The direct extension theorem

Joseph Ayoub

25th March 2004

Abstract

The problem of group extension can be divided into two sub-problems. The first is to find all the possible extensions of H by K. The second is to find the different ways a group G can arise as an extension of H by K. Here we prove that the direct product $H \times K$ can arise as an extension of H by K in an essentially unique way: that is the direct extension. I would like to thank Yacine Dolivet for drawing my attention to the direct "extension theorem", Anne-Marie Aubert as well as Charles-Antoine Louet for their support and Robert Guralnick for suggesting me better proofs of propositions 2.3 and 3.1

Contents

NOTATIONS:

- 1: *is the identity group*
- G, H, K, L... *will denote finite groups*
- $H \leqslant G$: *"H is a subgroup of G"*
- $H < G$: *"H is a subgroup of G distinct from G"*
- $H \trianglelefteq G$: *"H is a normal subgroup of G"*
- $\mathcal{Z}(G)$: *is the center of G*
- G': *is the derived group of G*
- $\mathrm{Card}(G)$ *or* $|G|$: *is the order of the finite group G*

1 Statement of the theorem

In this paper, we prove the following theorem, which we call the "direct extension" theorem.

Theorem 1.1 — *Let G, H and K be three finite groups. If G and $H \times K$ are isomoprhic, then every group extension $1 \longmapsto H \longmapsto G \longmapsto K \longmapsto 1$ is a direct extension.*

We may reformulate the theorem as follows.

Theorem 1.2 — *Let $G = H.K$ be a decomposition of the finite group G into direct factors and H_0 be a normal subgroup of G. Assume that H_0 and H are isomorphic, as well as G/H_0 and K. Then H_0 is a direct factor of G (that is, there exists $K_0 \trianglelefteq G$ such that $G = H_0.K_0$ and $H_0 \cap K_0 = 1$).*

We use the latter statement of the theorem in the proof. Assume that the theorem does not hold, then there is counter-example $(G = H.K; H_0)$, which is minimal with respect to $\operatorname{Card} G$. We shall derive a contradiction from the existence of G.

2 A few preliminary general results

In this section, G is any finite group, not necesseraly the group that appears in the theorem.

2.1 Subgroups of a direct product $G = H.K$

We give here some useful simple results:

Proposition 2.1 — *Let L be a subgroup of G, which we do not assume to be normal, and $G = H.K$ a decomposition of G into direct factors. Assume that $H \subseteq L$. Then $L = H.(L \cap K)$. In particular, H is a direct factor of L.*

Proof — It is clear that $H.(L \cap K) \subseteq L$. Now, every $l \in L$ may be written $l = h.k$, with $h \in H$ and $k \in K$. $H \subseteq L$, which shows that $h \in L$. Thus $k \in L \cap K$ and $l \in H.(L \cap K)$. **Q.E.D.**

Proposition 2.2 — *Let $G = H.K$ be a decomposition of G into direct factors. Then $G' = H'.K'$ and $\mathcal{Z}(G) = \mathcal{Z}(H).\mathcal{Z}(K)$.*

Proof — The first assertion comes from the following formula $[h_1.k_1, h_2.k_2] = [h_1, h_2].[k_1, k_2]$ which is true for $h_i \in H$ and $k_i \in K$. The second assertion is trivial. **Q.E.D.**

2.2 Coprime direct factors of a finite group G

In this paragraph, we make use of the famous Remak-Krull-Schmidt theorem on the decomposition of finite groups into indecomposable direct factors.

Definition 2.1 — *Let A and B be two finite groups. A and B are said to be factor coprime if no non-trivial direct factor of A is isomorphic to a direct factor of B.*

Proposition 2.3 — *Let A and B be two direct factors of the finite group G. Assume that A and B are factor coprime. Then $A \cap B = 1$ and $A.B$ is a direct factor of G.*

Proof — Let $K = AB$. We show that $K = A \times B$ (and this implies that $G = K \times L$ some L, for let e and f be projections of G onto A and B resp., then the map $e + f : G \longmapsto K$ defined by $(e + f)(g) = e(g)f(g)$ is a homomorphism since the images of e and f commute and the image is K and $(e + f)^2 = e + f$ (since $ef = fe = 0$)).

