MAXIMAL REGULARITY AND QUASILINEAR PARABOLIC BOUNDARY VALUE PROBLEMS

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There is given a sharp existence, uniqueness, and continuity theorem for quasilinear parabolic evolution equations, based on the concept of maximal Sobolev regularity. Its power is illustrated by applications to some model problems which are nonlocal in space and/or time.

Introduction

In this paper we consider quasilinear parabolic evolution equations of the form

$$\dot{u} + A(u)u = f(u) \text{ in } \mathring{J}\,, \qquad u(0) = u^0\,, \tag{1}$$

where $J := J_{T_0} := [0, T_0)$ for some fixed positive T_0. We study (1) under maximal regularity assumptions on the leading part $\partial_t + A(u)$ which, roughly speaking, say that, given any v in a suitable function space $\mathbb{W}(J)$ on J, the linear problem

$$\partial_t w + A(v)w = f(v) \text{ in } \mathring{J}\,, \qquad w(0) = 0$$

has a unique solution $w = w(v) \in \mathbb{W}(J)$. Clearly, a fixed point of the map $v \mapsto w(v)$ is a solution of (1). Fixed point arguments of this type are, of course, omnipresent in the study of evolution equations of type (1). The new feature of our work, which distinguishes it from all previous investigations, is that we assume that $\mathbb{W}(J)$ is a maximal regularity space and that $A(\cdot)$ is only defined on $\mathbb{W}(J)$. In all the works published so far, it is always assumed that the domain of $A(\cdot)$ is a larger space than $\mathbb{W}(J)$ on which the problem is relatively easy to handle. Our approach gives great flexibility in applications. In particular, we can deal with parabolic problems which are nonlocal in time, for example, with retarded equations, where the time lag occurs in the top order coefficients via nonlinear dependence on the solution at an earlier instant.

In the following section we give precise definitions and formulate the main abstract theorem. The rest of this paper is devoted to applications to a variety of nonlinear local and nonlocal parabolic initial boundary value problems, demonstrating the power of our abstract work.

1. Abstract theory

Let E_0 and E_1 be Banach spaces such that $E_1 \overset{d}{\hookrightarrow} E_0$ and suppose that $1 < p < \infty$. Put

$$\mathbb{W}_p^1(J) := \mathbb{W}_p^1\bigl(J, (E_1, E_0)\bigr) := L_p(J, E_1) \cap W_p^1(\mathring{J}, E_0)\,.$$

Then

$$B \in L_\infty\bigl(J, \mathcal{L}(E_1, E_0)\bigr)$$

possesses the property of **maximal L_p regularity** on J with respect to (E_1, E_0) if the map

$$\mathbb{W}_p^1(J) \to L_p(J, E_0) \times E\,, \quad u \mapsto \bigl(\dot{u} + Bu, u(0)\bigr)$$

is a bounded isomorphism, where the overdot denotes the distributional derivative on $\mathring{J}$, and E is the real interpolation space $(E_0, E_1)_{1/p',p}$. Since (e.g., Theorem III.4.10.2 in [1])

$$\mathbb{W}_p^1(J) \hookrightarrow C(\overline{J}, E), \tag{2}$$

$u(0)$ is well defined. The set of all such maps B is denoted by

$$\mathcal{MR}_p(J) := \mathcal{MR}_p\bigl(J, (E_1, E_0)\bigr)\,.$$

We also write $\mathcal{MR} := \mathcal{MR}(E_1, E_0)$ for the set of all $C \in \mathcal{L}(E_1, E_0)$ such that the constant map $t \mapsto C$ belongs to $\mathcal{MR}_p(J)$. Since the latter property is independent of p and the (bounded) interval (e.g., [2]), this notation is justified.

We are interested in the quasilinear evolution equation (1). By a solution on J_T, where $0 < T \leq T_0$, we mean a $u \in \mathbb{W}_{p,\mathrm{loc}}^1(J_T)$ satisfying (1) in the sense of distributions on $\mathring{J}_T$ or, equivalently, a.e. on J_T.

Henceforth, we write $C^{1\text{-}}$ for spaces of locally Lipschitz continuous maps, and $\mathcal{C}^{1\text{-}}$ if the Lipschitz continuity is uniform on bounded subsets of the domain (which is always the case if the latter is finite dimensional).

Due to (2) it is natural to assume that

$$(A, f) \in \mathcal{C}^{1\text{-}}\big(E, \mathcal{L}(E_1, E_0) \times E\big). \tag{3}$$

Indeed, this type of assumption has been used in practically all investigations. In particular, Clément and Li [3] were the first to study (1) — in a concrete setting — by imposing the maximal regularity hypothesis that $A(e) \in \mathcal{MR}$ for each $e \in E$. Recently, Prüss [4] has extended this method to a nonautonomous abstract setting.

An assumption like (3) uses only part of the information contained in the statement: $u \in \mathbb{W}_p^1(J)$. Consequently, it imposes stronger restrictions on (A, f) than the hypothesis that this map be defined on $\mathbb{W}_p^1(J)$, which, after all, is the space in which solutions live.

