The motivic Thom spectrum $MGL$ and the algebraic cobordism $\Omega^*(-)$

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This is a report on a "work in progress" of F. Morel and M. J. Hopkins. Their work is a step toward the identification of the motivically defined theory $MGL^{2*,*}(-)$ with the geometrically defined one $\Omega^*(-)$. Namely, they prove:

**Theorem 1.** For any smooth $k$-variety $X$ the natural graded homomorphism $MGL^{2*,*}(X) \rightarrow \Omega^*(X)$ is surjective.

The plan of the lecture was:

1. Some basic properties of $MGL$.
2. The computation of $MGL^{2*,*}(k)$.
3. Proof of the main theorem.

From now on, the base field $k$ is fixed and all our varieties will be $k$-varieties. For simplicity we shall assume $k$ to be of characteristic zero.

1. **Some basic properties of $MGL$**

In this first part, we transpose from the topological to the motivic context some classical properties of the Thom spectrum. We denote by $T = \mathbb{A}^1/Gm$ one of the motivic spheres. When speaking about spectra, we shall always mean $T$-spectra.

The $\mathbb{A}^1$-homotopy category of spectra is a triangulated category denoted by $\text{SH}(k)$ (cf. Morel [1]).

Let us recall that as in algebraic topology, the motivic Thom spectrum is defined by the collection: $(S^0, Th(\gamma_1), \ldots, Th(\gamma_n), \ldots)$ together with the usual assembly maps. Here $\gamma_n$ is the tautological vector bundle on the infinite Grassmanian of $n$-planes. For a smooth variety $X$, we put $MGL^{p,q}(X) = [X+, MGL^{*} \wedge T^q[p-2q]]$.

**Lemma 2.** $MGL$ is an oriented ring spectrum.

The proof is exactly the same as the classical one. It is based on the identification of $Th(\gamma_1)$ with the pointed space $(\mathbb{P}^\infty, \ast)$.

As a consequence, we can define for a line bundle $L$ on $X$ a first Chern class $c_1(L) \in MGL^{2,1}(X)$ by the composition: $X \xrightarrow{[L]} \mathbb{P}^\infty \xrightarrow{\ast} MGL \wedge T$. Using this, one obtain a projective bundle formula by the usual method, and then the other Chern classes for vector bundles. This can be used to define the Thom classes:

**Definition-Construction 3.** Let $V/X$ be a vector bundle of rank $r$. The Thom class $t(V)$ of $V$ lives in $MGL^{2*r}(Th(V))$. It is defined in the following manner: Recall that one model of $Th(V)$ is $\mathbb{P}(V + 1)/\mathbb{P}(V)$. Thus one have a long exact sequence (which breaks into short ones):

$$
MGL^{*,*}(Th(V)) \longrightarrow MGL^{*,*}(\mathbb{P}(V + 1)) \longrightarrow MGL^{*,*}(\mathbb{P}(V))
$$

We then define $t(V)$ to be the element of the middle group equal to $u^r - c_1 u^{r-1} + \cdots + (-1)^r c_r$ where $c_i$ are such that the image of $t(V)$ became zero in the last
group. The exactness of the sequence give us a unique antecedent of $t(V)$ in the first group. This is the Thom class.

A consequence of this construction is:

**Lemma 4.** $M\ell$ is the universal oriented ring spectrum.

Indeed let $E$ be such a spectrum. The construction above still make sens for $E$. In particular if we take the Thom classes of $\gamma_n$ we get maps: $Th(\gamma_n) \longrightarrow E \wedge T^n$ yielding the unique map of spectra $M\ell \longrightarrow E$. Later on, we shall apply this to $E = H\mathbb{Z}$, the motivic cohomology spectrum, to get the morphism: $M\ell \longrightarrow H\mathbb{Z}$.

The next step of our study is the Thom isomorphism. Let $V/X$ be a vector bundle of rank $r$. Define (as in topology) the reduced diagonal: $Th(V) \longrightarrow Th(V) \wedge X$ yields the unique map of spectra $M\ell \longrightarrow E$. Later on, we shall apply this to $E = H\mathbb{Z}$, the motivic cohomology spectrum, to get the morphism: $M\ell \longrightarrow H\mathbb{Z}$.

**Theorem-Definition 5.** For any oriented ring spectrum $E$, the following composition:

$$E \wedge Th(V) \longrightarrow E \wedge Th(V) \wedge X \longrightarrow E \wedge E \wedge T^r \wedge X \longrightarrow E \wedge T^r \wedge X$$

is an isomorphism. It is called the Thom isomorphism.

Roughly speaking, the above result says that an oriented ring spectrum does not make the difference between the Thom space of a non trivial vector bundle and the Thom space of a trivial one with the same rank. A consequence of that is a natural isomorphism: $E^{*,*}(Th(V)) = E^{*-2r,*,*}(X)$.