Write $K = A \times C$, then by Remak-Krull-Schmidt, since B is a direct factor of K, B is isomorphic to a direct summand of C and since $|C| = |B : A \cap B|$, B and C are isomorphic. It follows that the projection map from B to C is onto, whence $B = \{(f(c), c); c \in C\}$ for some homomorphism $f : C \longmapsto A$ and since B is normal, $f(c)$ is in the center of A and so $K = A \times B$ as well. **Q.E.D.**

Corollary 1 — *Let $G = B.C$ be a decomposition of G into direct factors. Let A be a direct factor of G, such that A and B are coprime. Then the projection of A onto C is a direct factor of G.*

Proof — We know that $B.A$ is a direct factor of G. But $B.A = B.p_C(A)$, where p_C is the projection onto C with respect to B. This shows that $p_C(A)$ is a direct factor of $B.A$, hence also of G. **Q.E.D.**

2.3 Strongly decomposable subgroups of G

In this paragraph, we define the concept of strongly decomposable subgroups, and show two propositions that will be needed later on.

Definition 2.2 — *Let G be a finite group and D a subgroup of G. D is said to be strongly decomposable in G if $D = (H \cap D).(K \cap D)$ for every decomposition $G = H.K$ of G into direct factors.*

Proposition 2.4 — *If D is strongly decomposable in G, then for every decomposition $G = H_1...H_m$ of G into direct factors we have $D = (H_1 \cap D)...(H_m \cap D)$. Furthermore, if D is a normal subgroup of G, $G/D = ((H_1.D)/D)...((H_m.D)/D)$ is a decomposition of G/D into direct factors.*

Proof — We start with the proof of the first part of the statement. Consider a fixed decomposition $G = H_1...H_m$ of G into direct factors . It is easy to see that $D = (H_1 \cap D)...(H_m \cap D)$ is equivalent to the following statement : for all $d = h_1...h_m \in D$ with $h_i \in H_i$, $h_i \in D$. Let $d = h_1...h_m \in D$. As D is strongly decomposable in G, $D = (H_i \cap D).((H_1...H_{i-1}.H_{i+1}...H_m) \cap D)$ for all $1 \leqslant i \leqslant m$. But then $h_i \in H_i \cap D$.

Assume now that D is normal in G. We have $G/D = ((H_1.D)/D)...((H_m.D)/D)$. Since $(H_i.D)/D \simeq H_i/(H_i \cap D)$, it is true that

$$|(H_1.D)/D| \times ... \times |(H_m.D)/D| = (\frac{|H_1|}{|H_1 \cap D|}) \times ... \times (\frac{|H_m|}{|H_m \cap D|}) = |G|/|D| = |G/D|,$$

which shows that $G/D = ((H_1.D)/D)...((H_m.D)/D)$ is necessarily a decomposition of G/D into direct factors. This completes the proof of our statement. **Q.E.D.**

Proposition 2.5 — *If T is a normal subgroup of G such that $T' = T \cap G'$, then T' is strongly decomposable in G.*

Proof — Let $G = L.M$ be a decomposition of G into direct factors. Write $T = (\overline{A}, A; \overline{B}, B)_\varphi$. Clearly $T \subseteq \overline{A}.\overline{B}$, so $T' \subseteq \overline{A}'.\overline{B}'$. But T is a normal subgroup of G so according to paragraph 2.1 $\overline{A}' \subseteq A$ and $\overline{B}' \subseteq B$. Thus, using $A.B \subseteq T$, we get the following chain of inclusions:

$$T' \subseteq \overline{A}'.\overline{B}' \subseteq (A.B) \cap G \subseteq T \cap G' = T'.$$

These inclusions are then necessarily equalities, so we have $T' = \overline{A}'.\overline{B}' = (T' \cap L).(T' \cap M)$. This shows that the subgroup is strongly decomposable. **Q.E.D.**

3 Two special cases of the theorem

In this section, we prove the theorem in the two special cases:

- G is a commutative group,
- G', the derived subgroup of G, is equal to G.

As in part 2, G stands for any finite group.

3.1 The case of commutative groups

We show the following result wich is a special case of [4] or [5]:

Proposition 3.1 — *Let G be a commutative finite group and $G = H.K$ a decomposition of G into direct factors. Let H_0 be a subgroup of G, such that H and H_0 are isomorphic, as well as G/H_0 and K. Then H_0 is a direct factor of G.*

Proof — Actually, me only need to assume that H is abelian. Consider the obvious map $r : \hom(G, H_0) \longmapsto \hom(H_0, H_0)$. Since H_0 is abelian, the sets $\hom(-, H_0)$ are abelian groups and r is a group homomorphism. The kernel of r is $\hom(G/H_0, H_0)$ so that we have an exact sequence of abelian groups:

$$0 \longmapsto \hom(G/H_0, H_0) \longmapsto \hom(G, H_0) \longmapsto \hom(H_0, H_0)$$

