

THE n -MOTIVIC t -STRUCTURES FOR $n = 0, 1$ AND 2

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ABSTRACT. For a field k and an integer $n \in \{0, 1, 2\}$, we construct a t -structure $({}^n\mathcal{T}_{\geq 0}^{\mathcal{M}}(k), {}^n\mathcal{T}_{\leq 0}^{\mathcal{M}}(k))$ on Voevodsky's triangulated category of motives $\mathbf{DM}_{\text{eff}}(k)$, which we call the n -motivic t -structure. When $n = 0$, this is simply the usual homotopy t -structure, but for $n \in \{1, 2\}$, these are new t -structures. We will show that the category of Deligne's 1-motives can be embedded as a full subcategory in the heart of the 1-motivic t -structure. By a rather straightforward analogy, we are led to specify a class of objects in the heart of the 2-motivic t -structure which we call *mixed 2-motives*. We will also check that these objects form an abelian category.

Keywords: Voevodsky's motives; mixed motives; motivic t -structures.

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Introduction. A major open problem in the theory of motives is the construction of a *motivic t-structure* on Voevodsky’s triangulated category $\mathbf{DM}_{\text{eff}}(k)$ whose heart would be the awaited abelian category of mixed motives. This motivic *t-structure* should be very different from the existing *homotopy t-structure* which is an outcome of the construction of $\mathbf{DM}_{\text{eff}}(k)$ and the study of homotopy invariant presheaves with transfers [7]. However, one can speculate about the existence of a sequence of *n-motivic t-structures* $({}^n\mathcal{T}_{\geq 0}^{\mathcal{M}}(k), {}^n\mathcal{T}_{\leq 0}^{\mathcal{M}}(k))$ on $\mathbf{DM}_{\text{eff}}(k)$, which interpolate between the homotopy *t-structure* and the motivic *t-structure*. More precisely, we expect the following to hold.

- (1) $({}^0\mathcal{T}_{\geq 0}^{\mathcal{M}}(k), {}^0\mathcal{T}_{\leq 0}^{\mathcal{M}}(k))$ is the homotopy *t-structure*.
- (2) ${}^n\mathcal{T}_{\geq 0}^{\mathcal{M}}(k) \subset {}^{n+i}\mathcal{T}_{\geq 0}^{\mathcal{M}}(k)$ and ${}^n\mathcal{T}_{\leq 0}^{\mathcal{M}}(k) \supset {}^{n+i}\mathcal{T}_{\leq 0}^{\mathcal{M}}(k)$ for $i \geq 0$.
- (3) Denote ${}^\infty\mathcal{T}_{\leq 0}^{\mathcal{M}}(k) = \bigcap_{n \in \mathbb{N}} {}^n\mathcal{T}_{\leq 0}^{\mathcal{M}}(k)$ and ${}^\infty\mathcal{T}_{\geq 0}^{\mathcal{M}}(k) \subset \mathbf{DM}_{\text{eff}}(k)$ the full subcategory of $P \in \mathbf{DM}_{\text{eff}}(k)$ such that

$$\text{hom}_{\mathbf{DM}_{\text{eff}}(k)}(P, N[-1]) = 0$$

for all $N \in {}^\infty\mathcal{T}_{\leq 0}^{\mathcal{M}}(k)$. Then $({}^\infty\mathcal{T}_{\geq 0}^{\mathcal{M}}(k), {}^\infty\mathcal{T}_{\leq 0}^{\mathcal{M}}(k))$ is the motivic *t-structure*.

- (4) ${}^n\mathcal{T}_{\geq 0}^{\mathcal{M}}(k) \cap {}^{n+i}\mathcal{T}_{\leq 0}^{\mathcal{M}}(k)$ is independent of $i \geq 1$ and is the abelian category of *mixed n-motives*.

The last property above, justifies our terminology. For us, a mixed *n-motive* is an object of the heart of the motivic *t-structure* which is also in the smallest triangulated subcategory of $\mathbf{DM}_{\text{eff}}(k)$ stable under small sums and containing the motives of smooth varieties of dimension at most n . In particular, contrary to the usual, we allow non-geometric (or non-constructible) objects.

In this paper, we propose a definition of the *n-motivic t-structure* for $n \in \{0, 1, 2\}$. Our construction relies on [2]. We will see that objects in ${}^0\mathcal{T}_{\leq 0}^{\mathcal{M}}(k) \cap {}^1\mathcal{T}_{\leq 1}^{\mathcal{M}}(k)$ are (possibly non-compact) 0-motives. We also see that objects in ${}^0\mathcal{T}_{\leq 1}^{\mathcal{M}}(k) \cap {}^1\mathcal{T}_{\leq 2}^{\mathcal{M}}(k)$ are (possibly non-compact) Deligne’s 1-motives. Finally, we specify a class of objects in the heart of the 2-motivic *t-structure* which we call mixed 2-motives. We check that the category of mixed 2-motives is abelian.

Acknowledgment. I wish to thank Luca Barbieri-Viale and Bruno Kahn for their encouragement and interest in this work. An incomplete definition of mixed 2-motives, very similar to the one proposed in this paper, appeared to me sometimes ago. However, I never dared to take it seriously. Especially, the problem of showing that these mixed 2-motives form an abelian category seemed, at a first sight, out of reach. After I shared my definition with Luca, he kept telling me that I should try to do something with it, till one evening (during the “Algebraic *K*-Theory and Motivic Cohomology” workshop in Oberwolfach (June 28th - July 4th, 2009)), I decided to follow his advise. I then realized that some progress was possible, and this paper was conceived.

1. PRELIMINARIES

1.1. Notation and general facts.

If not otherwise stated, we work with rational coefficients. In particular, our sheaves take values in the category of \mathbb{Q} -vectorspaces and we think of an isogeny of semi-abelian group-schemes as an invertible morphism (this will be made more precise later).

Fix a ground field k and denote by Sm/k the category of smooth k -schemes. Given two smooth k -schemes X and Y , we denote by $\text{Cor}(X, Y)$ the group of *finite correspondences*, i.e., the \mathbb{Z} -module freely generated by closed and integral subschemes $Z \subset X \times_k Y$ which are finite and surjective on X . There is an additive category $\text{Cor}(k)$ whose objects are smooth k -schemes and whose morphisms are finite correspondences (see [7, Lect. 1]). The graph of a morphism yields an inclusion $Sm/k \hookrightarrow \text{Cor}(k)$.

A *presheaf with transfers* (on Sm/k) is an additive contravariant functor F from $\text{Cor}(k)$ to the category of abelian groups. F is called a *Nisnevich* (resp. an *étale*) *sheaf with transfers* if its restriction to Sm/k is a sheaf for the Nisnevich (resp. the étale) topology. If not otherwise stated, a presheaf with transfers F is assumed to be uniquely divisible, i.e., takes values in the category of \mathbb{Q} -vectorspaces. Under this assumption, the restriction of F to Sm/k is a Nisnevich sheaf if and only if it is an étale sheaf. Thus, there will be no ambiguity in saying: F is a *sheaf with transfers*. We denote $Shv_{tr}(k)$ the abelian category of sheaves with transfers on Sm/k . There is an embedding $\mathbb{Q}_{tr}(-) : \text{Cor}(k) \hookrightarrow Shv_{tr}(k)$ which takes a smooth k -scheme X to the sheaf with transfers $\mathbb{Q}_{tr}(X) = \text{Cor}(-, X) \otimes \mathbb{Q}$ represented by X .

We denote by $\mathbf{K}(Shv_{tr}(k))$ the category of complexes of sheaves with transfers endowed with its *injective model structure* (i.e., $\mathbf{W} = \{\text{quasi-isomorphisms}\}$ and $\mathbf{Cof} = \{\text{monomorphisms}\}$). The homotopy category of this model structure is the derived category $\mathbf{D}(Shv_{tr}(k))$. Following Voevodsky [7], we define $\mathbf{DM}_{\text{eff}}(k)$ to be the homotopy category of the Bousfield localization of $\mathbf{K}(Shv_{tr}(k))$ with respect to the class of arrows $\mathbb{Q}_{tr}(\mathbb{A}_X^1)[n] \rightarrow \mathbb{Q}_{tr}(X)[n]$ for $X \in Sm/k$ and $n \in \mathbb{Z}$. Given a smooth k -scheme X , we denote by $M(X)$ the complex $\mathbb{Q}_{tr}(X)[0]$ considered as an object of $\mathbf{DM}_{\text{eff}}(k)$. This is the motive of X . From the general theory of Bousfield localizations (see [5]), we may identify (up to an equivalence) $\mathbf{DM}_{\text{eff}}(k)$ with the triangulated subcategory of $\mathbf{D}(Shv_{tr}(k))$ whose objects are the \mathbb{A}^1 -local complexes. Recall that $K \in \mathbf{K}(Shv_{tr}(k))$ is \mathbb{A}^1 -local if the natural homomorphism

$$H^n(X, K) \longrightarrow H^n(\mathbb{A}_X^1, K)$$

is invertible for all $n \in \mathbb{Z}$ and $X \in Sm/k$. (Here, $H^n(-, K)$ stands for the Nisnevich (or equivalently the étale) hypercohomology with values in K .) A central result of Voevodsky [7, Th. 24.1] asserts that the previous condition holds if and only if the homology sheaves $H_i(K)$ are homotopy invariant for all $i \in \mathbb{Z}$. In particular, this implies that the canonical t -structure on $\mathbf{D}(Shv_{tr}(k))$ restricts to a t -structure on $\mathbf{DM}_{\text{eff}}(k)$. This is the so-called *homotopy t -structure* whose heart is identified with the category $\mathbf{HI}(k)$ of homotopy invariant sheaves with transfers.

N.B. — In this paper, we will use the expression *\mathcal{H} -sheaf* as a shorthand for: “homotopy invariant sheaf with transfers”.

1.2. Some recollection from [2].

For $n \in \mathbb{N}$, we denote $Sm/k_{\leq n}$ the full subcategory of Sm/k whose objects are the smooth k -schemes of dimension less or equal to n . Similarly, we denote $\text{Cor}(k_{\leq n})$ the full-subcategory of $\text{Cor}(k)$ having the same objects as $Sm/k_{\leq n}$. A presheaf with transfers on $Sm/k_{\leq n}$ is an additive contravariant functor F from $\text{Cor}(k_{\leq n})$ to the category of \mathbb{Q} -vectorspaces. F is a sheaf with transfers if its restriction to $Sm/k_{\leq n}$ is a sheaf for the Nisnevich topology (or equivalently, for the étale topology). We

denote by $Shv_{tr}(k_{\leq n})$ the category of étale sheaves with transfers on $Sm/k_{\leq n}$. There is an adjunction [2, Lem. 1.1.12]:

$$Shv_{tr}(k_{\leq n}) \xrightleftharpoons[\sigma_{n*}]{\sigma_n^*} Shv_{tr}(k). \quad (1)$$

DEFINITION 1.1 — *An \mathcal{H} -sheaf $F \in \mathbf{HI}(k)$ is n -presented if the obvious morphism*

$$h_0 \sigma_n^* \sigma_{n*} F \rightarrow F$$

is an isomorphism. (Here, h_0 is the left adjoint to the inclusion $\mathbf{HI}(k) \hookrightarrow Shv_{tr}(k)$.)

Remark 1.2 — In [2] (see Def. 1.1.20 of *loc. cit.*) n -presented \mathcal{H} -sheaves were called “ n -motivic sheaves”. In this paper, we use a different terminology because of an eventual conflict with the notion of (n, \mathcal{H}) -sheaf which will be introduced later (for $n \in \{0, 1, 2\}$).

Let $\mathbf{HI}_{\leq n}(k)$ denotes the full-subcategory of $\mathbf{HI}(k)$ whose objects are the n -presented \mathcal{H} -sheaves. This is an abelian category, and the inclusion $\mathbf{HI}_{\leq n}(k) \hookrightarrow \mathbf{HI}(k)$ is right exact. It is conjectured that this inclusion is also left exact (see [2, Cor. 1.4.5] for a conjectural proof). This conjecture is known to hold for $n = 0, 1$ due to the following result (see [2, Cor. 1.2.5 and Prop. 1.3.11]).

