Motives of rigid varieties and the motivic nearby functor

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Constructing the category of motives of rigid varieties

Let k be a complete field for a non-archimedean norm $|.| : k \to \mathbb{R}_+$. Denote $RigVar/k$ the category of rigid varieties over k and $RigSm/k$ its sub-category of smooth varieties. One can construct out of $RigSm/k$ a triangulated category $\mathbf{RigDM}_{\mathsf{eff}}(k)$ of rigid motives in the same way as Voevodsky constructed the category $\mathbf{DM}_{\mathsf{eff}}(k)$:

Step 1: Define an additive category $\mathbf{RigCor}(k)$ with the same objects as $RigSm/k$ and morphisms $RigCor(X, Y)$ the free abelian group on closed and irreducible sub-varieties $Z \subset X \times Y$ which are finite and surjective over a connected component of X. The composition of finite correspondences is given as usual using the Serre's multiplicity formula.

Step 2: Denote $\mathbf{RigPST}(k)$ the category of contravariant additive functors:

$$\mathbf{RigCor}(k) \to \mathcal{A}b$$

Object of this category are called pre-sheaves with transfers. We have a Yoneda embedding:

$$\mathbf{RigCor}(k) \subset \mathbf{RigPST}(k)$$

We denote $\mathbb{Z}_{tr}(X)$ the pre-sheaf with transfers represented by X.

Definition: A morphism of k-affinoids $U \to X$ is called a *weak Nisnevich cover* if it is étale and every closed point $x \in X$ admits a lifting to U. The map $U \to X$ is called a *Nisnevich cover* if it is universally a weak Nisnevich cover. Here universally stands for the change of the base field along extensions of complete normed fields $k \subset K$. The *Nisnevich topology* on $RigSm/k$ is the topology generated by the usual topology and the Nisnevich covers.

Example: Let $\mathcal{X}/k^o$ be a finite type adic formal scheme and let $U \to \mathcal{X}_s$ be an (algebraic) Nisnevich cover. The étale morphism $U \to \mathcal{X}_s$ extends uniquely to an étale morphism of formal schemes $\mathcal{U} \to \mathcal{X}$. Further more the Raynaud generic fiber $\mathcal{U}_\eta \to \mathcal{X}_\eta$ is a (rigid) Nisnevich cover. Moreover every Nisnevich cover can be refined in a one coming in this way.

Step 4: Consider the sub-category:

$$\mathbf{RigShTr}(k) \subset \mathbf{RigPST}(k)$$

of pre-sheaves with transfers which are also Nisnevich sheaves on $RigSm/k$. This is an abelian category of Grothendieck. Indeed, one can show (following the proof of Voevodsky in the algebraic context) that the Nisnevich sheaf associated to a pre-sheaf with transfers has a canonical action by correspondences.

Remark that $\mathbb{Z}_{tr}(X)$ is a Nisnevich sheaf with transfers.

Definition: Define

$$\mathbf{RigDM}_{\mathsf{eff}}(k) \subset D(\mathbf{RigShTr}(k))$$

to be the triangulated sub-category whose objects are complexes of sheaves with transfers $K_\bullet$ such that for any smooth X one has:

$$\text{hom}(\mathbb{Z}_{tr}(X), K_\bullet[n]) = \text{hom}(\mathbb{Z}_{tr}(\mathbb{B}^1 \times X), K_\bullet[n])$$

Where $\mathbb{B}^1$ is the Tate ball $\text{Spm}(k\{t\})$. Such complexes are called $\mathbb{B}^1$-local.

As in the algebraic context one has:

Lemma: The obvious inclusion has a left adjoint $\text{Loc}_{\mathbb{B}^1} : D(\mathbf{RigShTr}(k)) \to \mathbf{RigDM}_{\mathsf{eff}}(k)$. The motive $\text{M}_{\text{rig}}(X)$ of X is by definition the complex $\text{Loc}_{\mathbb{B}}(\mathbb{Z}_{tr}(X))$. Moreover, the category $\mathbf{RigDM}_{\mathsf{eff}}(k)$ is compactly generated by the $\text{M}_{\text{rig}}(X)$ for X smooth affinoids.