Since $G = H \times K$, we have $\hom(G, H_0) = \hom(H, H_0) \times \hom(K, H_0)$ so that:

$$|\hom(G, H_0)| = |\hom(H, H_0)||\hom(G/H_0, H_0)|$$

and we see that r is onto. So there is a homomorphism $f : G \longmapsto H_0$ that is the identity on H_0. The kernel of f give the desired complement. **Q.E.D.**

3.2 The case where $G' = G$

Proposition 3.2 — *Let G be a finite group such that G is equal to the derived group G' and $G = H.K$ a decomposition of G into direct factors. Let $H_0 \trianglelefteq G$ be isomorphic to H and such that G/H_0 is isomorphic to K. Then H_0 is a direct factor of G.*

Proof — Consider a minimal counter-example $(G = H.K; H_0)$.

We show that H_0 is strongly decomposable in G. $H_0' = H_0$ because H_0 is isomorphic to H and $G' = H'.K' = H.K$. It follows that $H_0' = H_0 \cap G = H_0 \cap G'$. Proposition 2.5. then gives the result.

Moreover, H_0 does not contain a non-trivial direct factor of G. It is an exercise to show that, proceeding in rather the same way as in lemma 3.1.

Now, let $G = H_1...H_m.K_1...K_n$ be a decomposition of G into indecomposable direct factors such that $H = H_1...H_m$ and $K = K_1...K_n$. As H_0 is strongly decomposable in G, proposition 2.4 shows that

$$G/H_0 \simeq (H_1/(H_1 \cap H_0)) \times ... \times (H_m/(H_m \cap H_0)) \times (K_1/(K_1 \cap H_0)) \times ... \times (K_n/(K_n \cap H_0)).$$

But none of the $H_i/(H_i \cap H_0)$ and none of the $K_j/(K_j \cap H_0)$ are trivial. So $G/H_0 \simeq K$ contains at least $n+m$ indecomposable direct factors in a decomposition into irreducible direct factors. We deduce that $n + m \leqslant n$, and $m \leqslant 0$. We have reached a contradiction and out proposition is proved. **Q.E.D.**

We have shown that if $(G = H.K; H_0)$ is a counter-example to the theorem, then $1 < G' < G$. It is the starting point of our proof of the theorem.

4 A few preliminary lemmas

From now on, $(G = H.K; H_0)$ is our minimal counter-example to the theorem. A few lemmas follow, which are useful to describe H_0' and $\mathcal{Z}(H_0)$ in G.

Lemma 4.1 — *We have the following properties.*

1. *$H_0' = H_0 \cap G'$,*
2. *There exists $M \trianglelefteq G$ such that $G = M.H_0$ and $M \cap H_0 = H_0'$,*
3. *G/H_0' and G/H' are isomorphic, as well as M/H_0' and K,*
4. *$G/H_0' = (H_0/H_0').(M/H_0')$ is a decomposition of G/H_0' into direct factors.*

In what follows, we fix a subgroup M once and for all, which complies with point 3 of lemma 4.1.We now prove the lemma.

Proof — Let us start with point 1. Clearly $G' = H'.K'$ is a decomposition of G' into direct factors. Therefore $G'/H' \simeq K' \simeq (G/H_0)'$. But $(G/H_0)' = (G'.H_0)/H_0 \simeq G'/(H_0 \cap G')$. It follows that $\mathrm{Card}\, H_0 \cap G' = \mathrm{Card}\, H' = \mathrm{Card}\, H_0'$. But $H_0' \subseteq H_0 \cap G'$, so the two subgroups are actually equal.

To show point 2, notice two things. First, $G/G' = (HG'/G').(KG'/G') \simeq (H/H') \times (K/K')$. Then $(G/G')/((H_0.G')/G') \simeq (G/(H_0.G')) \simeq ((G/H_0)/(G'.H_0)/H_0) = (G/H_0)/(G/H_0)' \simeq K/K'$. As G/G' is commutative, we may use part 3 to show that there exists a normal subgroup M of G containing G' such that $G/G' = ((H_0.G')/G').(M/G')$ is a decomposition of G/G' into direct factors. This proves point 2.