PROPOSITION 1.3 — *For $n = 0, 1$, the inclusions $\mathbf{HI}_{\leq n}(k) \hookrightarrow Shv_{tr}(k)$ admit left adjoints denoted respectively by*

$$\pi_0 : Shv_{tr}(k) \longrightarrow \mathbf{HI}_{\leq 0}(k) \quad \text{and} \quad \text{Alb} : Shv_{tr}(k) \longrightarrow \mathbf{HI}_{\leq 1}(k). \quad (2)$$

DEFINITION 1.4 — *An \mathcal{H} -sheaf $F \in \mathbf{HI}(k)$ is called 0-connected if $\pi_0(F) = 0$. It is called 1-connected if $\text{Alb}(F) = 0$.*

By [2, Prop. 2.3.2 and Th. 2.4.1], the functors π_0 and Alb from (2) can be left derived, yielding functors

$$L\pi_0 : \mathbf{D}(Shv_{tr}(k)) \longrightarrow \mathbf{D}(\mathbf{HI}_{\leq 0}(k)) \quad \text{and} \quad L\text{Alb} : \mathbf{D}(Shv_{tr}(k)) \longrightarrow \mathbf{D}(\mathbf{HI}_{\leq 1}(k)).$$

The above functors pass to the Bousfield localization, yielding functors

$$L\pi_0 : \mathbf{DM}_{\text{eff}}(k) \longrightarrow \mathbf{D}(\mathbf{HI}_{\leq 0}(k)) \quad \text{and} \quad L\text{Alb} : \mathbf{DM}_{\text{eff}}(k) \longrightarrow \mathbf{D}(\mathbf{HI}_{\leq 1}(k))$$

which are left adjoints to the obvious functors

$$\mathbf{D}(\mathbf{HI}_{\leq 0}(k)) \longrightarrow \mathbf{DM}_{\text{eff}}(k) \quad \text{and} \quad \mathbf{D}(\mathbf{HI}_{\leq 1}(k)) \longrightarrow \mathbf{DM}_{\text{eff}}(k).$$

These are fully faithful embeddings with essential images $\mathbf{DM}_{\leq 0}(k)$ and $\mathbf{DM}_{\leq 1}(k)$ respectively. Recall that $\mathbf{DM}_{\leq n}(k)$ (with $n \in \mathbb{N}$) is the smallest triangulated subcategory of $\mathbf{DM}_{\text{eff}}(k)$ stable under small sums and containing $M(X)$ with $X \in Sm/k$ of dimension at most n . It follows that the obvious inclusions $\mathbf{DM}_{\leq 0}(k) \hookrightarrow \mathbf{DM}_{\text{eff}}(k)$ and $\mathbf{DM}_{\leq 1}(k) \hookrightarrow \mathbf{DM}_{\text{eff}}(k)$ have left adjoints, which we also denote as follows:

$$L\pi_0 : \mathbf{DM}_{\text{eff}}(k) \longrightarrow \mathbf{DM}_{\leq 0}(k) \quad \text{and} \quad L\text{Alb} : \mathbf{DM}_{\text{eff}}(k) \longrightarrow \mathbf{DM}_{\leq 1}(k).$$

1.3. Generating t -structures.

Let \mathcal{T} be a triangulated category. Recall from [3] that a t -structure on \mathcal{T} is a couple of full subcategories $(\mathcal{T}_{\geq 0}, \mathcal{T}_{\leq 0})$ satisfying three simple axioms. Contrary to *loc. cit.*, we will use the homological convention for t -structures. One passes back and forth between the homological and cohomological conventions via the usual rule: $\mathcal{T}_{\geq n} = \mathcal{T}^{\leq -n}$ and $\mathcal{T}_{\leq n} = \mathcal{T}^{\geq -n}$.

In this paragraph we recall the technique of generating t -structures which is described in [1, §2.1.3]. Let \underline{G} a class of objects in \mathcal{T} .

DEFINITION 1.5 — (*Compare with [1, Déf. 2.1.68].*)

(a) An object $N \in \mathcal{T}$ is \underline{G} -negative if for every $A \in \underline{G}$ and $n \in \mathbb{N}$, we have

$$\mathrm{hom}_{\mathcal{T}}(A[n+1], N) = 0.$$

We denote $\mathcal{T}_{\leq 0}^{\underline{G}}$ the full subcategory of \underline{G} -negative objects and set $\mathcal{T}_{\leq d}^{\underline{G}} = \mathcal{T}_{\leq 0}^{\underline{G}}[d]$ for $d \in \mathbb{Z}$.

(b) An object $P \in \mathcal{T}$ is \underline{G} -positive if for every $N \in \mathcal{T}_{\leq -1}^{\underline{G}}$, we have

$$\mathrm{hom}_{\mathcal{T}}(P, N) = 0.$$

We denote $\mathcal{T}_{\geq 0}^{\underline{G}}$ the full subcategory of \underline{G} -positive objects and set $\mathcal{T}_{\geq d}^{\underline{G}} = \mathcal{T}_{\geq 0}^{\underline{G}}[d]$ for $d \in \mathbb{Z}$.

Recall that an object E of \mathcal{T} is said to be an extension of E' and E'' if there exists a distinguished triangle in \mathcal{T} :

$$E' \longrightarrow E \longrightarrow E'' \longrightarrow E'[1].$$

We record the following fact (see [1, Prop. 2.1.70]).

PROPOSITION 1.6 — *Assume that \mathcal{T} has small sums and that \underline{G} is essentially small (i.e., the isomorphism classes of objects in \underline{G} form a set) and consists of compact objects. Then $(\mathcal{T}_{\geq 0}^{\underline{G}}, \mathcal{T}_{\leq 0}^{\underline{G}})$ is a t -structure on \mathcal{T} . Moreover, $\mathcal{T}_{\geq 0}^{\underline{G}}$ is the smallest full subcategory of \mathcal{T} containing \underline{G} , and stable under small sums, suspensions and extensions.*

The t -structure $(\mathcal{T}_{\geq 0}^{\underline{G}}, \mathcal{T}_{\leq 0}^{\underline{G}})$ is said to be *generated* by \underline{G} . Clearly $\underline{G} \subset \mathcal{T}_{\geq 0}^{\underline{G}}$, and $(\mathcal{T}_{\geq 0}^{\underline{G}}, \mathcal{T}_{\leq 0}^{\underline{G}})$ is the universal t -structure with this property in the following sense (see [1, Lem. 2.1.78]).

LEMMA 1.7 — *Keep the hypothesis in Proposition 1.6. Let \mathcal{S} be a triangulated category endowed with a t -structure $(\mathcal{S}_{\geq 0}, \mathcal{S}_{\leq 0})$. Let $F : \mathcal{T} \rightarrow \mathcal{S}$ be a triangulated functor. We assume that F commutes with small sums and that $F(\underline{G}) \subset \mathcal{S}_{\geq 0}$. Then F is t -positive, i.e., takes an object in $\mathcal{T}_{\geq 0}^{\underline{G}}$ to an object in $\mathcal{S}_{\geq 0}$.*

2. PERVERTING t -STRUCTURES

2.1. The abstract construction.

In this paragraph we present a simple way to construct new t -structures out of olds. This will be applied in the next section. We begin by describing the abstract setting.

Let \mathcal{T} be a triangulated category endowed with a t -structure $(\mathcal{T}_{\geq 0}, \mathcal{T}_{\leq 0})$. For $n \in \mathbb{Z}$, we denote by $\tau_{\geq n}$ and $\tau_{\leq n}$ the truncation functors with respect to this t -structures.

Thus, we have a canonical distinguished triangle

$$\tau_{\geq n}(A) \longrightarrow A \longrightarrow \tau_{\leq n-1}(A) \longrightarrow \tau_{\geq n}(A)[1]$$

for every $A \in \mathcal{T}$. We also set $H_n(A) = \tau_{\geq n} \circ \tau_{\leq n}(A)[-n]$. This is an object of the heart $\mathcal{H}_{\mathcal{T}} = \mathcal{T}_{\geq 0} \cap \mathcal{T}_{\leq 0}$.

Let $\mathcal{A} \subset \mathcal{H}_{\mathcal{T}}$ be a full subcategory. We assume the following.

HYPOTHESIS 2.1 —

- (i) \mathcal{A} is a thick abelian subcategory of $\mathcal{H}_{\mathcal{T}}$, i.e., stable under extensions, subobjects and quotients.
- (ii) The inclusion $\mathcal{A} \hookrightarrow \mathcal{H}_{\mathcal{T}}$ admits a left adjoint $F : \mathcal{H}_{\mathcal{T}} \rightarrow \mathcal{A}$.
- (iii) Let

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0,$$

be a short exact sequence in $\mathcal{H}_{\mathcal{T}}$. If $A'' \in \mathcal{A}$, then $F(A') \rightarrow F(A)$ is a monomorphism.

Remark 2.2 —

- (a) It follows from (i) that the inclusion $\mathcal{A} \hookrightarrow \mathcal{H}_{\mathcal{T}}$ is an exact functor. As it is also a full embedding, the unit of the adjunction $\phi_A : A \rightarrow F(A)$ is the universal morphism from $A \in \mathcal{H}_{\mathcal{T}}$ to an object in \mathcal{A} . In particular, when $A \in \mathcal{A}$, ϕ_A is invertible. For general $A \in \mathcal{H}_{\mathcal{T}}$, we claim that ϕ_A is surjective. Indeed, $im(\phi_A)$ is a sub-object of $F(A)$. Hence by (i), it is in \mathcal{A} . Applying the universal property to $A \rightarrow im(\phi_A)$, we get a retraction $F(A) \rightarrow im(\phi_A)$. As the composition $F(A) \rightarrow im(\phi_A) \hookrightarrow F(A)$ is the identity, $im(\phi_A) \rightarrow F(A)$ is an isomorphism.
- (b) Being a left adjoint, the functor F is right exact. Under the conditions of (iii), we thus have a short exact sequence in \mathcal{A} :

$$0 \longrightarrow F(A') \longrightarrow F(A) \longrightarrow A'' \longrightarrow 0. \quad (3)$$

(Here we use that $A'' \simeq F(A'')$.) For $A \in \mathcal{H}_{\mathcal{T}}$, we set

$$G(A) = \ker\{A \rightarrow F(A)\}. \quad (4)$$

Thus, we have a canonical exact sequence in $\mathcal{H}_{\mathcal{T}}$:

$$0 \longrightarrow G(A) \longrightarrow A \longrightarrow F(A) \longrightarrow 0. \quad (5)$$

It follows from the exact sequence (3) that $F(G(A)) = 0$.

DEFINITION 2.3 — *An object $A \in \mathcal{H}_{\mathcal{T}}$ is said to be F-connected if $F(A) = 0$. Equivalently, any morphism from A to an object of \mathcal{A} is zero.*

We have seen that for any $A \in \mathcal{H}_{\mathcal{T}}$, $G(A)$ is F-connected. Moreover, this is the largest F-connected subobject of A . Indeed, let $a : B \rightarrow A$ be a morphism in $\mathcal{H}_{\mathcal{T}}$ from an F-connected object B . Then the composition $\phi_A \circ a : B \rightarrow F(A)$ is zero and hence, a factors through $G(A)$. This also prove that G is the right adjoint to the inclusion of the full subcategory of F-connected objects in $\mathcal{H}_{\mathcal{T}}$.

We now come the main construction of this paragraph.

PROPOSITION 2.4 — *We define a t-structure $(\mathcal{T}_{\geq 0}, \mathcal{T}_{\leq 0})$ on \mathcal{T} as follows:*

- $\mathcal{T}_{\geq 0}$ is the full subcategory of $P \in \mathcal{T}$ such that $H_i(P) = 0$ for $i < -1$ and $H_{-1}(P)$ is F-connected.

- ‘ $\mathcal{T}_{\leq 0}$ is the full subcategory of $N \in \mathcal{T}$ such that $H_i(N) = 0$ for $i > 0$ and $H_0(N) \in \mathcal{A}$.

Proof. As usual, we set ‘ $\mathcal{T}_{\geq n} = \mathcal{T}_{\geq 0}[n]$ and ‘ $\mathcal{T}_{\leq n} = \mathcal{T}_{\leq 0}[n]$ for $n \in \mathbb{Z}$. We clearly have ‘ $\mathcal{T}_{\geq 1} \subset \mathcal{T}_{\geq 0}$ and ‘ $\mathcal{T}_{\leq 1} \supset \mathcal{T}_{\leq 0}$.

Let $P \in \mathcal{T}_{\geq 0}$ and $N \in \mathcal{T}_{\leq -1}$. Then $P \in \mathcal{T}_{\geq -1}$ and $N \in \mathcal{T}_{\leq -1}$. Thus, we have

$$\begin{aligned} \mathrm{hom}_{\mathcal{T}}(P, N) &\simeq \mathrm{hom}_{\mathcal{T}}(\tau_{\leq -1}(P), N) \simeq \mathrm{hom}_{\mathcal{T}}(\tau_{\leq -1}(P), \tau_{\geq -1}(N)) \\ &\simeq \mathrm{hom}_{\mathcal{H}_{\mathcal{T}}}(\mathrm{H}_{-1}(P), \mathrm{H}_{-1}(N)). \end{aligned}$$

As $\mathrm{H}_{-1}(P)$ is F-connected and $\mathrm{H}_{-1}(N) \in \mathcal{A}$, every morphism from $\mathrm{H}_{-1}(P)$ to $\mathrm{H}_{-1}(N)$ is zero. This shows that $\mathrm{hom}_{\mathcal{T}}(P, N) = 0$.