A map f of complexes in $D(\mathbf{RigShTr}(k))$ is called a $\mathbb{B}^1$-weak equivalence if $\text{Loc}_{\mathbb{B}^1}(f)$ is an isomorphism. One has:

$$\mathbf{RigDM}_{\mathsf{eff}}(k) \simeq D(\mathbf{RigShTr}(k))[(\mathbb{B}^1 - w.e.)^{-1}]$$

Remark: The rigid motive of the analytification of the pointed affine line $(\mathbb{A}^1, 0)$ is zero as one can write $\mathbb{A}^1 = \cup_{r>0}\mathbb{B}^1(0, r)$. It follows that the morphism of site $an : RigSm/k \to Sm/k$ induces a functor

$$\mathrm{Rig}^* = \mathrm{Loc}_{\mathbb{B}^1} \circ an^* : \mathbf{DM}_{\mathrm{eff}}(k) \to \mathbf{RigDM}_{\mathrm{eff}}(k)$$

One can think about Rig^* as a realization functor.

Example: The rigid motives of the following varieties:

$$\mathbb{G}_m,\ \mathbb{B}^1 - 0,\ \partial\mathbb{B}^1,\ C(a) = \mathrm{Spm}\left(\frac{k\{t_1, t_2\}}{t_1.t_2 - a}\right)$$

with $0 < |a| < 1$ are canonically isomorphic. This follows from the computation of the motive of $\mathbb{P}^1$ in different ways applying Mayer-Vietoris to one of the covers:

- $\mathbb{P}^1 = \mathbb{A}^1 \cup \mathbb{A}^1$ and $\mathbb{A}^1 \cap \mathbb{A}^1 = \mathbb{G}_m$,
- $\mathbb{P}^1 = \mathbb{A}^1 \cup \mathbb{B}^1$ and $\mathbb{A}^1 \cap \mathbb{B}^1 = \mathbb{B}^1 - 0$,
- $\mathbb{P}^1 = \mathbb{B}^1 \cup \mathbb{B}^1$ with different radius so that $\mathbb{B}^1 \cup \mathbb{B}^1 = \partial\mathbb{B}^1$ or $C(a)$.

First main result:
The generation theorem

From now on, we work (for simplicity) with rational coefficients.

Definition: An affinoid X has *good reduction* if it is the Raynaud generic fiber of an adic smooth formal scheme over k^o. It has *potentially good reduction* if $X \otimes_k k'$ has good reduction for some finite extension $k \subset k'$.

Theorem A: *The triangulated category with arbitrary sums* $\mathbf{RigDM}(k)$ *is compactly generated by* $\mathsf{M}_{\mathsf{rig}}(X)$ *for* X *affinoid varieties with potentially good reduction.*

We have seen that $\mathsf{M}_{\mathsf{rig}}(\mathsf{Spm}(\frac{k\{t_0,t_1\}}{t_0.t_1-a}))$ is isomorphic to $\mathsf{M}_{\mathsf{rig}}(\mathsf{Spm}(\frac{k\{t_0,t_1\}}{t_0.t_1-1}))$. This a confirmation of the above theorem. More generally we have:

lemma: *The rigid motive of the affinoid:*

$$S_n = \mathsf{Spm}(\frac{k\{t_0, \ldots, t_n\}}{t_0 \ldots t_n - a})$$

with $0 < |a| < 1$ *is isomorphic to the rigid motive of* $\partial\mathbb{B}^1 \times \cdots \times \partial\mathbb{B}^1$ *that is to the motive of*

$$\mathsf{Spm}(\frac{k\{t_0, \ldots, t_n\}}{t_0 \ldots t_n - 1})$$

For simplicity we assume that k is of equi-characteristic zero. Let $\mathbf{RigDM}_{pgr}(k)$ be the triangulated sub-category of $\mathbf{RigDM}_{\mathsf{eff}}(k)$ generated by the motives of k-affinoids with potentially good reduction. We need to show that $\mathsf{M}_{\mathsf{rig}}(X) \in \mathbf{RigDM}_{pgr}(k)$ for any smooth affinoid X.

