n-Motivic Sheaves

Joseph Ayoub

This talk is based on our joint paper [1] with L. Barbieri-Viale. We fix a ground field k which we assume, for simplicity, to be of characteristic zero. Also for simplicity, we will work with rational coefficients. In the sequel, *motivic sheaf* is a shorthand for homotopy invariant sheaf with transfers [3], i.e., a motivic sheaf $\mathcal{F}$ is an additive contravariant functor from the category of smooth correspondences $\mathbf{Cor}(k)$ (see [3, Def. 1.5]) to the category of $\mathbb{Q}$-vector spaces such that:

(a) for every smooth k-scheme X, $\mathcal{F}(X) \to \mathcal{F}(\mathbb{A}^1_X)$ is invertible.

(b) the restriction of $\mathcal{F}$ to the category Sm/k of smooth k-schemes is a Nisnevich (or equivalently, an étale) sheaf with transfers.

If $\mathcal{F}$ satisfy (b) but not necessarily (a), we call it a sheaf with transfers. The category of sheaves with transfers will be denoted by $Str(k)$. We denote $\mathbf{HI}(k)$ its full subcategory of motivic sheaves. The obvious inclusion admits a left adjoint $h_0 : Str(k) \to \mathbf{HI}(k)$. It follows from [3, Th. 22.3] that h_0 is the given by the Nisnevich sheaf of the associated homotopy invariant presheaf with transfers. In particular, $\mathbf{HI}(k)$ is an abelian category and the inclusion $\mathbf{HI}(k) \hookrightarrow Str(k)$ is exact. In fact, there is a natural t-structure on Voevodsky's category $\mathbf{DM}_{\text{eff}}(k)$ whose heart is canonically equivalent to $\mathbf{HI}(k)$. This gives a hint why motivic sheaves are important objects to study. Important examples include the following.

Example 1: Let X be a smooth k-scheme. We denote by $\widetilde{\mathrm{CH}}^p(X)$ the sheaf associated to the presheaf $U \rightsquigarrow \mathrm{CH}^p(U \times_k X)$. This is a motivic sheaf.

We recall the notion of an n-motivic sheaf from [1]. Fix an integer $n \in \mathbb{N}$ and let $\mathbf{Cor}(k_{\leq n}) \subset \mathbf{Cor}(k)$ be the full subcategory whose objects are the smooth k-schemes of dimension less than n. Let $Str(k_{\leq n})$ be the category of contravariant functors from $\mathbf{Cor}(k_{\leq n})$ to the category of $\mathbb{Q}$-vector spaces. There is an obvious restriction functor $\sigma_{n*} : Str(k) \to Str(k_{\leq n})$ which has a left adjoint σ_n^*.

Definition 2: *An object $\mathcal{F} \in \mathbf{HI}(k)$ is an n-*motivic *sheaf if the obvious morphism*

$$h_0\sigma_n^*\sigma_{n*}\mathcal{F} \to \mathcal{F}$$

is invertible. We denote by $\mathbf{HI}_{\leq n}(k) \subset \mathbf{HI}(k)$ the full subcategory of n-motivic sheaves.

It is formal to prove that $\mathbf{HI}_{\leq n}(k)$ is an abelian category. Given a morphism of n-motivic sheaves $a : \mathcal{F} \to \mathcal{G}$, $coker(a)$ is again an n-motivic sheaf and gives the cokernel of a in $\mathbf{HI}_{\leq n}(k)$. In other words, the inclusion $\mathbf{HI}_{\leq n}(k) \hookrightarrow \mathbf{HI}(k)$ is right exact. Unfortunately, it is an open problem whether or not this inclusion is left exact. In other words, we don't know that $ker(a)$ is n-motivic, and the kernel of a in $\mathbf{HI}(k)$ is a priori given by $h_0\sigma_n^*\sigma_{n*}ker(a)$. In fact, we conjecture much more than the left exactness of the inclusion $\mathbf{HI}_{\leq n}(k) \hookrightarrow \mathbf{HI}(k)$, namely:

Conjecture 3: *There is a functor $(-)^{\leq n} : \mathbf{HI}(k) \to \mathbf{HI}_{\leq n}(k)$ which is a left adjoint to the obvious inclusion.*