To end the proof, we still need to check axiom (iii) of [3, Déf. 1.3.1]. Let $A \in \mathcal{T}$. There is a distinguished triangle

$$P_0 \xrightarrow{u_0} A \xrightarrow{v_0} N_0 \longrightarrow P_0[1]$$

where $P_0 \in \mathcal{T}_{\geq -1}$ and $N_0 \in \mathcal{T}_{\leq -2}$. Consider the composition

$$t : P_0 \longrightarrow \mathrm{H}_{-1}(P_0)[-1] \longrightarrow \mathrm{F}(\mathrm{H}_{-1}(P_0))[-1],$$

and form a distinguished triangle

$$P \xrightarrow{s} P_0 \xrightarrow{t} \mathrm{F}(\mathrm{H}_{-1}(P_0))[-1] \longrightarrow P[1].$$

Let $u = u_0 \circ s : P \rightarrow A$. Clearly $H_i(P) = 0$ for $i < -1$, and we have an isomorphism $H_i(u) : H_i(P) \xrightarrow{\sim} H_i(A)$ for $i > -1$. Moreover, there is a short exact sequence

$$0 \longrightarrow \mathrm{H}_{-1}(P) \xrightarrow{\mathrm{H}_{-1}(u)} \mathrm{H}_{-1}(A) \longrightarrow \mathrm{F}(\mathrm{H}_{-1}(A)) \longrightarrow 0.$$

In particular $\mathrm{H}_{-1}(P)$ is F-connected. It follows that $P \in \mathcal{T}_{\geq 0}$.

Now, form a distinguished triangle

$$P \xrightarrow{u} A \xrightarrow{v} N \longrightarrow P[+1].$$

Then $H_i(N) = 0$ for $i \leq 0$ and

$$\mathrm{H}_{-1}(N) \simeq \mathrm{coker}\{\mathrm{H}_{-1}(P) \rightarrow \mathrm{H}_{-1}(A)\} \simeq \mathrm{F}(\mathrm{H}_{-1}(A)).$$

This shows that $N \in \mathcal{T}_{\leq -1}$. The proposition is proven. \square

DEFINITION 2.5 — *Keep the above notation and assumption. The t -structure ‘ $\mathcal{T}_{\geq 0}, \mathcal{T}_{\leq 0}$ is called the \mathcal{A} -perverted t -structure.*

Remark 2.6 — We denote ‘ $\mathcal{H}_{\mathcal{T}} = \mathcal{T}_{\geq 0} \cap \mathcal{T}_{\leq 0}$ the heart of the \mathcal{A} -perverted t -structure. Clearly, and object $A \in \mathcal{T}$ is in ‘ $\mathcal{H}_{\mathcal{T}}$ if and only if it satisfies the following properties:

- (1) $H_i(A) = 0$ for $i \notin \{0, -1\}$;
- (2) $H_0(A) \in \mathcal{A}$;
- (3) $\mathrm{H}_{-1}(A)$ is F-connected.

From this, it follows immediately that $\mathcal{A} = \mathcal{H}_{\mathcal{T}} \cap \mathcal{H}_{\mathcal{T}}$.

2.2. The case of a generated t -structure.

We keep the notation and assumption of §2.1. Suppose that the t -structure $(\mathcal{T}_{\geq 0}, \mathcal{T}_{\leq 0})$ is generated by an essentially small class \underline{G} of compact objects in \mathcal{T} . Assume that for every $A \in \underline{G}$, we can find a distinguished triangle

$$A^\perp \longrightarrow A \longrightarrow A^F \longrightarrow A^\perp[1]$$

such that A^F is compact and t -positive, $H_0(A^F) \in \mathcal{A}$, and the obvious morphism $F(H_0(A)) \rightarrow H_0(A^F)$ is invertible. Let $\underline{G}^\perp[-1] = \{A^\perp[-1] \mid A \in \underline{G}\}$. We choose the above triangles so that $\underline{G}^\perp[-1]$ is again essentially small (this is clearly possible). Remark also that $\underline{G}^\perp[-1]$ consists of compact objects.

PROPOSITION 2.7 — *The t -structure $(\mathcal{T}_{\geq 0}, \mathcal{T}_{\leq 0})$ is generated by the essentially small class of compact objects $\mathcal{G} = \underline{G} \cup \underline{G}^\perp[-1]$.*

Proof. Denote by $(\mathcal{T}'_{\geq 0}, \mathcal{T}'_{\leq 0})$ the t -structure on \mathcal{T} generated by \mathcal{G} . It suffices to check that $\mathcal{T}'_{\geq 0} \subset \mathcal{T}_{\geq 0}$ and $\mathcal{T}'_{\leq 0} \subset \mathcal{T}_{\leq 0}$. It is easy to see that $\mathcal{G} \subset \mathcal{T}'_{\geq 0}$. We thus have $\mathcal{T}'_{\geq 0} \subset \mathcal{T}_{\geq 0}$ by Lemma 1.7. To check the second inclusion, we fix $N \in \mathcal{T}'_{\leq 0}$. As $\underline{G} \subset \mathcal{G}$, we have $\mathcal{T}'_{\leq 0} \subset \mathcal{T}_{\leq 0}$ and thus $H_i(N) = 0$ for $i > 0$. It remains to show that $H_0(N) \in \mathcal{A}$.

Let $A \in \underline{G}$. Clearly, A^\perp is t -positive. It follows that

$$\mathrm{hom}_{\mathcal{T}}(A^\perp, N) \simeq \mathrm{hom}_{\mathcal{H}_{\mathcal{T}}}(\mathrm{H}_0(A^\perp), \mathrm{H}_0(N)).$$

On the other hand, $\mathrm{hom}_{\mathcal{T}}(A^\perp, N) = 0$ by the definition of the class of \mathcal{G} -negative objects. Thus we get $\mathrm{hom}_{\mathcal{H}_{\mathcal{T}}}(\mathrm{H}_0(A^\perp), \mathrm{H}_0(N)) = 0$.

From the statement of the proposition, we have an exact sequence

$$\mathrm{H}_0(A^\perp) \longrightarrow \mathrm{H}_0(A) \longrightarrow F(\mathrm{H}_0(A)) \longrightarrow 0,$$

and hence a surjective morphism $\mathrm{H}_0(A^\perp) \twoheadrightarrow G(\mathrm{H}_0(A))$. We deduce from this an inclusion

$$\mathrm{hom}_{\mathcal{H}_{\mathcal{T}}}(G(\mathrm{H}_0(A)), \mathrm{H}_0(N)) \hookrightarrow \mathrm{hom}_{\mathcal{H}_{\mathcal{T}}}(\mathrm{H}_0(A^\perp), \mathrm{H}_0(N)).$$

This shows that

$$\mathrm{hom}_{\mathcal{H}_{\mathcal{T}}}(G(\mathrm{H}_0(A)), \mathrm{H}_0(N)) = 0. \quad (6)$$

Now, the abelian category $\mathcal{H}_{\mathcal{T}}$ is generated by $\mathrm{H}_0(A)$ for $A \in \underline{G}$. Thus, we may find a family $(A_i)_{i \in I} \in \underline{G}^I$ and a surjective morphism $\bigoplus_{i \in I} \mathrm{H}_0(A_i) \twoheadrightarrow \mathrm{H}_0(N)$. Consider the induced morphism of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bigoplus_{i \in I} G(\mathrm{H}_0(A_i)) & \longrightarrow & \bigoplus_{i \in I} \mathrm{H}_0(A_i) & \longrightarrow & \bigoplus_{i \in I} F(\mathrm{H}_0(A_i)) \longrightarrow 0 \\ & & \alpha \downarrow & & \downarrow & & \downarrow \beta \\ 0 & \longrightarrow & G(\mathrm{H}_0(N)) & \longrightarrow & \mathrm{H}_0(N) & \longrightarrow & F(\mathrm{H}_0(N)) \longrightarrow 0. \end{array}$$

As $G(\mathrm{H}_0(N)) \rightarrow \mathrm{H}_0(N)$ is injective, we deduce from (6) that $\alpha = 0$. By the Snake Lemma, we have a surjective morphism $\ker(\beta) \twoheadrightarrow G(\mathrm{H}_0(N))$. As $\ker(\beta) \in \mathcal{A}$, we get from Hypothesis 2.1, (i) that $G(\mathrm{H}_0(N)) \in \mathcal{A}$. This implies that $G(\mathrm{H}_0(N)) = 0$. Indeed, the identity morphism of an F -connected object which is in \mathcal{A} is necessarily zero. The proposition is proven. \square

2.3. Perverting subcategories of the heart.

We keep the notation and assumption of §2.1. Let $\mathcal{B} \subset \mathcal{H}_{\mathcal{T}}$ be a full subcategory satisfying to the following conditions.

HYPOTHESIS 2.8 —

- (i) \mathcal{B} contains \mathcal{A} . Moreover, if $B \in \mathcal{B}$, then $G(B)$ is also in \mathcal{B} .
- (ii) The category \mathcal{B} is abelian and the inclusion $\mathcal{B} \hookrightarrow \mathcal{H}_{\mathcal{T}}$ admits a right adjoint $Q : \mathcal{H}_{\mathcal{T}} \rightarrow \mathcal{B}$.

It follows from (ii) that the inclusion $\mathcal{B} \hookrightarrow \mathcal{H}_{\mathcal{T}}$ is right exact. Given a morphism $b : B \rightarrow C$ in \mathcal{B} , its cokernel taken in \mathcal{B} coincides with its cokernel taken in $\mathcal{H}_{\mathcal{T}}$. It will be denoted by $\text{coker}(b)$. This is a priori not the case for kernels. We will reserve the notation $\text{ker}(b)$ for the kernel taken in $\mathcal{H}_{\mathcal{T}}$ and denote $\text{ker}_{\mathcal{B}}(b)$ the kernel taken in \mathcal{B} . We have a canonical isomorphism $\text{ker}_{\mathcal{B}}(b) \simeq Q(\text{ker}(b))$.

DEFINITION 2.9 — Let $\mathcal{B} \subset \mathcal{T}$ be the full subcategory whose objects are the $A \in \mathcal{T}$ such that:

- (1) $H_i(A) = 0$ for $i \neq \{0, -1\}$;
- (2) $H_0(A) \in \mathcal{A}$;
- (3) $H_{-1}(A) \in \mathcal{B}$ and is F -connected.

We call \mathcal{B} the \mathcal{A} -perverted subcategory associated to \mathcal{B} . Clearly, \mathcal{B} is contained in $\mathcal{H}_{\mathcal{T}}$.

LEMMA 2.10 — The inclusion $\mathcal{B} \hookrightarrow \mathcal{H}_{\mathcal{T}}$ has a right adjoint ${}^{\prime}Q : \mathcal{H}_{\mathcal{T}} \rightarrow \mathcal{B}$. Moreover, for $A \in \mathcal{H}_{\mathcal{T}}$ the counit of the adjunction ${}^{\prime}Q(A) \rightarrow A$ induces isomorphisms $H_0({}^{\prime}Q(A)) \simeq H_0(A)$ and $H_{-1}({}^{\prime}Q(A)) \simeq G \circ Q(H_{-1}(A))$.

Proof. It suffices to construct for every $A \in \mathcal{H}_{\mathcal{T}}$ a universal morphism ${}^{\prime}Q(A) \rightarrow A$ from an object ${}^{\prime}Q(A) \in \mathcal{B}$. We have a functorial distinguished triangle

$$H_0(A) \longrightarrow A \longrightarrow H_{-1}(A)[-1] \longrightarrow H_0(A)[1].$$

As both $H_0(A)$ and $H_{-1}(A)[-1]$ are in $\mathcal{H}_{\mathcal{T}}$, this determines a functorial short exact sequence

$$0 \longrightarrow H_0(A) \longrightarrow A \longrightarrow H_{-1}(A)[-1] \longrightarrow 0.$$

Consider $G \circ Q(H_{-1}(A))[-1]$. This is an object of $\mathcal{H}_{\mathcal{T}}$. We define

$${}^{\prime}Q(A) = A \times_{H_{-1}(A)[-1]} (G \circ Q(H_{-1}(A))[-1]), \quad (7)$$

the fiber product being taken in the abelian category $\mathcal{H}_{\mathcal{T}}$. We thus have a cartesian square in $\mathcal{H}_{\mathcal{T}}$:

$$\begin{array}{ccc} {}^{\prime}Q(A) & \longrightarrow & G \circ Q(H_{-1}(A))[-1] \\ \downarrow & & \downarrow \\ A & \longrightarrow & H_{-1}(A)[-1]. \end{array}$$

From the construction, $H_0({}^{\prime}Q(A)) \simeq H_0(A)$ and $H_{-1}({}^{\prime}Q(A)) \simeq G \circ Q(H_{-1}(A))$. In particular, ${}^{\prime}Q(A) \in \mathcal{B}$ as it follows from Hypothesis 2.8, (i).

