This talk is based on our joint paper [1] with L. Barbieri-Viale. We fix a ground field \( k \) which we assume, for simplicity, to be of characteristic zero. Also for simplicity, we will work with rational coefficients. In the sequel, motivic sheaf is a shorthand for homotopy invariant sheaf with transfers [3], i.e., a motivic sheaf \( \mathcal{F} \) is an additive contravariant functor from the category of smooth correspondences \( \text{Cor}(k) \) (see [3, Def. 1.5]) to the category of \( \mathbb{Q} \)-vector spaces such that:

(a) for every smooth \( k \)-scheme \( X \), \( \mathcal{F}(X) \to \mathcal{F}(\mathbb{A}^1_X) \) is invertible.
(b) the restriction of \( \mathcal{F} \) to the category \( \text{Sm}/k \) of smooth \( k \)-schemes is a Nisnevich (or equivalently, an étale) sheaf with transfers.

If \( \mathcal{F} \) satisfy (b) but not necessarily (a), we call it a sheaf with transfers. The category of sheaves with transfers will be denoted by \( \text{Str}(k) \). We denote \( \text{HI}(k) \) its full subcategory of motivic sheaves. The obvious inclusion admits a left adjoint \( h^0 : \text{Str}(k) \to \text{HI}(k) \). It follows from [3, Th. 22.3] that \( h^0 \) is the given by the Nisnevich sheaf of the associated homotopy invariant presheaf with transfers. In particular, \( \text{HI}(k) \) is an abelian category and the inclusion \( \text{HI}(k) \to \text{Str}(k) \) is exact.

In fact, there is a natural \( t \)-structure on Voevodsky’s category \( \text{DM}_{\text{eff}}(k) \) whose heart is canonically equivalent to \( \text{HI}(k) \). This gives a hint why motivic sheaves are important objects to study. Important examples include the following.

**Example 1:** Let \( X \) be a smooth \( k \)-scheme. We denote by \( \widetilde{\text{CH}}^p(X) \) the sheaf associated to the presheaf \( U \mapsto \text{CH}^p(U \times_k X) \). This is a motivic sheaf.

We recall the notion of an \( n \)-motivic sheaf from [1]. Fix an integer \( n \in \mathbb{N} \) and let \( \text{Cor}(k_{\leq n}) \subset \text{Cor}(k) \) be the full subcategory whose objects are the smooth \( k \)-schemes of dimension less than \( n \). Let \( \text{Str}(k_{\leq n}) \) be the category of contravariant functors from \( \text{Cor}(k_{\leq n}) \) to the category of \( \mathbb{Q} \)-vector spaces. There is an obvious restriction functor \( \sigma_n^* : \text{Str}(k) \to \text{Str}(k_{\leq n}) \) which has a left adjoint \( \sigma_n^* \).

**Definition 2:** An object \( \mathcal{F} \in \text{HI}(k) \) is an \( n \)-motivic sheaf if the obvious morphism

\[
h_0 \sigma_n^* \sigma_n^* \mathcal{F} \to \mathcal{F}
\]

is invertible. We denote by \( \text{HI}_{\leq n}(k) \subset \text{HI}(k) \) the full subcategory of \( n \)-motivic sheaves.

It is formal to prove that \( \text{HI}_{\leq n}(k) \) is an abelian category. Given a morphism of \( n \)-motivic sheaves \( a : \mathcal{F} \to \mathcal{G} \), \( \text{coker}(a) \) is again an \( n \)-motivic sheaf and gives the cokernel of \( a \) in \( \text{HI}_{\leq n}(k) \). In other words, the inclusion \( \text{HI}_{\leq n}(k) \to \text{HI}(k) \) is right exact. Unfortunately, it is an open problem whether or not this inclusion is left exact. In other words, we don’t know that \( \text{ker}(a) \) is \( n \)-motivic, and the kernel of \( a \) in \( \text{HI}(k) \) is a priori given by \( h_0 \sigma_n^* \sigma_n^* \text{ker}(a) \). In fact, we conjecture much more than the left exactness of the inclusion \( \text{HI}_{\leq n}(k) \to \text{HI}(k) \), namely:

**Conjecture 3:** There is a functor \( (-)^{\leq n} : \text{HI}(k) \to \text{HI}_{\leq n}(k) \) which is a left adjoint to the obvious inclusion.


Unfortunately, the previous conjecture seems out of reach for $n \geq 2$. When $n = 0$ or $n = 1$, the situation is much easier and the functors $(-)_{\leq n}$ exist and are denoted respectively by $\pi_0$ and Alb. One can even write formulas:

$$\pi_0(F) = \colim_{X \to F} \mathbb{Q}_{tr}(\pi_0(X)) \quad \text{and} \quad \Alb(F) = \colim_{X \to F} \Alb(X)$$

where $\pi_0(X)$ is the étale $k$-scheme of connected components of $X$ and $\Alb(X)$ is the Albanese scheme of $X$ considered as a sheaf with transfers.

**Example 4:** Assume that $k$ is algebraically closed and let $X$ be a smooth $k$-scheme. Then one can prove that $\pi_0(\tilde{CH}^p(X))$ is the constant sheaf with value $\text{NS}_p(X)$, the Neron-Severi group of codimension $p$-cycles up to algebraic equivalence.

We also address a (hopefully easy) conjecture.

**Conjecture 5:** Let $X$ be a complex algebraic variety. Then $\Alb(\tilde{CH}^p(X))(\mathbb{C})$ is canonically isomorphic to target of Walter’s morphic Abel-Jacoby map (see [2]).

In fact, $\pi_0$ and $\Alb$ are defined on the hole category $\text{Str}(k)$ by the same formulas.

Moreover, these two functors pass to the $\mathbb{A}^1$-localization yielding two functors

$$L\pi_0 : \mathcal{D}(\text{Str}(k)) \to \mathcal{D}(\mathcal{H}^{\leq 0}(k)) \quad \text{and} \quad L\Alb : \mathcal{D}(\text{Str}(k)) \to \mathcal{D}(\mathcal{H}^{\leq 1}(k)).$$

which are left adjoint to the obvious inclusions.

We now give two applications. The first one gives an extension of the classical Neron-Severi groups to a bigraded cohomology theory.

**Definition 6:** Let $X$ be a smooth $k$-scheme. We set

$$\text{NS}_p(X,q) = L_q\pi_0(\text{Hom}(X, \mathbb{Q}(p)[2p]))(k).$$

Then, $\text{NS}_p(X,0)$ is the classical Neron-Severi group $\text{NS}_p(X)$ and we have a canonical morphism from Bloch’s higher Chow groups:

$$\text{CH}^p(X, q) \to \text{NS}_p(X, q).$$

Except for $q = 0$, we do not expect this map to be surjective in general.

As a second application, we propose a definition of 2-motives.

**Definition 7:** A 2-motive is an object $M \in \mathcal{D}M_{\text{eff}}(k)$ satisfying the following properties.

(a) $h_i(M) = 0$ for $i \notin \{0, -1, -2\}$.
(b) $h_0(M)$ is a 0-motivic sheaf.
(c) $h_{-1}(M)$ is a 1-motivic sheaf.
(d) $h_{-2}(M)$ is a 1-connected 2-motivic sheaf.
(e) $M[1]$ doesn’t contain a non-zero direct summand which is a 0-motivic sheaf.
REFERENCES

