm{Sym}^r\mathcal{A})^{\leq n}, -). \tag{5}$$

Now consider the diagonal embedding of cosimplicial schemes $\mathcal{A}^\bullet \longrightarrow (\mathcal{A}^{\otimes r})^\bullet$. One easily sees that it is Σ_r-equivariant. So it factors uniquely through

$$\mathcal{A}^\bullet \longrightarrow (\mathrm{Sym}^r \mathcal{A})^\bullet.$$

This gives us a morphism of pro-objects

$$c\mathcal{A}^{\leq n} \longrightarrow c(\mathrm{Sym}^r \mathcal{A})^{\leq n}$$

and a natural transformation of functors

$$\begin{array}{c} \mathsf{Colim}_{n,r}\ i^* j_* \underline{\mathsf{Hom}}(f_\eta^* c(\mathrm{Sym}^r \mathcal{A})^{\leq n}, -) \\ \big\downarrow \\ \mathsf{Colim}_n\ i^* j_* \underline{\mathsf{Hom}}(f_\eta^* c\mathcal{A}^{\leq n}, -) = \Upsilon_f. \end{array}$$

Composing with (5), we finally get the natural transformation

$$\gamma_f :\ \mathsf{log}_f \longrightarrow \Upsilon_f.$$

We leave the verification of the following lemma to the reader:

Lemma 4.41 — *The family of natural transformations (γ_f) is a morphism of specialization systems. Moreover, we have a commutative triangle*

$$\begin{tikzcd} \chi \arrow[r] \arrow[dr] & \mathsf{log} \arrow[d, "\gamma"] \\ & \Upsilon. \end{tikzcd}$$

The rest of this section is mainly devoted to the proof of the following result:

Proposition 4.42 — *The morphism $\gamma :\ \mathsf{log} \longrightarrow \Upsilon$ is an isomorphism.*

We break up the proof into several lemmas, which are of independent interest:

Lemma 4.43 — *For every non-zero natural number n, the composition*

$$\mathbb{Q} \longrightarrow \chi_{e_n}\mathbb{Q} \longrightarrow \Upsilon_{e_n}\mathbb{Q}$$

is an isomorphism.

Proof. This is a generalization of Proposition 4.9 that holds when working in $\mathbf{DM}_{\mathbb{Q}}(-)$. We repeat exactly the same proof of 4.9, replacing everywhere $\mathcal{A}^\bullet$ with $(e_n)_\eta^* \mathcal{A}^\bullet$. We end up with the following problem: is the morphism

$$\mathbb{Q} \longrightarrow \mathsf{Tot}(\mathbb{G}_{\mathrm{m}} \tilde{\times}_{\mathbb{G}_{\mathrm{m}},(e_n)_\eta} k)$$

invertible in $\mathbf{DM}_{\mathbb{Q}}(k)$? The difference with 4.9 is that the cobar cosimplicial object on the right hand side is the one obtained by applying Lemma 4.2 to

$$\mathbb{G}_{\mathrm{m}} \xrightarrow{(e_n)_\eta} \mathbb{G}_{\mathrm{m}} \xleftarrow{1} k\ .$$

To answer this question, we look at the obvious morphism of cosimplicial objects

$$\mathbb{G}_{\mathrm{m}} \tilde{\times}_{\mathbb{G}_{\mathrm{m}},(e_n)_\eta} k \longrightarrow \mathbb{G}_{\mathrm{m}} \tilde{\times}_{\mathbb{G}_{\mathrm{m}}} k$$

and check that it is level-wise an $\mathbb{A}^1$-weak equivalence (up to torsion). On the level $\underline{\mathbf{i}}$ this morphism is given by

$$(e_n)_\eta \times \mathrm{id}^{\times i} : \mathbb{G}_m{}^{i+1} \longrightarrow \mathbb{G}_m{}^{i+1}.$$

It is well–known that elevation to the n-th power on $\mathbb{G}_m$ induces the identity on $\mathbb{Q}$ and multiplication by n on $\mathbb{Q}(1)[1]$ modulo the decomposition $M(\mathbb{G}_m) \simeq \mathbb{Q} \oplus \mathbb{Q}(1)[1]$ in $\mathbf{DM}_{\mathbb{Q}}(k)$. $\square$

Lemma 4.44 — *The motives $\mathcal{K}$ and $(e_n)^*_\eta\mathcal{K}$ are isomorphic.*

Proof. Indeed, the motive $(e_n)^*_\eta\mathcal{K}$ corresponds to the extension

$$\mathbb{Q}(1) \longrightarrow (e_n)^*_\eta\mathcal{K} \longrightarrow \mathbb{Q}(0) \xrightarrow{n.\theta} \mathbb{Q}(1)[1].$$

We have a commutative square

$$\begin{tikzcd} \mathbb{Q}(0) \arrow[r, "\theta"] \arrow[d, equal] & \mathbb{Q}(1)[1] \arrow[d, "\times n"] \\ \mathbb{Q}(0) \arrow[r, "n.\theta"] & \mathbb{Q}(1)[1] \end{tikzcd}$$

which we extend into a morphism of distinguished triangles

$$\begin{tikzcd} \mathbb{Q}(1) \arrow[r] \arrow[d, "\times n"] & \mathcal{K} \arrow[r] \arrow[d, "a"] & \mathbb{Q}(0) \arrow[r, "\theta"] \arrow[d, equal] & \mathbb{Q}(1)[1] \arrow[d, "\times n"] \\ \mathbb{Q}(1) \arrow[r] & (e_n)^*_\eta\mathcal{K} \arrow[r] & \mathbb{Q}(0) \arrow[r, "n.\theta"] & \mathbb{Q}(1)[1]. \end{tikzcd}$$