We claim that ${}^{\prime}Q(A) \rightarrow A$ is the universal morphism from an object of \mathcal{B} . Indeed, let B be an object of \mathcal{B} . With ${}^{\prime}A = {}^{\prime}Q(A)$, we have a commutative diagram of

abelian groups

$$\begin{array}{ccccccc}
0 & \rightarrow & \mathrm{hom}_{\mathcal{T}}(B, \mathrm{H}_0('A)) & \rightarrow & \mathrm{hom}_{\mathcal{T}}(B, 'A) & \rightarrow & \mathrm{hom}_{\mathcal{T}}(B, \mathrm{H}_{-1}('A)[-1]) & \rightarrow & \mathrm{hom}_{\mathcal{T}}(B, \mathrm{H}_0('A)[1]) \\
& & \downarrow \sim & & \downarrow & & \downarrow & & \downarrow \sim \\
0 & \rightarrow & \mathrm{hom}_{\mathcal{T}}(B, \mathrm{H}_0(A)) & \rightarrow & \mathrm{hom}_{\mathcal{T}}(B, A) & \rightarrow & \mathrm{hom}_{\mathcal{T}}(B, \mathrm{H}_{-1}(A)[-1]) & \rightarrow & \mathrm{hom}_{\mathcal{T}}(B, \mathrm{H}_0(A)[1])
\end{array}$$

with exact rows. By the Five Lemma, we are reduced to show that third vertical homomorphism is bijective. The latter can be identified with

$$\mathrm{hom}_{\mathcal{H}_{\mathcal{T}}}(\mathrm{H}_{-1}(B), \mathrm{G} \circ \mathrm{Q}(\mathrm{H}_{-1}(A))) \longrightarrow \mathrm{hom}_{\mathcal{H}_{\mathcal{T}}}(\mathrm{H}_{-1}(B), \mathrm{H}_{-1}(A)).$$

This is a bijection as $\mathrm{H}_{-1}(B)$ is F-connected and in \mathcal{B} . \square

PROPOSITION 2.11 — *Keep the above notation and assumptions. The category $'\mathcal{B}$ is abelian.*

Proof. We split the proof in two parts.

Part A: Let $a : A \rightarrow B$ be a morphism in $'\mathcal{B}$. Here we prove that $\mathrm{coker}(a)$, taken in $'\mathcal{H}_{\mathcal{T}}$ is an object of $'\mathcal{B}$. Denote $N = \ker(a)$, $C = \mathrm{im}(a)$ and $D = \mathrm{coker}(a)$, all taken in $'\mathcal{H}_{\mathcal{T}}$. From the two short exact sequences:

$$\begin{array}{ccccccc}
0 & \longrightarrow & N & \longrightarrow & A & \longrightarrow & C & \longrightarrow & 0 \\
0 & \longrightarrow & C & \longrightarrow & B & \longrightarrow & D & \longrightarrow & 0
\end{array}$$

we deduce two exact sequences in $\mathcal{H}_{\mathcal{T}}$:

$$\begin{array}{ccccccc}
\mathrm{H}_{-1}(N) & \longrightarrow & \mathrm{H}_{-1}(A) & \longrightarrow & \mathrm{H}_{-1}(C) & \longrightarrow & 0 \\
\mathrm{H}_{-1}(C) & \longrightarrow & \mathrm{H}_{-1}(B) & \longrightarrow & \mathrm{H}_{-1}(D) & \longrightarrow & 0
\end{array}$$

which can be put together to get another exact sequence:

$$\mathrm{H}_{-1}(A) \longrightarrow \mathrm{H}_{-1}(B) \longrightarrow \mathrm{H}_{-1}(D) \longrightarrow 0.$$

Now, as $\mathrm{H}_{-1}(A)$ and $\mathrm{H}_{-1}(B)$ are in \mathcal{B} , and the inclusion $\mathcal{B} \hookrightarrow \mathcal{H}_{\mathcal{T}}$ is right exact, we deduce that $\mathrm{H}_{-1}(D) \in \mathcal{B}$. This shows that $\mathrm{coker}(a) \in ' \mathcal{B}$.

Part B: The previous part shows that cokernels exist in $'\mathcal{B}$ and can be computed in $\mathcal{H}_{\mathcal{T}}$. On the other hand, by Lemma 2.10, the kernels also exist in $'\mathcal{B}$. Indeed, if $a : A \rightarrow B$ is a morphism in $'\mathcal{B}$, then its kernel in $'\mathcal{B}$ is $\ker_{'\mathcal{B}}(a) = ' \mathrm{Q}(\ker(a))$. It remains to show that images and coimages coincide in $'\mathcal{B}$.

Fix a morphism $a : A \rightarrow B$ in $'\mathcal{B}$. $'\mathcal{H}_{\mathcal{T}}$ being an abelian category, the canonical morphism

$$\mathrm{coker}\{\ker(a) \rightarrow A\} \longrightarrow \ker\{B \rightarrow \mathrm{coker}(a)\}$$

is invertible. Applying $'\mathrm{Q}$, we get an isomorphism

$$' \mathrm{Q}(\mathrm{coker}\{\ker(a) \rightarrow A\}) \xrightarrow{\sim} ' \mathrm{Q}(\ker\{B \rightarrow \mathrm{coker}(a)\}).$$

$'\mathrm{Q}$ being a left adjoint, it commutes with cokernels. It follows that:

$$' \mathrm{Q}(\mathrm{coker}\{\ker(a) \rightarrow A\}) \simeq \mathrm{coker}\{' \mathrm{Q}(\ker(a)) \rightarrow ' \mathrm{Q}(A)\} \simeq \mathrm{coker}\{\ker_{'\mathcal{B}}(a) \rightarrow A\}.$$

Thus the obvious morphism

$$\mathrm{coker}\{\ker_{'\mathcal{B}}(a) \rightarrow A\} \longrightarrow \ker_{'\mathcal{B}}\{B \rightarrow \mathrm{coker}(a)\}$$

is invertible. This finishes the proof of the proposition. \square

3. THE n -MOTIVIC t -STRUCTURE FOR $n = 0$ AND 1 3.1. The 0-motivic t -structure.

We bring in the notation from §1.1. From the introduction, we are led to make the following definition.

DEFINITION 3.1 — *The 0-motivic t -structure $({}^0\mathcal{T}_{\geq 0}^{\mathcal{M}}(k), {}^0\mathcal{T}_{\leq 0}^{\mathcal{M}}(k))$ on $\mathbf{DM}_{\text{eff}}(k)$ is the usual homotopy t -structure. An object in ${}^0\mathcal{T}_{\geq 0}^{\mathcal{M}}(k)$ will be called ${}^0t^{\mathcal{M}}$ -positive. An object in ${}^0\mathcal{T}_{\leq 0}^{\mathcal{M}}(k)$ will be called ${}^0t^{\mathcal{M}}$ -negative. For $n \in \mathbb{Z}$, we denote ${}^0\tau_{\geq n}^{\mathcal{M}}$ and ${}^0\tau_{\leq n}^{\mathcal{M}}$ the truncation functors, and ${}^0\mathbf{H}_n^{\mathcal{M}}(-) = {}^0\tau_{\geq n}^{\mathcal{M}} \circ {}^0\tau_{\leq n}^{\mathcal{M}}(-)[-n]$. We also denote ${}^0\mathcal{H}^{\mathcal{M}}(k) = {}^0\mathcal{T}_{\geq 0}^{\mathcal{M}}(k) \cap {}^0\mathcal{T}_{\leq 0}^{\mathcal{M}}(k)$, the heart of 0-motivic t -structure. An object of ${}^0\mathcal{H}^{\mathcal{M}}(k)$ is called a $(0, \mathcal{H})$ -sheaf.*

Strictly speaking, the category ${}^0\mathcal{H}^{\mathcal{M}}(k)$ is equivalent (and not isomorphic) to the category $\mathbf{HI}(k)$ of \mathcal{H} -sheaves (i.e., homotopy invariant sheaves with transfers). This equivalence takes a $(0, \mathcal{H})$ -sheaf to its zero homology \mathcal{H} -sheaf. However, it is safe enough to identify both categories, and we will often do this.

Remark 3.2 — In the sequel, we will keep using the notation $\tau_{\leq n}$, $\tau_{\geq n}$, \mathbf{H}_n and $\mathbf{HI}(k)$ relative to the homotopy t -structure. In fact, the only reason we introduced the new terminology in Definition 3.1, is to stress the analogy between the 0-motivic, 1-motivic and 2-motivic t -structures.

PROPOSITION 3.3 — *The 0-motivic t -structure on $\mathbf{DM}_{\text{eff}}(k)$ is generated by the essentially small class $\{\mathbf{M}(X) \mid X \in \text{Sm}/k\}$.*

Proof. This is well-known (see [8]). For the sake of completeness, we provide an argument. For $K \in \mathbf{K}(\text{Shv}_{\text{tr}}(k))$, the following two conditions are equivalent:

- (i) The homology sheaves $\mathbf{H}_i(K)$ are zero for $i < 0$;
- (ii) The (Nisnevich) hyper-cohomology groups $\mathbf{H}^{-i}(X, K)$ are zero for $i > 0$ and $X \in \text{Sm}/k$.

If moreover we assume that K is \mathbb{A}^1 -local, the second condition can be rewritten as follows:

- (ii') The groups $\text{hom}_{\mathbf{DM}_{\text{eff}}(k)}(\mathbf{M}(X)[i], K)$ are zero for $i > 0$ and $X \in \text{Sm}/k$.

In other words, K is $\{\mathbf{M}(X) \mid X \in \text{Sm}/k\}$ -negative. This proves the proposition. \square

DEFINITION 3.4 — *We denote by $\mathbf{M}_0(k) \subset {}^0\mathcal{H}^{\mathcal{M}}(k)$ the full subcategory whose objects are the 0-presented \mathcal{H} -sheaves which we will also call mixed 0-motives (or simply 0-motive).*

In [2], the category $\mathbf{M}_0(k)$ is denoted by $\mathbf{HI}_{\leq 0}(k)$ and their objects were called 0-motivic sheaves. It is the heart of the restriction of the homotopy t -structure on $\mathbf{DM}_{\leq 0}(k)$. By [2, Lem. 1.2.2], $\mathbf{M}_0(k)$ is canonically equivalent to the category $\text{Shv}_{\text{tr}}(k_{\leq 0})$. The latter is equivalent to the category of \mathbb{Q} -linear representations V of the absolute Galois group $\text{Gal}(k^s/k)$ of k such that the stabilizer of each element of V is open (i.e., of finite index). This justifies our terminology.

3.2. The 1-motivic t -structure.

The subcategory $\mathbf{M}_0(k) \subset {}^0\mathcal{H}^{\mathcal{M}}(k)$ satisfies Hypothesis 2.1. Indeed, (i) and (ii) are contained in [2, Cor. 1.2.5 and Prop. 1.2.7]. To check (iii), we use [2, Cor. 2.3.3].

It asserts that the left adjoint $\pi_0 : {}^0\mathcal{H}^{\mathcal{M}} \rightarrow \mathbf{M}_1(k)$ is induced on the hearts by a t -positive triangulated functor $L\pi_0 : \mathbf{DM}_{\text{eff}}(k) \rightarrow \mathbf{DM}_{\leq 0}(k)$. Given an exact sequence of \mathcal{H} -sheaves

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

we deduce a distinguished triangle

$$L\pi_0(M') \longrightarrow L\pi_0(M) \longrightarrow L\pi_0(M'') \longrightarrow L\pi_0(M')[1],$$

and thus an exact sequence of 0-motives

$$H_1(L\pi_0(M'')) \longrightarrow \pi_0(M') \longrightarrow \pi_0(M).$$

Now, assume that M'' is a 0-motive. Then $M'' \simeq L\pi_0(M'')$ and $H_1(L\pi_0(M'')) = 0$. This proves that $\pi_0(M') \rightarrow \pi_0(M)$ is injective.

We are now in position to apply the construction from §2.1.

DEFINITION 3.5 — *The 1-motivic t -structure $({}^1\mathcal{T}_{\geq 0}^{\mathcal{M}}(k), {}^1\mathcal{T}_{\leq 0}^{\mathcal{M}}(k))$ on $\mathbf{DM}_{\text{eff}}(k)$ is the $\mathbf{M}_0(k)$ -perverted t -structure associated to the 0-motivic t -structure. An object in ${}^1\mathcal{T}_{\geq 0}^{\mathcal{M}}(k)$ will be called ${}^1t^{\mathcal{M}}$ -positive. An object in ${}^1\mathcal{T}_{\leq 0}^{\mathcal{M}}(k)$ will be called ${}^1t^{\mathcal{M}}$ -negative. For $n \in \mathbb{Z}$, we denote ${}^1\tau_{\geq n}^{\mathcal{M}}$ and ${}^1\tau_{\leq n}^{\mathcal{M}}$ the truncation functors and ${}^1H_n^{\mathcal{M}}(-) = {}^1\tau_{\geq n}^{\mathcal{M}} \circ {}^1\tau_{\leq n}^{\mathcal{M}}(-)[-n]$. We also denote ${}^1\mathcal{H}^{\mathcal{M}}(k) = {}^1\mathcal{T}_{\geq 0}^{\mathcal{M}}(k) \cap {}^1\mathcal{T}_{\leq 0}^{\mathcal{M}}(k)$ the heart of 1-motivic t -structure. An object of ${}^1\mathcal{H}^{\mathcal{M}}(k)$ is called a $(1, \mathcal{H})$ -sheaf.*

Remark 3.6 — From the construction, we have the following description of the 1-motivic t -structure.

- (1) An object $P \in \mathbf{DM}_{\text{eff}}(k)$ is ${}^1t^{\mathcal{M}}$ -positive if and only if it satisfies:
 - (a) $H_n(P) = 0$ for $n < -1$;
 - (b) $H_{-1}(P)$ is a 0-connected \mathcal{H} -sheaf.
- (2) An object $N \in \mathbf{DM}_{\text{eff}}(k)$ is ${}^1t^{\mathcal{M}}$ -negative if and only if it satisfies:
 - (a) $H_n(N) = 0$ for $n > 0$;
 - (b) $H_0(N)$ is a 0-motive.
- (3) An object $M \in \mathbf{DM}_{\text{eff}}(k)$ is a $(1, \mathcal{H})$ -sheaf if and only if it satisfies:
 - (a) $H_i(M) = 0$ for $i \notin \{0, -1\}$;
 - (b) $H_0(M)$ is a 0-motive;
 - (c) $H_{-1}(M)$ is a 0-connected \mathcal{H} -sheaf.