By the semi-stable reduction theorem, we may assume that X has a model $\mathcal{X}$ which is smooth over:

$$\mathcal{S}_n = \mathsf{Spf}(\frac{k^o[[t_0, \ldots, t_n]]}{t_0 \ldots t_n - a})$$

with $0 < |a| < 1$.

We argue by induction on n. Let $o \in \mathcal{S}_n$ be the intersection of all branches. We may assume that the fiber $\mathcal{X}_o$ is non-empty.

We may further assume that $\mathcal{X}_o$ admits a Nisnevich neighborhood:

$$\mathcal{X}_o \to \mathcal{V} \to \mathcal{X}$$

which is also a Nisnevich neighborhood of $\mathcal{X}_o$ in $\mathcal{U} \times_{k^o} \mathcal{S}_n$:

$$\mathcal{X}_o \to \mathcal{V} \to \mathcal{U} \times_{k^o} \mathcal{S}_n$$

with $\mathcal{U}$ a smooth adic formal scheme with special fiber $\mathcal{X}_o$.

It follows by Nisnevich excision that:

$$\mathbb{Z}_{tr}\left[\frac{\mathcal{X}_\eta}{(\mathcal{X} - \mathcal{X}_0)_\eta}\right] \simeq \mathbb{Z}_{tr}\left[\frac{\mathcal{V}_\eta}{(\mathcal{V} - \mathcal{X}_0)_\eta}\right]$$

$$\simeq \mathbb{Z}_{tr}\left[\frac{\mathcal{U}_\eta \times (\mathcal{S}_n)_\eta}{\mathcal{U}_\eta \times (\mathcal{S}_n - o)_\eta}\right]$$

as Nisnevich sheaves.

By induction the motives $\mathsf{M}_{\mathsf{rig}}((\mathcal{X}-\mathcal{X}_0)_\eta)$ and $\mathsf{M}_{\mathsf{rig}}((\mathcal{S}_n-o)_\eta)$ are in

$$\mathbf{RigDM}_{pgr}(k)$$

It suffices to show that

$$\mathsf{M}_{\mathsf{rig}}((\mathcal{S}_n)_\eta) \in \mathbf{RigDM}_{pgr}(k)$$

(Note that the category $\mathbf{RigDM}_{pgr}(k)$ is stable by tensor product). But we saw that the rigid motive of $S_n = (\mathcal{S}_n)_\eta$ is isomorphic to:

$$\mathsf{Spm}(\frac{k\{t_0,\ldots,t_n\}}{t_0\ldots t_n - 1})$$

which is clearly an affinoid with good reduction.

Theorem **A** is proved.

Second main result:

An equivalence of categories

In the equi-characteristic zero situation, it is possible to prove a much more precise result than theorem **A**. To state it we introduce some notations. Let $\tilde{k} = k^o/k^\vee$ be the residue field of k. We assume $|.|$ to be discrete and fix $\pi \in k^o$ a uniformizer so that $k = \tilde{k}((\pi))$.

Definition: Let $qc\mathbf{DM}_{\mathsf{eff}}((\mathbb{G}_m)_{\tilde{k}})$ be the triangulated sub-category of $\mathbf{DM}_{\mathsf{eff}}((\mathbb{G}_m)_{\tilde{k}})$ having all sums and generated by $\mathrm{M}(X_{c,n})$ with $X_{c,n}$ the $(\mathbb{G}_m)_{\tilde{k}}$-scheme:

$$X \times_{\tilde{k}} (\mathbb{G}_m)_{\tilde{k}} \to (\mathbb{G}_m)_{\tilde{k}} \xrightarrow{(.)^n} (\mathbb{G}_m)_{\tilde{k}}$$

where X are smooth $\tilde{k}$-schemes.