The morphism a is clearly invertible. $\square$

Lemma 4.45 — *Denote by q the projection $\mathbb{G}_m \longrightarrow k$. For every $n \in \mathbb{N}$, there is a canonical distinguished triangle which splits:*

$$\mathbb{Q}(n+1)[1] \longrightarrow q_\# \mathcal{L}og^n \longrightarrow \mathbb{Q}(0) \longrightarrow$$

Moreover, the diagram

$$\begin{tikzcd} \mathbb{Q}(n+2)[1] \arrow[r] \arrow[d, "0"'] & q_\# \mathcal{L}og^{n+1} \arrow[r] \arrow[d] & \mathbb{Q}(0) \arrow[r] \arrow[d, equal] & {} \\ \mathbb{Q}(n+1)[1] \arrow[r] & q_\# \mathcal{L}og^{n} \arrow[r] & \mathbb{Q}(0) \arrow[r] & {} \end{tikzcd}$$

is a morphism of distinguished triangles.

Proof. This is a well-known fact to people working on Polylogarithms. The simplest way to prove it is to work over a number field and in the abelian category of mixed Tate motives $\mathbf{MTM}(\mathbb{G}_m)$. We gave an elementary proof in the third chapter of [3]. We use the finite filtration: $\mathcal{L}og^{n-i}(i) \subset \mathcal{L}og^n$ to produce a spectral sequence of mixed Tate motives

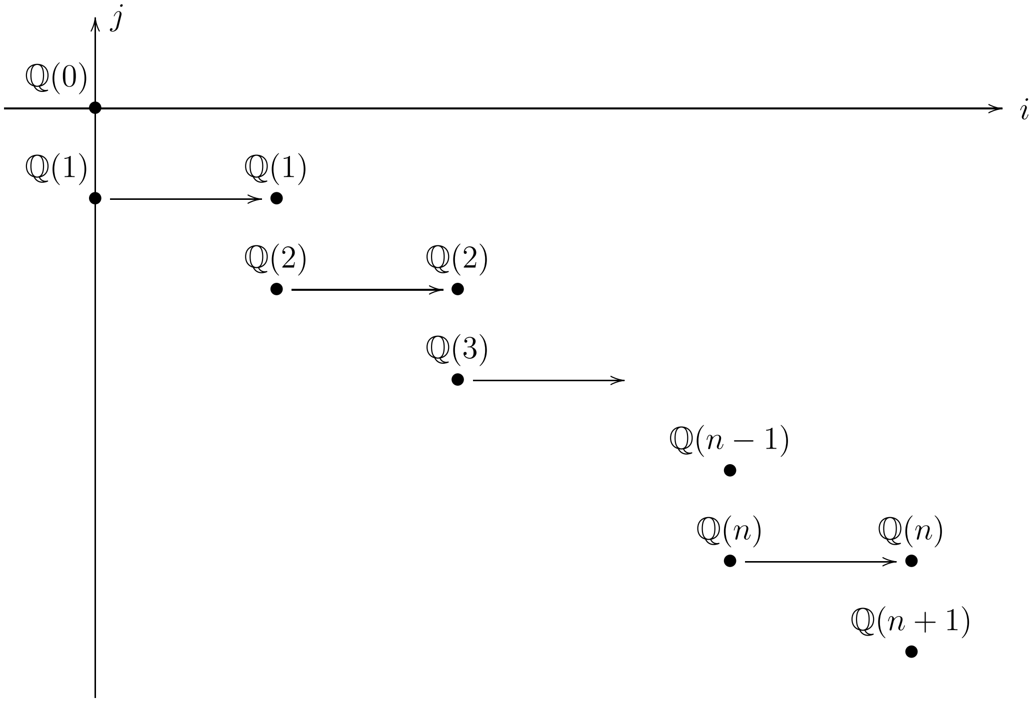
$$E_1^{i,j} = \mathrm{h}^{i+j}_{\mathcal{M}} q_\# \mathbb{Q}(i) \Longrightarrow \mathrm{h}^{i+j}_{\mathcal{M}} q_\# \mathcal{L}og^n$$

where $\mathrm{h}^{\bullet}_{\mathcal{M}}$ is the truncation with respect to the motivic t-structure. We have:

- $\mathrm{h}^r_{\mathcal{M}} q_\# \mathbb{Q}(i) = 0$ except for $r = -1,\ 0,$

- $\mathrm{h}^{-1}_{\mathcal{M}} q_{\#}\mathbb{Q}(i) = \mathbb{Q}(i+1)$ and $\mathrm{h}^{0}_{\mathcal{M}} q_{\#}\mathbb{Q}(i) = \mathbb{Q}(i)$.

So our spectral sequence on $\mathbf{MTM}(k)$ looks like:



It is easy to show that the non-zero differentials of the E_1-page are the identity. Indeed they are given by

$$\mathrm{h}^{-1}_{\mathcal{M}} q_{\#}\mathbb{Q}(i) \longrightarrow \mathrm{h}^{-1}_{\mathcal{M}} q_{\#}\mathbb{Q}(i+1)[+1]$$

where $\mathbb{Q}(i) \longrightarrow \mathbb{Q}(i+1)[1]$ is the motivic fundamental class $\mathbb{Q}(0) \longrightarrow \mathbb{Q}(1)[1]$ twisted by $\mathbb{Q}(i)$ (due to Lemma 4.36). So it suffices to compute the restriction of $q_{\#}\theta$: $q_{\#}\mathbb{Q}(0) \longrightarrow q_{\#}\mathbb{Q}(1)[1]$ to $\mathbb{Q}(1)[1]$. By definition $q_{\#}\theta$ is the diagonal embedding

$$\mathbb{G}_{\mathrm{m}} \longrightarrow \mathbb{G}_{\mathrm{m}} \times (\mathbb{G}_{\mathrm{m}}, 1)\ .$$

This shows that $q_{\#}$ induces the identity on $\mathbb{Q}(1)[1]$. In particular, our spectral sequence degenerates at E_2 and the only nonzero terms that we get are $\mathbb{Q}(0)$ and $\mathbb{Q}(n+1)$. This proves the lemma. □

Corollary 4.46 — *For every nonzero natural number n, the composition*

$$\mathbb{Q} \longrightarrow \chi_{e_n}\mathbb{Q} \longrightarrow \mathsf{log}_{e_n}\mathbb{Q}$$

is an isomorphism.