For $X \in Sm/k$ we choose a distinguished triangle

$$M_{\geq 1}(X) \longrightarrow M(X) \longrightarrow L\pi_0(M(X)) \longrightarrow M_{\geq 1}(X)[1].$$

From the construction in [2, §2.3], we have $L\pi_0(M(X)) \simeq M(\pi_0(X))$ where $\pi_0(X)$ is the étale k -scheme of connected components of X . It follows that $M_{\geq 1}(X)$ is isomorphic in $\mathbf{DM}_{\text{eff}}(k)$ to $\ker\{\mathbb{Q}_{tr}(X) \rightarrow \mathbb{Q}_{tr}(\pi_0(X))\}[0]$. The following result is a direct consequence of Proposition 2.7.

PROPOSITION 3.7 — *The 1-motivic t -structure on $\mathbf{DM}_{\text{eff}}(k)$ is generated by the essentially small class $\{M(X), M_{\geq 1}(X)[-1] \mid X \in Sm/k\}$.*

3.3. Mixed 1-motives.

DEFINITION 3.8 — *An object $M \in \mathbf{DM}_{\text{eff}}(k)$ is called a mixed 1-motive if it satisfies the following conditions:*

- (1) $H_i(M) = 0$ for $i \notin \{0, -1\}$;
- (2) $H_0(M)$ is a 0-motivic sheaf;

(3) $H_{-1}(M)$ is a 0-connected 1-presented \mathcal{H} -sheaf.

The full subcategory of mixed 1-motives will be denoted by $\mathbf{M}_1(k)$.

Clearly, $\mathbf{M}_1(k)$ is the $\mathbf{M}_0(k)$ -perverted subcategory associated to $\mathbf{HI}_{\leq 1}(k) \subset \mathbf{HI}(k) \simeq {}^0\mathcal{H}^{\mathcal{M}}(k)$. In particular, it is abelian. In fact, we have more as the following result shows.

PROPOSITION 3.9 — $\mathbf{M}_1(k)$ is a thick abelian subcategory of ${}^1\mathcal{H}^{\mathcal{M}}(k)$, i.e., stable under extensions, subobjects and quotients.

Proof. Indeed, consider a short exact sequence of $(1, \mathcal{H})$ -sheaves:

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0. \quad (8)$$

It suffices to show that M is a mixed 1-motive if and only if M' and M'' are mixed 1-motives. In other words, we need to show that the \mathcal{H} -sheaf $H_{-1}(M)$ is 1-presented if and only if the \mathcal{H} -sheaves $H_{-1}(M')$ and $H_{-1}(M'')$ are 1-presented.

From (8), we get an exact sequence of \mathcal{H} -sheaves:

$$0 \longrightarrow \text{im}\{H_0(M) \rightarrow H_0(M'')\} \longrightarrow H_{-1}(M') \longrightarrow H_{-1}(M) \longrightarrow H_{-1}(M'') \longrightarrow 0.$$

The \mathcal{H} -sheaf $\text{im}\{H_0(M) \rightarrow H_0(M'')\}$ is 0-presented and hence 1-presented. The lemma follows now as $\mathbf{HI}_{\leq 1}(k)$ is a thick abelian subcategory of $\mathbf{HI}(k)$ (see [2, Cor. 1.3.5]). \square

LEMMA 3.10 — Let M be a mixed 1-motive. Then, M decomposes into a direct sum

$$M \simeq H_0(M)[0] \bigoplus H_{-1}(M)[-1].$$

Proof. We have a distinguished triangle in $\mathbf{DM}_{\text{eff}}(k)$:

$$H_0(M) \longrightarrow M \longrightarrow H_{-1}(M)[-1] \xrightarrow{\epsilon} H_0(M)[1]. \quad (9)$$

We need to show that ϵ is zero. By [2, Th. 2.4.1, (i)], $\mathbf{HI}_{\leq 1}(k)$ is contained in $\mathbf{DM}_{\leq 1}(k)$, and hence $M \in \mathbf{DM}_{\leq 1}(k)$. Also, by [2, Th. 2.4.1, (i)], we have an equivalence of categories $\mathbf{D}(\mathbf{HI}_{\leq 1}(k)) \simeq \mathbf{DM}_{\leq 1}(k)$. Thus, we may view (9) as a triangle in the derived category $\mathbf{D}(\mathbf{HI}_{\leq 1}(k))$ and ϵ as a element of:

$$\text{hom}_{\mathbf{D}(\mathbf{HI}_{\leq 1}(k))}(H_{-1}(M)[-1], H_0(M)[1]) \simeq \text{ext}_{\mathbf{HI}_{\leq 1}(k)}^2(H_{-1}(M), H_0(M)).$$

On the other hand, the cohomological dimension of $\mathbf{HI}_{\leq 1}(k)$ is 1 by [2, Prop. 2.4.10]. This shows that $\text{ext}_{\mathbf{HI}_{\leq 1}(k)}^2(H_{-1}(M), H_0(M)) = 0$, and hence $\epsilon = 0$. \square

In the reminder of this paragraph, we describe the link between our notion of mixed 1-motives and the classical notion of Deligne's 1-motives. We do this in order to justify our terminology. However, this material will not be used elsewhere in the paper and can safely be skipped by the reader.

Recall (cf. [4]) that a Deligne 1-motive is a morphism of group-schemes $[L \xrightarrow{u} G]$ with L a lattice (i.e., locally for the étale topology isomorphic to \mathbb{Z}^r) and G a semi-abelian variety. We denote by $\mathcal{M}_1(k)$ the category of 1-motives. Given two 1-motives $M_1 = [L_1 \xrightarrow{u_1} G_1]$ and $M_2 = [L_2 \xrightarrow{u_2} G_2]$, we have

$$\text{hom}_{\mathcal{M}_1}(M_1, M_2) = \{(a : L_1 \rightarrow L_2, b : G_1 \rightarrow G_2) \mid b \circ u_1 = u_2 \circ a\} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

(Where a and b above are morphisms of group-schemes.) There is a functor

$$T : \mathcal{M}_1(k) \rightarrow \mathbf{DM}_{\text{eff}}(k)$$

which takes a 1-motive $[L \rightarrow G]$ to the complex

$$[\cdots \rightarrow 0 \rightarrow L \otimes \mathbb{Q} \rightarrow G \otimes \mathbb{Q} \rightarrow 0 \rightarrow \cdots]$$

where L and G are identified with the sheaves they represents and $L \otimes \mathbb{Q}$ is placed in degree 0.

PROPOSITION 3.11 — *The functor T induces an exact full embedding of $\mathcal{M}_1(k)$ into $\mathbf{M}_1(k)$.*

Proof. This is a special case of the main result in [9]. For the sake of completeness, we give a proof. Clearly, the image of T is contained in $\mathbf{M}_1(k)$. Let $M_i = [L_i \rightarrow G_i]$ (for $i \in \{1, 2\}$) be two Deligne 1-motives. We need to show that

$$\text{hom}_{\mathcal{M}_1}(M_1, M_2) \longrightarrow \text{hom}_{\mathbf{M}_1(k)}(T(M_1), T(M_2)) \quad (10)$$

is a bijection. We can decompose M_i as follows

$$M_i \simeq L'_i \oplus [L''_i \rightarrow G_i]$$

where L'_i and L''_i are sub-lattices of L_i such that:

- $L'_i \cap L''_i = 0$ and $L'_i + L''_i$ is of finite index in L_i ;
- $L''_i \rightarrow G_i$ is injective.

We are then reduced to check that (10) is bijective in the following cases:

- (a) G_i is zero for $i \in \{1, 2\}$;
- (b) G_1 is zero and $L_2 \hookrightarrow G_2$ is injective;
- (c) $L_1 \hookrightarrow G_1$ is injective and $G_2 = 0$;
- (d) $L_i \hookrightarrow G_i$ is injective for $i \in \{1, 2\}$.

Case (a) is easy. In case (b), both sides of (10) are zero. In case (c), both sides of (10) are canonically isomorphic to $\text{hom}(L_1, L_2) \otimes \mathbb{Q}$. Finally, in case (d), both sides of (10) are given by the sub-vectorspace of $e \in \text{hom}(G_1, G_2) \otimes \mathbb{Q}$ such that $e(L_1 \otimes \mathbb{Q}) \subset e(L_2 \otimes \mathbb{Q})$. \square

Remark 3.12 — Using [2, Th. 1.3.10] and Lemma 3.10, it is possible to show that $\mathbf{M}_1(k)$ is equivalent to the category of ind-objects in $\mathcal{M}_1(k)$. We leave the details to the reader.

3.4. n -Presented $(1, \mathcal{H})$ -sheaves.

DEFINITION 3.13 — *Let $n \geq 0$ be an integer. A $(1, \mathcal{H})$ -sheaf M is n -presented if the \mathcal{H} -sheaf $H_{-1}(M)$ is n -presented. We denote by ${}^1\mathcal{H}_{\leq n}^{\mathcal{M}}(k) \subset {}^1\mathcal{H}^{\mathcal{M}}(k)$ the full subcategory of n -presented $(1, \mathcal{H})$ -sheaves.*

Remark 3.14 — A $(1, \mathcal{H})$ -sheaf M is 0-presented if and only if $H_{-1}(M) = 0$. It follows that ${}^1\mathcal{H}_{\leq 0}^{\mathcal{M}}(k) = \mathbf{M}_0(k)$. Also, 1-presented $(1, \mathcal{H})$ -sheaves are exactly the mixed 1-motives, i.e., ${}^1\mathcal{H}_{\leq 1}^{\mathcal{M}}(k) = \mathbf{M}_1(k)$.

Clearly, ${}^1\mathcal{H}_{\leq n}^{\mathcal{M}}(k)$ is the $\mathbf{M}_0(k)$ -perverted subcategory of ${}^1\mathcal{H}^{\mathcal{M}}(k)$ associated to the full subcategory ${}^0\mathcal{H}_{\leq n}^{\mathcal{M}}(k) = \mathbf{HI}_{\leq n}(k)$ of ${}^0\mathcal{H}^{\mathcal{M}}(k)$. It is easy to check Hypothesis 2.8 for the 0-motivic t -structure with $\mathcal{A} = \mathbf{M}_0(k)$ and $\mathcal{B} = \mathbf{HI}_{\leq n}(k)$. Indeed,

the inclusion $\mathbf{HI}_{\leq n}(k) \hookrightarrow \mathbf{HI}(k)$ admits a right adjoint given by $Q_n = h_0\sigma_n^*\sigma_{n*}$. Moreover, a short exact sequence of \mathcal{H} -sheaves

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$$

such that F'' is 0-presented has a splitting. This clearly implies that $Q_n(F') \rightarrow Q_n(F)$ is injective. From Lemma 2.10 and Proposition 2.11, we deduce the following result.

COROLLARY 3.15 — ${}^1\mathcal{H}_{\leq n}^{\mathcal{M}}(k)$ is an abelian category and there is a functor

$${}^1Q_n : {}^1\mathcal{H}^{\mathcal{M}}(k) \longrightarrow {}^1\mathcal{H}_{\leq n}^{\mathcal{M}}(k),$$

which is a right adjoint to the obvious inclusion.

4. THE 2-MOTIVIC t -STRUCTURE

4.1. The construction.

By Proposition 3.9, $\mathbf{M}_1(k) \subset {}^1\mathcal{H}^{\mathcal{M}}(k)$ is a thick abelian subcategory. Thus the first condition in Hypothesis 2.1 is satisfied. We will see in a moment that the two other conditions are satisfied as well. First, we note the following result which is of independent interest.

PROPOSITION 4.1 — *The 1-motivic t -structure restricts to a t -structure on $\mathbf{DM}_{\leq 1}(k)$ whose heart is $\mathbf{M}_1(k)$.*

Proof. We know that the homotopy t -structure restricts to a t -structure on $\mathbf{DM}_{\leq 1}(k)$ whose heart is $\mathbf{HI}_{\leq 1}(k)$. The subcategory $\mathbf{M}_0(k) \subset \mathbf{HI}_{\leq 1}(k)$ satisfies the conditions in Hypothesis 2.1. Thus, we may consider the $\mathbf{M}_0(k)$ -perverted t -structure on $\mathbf{DM}_{\leq 1}(k)$ associated to the homotopy t -structure. By a straightforward inspection, we see that the inclusion $\mathbf{DM}_{\leq 1}(k) \hookrightarrow \mathbf{DM}_{\text{eff}}(k)$ is exact with respect to the $\mathbf{M}_0(k)$ -perverted t -structures. This proves the proposition. \square

DEFINITION 4.2 — *The restriction of the 1-motivic t -structure to $\mathbf{DM}_{\leq 1}(k)$ is also called the 1-motivic t -structure.*

LEMMA 4.3 — *The functor $\text{LAlb} : \mathbf{DM}_{\text{eff}}(k) \rightarrow \mathbf{DM}_{\leq 1}(k)$ is ${}^1t^{\mathcal{M}}$ -positive (i.e., t -positive with respect to the 1-motivic t -structures).*

Proof. This is clear as LAlb is the left adjoint to the inclusion $\mathbf{DM}_{\leq 1}(k) \hookrightarrow \mathbf{DM}_{\text{eff}}(k)$ which is ${}^1t^{\mathcal{M}}$ -exact. \square

We are now in position to check Hypothesis 2.1, (ii) and (iii) for $\mathbf{M}_1(k) \subset {}^1\mathcal{H}^{\mathcal{M}}(k)$.