Unfortunately, the previous conjecture seems out of reach for $n \geq 2$. When $n = 0$ or $n = 1$, the situation is much easier and the functors $(-)^{\leq n}$ exist and are denoted respectively by π_0 and Alb. One can even write formulas:

$$\pi_0(\mathcal{F}) = \operatorname*{colim}_{X \to \mathcal{F}} \mathbb{Q}_{tr}(\pi_0(X)) \quad \text{and} \quad \mathrm{Alb}(\mathcal{F}) = \operatorname*{colim}_{X \to \mathcal{F}} \mathrm{Alb}(X)$$

where $\pi_0(X)$ is the étale k-scheme of connected components of X and $\mathrm{Alb}(X)$ is the Albanese scheme of X considered as a sheaf with transfers.

Example 4: Assume that k is algebraically closed and let X be a smooth k-scheme. Then one can prove that $\pi_0(\widetilde{\mathrm{CH}}^p(X))$ is the constant sheaf with value $\mathrm{NS}^p(X)$, the Neron-Severi group of codimension p-cycles up to algebraic equivalence.

We also address a (hopefully easy) conjecture.

Conjecture 5: *Let X be a complex algebraic variety. Then* $\mathrm{Alb}(\widetilde{\mathrm{CH}}^p(X))(\mathbb{C})$ *is canonically isomorphic to target of Walter's morphic Abel-Jacoby map (see* [2]*).*

In fact, π_0 and Alb are defined on the hole category $Str(k)$ by the same formulas. An important issue is that these functors can be left derived, yielding two functors

$$\mathrm{L}\pi_0 : \mathbf{D}(Str(k)) \to \mathbf{D}(\mathbf{HI}_{\leq 0}(k)) \quad \text{and} \quad \mathrm{LAlb} : \mathbf{D}(Str(k)) \to \mathbf{D}(\mathbf{HI}_{\leq 1}(k)).$$

Moreover, these two functors pass to the $\mathbb{A}^1$-localization yielding two functors

$$\mathrm{L}\pi_0 : \mathbf{DM}_{\mathrm{eff}}(k) \to \mathbf{D}(\mathbf{HI}_{\leq 0}(k)) \quad \text{and} \quad \mathrm{LAlb} : \mathbf{DM}_{\mathrm{eff}}(k) \to \mathbf{D}(\mathbf{HI}_{\leq 1}(k))$$

which are left adjoint to the obvious inclusions.

We now give two applications. The first one gives an extension of the classical Neron-Severi groups to a bigraded cohomology theory.

Definition 6: *Let X be a smooth k-scheme. We set*

$$\mathrm{NS}^p(X, q) = \mathrm{L}_q\pi_0(\underline{\mathrm{Hom}}(X, \mathbb{Q}(p)[2p]))(k).$$

Then, $\mathrm{NS}^p(X, 0)$ is the classical Neron-Severi group $\mathrm{NS}^p(X)$ and we have a canonical morphism from Bloch's higher Chow groups:

$$\mathrm{CH}^p(X, q) \to \mathrm{NS}^p(X, q).$$

Except for $q = 0$, we do not expect this map to be surjective in general.

As a second application, we propose a definition of 2-motives.

Definition 7: *A* 2-motive *is an object $M \in \mathbf{DM}_{\mathrm{eff}}(k)$ satisfying the following properties.*

(a) $h_i(M) = 0$ *for* $i \notin \{0, -1, -2\}$.
(b) $h_0(M)$ *is a* 0*-motivic sheaf.*
(c) $h_{-1}(M)$ *is a* 1*-motivic sheaf.*
(d) $h_{-2}(M)$ *is a* 1*-connected* 2*-motivic sheaf.*
(e) $M[+1]$ *doesn't contains a non-zero direct summand which is a* 0*-motivic sheaf.*

References

[1] J. Ayoub and L. Barbieri-Viale, 1-*Motivic sheaves and the Albanese functor*, Journal of Pure and Applied Algebra **213** (2009), 809–839.

[2] M. Walker, *The morphic Abel-Jacobi map*, Preprint.

[3] C. Mazza, V. Voevodsky and C. Weibel, *Lecture notes on motivic cohomology*, Clay Mathematics Monographs, Volume 2.