We end this section by constructing transfers maps for $M\ell^{2*,*}(\cdot)$. It is sufficient to consider the case of a closed immersion and the projection of a projective space over $X$. The second case follow easily from the projective bundle formula. For a closed immersion we need to use the Thom isomorphism. Indeed, let $i: Y \subset X$ be a closed immersion. We denote by $\nu_i$, $\nu_X$ and $\nu_Y$ the normal bundles of $i$, $X$ and $Y$. Note that $\nu_X$ and $\nu_Y$ are not vector bundles in the usual sens but only virtual one (that is of negative rank). As in topology, we can form the composition in $\text{SH}(k)$:

$$Th(\nu_X) \longrightarrow Th(i^*\nu_X \oplus \nu_i) \longrightarrow Th(\nu_Y)$$

When applying $E^{*,*}$ we get a map in the opposite direction: $E^{*,*}(Th(\nu_Y)) \longrightarrow E^{*,*}(Th(\nu_X))$. Now using the Thom isomorphism, we have the identifications

$$E^{*,*}Th(\nu_Y) \simeq E^{*-2d_Y,*,*+d_Y}(Y) \quad \text{and} \quad E^{*,*}Th(\nu_X) \simeq E^{*-2d_X,*,*+d_X}(X)$$

Where $d_X$ and $d_Y$ are the dimension of $X$ and $Y$. Then denoting $c = d_X - d_Y$ the codimension of $Y$ in $X$, we obtain the wanted transfer map: $E^{*,*}(Y) \longrightarrow E^{*,*,2c,*,*+c}(X)$. As a consequence, $E^{2*,*}(\cdot)$ is an oriented Borel-Moore cohomology theory. In particular using the universality of $\Omega^* (\cdot)$ we get the natural homomorphism in theorem 1.
2. The computation of $\text{MGL}^{2*,*}(k)$

The main step of the proof of theorem 1 is the following proposition:

**Proposition 6.** The canonical homomorphism given by the formal group law:

$L_* \mapsto \text{MGL}^{-2s,-*}(k)$ is an isomorphism.

The injectivity of the above homomorphism is easy: one use for example a complex realization. There is also a purely algebraic proof based on a Quillen trick... The main difficulty is to show the surjectivity. For this one need a difficult lemma:

**Lemma 7.** The canonical morphism of spectra:

$\text{MGL} \mapsto H\mathbb{Z}$ induce an isomorphism:

$\text{MGL}/(x_1, \ldots, x_n, \ldots) \xrightarrow{\sim} H\mathbb{Z}$

Where $x_i$ are generator of the Lazard ring.

Assuming lemma 7, the proof of proposition 6 goes by induction on $\ast$. The point is that for $N > 0$, one have $[T^N, H\mathbb{Z}] = 0$ by Voevodsky cancellation theorem. Then if we apply $[T^N, -]$ to the distinguished triangle:

$\text{MGL}/(x_1, \ldots, x_{N-1}) \wedge T^N \xrightarrow{x_N} \text{MGL}/(x_1, \ldots, x_{N-1}) \rightarrow \text{MGL}/(x_1, \ldots, x_N) \rightarrow$

we get a surjection: $x_N : Z \rightarrow \text{MGL}^{-2N,-N}/(x_1, \ldots, x_{N-1})(k)$. Using the induction hypothesis, one deduce that: $L_N \mapsto \text{MGL}^{-2N,-N}(k)$ is indeed a surjection.

3. Proof of the main theorem

A consequence of proposition 6 is that $\text{MGL}^{2s,*}(-)$ is generically constant. Moreover, we have a weak form of the localization property, namely: given a smooth pair $(Y \subset X)$ with $Y$ of codimension $c$, one have an exact sequence:

$\text{MGL}^{2s-2c,*-c}(Y) \xrightarrow{\sim} \text{MGL}^{2s,*}(X) \xrightarrow{\sim} \text{MGL}^{2s,*}(X - Y)$

These properties suffices to derive a generalized degree formula (see [2], [3]) for $\text{MGL}^{2s,*}(-)$. In particular, this implies that $\text{MGL}^{2s,*}(X)$ is generated as a $\text{MGL}^{2s,-*}(k) = L_\ast$-module by cobordism cycles: $[Z \mapsto X]$ with $Z$ a desingularization of a closed subset of $X$. This clearly implies Theorem 1.

References

http://www.ictp.trieste.it/~pub_off/lectures/vol15.html


http://www.math.neu.edu/~levine/publ/Publ.html

http://www.math.neu.edu/~levine/publ/Publ.html

$^1$The quotient ring spectrum $\text{MGL}/(x_1, \ldots, x_n, \ldots)$ is not so easy to construct. Some serious technical difficulties arise if one try to do this naively. One way to overcome these difficulties is to work in a category of $\text{MGL}$-modules.