LEMMA 4.4 — *The inclusion $\mathbf{M}_1(k) \hookrightarrow {}^1\mathcal{H}^{\mathcal{M}}(k)$ has a left adjoint*

$$\text{Alb}^{\mathcal{M}} : {}^1\mathcal{H}^{\mathcal{M}}(k) \longrightarrow \mathbf{M}_1(k).$$

Moreover, given an exact sequence of $(1, \mathcal{H})$ -sheaves

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

with M'' a mixed 1-motive, the morphism $\text{Alb}^{\mathcal{M}}(M') \rightarrow \text{Alb}^{\mathcal{M}}(M)$ is injective.

Proof. Given a $(1, \mathcal{H})$ -sheaf M , we set

$$\mathrm{Alb}^{\mathcal{M}}(M) = {}^1\mathrm{H}_0^{\mathcal{M}}(\mathrm{LAlb}(M)).$$

That this is a left adjoint to the obvious inclusion, follows immediately from Lemma 4.3. To prove the second part, we use the distinguished triangle

$$\mathrm{LAlb}(M') \longrightarrow \mathrm{LAlb}(M) \longrightarrow \mathrm{LAlb}(M'') \longrightarrow \mathrm{LAlb}(M')[1].$$

We deduce an exact sequence of mixed 1-motives

$${}^1\mathrm{H}_1^{\mathcal{M}}(\mathrm{LAlb}(M'')) \longrightarrow {}^1\mathrm{H}_0^{\mathcal{M}}(\mathrm{LAlb}(M')) \longrightarrow {}^1\mathrm{H}_0^{\mathcal{M}}(\mathrm{LAlb}(M)).$$

Now, if M'' is a mixed 1-motive, $M'' \simeq \mathrm{LAlb}(M'')$ and thus ${}^1\mathrm{H}_1^{\mathcal{M}}(\mathrm{LAlb}(M'')) = 0$. This finishes the proof of the lemma. \square

DEFINITION 4.5 — *The 2-motivic t -structure $({}^2\mathcal{T}_{\geq 0}^{\mathcal{M}}(k), {}^2\mathcal{T}_{\leq 0}^{\mathcal{M}}(k))$ on $\mathbf{DM}_{\mathrm{eff}}(k)$ is the $\mathbf{M}_1(k)$ -perverted t -structure associated to the 1-motivic t -structure. An object in ${}^2\mathcal{T}_{\geq 0}^{\mathcal{M}}(k)$ will be called ${}^2t^{\mathcal{M}}$ -positive. An object in ${}^2\mathcal{T}_{\leq 0}^{\mathcal{M}}(k)$ will be called ${}^2t^{\mathcal{M}}$ -negative. For $n \in \mathbb{Z}$, we denote ${}^2\tau_{\geq n}^{\mathcal{M}}$ and ${}^2\tau_{\leq n}^{\mathcal{M}}$ the truncation functors and ${}^2\mathcal{H}_n^{\mathcal{M}}(-) = {}^2\tau_{\geq n}^{\mathcal{M}} \circ {}^2\tau_{\leq n}^{\mathcal{M}}(-)[-n]$. We also denote ${}^2\mathcal{H}^{\mathcal{M}}(k) = {}^2\mathcal{T}_{\geq 0}^{\mathcal{M}}(k) \cap {}^2\mathcal{T}_{\leq 0}^{\mathcal{M}}(k)$ the heart of 2-motivic t -structure. An object of ${}^2\mathcal{H}^{\mathcal{M}}(k)$ is called a $(2, \mathcal{H})$ -sheaf.*

Remark 4.6 — We will say that a $(1, \mathcal{H})$ -sheaf M is 1-connected if $\mathrm{Alb}^{\mathcal{M}}(M) = 0$. From the construction, we have the following.

- (1) An object $P \in \mathbf{DM}_{\mathrm{eff}}(k)$ is ${}^2t^{\mathcal{M}}$ -positive if and only if it satisfies:
 - (a) ${}^1\mathrm{H}_n^{\mathcal{M}}(P) = 0$ for $n < -1$;
 - (b) ${}^1\mathrm{H}_{-1}^{\mathcal{M}}(P)$ is a 1-connected $(1, \mathcal{H})$ -sheaf.
- (2) An object $N \in \mathbf{DM}_{\mathrm{eff}}(k)$ is ${}^2t^{\mathcal{M}}$ -negative if and only if it satisfies:
 - (a) ${}^1\mathrm{H}_n^{\mathcal{M}}(N) = 0$ for $n > 0$;
 - (b) ${}^1\mathrm{H}_0^{\mathcal{M}}(N)$ is a mixed 1-motive.
- (3) An object $M \in \mathbf{DM}_{\mathrm{eff}}(k)$ is a $(2, \mathcal{H})$ -sheaf if and only if it satisfies:
 - (a) ${}^1\mathrm{H}_i^{\mathcal{M}}(M) = 0$ for $i \notin \{0, -1\}$;
 - (b) ${}^1\mathrm{H}_0^{\mathcal{M}}(M)$ is a 0-motive;
 - (c) ${}^1\mathrm{H}_{-1}^{\mathcal{M}}(M)$ is a 0-connected \mathcal{H} -sheaf.

In the next paragraph, we will give equivalent formulations of the above conditions in terms of the homotopy t -structure.

For $X \in \mathrm{Sm}/k$ we choose a distinguished triangle

$$\mathrm{M}_{\geq 2}(X) \longrightarrow \mathrm{M}(X) \longrightarrow \mathrm{LAlb}(\mathrm{M}(X)) \longrightarrow \mathrm{M}_{\geq 2}(X)[1].$$

We have the following result.

PROPOSITION 4.7 — *The 2-motivic t -structure on $\mathbf{DM}_{\mathrm{eff}}(k)$ is generated by the essentially small class $\{\mathrm{M}(X), \mathrm{M}_{\geq 1}(X)[-1], \mathrm{M}_{\geq 2}(X)[-2] \mid X \in \mathrm{Sm}/k\}$.*

Proof. For $X \in \mathrm{Sm}/k$, we choose a distinguished triangle

$$\mathrm{M}'_{\geq 2}(X) \longrightarrow \mathrm{M}_{\geq 1}(X) \longrightarrow \mathrm{LAlb}(\mathrm{M}_{\geq 1}(X)) \longrightarrow \mathrm{M}'_{\geq 2}(X)[1].$$

A direct application of Propositions 2.7 and 3.7 yields the following generating class for the 2-motivic t -structure:

$$\{\mathrm{M}(X), \mathrm{M}_{\geq 1}(X)[-1], \mathrm{M}_{\geq 2}(X)[-1], \mathrm{M}'_{\geq 2}(X)[-2] \mid X \in \mathrm{Sm}/k\}.$$

The proposition would follow if we prove that $M'_{\geq 2}(X) \simeq M_{\geq 2}(X)$. There is a morphism of distinguished triangles

$$\begin{array}{ccccccc} M'_{\geq 2}(X) & \longrightarrow & M_{\geq 1}(X) & \longrightarrow & \mathrm{LAlb}(M_{\geq 1}(X)) & \longrightarrow & M'_{\geq 2}(X)[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ M_{\geq 2}(X) & \longrightarrow & M(X) & \longrightarrow & \mathrm{LAlb}(M(X)) & \longrightarrow & M_{\geq 2}(X)[1]. \end{array}$$

By the Verdier's octahedral axiom (and the fact that LAlb is a triangulated functor), it suffices to show that

$$\mathrm{Cone}\{M_{\geq 1}(X) \rightarrow M(X)\} \simeq \mathrm{LAlb}(\mathrm{Cone}\{M_{\geq 1}(X) \rightarrow M(X)\}).$$

This is indeed the case as $\mathrm{Cone}\{M_{\geq 1}(X) \rightarrow M(X)\} \simeq \mathrm{L}\pi_0(M(X)) \in \mathbf{DM}_{\leq 0}(k) \subset \mathbf{DM}_{\leq 1}(k)$. The proposition is proven. \square

4.2. A more explicit description.

The description of the 2-motivic t -structure in term of the 1-motivic t -structure given in Remark 4.6 is rather abstract and not intuitive. Here we give a more down-to-earth description which only uses the homotopy t -structure.

PROPOSITION 4.8 —

- (1) An object $P \in \mathbf{DM}_{\mathrm{eff}}(k)$ is ${}^2t^{\mathcal{M}}$ -positive if and only if it satisfies:
 - (a') $H_i(P) = 0$ for $i < -2$;
 - (b') $H_{-2}(P)$ is a 1-connected \mathcal{H} -sheaf;
 - (c') For every 0-motive L , $\mathrm{hom}_{\mathbf{DM}_{\mathrm{eff}}(k)}(P, L[-1]) = 0$.
- (2) An object N in $\mathbf{DM}_{\mathrm{eff}}(k)$ is ${}^2t^{\mathcal{M}}$ -negative if and only if it satisfies:
 - (a') $H_i(N) = 0$ for $i > 0$;
 - (b') $H_0(N)$ is a 0-presented \mathcal{H} -sheaf;
 - (c') $H_{-1}(N)$ is a 1-presented \mathcal{H} -sheaf.

Proof. We will compare the conditions of the statement with those in Remark 4.6. We split the proof into four parts.

Part A: Let $P \in \mathbf{DM}_{\mathrm{eff}}(k)$ be a ${}^2t^{\mathcal{M}}$ -positive object. We will show that P satisfies conditions (1a'), (1b') and (1c').

We have a chain of inclusions ${}^2\mathcal{T}_{\geq 0}^{\mathcal{M}}(k) \subset {}^1\mathcal{T}_{\geq -1}^{\mathcal{M}}(k) \subset {}^0\mathcal{T}_{\geq -2}^{\mathcal{M}}(k)$. It follows that $H_i(P) = 0$ for $i < -2$. This is condition (1a') of the statement.

By Remark 4.6, (1b), the $(1, \mathcal{H})$ -sheaf ${}^1H_{-1}^{\mathcal{M}}(P)$ is 1-connected. As

$${}^1H_{-1}^{\mathcal{M}}(\mathrm{LAlb}(P)) \simeq \mathrm{Alb}^{\mathcal{M}}({}^1H_{-1}^{\mathcal{M}}(P)) = 0,$$

we deduce that $\mathrm{LAlb}(P)$ is ${}^1t^{\mathcal{M}}$ -positive. This implies that $\mathrm{LAlb}(P)[1]$ is t -positive. It follows that

$$\mathrm{Alb}(H_{-2}(P)) \simeq H_{-2}(\mathrm{LAlb}(P)) = 0.$$

Hence $H_{-2}(P)$ is a 1-connected \mathcal{H} -sheaf. This is condition (1b') of the statement.

Condition (1c') of the statement is clear as P is ${}^2t^{\mathcal{M}}$ -positive and $L[-1]$ is strictly ${}^2t^{\mathcal{M}}$ -negative.

Part B: Let $P \in \mathbf{DM}_{\mathrm{eff}}(k)$ satisfying conditions (1a'), (1b') and (1c'). We will show that P is ${}^2t^{\mathcal{M}}$ -positive.

The \mathcal{H} -sheaf $H_{-2}(P)$ being 1-connected, is also 0-connected. It follows that $P \in {}^1\mathcal{T}_{\geq -1}^{\mathcal{M}}(k)$, i.e., ${}^1H_i^{\mathcal{M}}(P) = 0$ for $i < -1$. It remains to show that the $(1, \mathcal{H})$ -sheaf ${}^1H_{-1}^{\mathcal{M}}(P)$ is 1-connected.

Consider $\mathrm{LAlb}(P)$. We have $H_{-2}(\mathrm{LAlb}(P)) \simeq \mathrm{Alb}(H_{-2}(P)) = 0$ by condition (1b'). It follows that $\mathrm{LAlb}(P)[1]$ is t -positive. On the other hand, we claim that $H_{-1}(\mathrm{LAlb}(P))$ is a 0-connected \mathcal{H} -sheaf. Indeed, given any 0-motive L , we have

$$\begin{aligned} \mathrm{hom}_{\mathbf{HI}(k)}(\pi_0(H_{-1}(\mathrm{LAlb}(P))), L) &\simeq \mathrm{hom}_{\mathbf{HI}(k)}(H_{-1}(\mathrm{LAlb}(P)), L) \\ &\simeq \mathrm{hom}_{\mathbf{DM}_{\mathrm{eff}}(k)}(\mathrm{LAlb}(P), L[-1]) \simeq \mathrm{hom}_{\mathbf{DM}_{\mathrm{eff}}(k)}(P, L[-1]) = 0. \end{aligned}$$

By Yoneda's Lemma, this shows that $\pi_0(H_{-1}(\mathrm{LAlb}(P))) = 0$. Thus, we have proven that $\mathrm{LAlb}(P) \in {}^1\mathcal{T}_{\geq 0}^{\mathcal{M}}(k)$. But, we have

$$\mathrm{Alb}^{\mathcal{M}}({}^1H_{-1}^{\mathcal{M}}(P)) \simeq {}^1H_{-1}^{\mathcal{M}}(\mathrm{LAlb}(P)) = 0.$$

This proves that P satisfies condition (1b) of Remark 4.6.