We now proceed to prove the two last statements. $G/H_0' = (H_0/H_0').(M/H_0')$ is a decomposition into direct factors. Moreover $M/H_0' \simeq (G/H_0')/(H_0/H_0') \simeq G/H_0 \simeq K$, and $H_0/H_0' \simeq H/H'$. So G/H_0' and G/H' are isomorphic. **Q.E.D.**

We now state a corollary of the above lemma, which is crucial in the proof of the "direct extension" theorem.

Corollary 1 — *H_0' is strongly decomposable in G.*

Proof — It is an immediate consequence of proposition 2.5 and the above lemma. **Q.E.D.**

The corollary shows that taking the quotient by H_0' is compatible with any decomposition of G into direct factors. More precisely, if $G = L.M$ is a decomposition of G into direct factors then $G/H_0' = ((L.H_0')/H_0').((M.H_0')/H_0')$ is also a decomposition into direct factors.

Lemma 4.2 — *We have the following properties:*

1. $\mathcal{Z}(H_0) = H_0 \cap \mathcal{Z}(G)$,
2. $\mathcal{Z}(H_0)$ *is a direct factor of* $\mathcal{Z}(G)$,
3. $\mathcal{Z}(M/H_0') = (M \cap (\mathcal{Z}(G).H_0))/H_0'$.

Proof — $\mathcal{Z}(H_0) \simeq \mathcal{Z}(H)$ and $\mathcal{Z}(K) \simeq \mathcal{Z}(G)/\mathcal{Z}(H)$. But $\mathcal{Z}(K) \simeq \mathcal{Z}(G/H_0) \supseteq (\mathcal{Z}(G).H_0)/H_0 \simeq \mathcal{Z}(G)/(H_0 \cap \mathcal{Z}(G))$. Therefore $|H_0 \cap \mathcal{Z}(G)| \geqslant |\mathcal{Z}(H)| = |\mathcal{Z}(H_0)|$. But we know that $H_0 \cap \mathcal{Z}(G) \subseteq \mathcal{Z}(H_0)$. Hence the equality of the two groups. We have also achieved $\mathcal{Z}(G/H_0) = (\mathcal{Z}(G).H_0)/H_0$. This completes the proof of the first point.

For point number 2, notice that $\mathcal{Z}(G)/\mathcal{Z}(H_0) = \mathcal{Z}(G)/(H_0 \cap \mathcal{Z}(G)) \simeq (\mathcal{Z}(G).H_0)/H_0 = \mathcal{Z}(G/H_0)$. As G is not equal to its center, it follows from the minimality of G that $\mathcal{Z}(H_0)$ is a direct factor of $\mathcal{Z}(G)$.

To prove the 3rd point, consider the natural isomorphism $\sigma : M/H_0' \longmapsto G/H_0$. We have

$$\mathcal{Z}(M/H_0') = \sigma^{-1}((\mathcal{Z}(G).H_0)/H_0) = \{x.H_0' \in M/H_0'/x.H_0 \in (\mathcal{Z}(G).H_0)/H_0\}$$

$$= \{x.H_0'/x \in \mathcal{Z}(G).H_0, x \in M\} = ((\mathcal{Z}(G).H_0) \cap M)/H_0'.$$

Hence the announced result. **Q.E.D.**

Lemma 4.3 — *H_0 does not contain a direct factor of G other than 1. Similarly, H_0 is not contained in a direct factor of G other than G.*

Proof — The first statement is left as an exercise. We prove the second one, which is as simple as the first one.

Let $G = L.N$ be a decomposition of G into direct factors. Assume $N > 1$ and $H_0 \subseteq L$. Clearly $G/H_0 = (L/H_0).((N.H_0)/H_0)$ is a decomposition of G/H_0 into direct factors. Since $K \simeq G/H_0$, $N \simeq (N.H_0)/H_0$ is isomorphic to a direct factor of K. We may therefore assume that $N \subset K$. We may then write $G/N \simeq H \times K/N$. But $H \simeq (H_0.N)/N$, and $(G/N)/((H_0.N)/N) \simeq (G/H_0)/((N.H_0)/H_0) \simeq K/N$, because $(N.H_0)/H_0$ is a direct factor of G/H_0 which is isomorphic to N. It follows that $(H_0.N)/N$ is a direct factor of G/N, using minimality of G. Thus we have a normal subgroup P of G containing N such that $G = (H_0.N).P$ with $(H_0.N) \cap P = N$. It is now clear that $G = H_0.P$ is a decomposition of G into direct factors. That contradicts our assumption on G. **Q.E.D.**

5 The proof of the theorem

We may now proceed with the actual proof of our theorem.