Proof. Due to Lemma 4.44, it suffices to consider the case $n = 1$. Once again we apply the argument in the proof of Proposition 4.9. We end up with the following question: Is the morphism

$$\mathbb{Q} \longrightarrow \mathsf{Colim}_n\ \underline{\mathsf{Hom}}(p_{\#}\mathcal{L}og^n, \mathbb{Q})$$

an isomorphism? The answer is yes by Lemma 4.45. □

Lemma 4.43 and Corollary 4.46 together imply that for any n, the morphism

$$\mathsf{log}_{e_n}\mathbb{Q} \longrightarrow \Upsilon_{e_n}\mathbb{Q}$$

is an isomorphism. This proves Proposition 4.42 by applying Theorem 3.9, part *2*. We have proved the first part of Theorem 4.28. For the nilpotency of N, first remark that due to Lemma 4.43 and Theorem 3.8 we know that Υ_f sends constructible objects to constructible objects. So we can apply the following general result:

Proposition 4.47 — *Let S be a scheme of finite type over a field k of characteristic zero. Let A and B be constructible objects in $\mathbf{DM}(S)$. Then for N large enough, the groups $\hom_{\mathbf{DM}(S)}(A, B(-N)[*])$ are zero.*

Proof. One may assume that A is the motive of a smooth S-scheme $f : U \longrightarrow S$. Then we see that

$$\hom(A, B(-N)[*]) = \hom(\mathbb{Z}_U, f^*B(-N)[*]) = \hom(\mathbb{Z}, (\pi_U)_*f^*B(-N)[*])$$

with π_U the projection of U to k. Thus it suffices to consider the case where $A = \mathbb{Z}$ and $B \in \mathbf{DM}^{ct}(k)$. Denoting $\mathrm{D} = \underline{\mathsf{Hom}}(-, \mathbb{Z})$ the duality operator, we have: $\hom(\mathbb{Z}, B(-N)[*]) = \hom(\mathrm{D}(B), \mathbb{Z}(-N)[*])$. We finally see that it suffices to prove that for a smooth variety V over k we have $\hom(V, \mathbb{Z}(-N)[*]) = 0$ for N large enough. But by the cancellation theorem of Voevodsky [38], we know that we can take any $N \geq 1$. □

5. Conservation conjecture. Application to Schur finiteness of motives

5.1. **The statement of the conjecture.** Recall that a category $\mathcal{C}$ is pointed if it has an initial and a terminal object that are isomorphic (via the unique map between them). The initial and terminal objects will be called zero objects. Usually a functor $F : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ between two abstract categories is said to be conservative if it detects isomorphisms, that is, an arrow f is an isomorphism if and only if $F(f)$ is an isomorphism. For our purposes, it will be more convenient to say that a functor $F : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ between two pointed categories is *conservative* if it detects the zero objects. That is, an object $A \in \mathcal{C}_1$ is zero if and only if $F(A)$ is zero. When F is a triangulated functor between two triangulated categories, then the two notions coincide. In general, they are quite different.

Conjecture 5.1 — *Let S be the spectrum of a geometric DVR of equi-characteristic zero. We denote as usual η and s the points of S, and we fix a uniformizer $\pi \in \Gamma(S, \mathscr{O}_S)$. Then the functor $\Psi_{id_S} : \mathbf{DM}^{ct}_{\mathbb{Q}}(\eta) \longrightarrow \mathbf{DM}^{ct}_{\mathbb{Q}}(s)$ is conservative.*

Remark 5.2 — We do not know if it is reasonable to expect that the functor $\Psi_{\mathrm{id}_S} : \mathbf{DM}^{ct}(\eta) \longrightarrow \mathbf{DM}^{ct}(s)$ is conservative without killing torsion.

Remark 5.3 — One can also ask if $\Psi_{\mathrm{id}_S} : \mathbf{SH}^{ct}_{\mathbb{Q}}(\eta) \longrightarrow \mathbf{SH}^{ct}_{\mathbb{Q}}(s)$ is conservative. When (-1) is a sum of squares in $\mathscr{O}_S$ this is equivalent to 5.1. Indeed, by Morel [27] the category $\mathbf{SH}_{\mathbb{Q}}(k)$ decomposes into a direct product of triangulated categories $\mathbf{DM}_{\mathbb{Q}}(k) \oplus ?(k)$ where $?(k)$ is zero if and only if (-1) is a sum of squares in k. When

(-1) is not a sum of squares in $\mathscr{O}_S$, the functor Ψ_{id_S} may fail to be conservative for obvious reasons. Indeed, one can prove that Ψ_{id_S} is compatible with the decomposition: $\mathbf{SH}_{\mathbb{Q}}(-) = \mathbf{DM}_{\mathbb{Q}}(-)\oplus ?(-)$. In particular, for a base S such that (-1) as a sum of squares over s but not over η, the functor Ψ_{id_S} takes the non-zero subcategory $?(\eta)$ to zero.