Part C: Let $N \in \mathbf{DM}_{\mathrm{eff}}(k)$ be a ${}^2t^{\mathcal{M}}$ -negative object. We will show that N satisfies conditions (2a'), (2b') and (2c').

We have a chain of inclusions ${}^2\mathcal{T}_{\leq 0}^{\mathcal{M}}(k) \subset {}^1\mathcal{T}_{\leq 0}^{\mathcal{M}}(k) \subset {}^0\mathcal{T}_{\leq 0}^{\mathcal{M}}(k)$. It follows that $H_i(N) = 0$ for $i > 0$. This is condition (2a') of the statement.

We have $H_0(N) \simeq H_0({}^1H_0^{\mathcal{M}}(N))$, and the latter is 0-presented because ${}^1H_0^{\mathcal{M}}(N)$ is a mixed 1-motive by Remark 4.6, (2b). This is condition (2b') of the statement.

We clearly have $H_{-1}(N) \simeq H_{-1}({}^1\tau_{\geq -1}^{\mathcal{M}}(N))$. Using the distinguished triangle

$${}^1H_0^{\mathcal{M}}(N) \longrightarrow {}^1\tau_{\geq -1}^{\mathcal{M}}(N) \longrightarrow {}^1H_{-1}^{\mathcal{M}}(N)[-1] \longrightarrow {}^1H_0^{\mathcal{M}}(N)[1],$$

we deduce an exact sequence of \mathcal{H} -sheaves

$$0 \longrightarrow H_{-1}({}^1H_0^{\mathcal{M}}(N)) \longrightarrow H_{-1}(N) \longrightarrow H_0({}^1H_{-1}^{\mathcal{M}}(N)). \quad (11)$$

This shows that the \mathcal{H} -sheaf $H_{-1}(N)$ is 1-presented. This is condition (2c') of the statement.

Part D: Let $N \in \mathbf{DM}_{\mathrm{eff}}(k)$ satisfying conditions (2a'), (2b') and (2c'). Then N is ${}^2t^{\mathcal{M}}$ -negative. Indeed, $N \in {}^1\mathcal{T}_{\leq 0}^{\mathcal{M}}(k)$. On the other hand, from (11), we deduce that ${}^1H_0^{\mathcal{M}}(N)$ is a mixed 1-motive. This finishes the proof of the proposition. \square

The next lemma shows that we may replace condition (1c') of Proposition 4.8 by a more concrete condition.

LEMMA 4.9 — *Let $P \in \mathbf{DM}_{\mathrm{eff}}(k)$ be an object satisfying (1a') and (1b') of Proposition 4.8. Then the following conditions are equivalent:*

- (c') For every 0-motive L , $\mathrm{hom}_{\mathbf{DM}_{\mathrm{eff}}(k)}(P, L[-1]) = 0$,
- (c'') For every 0-motive L , $\mathrm{ext}_{\mathbf{HI}(k)}^1(H_{-2}(P), L) = 0$ and $L[-1]$ is not a direct summand of P unless $L = 0$.

Proof. First, assume that P satisfies condition (c'). Then clearly, $L[-1]$ cannot be a direct summand of P unless $L = 0$.

On the other hand, from the distinguished triangle

$$\tau_{\geq -1}(P)[1] \longrightarrow P[1] \longrightarrow H_{-2}(P)[-1] \longrightarrow \tau_{\geq -1}(P)[2], \quad (12)$$

we deduce an exact sequence

$$\mathrm{hom}_{\mathbf{DM}_{\mathrm{eff}}(k)}(\tau_{\geq -1}(P)[2], L) \longrightarrow \mathrm{hom}_{\mathbf{DM}_{\mathrm{eff}}(k)}(H_{-2}(P)[-1], L) \longrightarrow \mathrm{hom}_{\mathbf{DM}_{\mathrm{eff}}(k)}(P[1], L).$$

Both extremal terms are zero; the left one is zero for degree reasons and the right one is zero by assumption. We are done as the middle term is isomorphic

$\text{ext}_{\mathbf{HI}(k)}^1(\mathbb{H}_{-2}(P), L)$. Indeed, both groups classify \mathcal{H} -sheaves which are extensions of $\mathbb{H}_{-2}(P)$ by L .

Conversely, assume that P satisfies condition (c''). We argue by contradiction. Thus, let $\alpha : P[1] \rightarrow L$ be a non-zero morphism. As L is a direct sum of indecomposable 0-motives, we may assume that L is itself indecomposable.

By the long exact sequence associated to the distinguished triangle (12), the obvious homomorphism

$$\text{hom}_{\mathbf{DM}_{\text{eff}}(k)}(P[1], L) \longrightarrow \text{hom}_{\mathbf{DM}_{\text{eff}}(k)}(\tau_{\geq -1}(P)[1], L)$$

is injective. (Here again, we use that $\text{hom}_{\mathbf{DM}_{\text{eff}}(k)}(\tau_{\geq -1}(P)[2], L)$ is zero for degree reasons.) In other words, the composition

$$\alpha' : \tau_{\geq -1}(P)[1] \longrightarrow P[1] \xrightarrow{\alpha} L$$

is also non-zero. Moreover, α' uniquely factors through $\pi_0(\mathbb{H}_{-1}(P))$ yielding a commutative diagram

$$\begin{array}{ccc} \tau_{\geq -1}(P)[1] & \xrightarrow{\quad\quad\quad} & P[1] \\ \downarrow & \searrow \alpha' & \downarrow \alpha \\ \mathbb{H}_{-1}(P) & \longrightarrow \pi_0(\mathbb{H}_{-1}(P)) \longrightarrow & L. \\ & & \nearrow \alpha'' \end{array}$$

As α'' is non-zero and L is indecomposable, we deduce that α'' is surjective. It follows that $\mathbb{H}_{-1}(P) \rightarrow L$ is also surjective. Hence, given a distinguished triangle

$$Q \longrightarrow \tau_{\geq -1}(P)[1] \xrightarrow{\alpha'} L \longrightarrow Q[1],$$

the object Q is t -positive and thus the morphism $L \rightarrow Q[1]$ is zero. This shows that α' has a section $\beta' : L \rightarrow \tau_{\geq -1}(P)[1]$. Clearly, the composition

$$\beta : L \xrightarrow{\beta'} \tau_{\geq -1}(P)[1] \longrightarrow P[1]$$

is a section to α . We have proven that L is a non-trivial direct summand of $P[1]$ which is a 0-motive. This contradicts (c''). \square

COROLLARY 4.10 — *An object $M \in \mathbf{DM}_{\text{eff}}(k)$ is a $(2, \mathcal{H})$ -sheaf if and only if it satisfies:*

- (a') $\mathbb{H}_i(M) = 0$ for $i \notin \{0, -1, -2\}$;
- (b') $\mathbb{H}_0(M)$ is a 0-presented \mathcal{H} -sheaf;
- (c') $\mathbb{H}_{-1}(M)$ is a 1-presented \mathcal{H} -sheaf;
- (d') $\mathbb{H}_{-2}(M)$ is a 1-connected \mathcal{H} -sheaf;
- (e') $M[1]$ does not have any non-trivial direct summand which is a 0-motive;
- (f') If L is a 0-presented \mathcal{H} -sheaf, then $\text{ext}_{\mathbf{HI}(k)}^1(\mathbb{H}_2(M), L) = 0$, i.e., every extension of $\mathbb{H}_{-2}(M)$ by L splits.

4.3. n -Presented $(2, \mathcal{H})$ -sheaves and mixed 2-motives.

PROPOSITION 4.11 — *Let $n \geq 1$ be an integer. Then Hypothesis 2.8 is satisfied for the 1-motivic t -structure with $\mathcal{A} = \mathbf{M}_1(k)$ and $\mathcal{B} = {}^1\mathcal{H}_{\leq n}^{\mathcal{M}}(k)$.*

Proof. By Corollary 3.15, we have a right adjoint ${}^1Q_n : {}^1\mathcal{H}^M(k) \rightarrow {}^1\mathcal{H}_{\leq n}^M(k)$ to the obvious inclusion. Next, consider a short exact sequence of $(1, \mathcal{H})$ -sheaves

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

where M'' is a mixed 1-motive. We need to show that ${}^1Q_n(M') \rightarrow {}^1Q_n(M)$ is injective. This is easily seen to be equivalent to the condition that

$$U = \ker\{{}^1Q_n(M) \rightarrow M''\}$$

is n -presented. Consider the exact sequence of \mathcal{H} -sheaves

$$0 \longrightarrow \operatorname{coker}\{H_0(M) \rightarrow H_0(M'')\} \longrightarrow H_{-1}(U) \longrightarrow H_{-1}({}^1Q_n(M)) \longrightarrow H_{-1}(M'') \longrightarrow 0.$$

By Lemma 4.12 below, $\ker\{H_{-1}({}^1Q_n(M)) \rightarrow H_{-1}(M'')\}$ is n -presented. It follows from [2, Lem. 1.1.22] that $H_{-1}(U)$ is also n -presented. This implies that U is an n -presented $(1, \mathcal{H})$ -sheaf. \square

LEMMA 4.12 — *Consider a short exact sequence of \mathcal{H} -sheaves:*

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0.$$

Assume that F is n -presented (with $n \geq 1$) and F'' is 1-presented. Then F' is n -presented.

Proof. Clearly, we can write F as a filtered colimit as follows:

$$F = \operatorname{colim}_{i \in I} F_i, \quad \text{where } F_i = \operatorname{coker}\{\alpha_i : h_0(Y_i) \rightarrow h_0(X_i)\}.$$

Above, I is a filtered ordered set, Y_i and X_i are smooth k -schemes of dimension at most n , and α_i is a morphism of sheaves. Let $F_i'' = \operatorname{im}\{F_i \rightarrow F''\}$ and $F_i' = \ker\{F_i \rightarrow F''\}$. Then $F' = \operatorname{colim}_{i \in I} F_i'$ (use that filtered colimits are exact) and we have short exact sequences

$$0 \longrightarrow F_i' \longrightarrow F_i \longrightarrow F_i'' \longrightarrow 0. \tag{13}$$

Clearly, it suffices to show that each F_i' is n -presented. As F_i'' is a subsheaf of F'' , it is 1-presented. Hence, the exact sequence (13) satisfies to the conditions of the statement. In other words, we may assume that

$$F = \operatorname{coker}\{\alpha : h_0(Y) \rightarrow h_0(X)\}$$

with X and Y of dimension at most n .

Given a smooth k -variety V , we denote by $h_0^{\geq 2}(V) = \ker\{h_0(V) \rightarrow \operatorname{Alb}(V)\}$. We also set

$$F^{\geq 2} = \operatorname{coker}\{h_0^{\geq 2}(Y) \rightarrow h_0^{\geq 2}(X)\}.$$

As F'' is 1-presented and $F^{\geq 2}$ is 1-connected, the composition $F^{\geq 2} \rightarrow F \rightarrow F''$ is zero. It follows that $F^{\geq 2} \rightarrow F$ factors through F' , yielding a morphism

$$\beta : F^{\geq 2} \longrightarrow F'.$$

The kernel and cokernel of β are 1-presented \mathcal{H} -sheaves. Indeed, $\ker(\beta)$ is a subquotient of $\operatorname{Alb}(Y)$ and $\operatorname{coker}(\beta)$ is a subquotient of $\operatorname{Alb}(X)$. Using [2, Lem. 1.1.22], we are reduced to show that $F^{\geq 2}$ is n -presented. By a second application of [2, Lem. 1.1.22], we are further reduced to check that $h_0^{\geq 2}(V)$ is n -presented for V a k -smooth scheme of dimension at most n .

One can find a smooth curve $C \subset V$ such that the composition

$$h_0(C) \rightarrow h_0(V) \rightarrow \text{Alb}(V)$$

is surjective. Let $E = \text{coker}\{h_0(C) \rightarrow h_0(V)\}$. This is an n -presented \mathcal{H} -sheaf. The morphism $\gamma : h_0^{\geq 2}(V) \rightarrow E$ is clearly surjective and its kernel is a subquotient of $h_0(C)$, and hence is 1-presented. We use again [2, Lem. 1.1.22] to conclude. \square

DEFINITION 4.13 — *Let $n \geq 1$ be an integer. We denote by ${}^2\mathcal{H}_{\leq n}^{\mathcal{M}}(k) \subset {}^2\mathcal{H}^{\mathcal{M}}(k)$ the $\mathbf{M}_1(k)$ -perverted subcategory associated to ${}^1\mathcal{H}_{\leq n}^{\mathcal{M}}(k) \subset {}^1\mathcal{H}^{\mathcal{M}}(k)$. A $(2, \mathcal{H})$ -sheaf which is in ${}^2\mathcal{H}_{\leq n}^{\mathcal{M}}(k)$ is called n -presented.*

Clearly, a 1-presented $(2, \mathcal{H})$ -sheaf is simply a mixed 1-motive, i.e., ${}^2\mathcal{H}_{\leq 1}^{\mathcal{M}}(k) = \mathbf{M}_1(k)$. By convention, a 0-presented $(2, \mathcal{H})$ -sheaf is a 0-motive and we set ${}^2\mathcal{H}_{\leq 0}^{\mathcal{M}}(k) = \mathbf{M}_0(k)$.

