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Arbeitsgemeinschaft mit aktuellem Thema: Polylogarithms

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Motivic version of the classical Polylogarithms
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We show that the mixed Hodge variation pol_H and the ℓ-adic sheaf pol_ℓ are realization of a same motivic object $pol_{\mathcal{M}}$ which live in the abelian category $\mathbf{MTM}(\mathbb{U})$ of mixed Tate motives over $\mathbb{U} = \mathbb{P}^1 - \{0, 1, \infty\}$.

(1) Categories of motives.
(2) Construction of $\mathcal{L}og_{\mathcal{M}}$ and $pol_{\mathcal{M}}$.
(3) Comparison with the realizations.

1. Categories of motives

Given a scheme X one have the Voevodsky's category $\mathbf{DM}(X)$ of triangulated motives over X (see [Vo]). Recall that objects of $\mathbf{DM}(X)$ are $\mathbb{G}\mathrm{m}$-spectra of complexes $(A^k_\bullet)_k$ where A^k_j are smooth X-schemes locally of finite type[1] and the differentials $A^k_{j+1} \to A^k_j$ as well as the assembly maps $\mathbb{G}\mathrm{m} \wedge A^k_j \to A^{k+1}_j$ are given by some kind of finite correspondences which behave well under composition. We put $\mathbb{Z}_X(1)[1] = [\mathrm{id}_X \to \mathbb{G}\mathrm{m}_X]$ where $\mathbb{G}\mathrm{m}_X$ is in degree zero. For every $n \in \mathbb{Z}$ we define the Tate object $\mathbb{Z}_X(n)$ by the usual formula and for any $A \in \mathbf{DM}(X)$ we put $A(n) = A \otimes \mathbb{Z}_X(n)$.

The Voevodsky's categories $\mathbf{DM}(X)$ like the Saito's categories of mixed Hodge modules ([Sa]) have the full Grothendieck formalism of the six operations. What we don't (yet) have in $\mathbf{DM}(X)$ is a motivic t-structure. Such a t-structure should play the role of the canonical t-structures in the classical theories (ℓ-adic sheaves and mixed Hodge modules...); in particular it's heart should contains at least the Tate objects $\mathbb{Q}(n)$. The existence of such a t-structure is very related to the Beilinson-Soulé Vanishing conjecture:

Conjecture: For every smooth scheme X over a field k, the motivic cohomology groups $H^p(X, \mathbb{Z}(q))$ vanish for $p < 0$.

Where $H^p(X, \mathbb{Z}(q))$ is defined to be the group $\hom_{\mathbf{DM}(k)}([X], \mathbb{Z}(q)[p])$. Unfortunately this conjecture remains wide open... It is only known in some very special cases: for example X the spectrum of a number field[2]. In particular if we restrict ourself to the sub-category $\mathbf{DTM}(U) \subset \mathbf{DM}(U)$ generated (as a triangulated category) by the Tate objects $\mathbb{Z}_U(n)$ for U a subscheme (open or closed) of $\mathbb{P}^1_{\mathbb{Q}}$ we got the:

Theorem: The category $\mathbf{DTM}(U)$ can be equipped with a motivic t-structure. The heart of this t-structure is the abelian category of mixed Tate motives $\mathbf{MTM}(U)$ generated by the $\mathbb{Z}_U(n)$.

Recall that $\mathbb{U}$ is $\mathbb{P}^1 - \{0, 1, \infty\}$. Our main result will be the construction of a pro-object $pol_{\mathcal{M}}$ in $\mathbf{MTM}(\mathbb{U})$.

[1] Infinite disjoint union of smooth varieties are allowed.
[2] This a consequence of Borel work on the K-theory of number fields.