The main reason why one believes in the conservation conjecture is because it is a consequence of the conservation of the realization functors. Indeed, assuming that the ℓ-adic realization functor $R_\ell : \mathbf{DM}^{ct}_{\mathbb{Q}}(-) \longrightarrow \mathsf{D}^+(-, \mathbb{Q}_\ell)$ (see [19]) is conservative for fields it is easy to deduce conjecture 5.1 using the commutative diagram (up to a natural isomorphism)

$$\begin{array}{ccc} \mathbf{DM}^{ct}_{\mathbb{Q}}(\eta) & \xrightarrow{\Psi} & \mathbf{DM}^{ct}_{\mathbb{Q}}(s) \\ {\scriptstyle R_\ell}\downarrow & & \downarrow{\scriptstyle R_\ell} \\ \mathsf{D}^+(\eta, \mathbb{Q}_\ell) & \xrightarrow{\Psi} & \mathsf{D}^+(s, \mathbb{Q}_\ell) \end{array}$$

that expresses the compatibility of our motivic nearby cycles functor with the classical ℓ-adic one.

Indeed, the functor Ψ on the level of continuous Galois modules is nothing but a forgetful functor which associates to a $\mathrm{Gal}(\bar{\eta}/\eta)$-module the $\mathrm{Gal}(\bar{s}/s)$-module with the same underlying $\mathbb{Q}_\ell$-vector space obtained by restricting the action using an inclusion $\mathrm{Gal}(\bar{s}/s) \subset \mathrm{Gal}(\bar{\eta}/\eta)$. The latter inclusion is obtained using the choice of a uniformizer (in equi-characteristic zero).

Maybe it is worth pointing that our conservation conjecture is weaker than the conservation of the realizations, which seems out of reach for the moment. Furthermore, the statement of 5.1 is completely motivic. So we hope it is easier to prove.

5.2. **About the Schur finiteness of motives.** Let us first recall the notion of Schur finiteness due to Deligne (see [7]). Let $(\mathcal{C}, \otimes)$ be a $\mathbb{Q}$-linear tensor category. For an object A of $\mathcal{C}$, the n-th symmetric group Σ_n acts on $A^{\otimes n} = A \otimes \cdots \otimes A$. By linearity, we get an action of the group algebra $\mathbb{Q}[\Sigma_n]$ on $A^{\otimes n}$. If $\mathcal{C}$ is pseudo-abelian, then for any idempotent p of $\mathbb{Q}[\Sigma_n]$ we can take its image in $A^{\otimes n}$ obtaining in this way an object $S_p(A) \in \mathcal{C}$.

Definition 5.4 — *An object A of $\mathcal{C}$ is said to be* Schur finite *if there exists an integer n and a non-zero idempotent p of the algebra $\mathbb{Q}[\Sigma_n]$ such that $S_p(A) = 0$.*

This notion is a natural generalization of the notion of finite dimensionality of vector spaces. Indeed a vector space E is of finite dimension if and only if for some $n \geq 0$, the n-th exterior product $\overset{n}{\Lambda} E$ is zero. The notion of Schur finiteness makes sense in many contexts. One can speak about Schur finiteness of mixed motives in $\mathbf{DM}_{\mathbb{Q}}(k)$. For more about this notion the reader can consult [22]. Another finiteness notion of the same spirit is the Kimura finiteness (see [21]). One of the reasons why Schur finiteness is more flexible than Kimura finiteness is the following striking result, proved in [22]:

Lemma 5.5 — *Suppose given a distinguished triangle in $\mathbf{DM}_{\mathbb{Q}}(k)$:*

$$A' \longrightarrow A \longrightarrow A'' \longrightarrow A[+1] \; .$$

If two of the three objects A, A' and A'' are Schur finite, then so is the third[6].

It is conjectured that any constructible object of $\mathbf{DM}_{\mathbb{Q}}(k)$ is Schur finite. For Kimura finiteness one can at most hope that this property holds for pure objects of $\mathbf{DM}_{\mathbb{Q}}(k)$, that is, objects coming from the fully faithful embedding (see [40])

$$\mathbf{Chow}(k)_{\mathbb{Q}} \longrightarrow \mathbf{DM}_{\mathbb{Q}}(k)$$

where $\mathbf{Chow}(k)$ is the category of Chow motives. The problem is that Lemma 5.5 fails for Kimura finiteness. Lots of unsolved problem would follow if one could prove that some motives are Schur or Kimura finite. For an overview, the reader may consult [1], [12] and [21]. Let us only mention the Bloch conjecture[7] for surfaces with $p_g = 0$. Unfortunately, the only way we know to construct Schur finite motives is by the following proposition (see [21]):

Proposition 5.6 — *If C is a smooth k-curve then its motive $M(C)$ is Schur (even Kimura) finite.*

It formally follows from Lemma 5.5 and the above proposition that all objects of the triangulated tensor subcategory $\mathbf{DM}_{\mathbb{Q}}^{Abelian}(k)$ of $\mathbf{DM}_{\mathbb{Q}}^{ct}(k)$ generated by motives of curves are Schur finite. It is remarkable that there is not a single motive that does not belong to $\mathbf{DM}_{\mathbb{Q}}^{Abelian}(k)$ for which Schur finiteness has been established.

One of the applications of the theory of vanishing cycles is the following reduction:

Proposition 5.7 — *Suppose that k is of infinite transcendence degree over $\mathbb{Q}$. To show that every constructible object of $\mathbf{DM}_{\mathbb{Q}}(k)$ is Schur finite, it suffices to check that for any $n \in \mathbb{N}$, and any general*[8] *smooth hypersurface H of $\mathbb{P}^{n+1}_k$, the motive $M(H)$ is Schur finite.*

In the rest of the paragraph we give a proof of 5.7. We work under the assumption that the motive $M(H)$ is Schur finite whenever H is a general smooth hypersurface of some $\mathbb{P}^r_k$. Remark that if k' is another field and H' is a general hypersurface of $\mathbb{P}^r_{k'}$ then the motive $M(H')$ is Schur finite in $\mathbf{DM}_{\mathbb{Q}}(k')$. Indeed, one can replace k' by an extension so that H' is isomorphic to the pull-back of H along some morphism $\mathsf{Spec}(k') \longrightarrow \mathsf{Spec}(k)$.