Proposition 5.1 — *H contains no non-trivial commutative direct factor.*

Proof — Again *ab absurdo*. Let A be a non-trivial commutative direct factor of H_0, which exists since $H \simeq H_0$. We know that $G/H_0' = (H_0/H_0').(M/H_0')$ is a decomposition of G/H_0' into direct factors. It is easy to see that $(A.H_0')/H_0'$ is a direct factor of H_0/H_0' and hence also of G/H_0'. Thus there exists a direct factor N/H_0' of G/H_0' which contains M/H_0' and is a supplementary subgroup of $(A.H_0')/H_0'$. But $G = A.N$ is then a decomposition of G into direct factors since $(A.H_0') \cap N = H_0'$ and it follows that $A \cap N \subseteq A \cap H_0' = 1$ (since A is a commutative direct factor of H_0). We have shown that $A \subset H_0$ is also a direct factor of G. This contradicts lemma 4.3. **Q.E.D.**

Proposition 5.2 — *If L is a non-commutative direct factor of G, then $L \cap H_0' > 1$.*

Proof — *Ab absurdo.* Let L be a non-commutative direct factor of G such that $L \cap H_0' = 1$. We may suppose that L is indecomposable. Then as H_0' is strongly decomposable in G, $(L.H_0')/H_0'$ is a direct factor of G/H_0' isomorphic to L. But L and H_0/H_0' are coprime because L is a non-commutative indecomposable group and H_0/H_0' is commutative. Then $((L.H_0)'/H_0').(H_0/H_0')$ is a direct factor of G/H_0' by proposition 2.3.

Now let $H_0' \leqslant P \trianglelefteq G$, such that $G/H_0' = ((L.H_0')/H_0').(H_0/H_0').(P/H_0')$ be a decomposition of G/H_0' into direct factors. Then we have $G = L.(P.H_0)$ and $L \cap (P.H_0) = L \cap (L.H_0') \cap H_0 = L \cap H_0' = 1$. We have reached a contradiction since this implies that $P.H_0$ is a direct factor of G distinct from G and containing H_0. **Q.E.D.**

Proposition 5.3 — *If A is a commutative direct factor of G then $H_0 \cap A = 1$*

Proof — Consider $H_0.A \subset G$. Clearly $\mathcal{Z}(H_0).A \subset \mathcal{Z}(G)$ and $\mathcal{Z}(H_0)$ is a direct factor of $\mathcal{Z}(G)$ (by lemma 4.2). Proposition 2.2 shows that there exists B a supplementary of $\mathcal{Z}(H_0)$ in $\mathcal{Z}(H_0).A$. It follows that $H_0.A{=}H_0.B$ with $H_0 \cap B = \mathcal{Z}(H_0) \cap B = 1$. Thus H_0 is a direct factor of $H_0.A$. In the same way, A is a direct factor of $H_0.A$ because it is a direct factor of G. But according to proposition 5.1 H_0 and A are coprime. Therefore $H_0 \cap A = 1$, using once again proposition 2.3. **Q.E.D.**

We may now prove the theorem. Clearly K is non-commutative, otherwise, $H_0 \cap K = 1$, because of the above proposition, and we would then have $G = H_0.K$. This shows that K contains at least one non-commutative indecomposable direct factor.

Let $\mathcal{X}$ be the class up to isomorphism of an indecomposable non-commutative direct factor of K of minimal order. If L is a direct factor of G which is a member of $\mathcal{X}$ then $L' \subset H_0'$. That is true because $L/(L \cap H_0')$ is isomorphic to a direct factor of $G/H_0' \simeq (H_0/H_0') \times K$. But $L \cap H_0' > 1$ according to the corollary of proposition 5.2, which shows that all the indecomposable direct factors of $L/(L \cap H_0')$ have strictly less elements than a member of $\mathcal{X}$. By construction of $\mathcal{X}$, all the indecomposable direct factors of $L/(L \cap H_0')$ are commutative, so $L/(L \cap H_0')$ is itself commutative. We have shown that $L' \subseteq H_0'$.

Let N be a direct factor of G isomorphic to a direct product of members of $\mathcal{X}$. Also assume that N is maximal in that respect. It now suffices to prove that N is isomorphic to a direct factor of H_0 (for then one can find a bigger such N), and we will have shown that a counter-example to our theorem cannot exist.