2. Construction of $\mathcal{L}og_{\mathcal{M}}$ and $pol_{\mathcal{M}}$

We adapt here the construction given in [Hu-Wi] to the motivic context. We first consider the Kummer mixed Tate motive $\mathcal{K} \in \mathbf{MTM}(\mathbb{G}\mathrm{m})$. It fits naturally in an exact sequence in $\mathbf{MTM}(\mathbb{G}\mathrm{m})$:

$$0 \longrightarrow \mathbb{Q}(1) \longrightarrow \mathcal{K} \longrightarrow \mathbb{Q}(0) \longrightarrow 0$$

Or equivalently in a distinguished triangle in $\mathbf{DTM}(\mathbb{G}\mathrm{m})$:

$$\mathbb{Q}(1) \longrightarrow \mathcal{K} \longrightarrow \mathbb{Q}(0) \xrightarrow{e} \mathbb{Q}(1)[1]$$

Thus $\mathcal{K}$ is uniquely (up to a unique isomorphism!) determined by the morphism e which can be constructed using the diagonal morphism $\mathbb{G}\mathrm{m} \longrightarrow \mathbb{G}\mathrm{m} \times \mathbb{G}\mathrm{m}$ considered as a morphism of $\mathbb{G}\mathrm{m}$-schemes.

Definition: $\mathcal{L}og^N_{\mathcal{M}} = \mathrm{Sym}^N(\mathcal{K})$ and $\mathcal{L}og_{\mathcal{M}}$ is the projective system $(\mathcal{L}og^{N+1}_{\mathcal{M}} \to \mathcal{L}og^N_{\mathcal{M}})_N$.

Note that we have exact sequences:

$$0 \longrightarrow \mathbb{Q}(N) \longrightarrow \mathcal{L}og^N_{\mathcal{M}} \longrightarrow \mathcal{L}og^{N-1}_{\mathcal{M}} \longrightarrow 0$$

Let us denote $(\mathcal{L}og_{\mathcal{M}})_{|\mathbb{U}}$ the pull-back of $\mathcal{L}og_{\mathcal{M}}$ by the inclusion $\mathbb{U} \subset \mathbb{G}\mathrm{m}$. The polylogarithmic mixed Tate motive will be defined as an extension:

$$0 \longrightarrow (\mathcal{L}og_{\mathcal{M}})_{|\mathbb{U}} \longrightarrow pol_{\mathcal{M}} \longrightarrow \mathbb{Q}(0) \longrightarrow 0$$

or equivalently as an element of $\mathcal{E}xt^1(\mathbb{Q}(0), (\mathcal{L}og_{\mathcal{M}})_{|\mathbb{U}})$. The main technical result is the identification of this ext-group with $\mathbb{Q}$ which allows us to make the definition:

Definition: $pol_{\mathcal{M}}$ correspond to 1 by the identification (still to be proven): $\mathcal{E}xt^1(\mathbb{Q}(0), (\mathcal{L}og_{\mathcal{M}})_{|\mathbb{U}}) = \mathbb{Q}$.

For the computation of our ext-group we consider the following commutative diagram:

$$\begin{tikzcd}
\mathbb{U} \arrow[r, "j"] \arrow[dr, "q"'] & \mathbb{G}\mathrm{m} \arrow[d, "p"] & 1 \arrow[l, "i"'] \arrow[dl, equal] \\
& \mathbb{Q} &
\end{tikzcd}$$

so we can write:

$$\begin{aligned}
\mathcal{E}xt^1(\mathbb{Q}(0), (\mathcal{L}og_{\mathcal{M}})_{|\mathbb{U}}) &= \hom_{\mathbf{DM}(\mathbb{U})}(q^*\mathbb{Q}, j^*\mathcal{L}og_{\mathcal{M}}[+1]) \\
&= \hom_{\mathbf{DM}(\mathbb{Q})}(\mathbb{Q}, p_*j_*j^*\mathcal{L}og_{\mathcal{M}}[+1])
\end{aligned}$$

The last equality comes from adjunction. Next we invoke the distinguished triangle:

$$i_*i^!\mathcal{L}og_{\mathcal{M}} \longrightarrow \mathcal{L}og_{\mathcal{M}} \longrightarrow j_*j^*\mathcal{L}og_{\mathcal{M}} \longrightarrow i_*i^!\mathcal{L}og_{\mathcal{M}}[+1]$$

The computation then splits into two parts:

- $p_*\mathcal{L}og_{\mathcal{M}} = \mathbb{Q}(-1)[-1]$.