The subcategory $\mathbf{DM}_{\mathbb{Q}}^{ct}(k)$ of constructible motives is generated (up to Tate twist and direct factors) by motives $M(X)$ with X a smooth projective variety (see [40]). By Lemma 5.5, we need only to check that these motives are Schur finite. We argue by induction on the dimension of X.

Let n be the dimension of X. Denote $\mathbf{DM}_{\mathbb{Q}}^{ct}(k)_{\leq n-1}$ the triangulated subcategory of $\mathbf{DM}_{\mathbb{Q}}^{ct}(k)$ generated by motives of smooth projective varieties of dimension $\leq n-1$ and their Tate twists. By induction, the objects of $\mathbf{DM}_{\mathbb{Q}}^{ct}(k)_{\leq n-1}$ are Schur finite.

Let X' be a smooth (possibly open) k-variety birational to X. Using:

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[6] This property is not specific to $\mathbf{DM}_{\mathbb{Q}}(k)$. It holds for any triangulated $\mathbb{Q}$-linear tensor category $\mathcal{T}$ coming from a monoidal Quillen model category (see [11]).

[7] Actually it is not clear that the Bloch conjecture follows from the Schur finiteness of the motives of surfaces, but it does follow from their Kimura finiteness. For more information the readers can consult [23].

[8] Here "general" means that the coefficients of the equation of H are algebraically independent in k.

- Resolution of singularities (see [14]) and the weak factorization theorem (see [41]),
- The blow up with smooth center formula for motives (see [40]),
- The Gysin distinguished triangle for the complement of a smooth closed subscheme (see [40]),

one sees that the motive of X is obtained from the motive of X' and some objects of $\mathbf{DM}^{ct}_{\mathbb{Q}}(k)_{\leq n-1}$ by successive fibers and cofibers. It follows that the Schur finiteness of X is equivalent to the Schur finiteness of X'.

We would like to deform X to a smooth hypersurface. This is impossible in general but we have:

Lemma 5.8 — *There exists a projective flat morphism $f\colon\ E \longrightarrow \mathbb{A}^1_k = \mathsf{Spec}(k[\pi])$ such that:*

1. *E is smooth,*
2. *The generic fiber of f is a general smooth hypersurface in $\mathbb{P}^{n+1}_{k(\pi)}$,*
3. *$E_0 = f^{-1}(0)$ is a reduced normal crossing divisor,*
4. *The fiber E_0 contains a branch D which is birational to X.*

Proof. The variety X is birational to a possibly singular hypersurface $X_0 \subset \mathbb{P}^{n+1}_k$ of degree d. By taking a general pencil of degree d hypersurfaces passing through X_0 we get a flat morphism $f' :\ E' \longrightarrow \mathbb{A}^1_k$, such that:

- The generic fiber of f' is a general smooth hypersurface in $\mathbb{P}^{n+1}_{k(\pi)}$,
- The fiber E'_0 is the reduced scheme X_0.

By pulling back the family f' along the elevation to the m-th power $e_m :\ \mathbb{A}^1_k \xrightarrow{(-)^m} \mathbb{A}^1_k$ (for some sufficiently divisible m) and resolving singularities we get by [29] a morphism $f :\ E \longrightarrow \mathbb{A}^1_k$ with semi-stable reduction. This f has the properties (1)-(3). Property (4) for f follows immediately from the fact that f' is smooth in the neighborhood of the generic point of X_0. □

Let $\eta = \mathsf{Spec}(k(\pi))$ be the generic point of $\mathbb{A}^1_k$, and denote by s its zero section. The motive $M(E_\eta) \in \mathbf{DM}_{\mathbb{Q}}(\eta)$ is Schur finite. Let us denote by $\Psi :\ \mathbf{DM}_{\mathbb{Q}}(\eta) \longrightarrow \mathbf{DM}_{\mathbb{Q}}(s)$ our nearby cycles functor. By Theorem 4.24 this functor is monoidal. It follows that the motive $\Psi(M(E_\eta)) \in \mathbf{DM}_{\mathbb{Q}}(s)$ is Schur finite. Proposition 5.7 is a consequence of the following result (which we state with integral coefficients):

Lemma 5.9 — *Let $f :\ E \longrightarrow \mathbb{A}^1$ be a flat projective morphism of relative dimension n. Suppose that E is smooth, and that $E_s = f^{-1}(s)$ is a reduced normal crossing divisor. Let us write $E_s = D_1 \cup \cdots \cup D_r$, where D_i are the smooth branches. We let D^0_i be the open scheme of D_i defined by $D_i - \cup_{j\neq i} D_j$. There is a distinguished triangle in $\mathbf{DM}(s)$:*

$$\oplus_i M(D^0_i) \longrightarrow \Psi(M(E_\eta)) \longrightarrow N \longrightarrow$$

with N in the triangulated subcategory $\mathbf{DM}^{ct}(s)_{\leq n-1} \subset \mathbf{DM}^{ct}(s)$ generated by Tate twists of motives of smooth projective varieties with dimension less than $n-1$.