Considering a map

$$(A, f):\ \mathbb{W}_p^1(J) \to L_\infty\big(J, \mathcal{L}(E_1, E_0)\big) \times L_p(J, E_0)$$

we say that it possesses the Volterra property if, given $u \in \mathbb{W}_p^1(J)$ and $0 < T < T_0$, the restriction of $(A, f)(u)$ to J_T depends on $u \,|\, J_T$ only. Now we can formulate our main result, whose proof is found in [5].

Theorem 1.1. *Suppose that*

- $A \in \mathcal{C}^{1\text{-}}\big(\mathbb{W}_p^1(J), \mathcal{MR}_p(J)\big)$;
- $f - f(0) \in \mathcal{C}^{1\text{-}}\big(\mathbb{W}_p^1(J), L_r(J, E_0)\big)$ *for some* $r \in (p, \infty]$, *and* $f(0) \in L_p(J, E_0)$;
- (A, f) *possesses the Volterra property;*
- $u^0 \in E$.

Then:

- *there exist a maximal* $T^* \in (0, T_0]$ *and a unique solution* u *of* (1) *on* $J^* := J_{T^*}$;
- *the map* $(A, f, u^0) \mapsto u$ *is locally Lipschitz continuous with respect to the natural Fréchet topologies of the spaces occurring above;*
- *if* $u \in \mathbb{W}_p^1(J^*)$, *then* $J^* = J$, *that is,* u *is global.*

The following proposition gives two important sufficient conditions for maximal regularity in the nonautonomous case.

Proposition 1.1.

(i) *If* $B \in C(\overline{J}, \mathcal{MR})$, *then* $B \in \mathcal{MR}_p(J)$.

(ii) *Let*
$V \overset{d}{\hookrightarrow} H \overset{d}{\hookrightarrow} V'$ *be real Hilbert spaces and let* $B \in L_\infty\bigl(J, \mathcal{L}(V,V')\bigr)$ *be such that there exist constants* $\alpha > 0$ *and* $\beta \geq 0$ *with*

$$\bigl\langle v, B(t)v \bigr\rangle + \beta \,\|v\|_H^2 \geq \alpha \,\|v\|_V^2 \,, \qquad a.a.\ t \in J \,, \quad v \in V \,,$$

where $\langle \cdot,\cdot \rangle : V \times V' \to \mathbb{R}$ *is the duality pairing. Then* B *belongs to* $\mathcal{MR}_2\bigl(J, (V,V')\bigr)$.

Proof. (i) has been shown in [6] by constructing an evolution family. A simple direct proof is given in [2].

(ii) is a consequence of the well known Galerkin approach to evolution equations in a variational setting, essentially due to J.-L. Lions (see [5] for details). □

2. Parabolic boundary value problems: weak settings

Throughout the rest of this paper we suppose (at least) that

$$\left.\begin{array}{l} \bullet\ \ \Omega \text{ is a bounded Lipschitz domain in } \mathbb{R}^n, \text{ where } n \geq 2\,; \\ \bullet\ \ \Gamma_0 \text{ is a measurable subset of its boundary } \Gamma\,. \end{array}\right\} \tag{4}$$

We put $\Gamma_1 := \Gamma \backslash \Gamma_0$ and denote by $\chi : \Gamma \to \{0,1\}$ the characteristic function of Γ_1 so that $\Gamma_j = \chi^{-1}(j)$ for $j = 0,1$. The pair (Ω,χ) is said to be **regular** if Γ is a C^2 manifold and χ is continuous. Note that then Γ_0 and Γ_1 are open as well as closed in Γ.

The trace operator for Γ is denoted by γ, and $\nu = (\nu^1,\ldots,\nu^n)$ is the outward unit normal. Moreover,

$$\langle u,v \rangle := \int_\Omega u \cdot v \, dx \,, \quad \langle \varphi,\psi \rangle_\Gamma := \int_\Gamma \varphi \cdot \psi \, d\sigma$$

with $d\sigma$ being the volume measure of Γ, whenever the right hand sides are well defined. We also set $Q := \Omega \times J$ and $\Sigma := \Gamma \times J$, as well as $\Sigma_j := \Gamma_j \times J$ for $j = 0,1$.

For $1 < q < \infty$ we put

$$H_{q,\mathcal{B}}^1 := \bigl\{\, u \in H_q^1(\Omega) \ ;\ (1-\chi)\gamma u = 0 \,\bigr\} = \bigl\{\, u \in H_q^1(\Omega) \ ;\ u \,|\, \Gamma_0 = 0 \,\bigr\}$$

and

$$H_{q,\mathcal{B}}^{-1} := (H_{q',\mathcal{B}}^1)' \,,$$

where the dual space is determined with respect to the duality pairing naturally induced by $\langle \cdot,\cdot \rangle$. If I is a nontrivial interval and $1 < p,q < \infty$, then

$$\mathcal{H}_{p,q}^{1,1}(I) := \mathbb{W}_p^1\bigl(I, (H_{q,\mathcal{B}}^1, H_{q,\mathcal{B}}^{-1})\bigr)$$

and $\mathcal{H}^{1,1}_{p,q} := \mathcal{H}^{1,1}_{p,q}(J)$. Moreover, $\mathcal{H}^{1,1}(I) := \mathcal{H}^{1,1}_{2,2}(I)$ and $H^s_{\mathcal{B}} := H^s_{2,\mathcal{B}}$.