LEMMA 4.14 — *Let $n \geq 1$ be an integer. A $(2, \mathcal{H})$ -sheaf M is n -presented if and only if the \mathcal{H} -sheaf $H_{-2}(M)$ is n -presented.*

Proof. Indeed, M is n -presented if and only if the $(1, \mathcal{H})$ -sheaf ${}^1H_{-1}^{\mathcal{M}}(M)$ is n -presented which is equivalent to $H_{-1}({}^1H_{-1}^{\mathcal{M}}(M)) \simeq H_{-2}(M)$ being an n -presented \mathcal{H} -sheaf. \square

PROPOSITION 4.15 — *${}^2\mathcal{H}_{\leq n}^{\mathcal{M}}(k)$ is an abelian category and there is a functor*

$${}^2Q_n : {}^2\mathcal{H}^{\mathcal{M}}(k) \rightarrow {}^2\mathcal{H}_{\leq n}^{\mathcal{M}}(k),$$

which is a right adjoint to the obvious inclusion.

We now come to our definition of mixed 2-motives.

DEFINITION 4.16 — *An object $M \in \mathbf{DM}_{\text{eff}}(k)$ is a mixed 2-motive if it satisfies the following conditions:*

- (a) $H_i(M) = 0$ for $i \notin \{0, -1, -2\}$;
- (b) $H_0(M)$ is a 0-presented \mathcal{H} -sheaf;
- (c) $H_{-1}(M)$ is a 1-presented \mathcal{H} -sheaf;
- (d) $H_{-2}(M)$ is a 1-connected and 2-presented \mathcal{H} -sheaf;
- (e) $M[1]$ does not have any non-trivial summand which is a 0-motive;
- (f) If L is a 0-presented \mathcal{H} -sheaf, then $\text{ext}_{\mathbf{HI}(k)}^1(H_2(M), L) = 0$, i.e., every extension of $H_{-2}(M)$ by L splits.

We denote by $\mathbf{M}_2(k)$ the full subcategory of mixed 2-motives.

Obviously, mixed 2-motives are exactly the 2-presented $(2, \mathcal{H})$ -sheaves, i.e., $\mathbf{M}_2(k) = {}^2\mathcal{H}_{\leq 2}^{\mathcal{M}}(k)$. In particular, $\mathbf{M}_2(k)$ is an abelian category.

4.4. Mixed 2-motives associated to surfaces.

Ideally, we should have the following.

CONJECTURE 4.17 — *Let S be a k -surface (possibly singular). Then ${}^2H_i^{\mathcal{M}}(M(S))$ is a mixed 2-motive for all $i \in \mathbb{Z}$.*

In fact, ${}^2H_i^{\mathcal{M}}(M(S))$ is expected to vanish for $i \notin \llbracket 0, 4 \rrbracket$ and more precisely, whenever the ℓ -adic homology group $H_i^{\text{ét}}(S \otimes_k k^s, \mathbb{Q}_{\ell})$ vanishes (here, k^s is a separable closure of the base field k and ℓ is a prime which is invertible in k). Unfortunately, Conjecture 4.17 seems out of reach of the actual techniques. However, it is possible

to attach to S a sequence $\{H_i^{\mathcal{M}}(S)\}$ of mixed 2-motives which hopefully coincide with those in Conjecture 4.17.

For simplicity, we assume that S is smooth and, except for the next result, we consider the cases where S is affine or S is projective.

LEMMA 4.18 — *Let S be a smooth surface. Then ${}^2H_n^{\mathcal{M}}(M(S))$ is a mixed n -motive for $n \in \{0, 1, 2\}$.*

Proof. Consider the distinguished triangle

$$M_{\geq 1}(S) \longrightarrow M(S) \longrightarrow M(\pi_0(S)) \longrightarrow M_{\geq 1}(S)[1].$$

Clearly, $M(\pi_0(S))$ is ${}^2t^{\mathcal{M}}$ -negative, whereas $M_{\geq 1}(S)$ is strictly ${}^2t^{\mathcal{M}}$ -positive. This shows that ${}^2H_0^{\mathcal{M}}(M(S)) \simeq M(\pi_0(S))$ is a 0-motive.

Similarly, consider the distinguished triangle

$$\tilde{M}_{\geq 2}(S)[-1] \longrightarrow M_{\geq 1}(S)[-1] \longrightarrow (\text{Alb}^0(S) \otimes \mathbb{Q})[-1] \longrightarrow \tilde{M}_{\geq 2}(S),$$

with $\text{Alb}^0(S)$ the connected component of the Albanese scheme of S . Clearly, $(\text{Alb}^0(S) \otimes \mathbb{Q})[-1]$ is ${}^2t^{\mathcal{M}}$ -negative, whereas $\tilde{M}_{\geq 2}(S)[-1]$ is strictly ${}^2t^{\mathcal{M}}$ -positive. This shows that ${}^2H_1^{\mathcal{M}}(M(S)) \simeq (\text{Alb}^0(S) \otimes \mathbb{Q})[-1]$ is a mixed 1-motive.

Finally, to prove that ${}^2H_2^{\mathcal{M}}(M(S))$ is a mixed 2-motives, it suffices to show that $H_{-2}({}^2H_2^{\mathcal{M}}(M(S)))$ is 2-presented. But, the latter is given by $\ker\{h_0(S) \rightarrow \text{Alb}(S)\}$. We conclude using Lemma 4.12. \square

Now, assume that S is affine. Then ${}^2H_i^{\mathcal{M}}(M(S))$ are expected to be zero for $i \notin \{0, 1, 2\}$. Thus, we can make the following definition.

DEFINITION 4.19 — *For $i \in \llbracket 0, 2 \rrbracket$, we set $\{H_i^{\mathcal{M}}(S) = {}^2H_i^{\mathcal{M}}(M(S))\}$. These are the (possibly non-zero) mixed 2-motives associated to S .*

Next, assume that S is projective. It is classical that the Chow motive of S admits a Künneth decomposition (see for example [6]). As the category of Chow motives is embedded into $\mathbf{DM}_{\text{eff}}(k)$, we deduce a decomposition

$$M(S) = \bigoplus_{i=0}^4 M_i(S)[i]$$

such that $M_i(S)$ corresponds under the ℓ -adic realization to $H_i^{\text{ét}}(S \otimes_k k^s, \mathbb{Q}_{\ell})$. We know that $M_0(S) = M(\pi_0(S))$ and that

$$M_4(S) = \underline{\text{Hom}}(M_0(S), \mathbb{Q}(2)) \simeq M_0(S)^{\vee}(2),$$

where $M_0(S)^{\vee} = \underline{\text{Hom}}(M_0(S), \mathbb{Q}(0))$ is the dual 0-motive to $M_0(S)$. Also, $M_1(S)$ is a pure 1-motive given by the complex $(\text{Alb}^0(S) \otimes \mathbb{Q})[-1]$ with $\text{Alb}^0(S)$ the connected component of the Albanese scheme of S . Moreover,

$$M_3(S) = \underline{\text{Hom}}(M_1(S), \mathbb{Q}(2)) \simeq M_1(S)^{\vee}(1),$$

with $M_1(S)^{\vee} = \underline{\text{Hom}}(M_1(S), \mathbb{Q}(1))$, the Cartier dual of the 1-motive $M_1(S)$.

LEMMA 4.20 — *$M_0(S)$ is a 0-motive and $M_1(S)$ is a 1-motive. Moreover, for $i \in \{2, 3, 4\}$, ${}^2H_0^{\mathcal{M}}(M_i(S))$ is a mixed 2-motive.*

Proof. The first statement is obvious. To prove the second statement, it suffices to show that for $i \in \{2, 3, 4\}$, the \mathcal{H} -sheaf

$$H_{-2}({}^2H_0^{\mathcal{M}}(M_i(S))) \simeq H_{-2}(M_i(S))$$

is 2-presented. By an easy inspection, we see that:

$$\begin{aligned} H_{-2}(M_2(S)) &= \ker\{h_0(S) \rightarrow \text{Alb}(S)\}, & H_{-2}(M_3(S)) &= K_1^M \otimes \text{Alb}^0(S)^\vee \\ \text{and} & & H_{-2}(M_4(S)) &= K_2^M \otimes \mathbb{Q}_{tr}(\pi_0(S))^\vee. \end{aligned}$$

Above, K_n^M is the n -th Milnor K -theory sheaf and the tensor product is taken in $\mathbf{HI}(k)$. This proves the lemma. \square

DEFINITION 4.21 — *Under the above hypothesis, we set $'H_i^{\mathcal{M}}(S) = {}^2H_0^{\mathcal{M}}(M_i(S))$ for $i \in \llbracket 0, 4 \rrbracket$. These are the (possibly non-zero) mixed 2-motives attached to S .*

Remark 4.22 — Let us consider the simplest projective surface, i.e., the projective plane \mathbb{P}_k^2 . We have $M(\mathbb{P}_k^2) = \mathbb{Z}(0) \oplus \mathbb{Z}(1)[2] \oplus \mathbb{Z}(2)[4]$. Then clearly, $'H_i^{\mathcal{M}}(\mathbb{P}_k^2) = 0$ for odd $i \in \llbracket 0, 4 \rrbracket$. Also, $'H_0^{\mathcal{M}}(\mathbb{P}_k^2) = \mathbb{Z}(0)$ and $'H_2^{\mathcal{M}}(\mathbb{P}_k^2) = \mathbb{Z}(1)$. For $i = 4$, we would like to write: “ $'H_4^{\mathcal{M}}(\mathbb{P}_k^2) = \mathbb{Z}(2)$ ”, but unfortunately we can't. Indeed, this requires to prove that $\mathbb{Z}(2)$ is a mixed 2-motive and, in particular, that the following properties hold true:

- (1) $H_i(\mathbb{Z}(2)) = 0$ for $i > 0$,
- (2) $H_0(\mathbb{Z}(2))$ is a 0-presented \mathcal{H} -sheaf,
- (3) $H_{-1}(\mathbb{Z}(2))$ is a 1-presented \mathcal{H} -sheaf.

In fact, these properties are sufficient for showing that $\mathbb{Z}(2)$ is a mixed 2-motive as it follows easily from Definition 4.16. Note also that (1) is a reformulation of the Beilinson-Soulé vanishing conjecture for the motivic cohomology groups $H^n(-, \mathbb{Z}(2))$ with $n < 0$.

4.5. Beyond the case $n = 2$.

From what we have learned in this paper, it is natural to expect that the 3-motivic t -structure $({}^3\mathcal{T}_{\geq 0}^{\mathcal{M}}(k), {}^3\mathcal{T}_{\leq 0}^{\mathcal{M}}(k))$ is the $\mathbf{M}_2(k)$ -perverted t -structure associated to the 2-motivic t -structure. In fact, we expect more generally that the n -motivic t -structure is obtained by perverting the $(n-1)$ -motivic t -structure with respect to a well-chosen category $\mathbf{M}_{n-1}(k)$ of mixed $(n-1)$ -motives, and this for all $n \in \mathbb{N} - \{0\}$. Unfortunately, even for $n = 3$, we have to assume some outstanding conjectures to ensure that $\mathbf{M}_2(k)$ satisfies the conditions in Hypothesis 2.1 which would enable us to pervert the 2-motivic t -structure as we did in case $n = 1$ and $n = 2$.

For instance, we need to know that $\mathbf{M}_2(k)$ is a thick abelian subcategory of ${}^2\mathcal{H}^{\mathcal{M}}(k)$. However, to prove this along the lines of Proposition 3.9, it seems necessary to assume that $\mathbf{HI}_{\leq 2}(k)$ is a thick abelian subcategory of $\mathbf{HI}(k)$. This is conjecturally true by [2, Cor. 1.4.5] (which relies on Conj. 1.4.1 of *loc. cit.*). Also, if we want to construct a left adjoint to the inclusion $\mathbf{M}_2(k) \subset {}^2\mathcal{H}^{\mathcal{M}}(k)$, it is certainly useful to have at our disposal a left adjoint to the inclusion $\mathbf{HI}_{\leq 2}(k) \subset \mathbf{HI}(k)$. Again, such an adjoint exists by [2, Prop. 1.4.6] assuming Conj. 1.4.1 of *loc. cit.*. (It is worth noting here that a similar left adjoint on the level of triangulated categories does not exist by [2, §2.5] as was claimed by Voevodsky in [10, §3.4].)

In any case, it is an interesting problem to give a conditional construction of all the n -motivic t -structures using Conj. 1.4.1 of [2]. We will not pursue this goal in this paper.

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