Proof. The main ingredient in the proof of this lemma is Theorem 3.11. We first work on E_s and then push everything down using $f_{s!}$. Let $w : \cup_i D_i^0 \longrightarrow E_s$ be the obvious inclusion, and denote $c : C \longrightarrow E_s$ the complement. We have an exact triangle in $\mathbf{DM}(E_s)$:

$$w_! w^* \Psi_f \mathbb{Z} \longrightarrow \Psi_f \mathbb{Z} \longrightarrow c_* c^* \Psi_f \mathbb{Z} .$$

Because f is projective, we have $f_{s!}\Psi_f = \Psi f_{\eta!}$. To show what we want, it suffices to prove that:

- $f_{\eta!}\mathbb{Z}$ is up to a twist $M(E_\eta)$,
- $f_{s!}w_!w^*\Psi_f\mathbb{Z}$ equals $\oplus_i M(D_i^0)$ up to a twist,
- $f_{s!}c_*c^*\Psi_f\mathbb{Z}$ is in the subcategory $\mathbf{DM}^{ct}(s)_{\leq n-1}$.

The first point is easy. Indeed we have $f_{\eta!}\mathbb{Z} = f_{\eta\#}\mathbb{Z}(-n)[-2n] = M(E_\eta)(-n)[-2n]$ because f is smooth (see [3], chapter I). The second point follows in the same way, using the equality $w^*\Psi_f\mathbb{Z} = \mathbb{Z}_{\cup_i D_i^0}$ and smoothness of $\cup_i D_i^0$. For the last point, it suffices to prove that $c^*\Psi_f\mathbb{I}$ is in the triangulated subcategory of $\mathbf{DM}^{ct}(C)$ generated by objects of the form $t_*\mathbb{I}(m)$ where $t : Z \longrightarrow C$ is a closed immersion, Z smooth and m an integer.

To do this, we need some notations. For non-empty $I \subset [1, r]$ we denote $C_I = \cap_{i\in I} D_i$ the closed subscheme of E_s and $c_I : C_I \longrightarrow E_s$ its inclusion. For $J \subset I$ we let $c_{I,J} : C_I \longrightarrow C_J$ be the obvious inclusions. When $\mathrm{card}(I) \geq 2$, the subscheme C_I is inside C. In this case, we call $d_I : C_I \longrightarrow C$ the inclusion. Note also the following commutative diagrams:

$$\begin{tikzcd} D_i^0 \arrow[r, "v_i"] & D_i \arrow[r, "u_i"] & E_s \\ & D_I \arrow[u, "c_{I,i}"] \arrow[ur, "c_I"'] & \end{tikzcd}$$

for $i \in I$.

The $d_I : C_I \longrightarrow C$ for $\mathrm{card}(I) = 2$ form a cover by closed subsets of C. By a variant of the Mayer-Vietoris distinguished triangle for covers by closed subschemes (see [3], chapter II), one proves that any object $A \in \mathbf{DM}(C)$ is in the triangulated subcategory generated by the set of objects

$$\{d_{I*}d_I^*A \mid I \subset [1, r] \text{ and } \mathrm{card}(I) \geq 2\}.$$

To finish the proof, we will show that for $\emptyset \neq I \subset [1, r]$ the object $c_I^*\Psi_f\mathbb{I}$ is in the triangulated subcategory generated by the set of objects

$$\{(c_{K,I})_*\mathbb{Z}(m) \mid I \subset K \subset [1, r] \text{ and } m \in \mathbb{Z}\}.$$

Using Theorem 3.11 one has: $c_I^*\Psi_f\mathbb{I} \simeq c_{I,i}^* u_i^*\Psi_f\mathbb{I} \simeq c_{I,i}^* v_{i*}\mathbb{I}$. It is well-known that $v_{i*}\mathbb{I}$ is in the triangulated subcategory generated by $(c_{K,i})_*\mathbb{I}(-m)$ for $K \subset [1, r]$ containing i and m an integer. This implies our claim. □

Remark 5.10 — The proof of Proposition 5.7 gives the following more precise statement: *the category* $\mathbf{DM}^{ct}(k)$ *is generated by the motives* $\Psi_{id}(M(H))$, H *a general hypersurface of* $\mathbb{P}^{n+1}_{k(\pi)}$. This fact is interesting for its own sake.

For instance, using the Chow-Kunneth decomposition for smooth hypersurfaces $M(H) = \mathbb{Q}(0)[0] \oplus \cdots \oplus \mathbb{Q}(n)[2n] \oplus \tilde{h}_n^{\mathcal{M}}(H)[n]$ we conclude that $\mathbf{DM}_{\mathbb{Q}}^{ct}(k)$ can be generated by the motives $\Psi_{id}(\tilde{h}_n^{\mathcal{M}}(H))$. Note that these generators have the following nice properties:

- They are in the heart of the conjectural motivic t-structure i.e. the realizations of $\Psi_{id}(\tilde{h}_n^{\mathcal{M}}(H))$ are concentrated in degree zero.
- They are equipped with a non-degenerate pairing

$$\Psi_{id}(\tilde{h}_n^{\mathcal{M}}(H)) \otimes \Psi_{id}(\tilde{h}_n^{\mathcal{M}}(H)) \longrightarrow \mathbb{Q}(2n)$$

  inducing an isomorphism $\Psi_{id}(\tilde{h}_n^{\mathcal{M}}(H)) \simeq \mathrm{D}(\Psi_{id}(\tilde{h}_n^{\mathcal{M}}(H)))(2n)$ where $\mathrm{D} = \underline{\mathsf{Hom}}(-, \mathbb{Q})$ is the duality functor.
- They are conjecturally Kimura finite (and not simply Schur finite).