Theorem B: *The composition:*

$$qc\mathbf{DM}_{\mathsf{eff}}((\mathbb{G}_m)_{\widetilde{k}}) \stackrel{\pi^*}{\to} \mathbf{DM}_{\mathsf{eff}}(k) \to \mathbf{RigDM}_{\mathsf{eff}}(k)$$

is an equivalence of categories.

We will apply the following lemma:

Lemma: *Let $F : \mathcal{T}_1 \to \mathcal{T}_2$ be a triangulated functor between two compactly generated triangulated categories. We suppose that:*

- *F commutes with sums,*
- *There exists a set of compact generators $\underline{G}_1 \subset \mathcal{T}_1$ such that $F(\underline{G}_1)$ is a set of compact generators of $\mathcal{T}_2$,*
- *For every $A, B \in \underline{G}_1$ and $p \in \mathbb{Z}$ we have an isomorphism:*

$$\mathsf{hom}(A, B[p]) \simeq \mathsf{hom}(F(A), F(B)[p])$$

Then F is an equivalence of categories.

We take $\underline{G}_1$ to be the class of $\mathsf{M}(X_{c,n})$ with X smooth over $\tilde{k}$. One can prove:

Lemma: *Denote $X^{an}_{c,n}$ the rigid variety which is the generic fiber of the completion along the special fiber of:*

$$X \times_{\tilde{k}} k^o[\pi^{1/n}]$$

Then $\mathsf{Rig}^*\pi^*\mathsf{M}(X_{c,n}) \simeq \mathsf{M}_{\mathsf{rig}}(X^{an}_{c,n})$.

It follows that $\mathsf{Rig}^*\pi^*\underline{G}_1$ is a set of generators of $\mathbf{RigDM}_{\mathsf{eff}}(k)$. Indeed, by theorem **A** $\mathbf{RigDM}_{\mathsf{eff}}(k)$ is generated by motives of affinoid having potentially good reduction. Such affinoids become isomorphic to a $X^{an}_{c,1}$ after finite extension of the base field.

We still need to check the equalities of Homs. We easily reduce to the case $n = 1$. We show more generally:

<u>Lemma:</u> *Let M be a motive in $\mathbf{DM}_{\mathsf{eff}}(\tilde{k})$ and denote M_k the pull-back of M along $\tilde{k} \subset k$. We have:*

$$\mathsf{hom}_{\mathbf{RigDM}_{\mathsf{eff}}(k)}(\mathsf{M}_{\mathsf{rig}}(X_c^{an}), \mathsf{Rig}^*(M_k))$$
$$= \mathsf{hom}_{\mathbf{DM}_{\mathsf{eff}}(\tilde{k})}(\mathsf{M}(X \times \mathbb{G}_m), M)$$

Idea of the proof: Let $an^* M_k = M_k^{an}$ be the analytification of the complex of Nisnevich sheaves with transfers M_k. We construct a complex of sheaves with transfers $\Phi(M)$ on $RigSm/k$ together with a map $M_k^{an} \to \Phi(M)$ in the following way.

Consider the diagram of $\tilde{k}$-schemes:

$$\mathcal{D} : \textit{Affinoid}/k \to Sch/\tilde{k}$$

with *Affinoid*$/k$ the category of k-affinoids and $\mathcal{D}(\mathsf{Spm}(A)) = \mathsf{Spec}(A^o)$. One has the following diagram:

$$\mathcal{D}_\eta \xrightarrow{j} \mathcal{D} \xleftarrow{i} \mathcal{D}_s$$

in the category of diagrams of $\tilde{k}$-schemes.