Clearly $N' \subseteq H_0'$. H_0/H_0' is a subset of the center of G/H_0' because it is a commutative direct factor of that group. Likewise, $(N.H_0')/H_0'$ is a commutative direct factor of G/H_0' because $N' \subseteq H_0'$. Therefore $(N.H_0)/H_0 \subseteq \mathcal{Z}(G/H_0') = (H_0/H_0').(((\mathcal{Z}(G).H_0) \cap M)/H_0')$ (by lemma 4.2). So $H_0.N \subset H_0.\mathcal{Z}(G)$.

Now $H_0.\mathcal{Z}(G) = H_0.S$, where S is a supplementary of $\mathcal{Z}(H_0)$ in $\mathcal{Z}(G)$. Clearly $H_0 \cap S = 1$ and therefore H_0 is a direct factor of $H_0.\mathcal{Z}(G) \supset H_0.N$. We have shown that H_0 is a direct factor of $H_0.N$ and that its supplementary is commutative. On the other hand, N is a direct factor of G and thereby also of $H_0.N$. We now use the Remak-Krull-Schmidt theorem on $H_0.N$. Notice that N has no non-trivial commutative direct factors to obtain that N is isomorphic to a direct factor of H_0. This is precisely what we have striven to show. Our theorem is now proven.

6 Some additional remarks

1. The theorem no longer holds if G is infinite. We give a simple counter-example. Let $G = (\mathbb{Z}/p.\mathbb{Z})^{\mathbb{N}} \times (\mathbb{Z}/p^2.\mathbb{Z})^{\mathbb{N}}$, H (resp. K) the subgroup of G consisting of all pairs (f, g) such that $f : \mathbb{N} \longmapsto \mathbb{Z}/p.\mathbb{Z}$ and $g : \mathbb{N} \longmapsto \mathbb{Z}/p^2.\mathbb{Z}$ with $f(2n) = g(n) = 0$ (resp. $f(2n+1) = 0$).

   Take H_0 the subgroup of G consisting of all pairs (f, g) with $g(2n) = 0$ and $g(2n+1) \in p.\mathbb{Z}/p^2.\mathbb{Z}$.

   Clearly, $G = H.K$ and $H \cap K = 1$. Moreover H_0 is not a direct factor of G. But $H \simeq H_0 \simeq (\mathbb{Z}/p.\mathbb{Z})^{\mathbb{N}}$ and $K \simeq G/H_0 \simeq (\mathbb{Z}/p.\mathbb{Z})^{\mathbb{N}} \times (\mathbb{Z}/p^2.\mathbb{Z})^{\mathbb{N}}$.

2. The following statement does not hold: "Let G be a finite group. If there exists a split extension $1 \longmapsto H \longmapsto G \longmapsto K \longmapsto 1$ then any extension $1 \longmapsto H \longmapsto G \longmapsto K \longmapsto 1$ also splits".

   We give a counter-example. Let A, B, C and D be four groups, each one of them isomorphic to $\mathbb{Z}/p.\mathbb{Z}$, and e_A a generator of A. A acts on $B \times C$ by $\phi : k.e_A \longmapsto [(b, c) \longmapsto (b, c + k.b)]$.

Set $G = (A \ltimes_\phi (B \times C)) \times D$. Clearly $B \times C$ is a normal subgroup of G, with $A \times D$ a supplementary subgroup. This defines a split extension $1 \longmapsto (\mathbb{Z}/p.\mathbb{Z})^2 \longmapsto G \longmapsto (\mathbb{Z}/p.\mathbb{Z})^2 \longmapsto 1$.

However $C \times D$ is in the center of G and $G/(C \times D)$ is isomorphic to $(\mathbb{Z}/p.\mathbb{Z})^2$. On the other hand $C \times D$ cannot have a supplementary because as it is in the center, the semi-direct product would be trivial and G and $(\mathbb{Z}/p.\mathbb{Z})^4$ would be isomorphic. So $1 \longmapsto (\mathbb{Z}/p.\mathbb{Z})^2 \longmapsto G \longmapsto (\mathbb{Z}/p.\mathbb{Z})^2 \longmapsto 1$ is an extension which does not split.

BIBLIOGRAPHY:

1. Group Theory I: *Michio Suzuki*
2. Group Theory : *W.R. Scott*
3. Maximal Subgroups of Direct Products: *Jacques Thèvenaz*
4. *Miyata Takehiko*: Note on direct sumands of modules. J. Math. Kyoto Univ. 7 1967 65-69.
5. *Guralnick Robert*: Roth's theorems and decomposition of modules. Linear Algebra Appl. 39 (1981), 155-165.