5.3. **The conservation conjecture implies the Schur finiteness of motives.** A way to prove the Schur finiteness of objects in $\mathbf{DM}_{\mathbb{Q}}^{ct}(k)$ is to prove the conservation conjecture. Indeed:

Proposition 5.11 — *Assume conjecture 5.1. Then every constructible motive of* $\mathbf{DM}_{\mathbb{Q}}(k)$ *is Schur finite.*

Proof. We have seen in 5.7 that to check the Schur finiteness of constructible motives, one only needs to consider the motive of a general smooth hypersurface of some projective space. Let $H \subset \mathbb{P}^{n+1}$ be a general smooth hypersurface of degree d. One can find a projective flat morphism $f : E \longrightarrow \mathbb{A}^1_k$ such that $E_0 = f^{-1}(0)$ is a Fermat hypersurface and $E_1 = f^{-1}(1)$ is H.

It is well known that the motive of a Fermat hypersurface is a direct factor of the motive of a product of projective smooth curves. It follows from Proposition 5.6 that $M(E_0)$ is Schur finite. Fix S_p, a non-zero projector of $\mathbb{Q}[\Sigma_m]$ such that $S_p M(E_0) = 0$.

Let us consider for $? \in \{0, 1\}$ the vanishing cycles functors

$$\Psi_? : \mathbf{DM}_{\mathbb{Q}}(\eta) \longrightarrow \mathbf{DM}_{\mathbb{Q}}(?).$$

We know that $\Psi_?(M(E_\eta)) = M(E_?)$ and that $\Psi_?$ is monoidal. We have

$$\Psi_0(S_p M(E_\eta)) = S_p \Psi_0(M(E_\eta)) = S_p M(E_0) = 0.$$

The conservation of Ψ_0 tell us that $S_p M(E_\eta) = 0$. Applying Ψ_1, we get:

$$0 = \Psi_1(S_p M(E_\eta)) = S_p \Psi_1(M(E_\eta)) = S_p M(E_1) = S_p(M(H)).$$

This proves that the motive of H is Schur finite. $\square$

Remark 5.12 — The proof of the above proposition was suggested to us by Kimura. Our original proof was more complicated and very similar to the proof of Proposition 5.7. It was by induction on the degree d. The idea was to degenerate a hypersurface of degree d to the union of two hypersurface of degree $d - 1$ and 1. This original proof was more elementary as it did not use Proposition 5.6.

5.4. **Some steps toward the Conservation conjecture.** In this final paragraph, we shall explain some reductions of the conservation conjecture. With our definition of Ψ, it seems too difficult to study the conservation conjecture. Our first result says that the conservation of Ψ is equivalent to the conservation of a simpler functor Φ already introduced in the beginning of section 4.

Let us recall the definition of the functor Φ. As in paragraph 4.3, we call $e_n : \mathbb{A}^1_k \longrightarrow \mathbb{A}^1_k$ the elevation to the n-th power. We let η be the generic point of $\mathbb{A}^1_k$ and s its zero section. We consider the commutative diagrams

$$\begin{tikzcd} \eta_n \arrow[r, "j"] \arrow[d, "(e_n)_\eta"'] & \mathbb{A}^1_k \arrow[d, "e_n"] & s \arrow[l, "i"'] \arrow[d, equal] \\ \eta \arrow[r, "j"] & \mathbb{A}^1_k & s. \arrow[l, "i"'] \end{tikzcd}$$

We then define $\Phi(A) = \mathsf{Colim}_{n \in \mathbb{N}^\times}\ i^* j_* (e_n)^*_\eta A$ for every object A of $\mathbf{DM}_{\mathbb{Q}}(\eta)$.

Proposition 5.13 — *The following two statements are equivalent:*

- *The functor* $\Psi : \mathbf{DM}^{ct}_{\mathbb{Q}}(\eta) \longrightarrow \mathbf{DM}^{ct}_{\mathbb{Q}}(s)$ *is conservative,*
- *The functor* $\Phi : \mathbf{DM}^{ct}_{\mathbb{Q}}(\eta) \longrightarrow \mathbf{DM}^{ct}_{\mathbb{Q}}(s)$ *is conservative.*

Proof. Indeed, let A be a finite type object of $\mathbf{DM}_{\mathbb{Q}}(\eta)$. Replacing A by a $(e_n)^*_\eta A$ with n sufficiently divisible ($(e_n)^*_\eta$ is a conservative functor), we may assume by Theorem 4.17 (and its variant for Φ and χ) that:

- $\Psi(A) = \Upsilon(A)$,
- $\Phi(A) = \chi(A) = i^* j_*(A)$.

By the monodromy Theorem 4.28 we have a distinguished triangle

$$\Upsilon(A)(-1)[-1] \longrightarrow \chi(A) \longrightarrow \Upsilon(A) \xrightarrow{N} \Upsilon(A)(-1).$$

Now, suppose that $\chi(A) = 0$. Then N is an isomorphism. But we know by the same theorem that N is nilpotent (because A is of finite type). This means that the zero map of $\Upsilon(A) \longrightarrow \Upsilon(A)(-m)$ is an isomorphism for sufficiently divisible m. This of course implies that $\Upsilon(A)$ is zero. On the other hand, if $\chi(A) = 0$ one sees that $\Upsilon(A) = 0$ by looking at the definition of $\Upsilon(A)$. Thus we have proved the equivalence

$$\Psi(A) = 0 \Longleftrightarrow \Phi(A) = 0\ .$$

This clearly implies the statement of the proposition. □

One can go further and prove that the conservation of Φ is a consequence of the conservation of a very concrete functor ϕ defined on the level of homotopy sheaves with transfers. Before doing this, we need to introduce a t-structure on $\mathbf{DM}_{\text{eff}}(k)$ and $\mathbf{DM}(k)$.