One can extend the construction of $\mathbf{DM}_{\mathsf{eff}}(-)$ and its stable version $\mathbf{DM}(-)$ from schemes to diagram of schemes in the obvious manner. In particular, we can consider the functors:

$$\mathbf{DM}(\mathcal{D}_\eta) \xrightarrow{j_*} \mathbf{DM}(\mathcal{D}) \xrightarrow{i^*} \mathbf{DM}(\mathcal{D}_s)$$

Define $\phi(M) = i^* j_*(M_{|\mathcal{D}_\eta})$; this a $\mathbb{G}_m$-spectrum of pre-sheaves on the category $Sm/\mathcal{D}_s$. Finally define:

$$\Phi(M)(\mathsf{Spm}(A)) = \mathsf{R}\Gamma(\mathsf{Spec}(\tilde{A}), \phi(M))$$

One easily checks that as a complex of pre-sheaves, $\Phi(M)$ is quasi-isomorphic to a complex of Nisnevich sheaves with transfers. Moreover we have:

Sub-Lemma 1: $\Phi(M)$ *is* $\mathbb{B}^1$*-local.*

Indeed, let $X = \mathsf{Spm}(A)$ be a smooth affinoid. We have by construction that

$$\mathsf{H}^*_{\mathsf{Nis}}(X, \Phi(M)) = \operatorname*{Colim}_{\mathcal{X}_\eta = X} \mathsf{H}^*_{\mathsf{Nis}}(\mathcal{X}_s, i^* j_*(M_{|A}))$$

Now, for the stable $\mathbf{DM}(-)$ we have the base change theorem for projective morphisms. This implies that:

$$\mathsf{H}^*_{\mathsf{Nis}}(\mathcal{X}_s, i^* j_*(M_{|A})) = \mathsf{H}^*_{\mathsf{Nis}}(\mathsf{Spec}(\tilde{A}), i^* j_*(M_{|A}))$$

So that the colimit stabilize in:

$$\mathsf{H}^*_{\mathsf{Nis}}(X, \Phi(M)) = \mathsf{H}^*_{\mathsf{Nis}}(\mathsf{Spec}(\tilde{A}), i^* j_*(M_{|A}))$$

This immediately implies that $\Phi(M)$ is $\mathbb{B}^1$-local as $i^* j_*(M_{|A})$ is $\mathbb{A}^1$-local.

Sub-Lemma 2: *The morphism $M_k^{an} \to \Phi(M)$ is a $\mathbb{B}^1$-weak equivalence. This identifies $\mathsf{Loc}_{\mathbb{B}^1}(M_k^{an})$ with $\Phi(M)$.*

This will implies theorem **B** as we have for an affinoid with good reduction $X = \mathsf{Spm}(A)$ that $\Phi(M)(X) = \mathsf{hom}(\mathsf{Spec}(\tilde{A}) \times \mathbb{G}_m, M)$.

Note that we have a factorization:

$$M_k^{an} \to C_*^{\mathbb{B}} M_k^{an} \to \Phi(M)$$

Where $C_*^{\mathbb{B}}$ is the rigid analogue of the Suslin-Voevodsky complex i.e. $C_n^{\mathbb{B}}(-) = \underline{Hom}(\Delta_{rig}^n, -)$ where Δ_{rig}^n is the rigid simplex:

$$\mathrm{Spm}(\frac{k\{t_0, \dots, t_n\}}{t_0 + \cdots + t_n = 1})$$

By a direct computation, one can prove that for $X = \mathrm{Spm}(A)$ an affinoid with good reduction $C_*^{\mathbb{B}}(M_k^{an})(X) \simeq C_*^{\mathbb{A}}(M)(\tilde{X} \times \mathbb{G}_m)$. This computation was already known to Marc Levine (in *Motivic Tubular Neighborhood*). The idea is to use the retraction $\mathrm{Spec}(A) \to \mathrm{Spec}(\tilde{A})$ to get some explicit homotopy equivalence of complexes.

Now the sub-lemma follows easily from the following proposition:

Proposition: *Let F be a pre-sheaf with transfers on $RigSm/k$ such that $F(X) = 0$ whenever X is a smooth affinoid with potentially good reduction. Then F_{Nis} is $\mathbb{B}^1$-weakly equivalent to zero.*

Denote S_n the affinoid $\mathrm{Spm}(\frac{k\{t_0,\ldots,t_n\}}{t_0\ldots t_n - a})$ with $0 < |a| < 1$. One reduces formally to the case F the quotient of $\mathbb{Z}_{tr}(S_n)$ by all the sections with value in potentially good reduction affinoids. By induction me may further assume that F is the quotient of $\mathbb{Z}_{tr}(S_n)$ by all the sections with value in affinoids having potentially semi-stable reduction with $< n$ branches.