- $i^! \mathcal{L}og_{\mathcal{M}} = i^* \mathcal{L}og_{\mathcal{M}}(-1)[-2] = \prod_{k \geq 0} \mathbb{Q}(k-1)[-2]$.

Which gives the exact sequence:

$$\hom(\mathbb{Q}, \mathbb{Q}(-1)[-1]) \to \mathcal{E}xt^1 \to \hom(\mathbb{Q}, \prod_{k \geq 0} \mathbb{Q}(k-1)) \to \hom(\mathbb{Q}, \mathbb{Q}(-1))$$

It is clear that the first and the last groups are zero. Thus we get our identification:

$$\mathcal{E}xt^1 = \hom(\mathbb{Q}(0), \prod_{k \geq 0} \mathbb{Q}(k-1)) = \hom(\mathbb{Q}(0), \mathbb{Q}(0)) = \mathbb{Q} \tag{1}$$

3. Compatibility with the realizations

We concentrate here on the Hodge realization: the ℓ-adic case is relatively easier. We assume that we have a realization functor from $\mathbf{DM}(X)$ to the category of Saito's mixed Hodge modules $\mathbf{MHM}(X)$ over a $\mathbb{C}$-scheme X[3]. This realization functor should be compatible with the six operations. On the other hand the computation carried out in the previous section can also be done in the context of mixed Hodge variations. In particular we get an element $pol'_{\mathcal{H}}$ in :

$$\mathcal{E}xt^1_{\mathbf{MHM}(\mathbb{U})}(\mathbb{Q}(0), (\mathcal{L}og)_{|\mathbb{U}}) = Q$$

and the realization of $pol_{\mathcal{M}}$ is exactly $pol'_{\mathcal{H}}$. So in order to prove that the Hodge realization of $pol_{\mathcal{M}}$ gives the classical polylogaritmic variation of mixed Hodge structure, we have to identify the class of the extension $pol_{\mathcal{H}}$ with the class of 1 (under the identification 1). To do this we recall that $pol_{\mathcal{H}}$ was associated to the following pro-matrix (see [BD2]):

$$\begin{pmatrix} 1 & 0 & 0 & \cdot\cdot \\ -\mathrm{li}_1 & 2\pi\mathbf{i} & 0 & \cdot\cdot \\ -\mathrm{li}_2 & (2\pi\mathbf{i})\log & (2\pi\mathbf{i})^2 & \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Denoting again $pol_{\mathcal{H}}$ the class of this extension in $\mathcal{E}xt^1_{\mathbf{MHV}(\mathbb{U})}(\mathbb{Q}(0), (\mathcal{L}og)_{|\mathbb{U}})$ and using the injectivity of the map: $\mathcal{E}xt^1_{\mathbb{U}}(\mathbb{Q}(0), (\mathcal{L}og)_{|\mathbb{U}}) \longrightarrow \mathcal{E}xt^1_{\mathbb{U}}(\mathbb{Q}(0), \mathcal{K}_{|\mathbb{U}})$, one see that it suffices to prove that the image of $pol_{\mathcal{H}}$ coïncide with the image of the Kummer torsor over $\mathbb{A}^1_{\mathbb{Q}} - \{1\}$ under: $\mathcal{E}xt^1_{\mathbb{U}}(\mathbb{Q}(0), \mathbb{Q}(1)) \longrightarrow \mathcal{E}xt^1_{\mathbb{U}}(\mathbb{Q}(0), \mathcal{K}_{|\mathbb{U}})$. This means that we have to show that the 2×2-sub-matrix:

$$\begin{pmatrix} 1 & 0 \\ -\mathrm{li}_1 & 2\pi\mathbf{i} \end{pmatrix}$$

define the expected Kummer torsor. This is obvious.

[3]Such a realisation functor has not been constructed yet!

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