Definition 5.14 — ***1-*** *The category* $\mathbf{DM}_{\text{eff}}(k)$ *is equipped with a natural* t*-structure called the* homotopy t-structure. *The heart of this* t*-structure is denoted by* $\mathbf{HI}(k)$. *The objects of* $\mathbf{HI}(k)$ *are the* homotopy invariant Nisnevich sheaves with transfers *on* Sm/k *(see* [40]*).* ***2-*** *The category* $\mathbf{DM}(k)$ *is equipped with a natural* t*-structure also called the* homotopy t-structure. *The heart of this* t*-structure is* *denoted by* $\mathbf{HIM}(k)$*. The objects of* $\mathbf{HIM}(k)$ *are modules on the the Milnor* K*-theory spectrum* $\mathcal{K}^M_*$*; we shall call them* $\mathbb{A}^1$-homotopy modules. *The category* $\mathbf{HIM}(k)$ *is equivalent to the category of* Rost modules *by a result of Deglise* [4].

Let us briefly explain what an $\mathbb{A}^1$-homotopy module is. An $\mathbb{A}^1$-homotopy module is a collection $(F_i)_{i\in\mathbb{Z}}$ of homotopy invariant sheaves with transfers on Sm/k together with assembly isomorphisms $F_i \xrightarrow{\sim} \underline{\mathsf{Hom}}(\mathcal{K}^M_1, F_{i+1})$. They are in some sense analogous to topological spectra, where the topological spheres are replaced by the Milnor K-theory sheaves.

Let us return to our specialization functors. The reason why the homotopy t-structure is interesting for the conservation conjecture is the following result, obtained in the second and third chapters of [3]:

Lemma 5.15 — *The two functors* Φ, Ψ : $\mathbf{DM}(\eta) \longrightarrow \mathbf{DM}(s)$ *are right* t*-exact with respect to the homotopy* t*-structures.*

This is a little bit surprising, because in étale cohomology or in Betti cohomology, these two functors turn out to be left exact with respect to the canonical t-structures. Of course, the point is that the homotopy t-structure is specific to motives and does not correspond via realization to any reasonable t-structure in the étale or the Betti context. Another way to say this is that the homotopy t-structure is not the dreamt of motivic t-structure.

Corollary 5.16 — *Let* ϕ : $\mathbf{HIM}(\eta) \longrightarrow \mathbf{HIM}(s)$ *be the functor defined by* $\phi(-) = \tau_{\leq 0}(\Phi(-))$ *with* $\tau_{\leq 0}$ *being the truncation with respect to the homotopy* t*-structure. Then* ϕ *is a right exact functor between abelian categories.*

There is a natural notion of *finite type* and *finitely presented* objects in $\mathbf{HIM}(k)$. The subcategory of finite type objects[9] in $\mathbf{HIM}(k)$ is denoted by $\mathbf{HIM}^{tf}(k)$. We conjecture that:

Conjecture 5.17 — *Suppose that* k *is of characteristic zero. The functor* ϕ : $\mathbf{HIM}^{tf}_{\mathbb{Q}}(\eta) \longrightarrow \mathbf{HIM}_{\mathbb{Q}}(s)$ *is conservative.*

The conservation of ϕ implies the conservation of Φ. Indeed, if A is a constructible object then $h_i(A) = 0$ for i small enough (where h_i means the homology object of A with respect to the homotopy t-structure). So if A is non zero, we can assume that $h_0(A) \neq 0$ and $h_i(A) = 0$ for $i < 0$. The constructibility of A implies that $h_0(A)$ is of finite type (and even finitely presented). But then we would have $\phi(h_0(A)) = h_0(\Phi(A))$. Thus if $\Phi(A) = 0$, then $h_0(A)$ would be zero, contradicting our assumption that A is non zero. It is also possible to consider the effective version Φ_{eff} : $\mathbf{DM}_{\mathrm{eff}}(\eta) \longrightarrow \mathbf{DM}_{\mathrm{eff}}(s)$ of Φ. We can still prove that Φ_{eff} is right exact. We let ϕ_{eff} : $\mathbf{HI}(\eta) \longrightarrow \mathbf{HI}(s)$ be the induced functor on the hearts. We think that it is easy to show that the functor ϕ is conservative (on objects of finite type and rational coefficients) if and only if its effective version ϕ_{eff} is conservative (on objects of finite type and rational coefficients). Such a reduction could be interesting. Indeed, the functor ϕ_{eff} is rather explicit and defined on sheaves. Unfortunately, we do not know how to prove that ϕ_{eff} : $\mathbf{HI}^{tf}_{\mathbb{Q}}(\eta) \longrightarrow \mathbf{HI}_{\mathbb{Q}}(s)$ is conservative. We should also

[9]Warning: this category is not abelian. Indeed, kernels are not necessarily of finite type.

say that Srinivas gave us a counterexample to the conservation of ϕ_{eff} for fields of positive characteristic. We end by recalling his example.

Example 5.18 — Let $e: E \longrightarrow B$ be the universal family of elliptic curves over a field of positive characteristic k. Fix $s \in B$ such that the fiber E_s is super-singular. Then define the relative surface S/B by a desingularization of $(E \times_B E)/(\mathbb{Z}/2\mathbb{Z})$ where the group $\mathbb{Z}/2\mathbb{Z}$ is acting by: $(x, y) \longmapsto (-x, -y)$. Finally let η be the generic point of B. Then it is known that:

- $\mathrm{CH}_0(S_\eta)$ is infinite dimensional (in the sense of Mumford),
- S_s is a uniruled surface.

In particular, the reduced Suslin homology sheaf $\tilde{h}_0(S_\eta)$ is non-zero, but $\tilde{h}_0(S_s) = 0$. Now, it is expected that $\phi_{\mathrm{eff}}(h_0(S_\eta)) = h_0(S_s)$. This means that ϕ_{eff} kills the non-zero object $\tilde{h}_0(S_\eta)$ (which is of finite type).

Consequently, any proof of 5.1 via the functor ϕ_{eff} should use in a non-trivial way the assumption that the base field is of characteristic zero.

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