One then construct a section of F_{Nis} with value in $S_n \times \mathbb{B}^1$ by gluing along the standard cover:

$$S_{n+1} \coprod_{S_n \times \partial\mathbb{B}^1} (S_n \times \mathbb{B}^1)$$

some well chosen section in $F(S_{n+1})$ with the zero section. This gives a homotopy $F \times \mathbb{B}^1 \to F_{\mathrm{Nis}}$ between the identity and the zero morphism.

Application to the motivic nearby functor

The computation of the $\mathbb{B}^1$-localization we made in the proof of theorem **B** extends to non-necessarily "constant" motives. In fact, for every $M \in \mathbf{DM}_{\mathsf{eff}}(k)$ the complex $\mathsf{Rig}^*(M)$ is quasi-isomorphic to $\Phi(M)$ which is given by:

$$\Phi(M)(\mathsf{Spm}(A)) = \mathsf{R}\Gamma(\mathsf{Spec}(\tilde{A}, i^*j_*(M_{\mathcal{D}_\eta}))$$

We deduce from this the following theorem:

Theorem C: *The functor:*

$$i^*j_* : \mathbf{DM}(k) \to \mathbf{DM}(\tilde{k})$$

is canonically isomorphic to the composition:

$$\mathbf{DM}(k) \xrightarrow{\mathsf{Rig}^*} \mathbf{RigDM}(k) \simeq qc\mathbf{DM}((\mathbb{G}_m)_{\tilde{k}}) \xrightarrow{q_*} \mathbf{DM}(\tilde{k})$$

Note that theorem **C** concerns the stable motives (i.e. the Tate motive is inverted). This is because we had to use i^*j_* between the stable categories so that we can apply the base change theorem for projective morphisms. We conjecture that theorem **C** is true for the "effective" i^*j_* but we couldn't find a proof.

It is natural to make the following definition:

Definition: The effective nearby cycles functor Ψ_{eff} is the composition:

$$\begin{array}{ccc} \mathbf{DM}_{\text{eff}}(k) \xrightarrow{Rig^*} \mathbf{RigDM}_{\text{eff}}(k) & & \\ \uparrow \wr & & \\ qc\mathbf{DM}_{\text{eff}}((\mathbb{G}_m)_{\tilde{k}}) \xrightarrow{1^*} \mathbf{DM}_{\text{eff}}(\tilde{k}) & & \end{array}$$

In our PhD thesis we defined in a different way a motivic nearby cycles functor $\Psi : \mathbf{DM}(k) \to \mathbf{DM}(\tilde{k})$. It is not difficult to show that the two definitions agree.

The following properties are obvious on the definition of Ψ_{eff}:

Property 1: Ψ_{eff} takes compact motives to compact motives.

Property 2: Ψ_{eff} is a monoidal functor.

Some open questions

Problem 1: Is it possible to formulate and prove an analogue of theorem **B** when $\tilde{k}$ is of positive characteristic? Can this be used to extend the definition of the motivic nearby functors for the non equi-characteristic zero case?

Problem 2: Over $\mathbb{C}$, one has the Betti realization functor:

$$\mathsf{Betti}^* : \mathbf{DM}_{\mathsf{eff}}(\mathbb{C}) \to D(\mathbb{Z} - \mathsf{mod})$$

Classical Hodge theory give a factorization:

$$\mathbf{DM}_{\mathsf{eff}}(\mathbb{C}) \longrightarrow D(\mathsf{MHS}) \longrightarrow D(\mathbb{Z} - \mathsf{mod})$$

Given $M \in \mathbf{DM}_{\mathsf{eff}}(k)$ is it possible to define a "motivic Hodge structure" on $\mathsf{Rig}^*(M)$ so that one get a factorization:

$$\mathbf{DM}_{\mathsf{eff}}(k) \to \mathbf{???} \to \mathbf{RigDM}_{\mathsf{eff}}(k)$$