If (Ω,χ) is regular and $1 < r < \infty$, then we put

$$B^s_{q,r,\mathcal{B}} := \begin{cases} \bigl\{\, u \in B^s_{q,r}(\Omega) \ ; \ (1-\chi)\gamma u = 0 \,\bigr\}\,, & 1/q < s < 1\,, \\ B^s_{q,r}(\Omega)\,, & -1+1/q < s < 1/q\,, \\ \bigl(B^{-s}_{q',r'}(\Omega)\bigr)'\,, & -1 < s < -1+1/q\,, \end{cases}$$

where $B^s_{q,r}(\Omega)$ are the usual Besov spaces on Ω and the dual space is again determined by $\langle\cdot,\cdot\rangle$. Interpolation theory implies

$$(H^{-1}_{q,\mathcal{B}}, H^1_{q,\mathcal{B}})_{1/p',p} \doteq B^{1-2/p}_{q,p,\mathcal{B}}\,, \qquad 2/p + 1/q \notin \{1,2\}, \tag{5}$$

where $\doteq$ means: equal except for equivalent norms.

If only assumption (4) is imposed, then

$$(H^{-1}_{\mathcal{B}}, H^1_{\mathcal{B}})_{1/2,2} \doteq L_2(\Omega) \doteq B^0_{2,2}(\Omega) =: B^0_{2,2,\mathcal{B}}. \tag{6}$$

Hence, by the trace theorem (e.g., Theorem III.4.10.2 in [1]),

$$\mathcal{H}^{1,1}_{p,q}(I) \hookrightarrow C_0\bigl(\overline{I}, B^{1-2/p}_{q,p,\mathcal{B}}\bigr)\,, \qquad 2/p + 1/q \notin \{1,2\}, \tag{7}$$

provided (Ω,χ) is regular if $(p,q) \neq (2,2)$, where C_0 is the space of continuous functions vanishing at infinity.

We shall apply Theorem 1.1 to quasilinear initial boundary value problems of the following prototypical form:

$$\left.\begin{aligned} \partial_t u + \boldsymbol{\mathcal{A}}(u)u &= \boldsymbol{f}(u) && \text{in } Q\,, \\ \boldsymbol{\mathcal{B}}(u)u &= \boldsymbol{g}(u) && \text{on } \Sigma\,, \\ u(\cdot,0) &= u^0 && \text{on } \Omega\,, \end{aligned}\right\} \tag{8}$$

where, using the summation convention,

$$\boldsymbol{\mathcal{A}}(u)v := -\partial_j\bigl(\boldsymbol{a}_{jk}(u)\partial_k v + \boldsymbol{a}_j(u)v\bigr) + \boldsymbol{c}_j(u)\partial_j v$$

and

$$\boldsymbol{\mathcal{B}}(u)v := \chi\nu^j\bigl(\boldsymbol{a}_{jk}(u)\partial_k v + \boldsymbol{a}_j(u)v\bigr) + (1-\chi)v\,.$$

Here $\boldsymbol{a}_{jk}$, $\boldsymbol{a}_j$, $\boldsymbol{c}_j$, $\boldsymbol{f}$, and $\boldsymbol{g}$ are real-valued functions whose regularity properties are specified below and the boundary operator $\boldsymbol{\mathcal{B}}(u)$ is to be understood in the sense of traces, of course.

We assume that

- $$\boldsymbol{a}_{jk} = \boldsymbol{a}_{kj} \in \begin{cases} \mathcal{C}^{1\text{-}}_{\mathrm{Volt}}\bigl(\mathcal{H}^{1,1}_{p,q}, C\bigl(\overline{J}, C(\overline{\Omega})\bigr)\bigr)\,, & \text{if } (\Omega,\chi) \text{ is regular}\,, \\ \mathcal{C}^{1\text{-}}_{\mathrm{Volt}}\bigl(\mathcal{H}^{1,1}, L_\infty\bigl(J, L_\infty(\Omega)\bigr)\bigr) & \text{otherwise}\,, \end{cases} \tag{9}$$

and

$$\bullet \quad \boldsymbol{a}_j, \boldsymbol{c}_j \in \mathcal{C}^{1\text{-}}_{\mathrm{Volt}}\big(\mathcal{H}^{1,1}_{p,q}, L_\infty\big(J, L_\infty(\Omega)\big)\big), \tag{10}$$

where $\mathcal{C}^{1\text{-}}_{\mathrm{Volt}}$ means that we consider only those elements in $\mathcal{C}^{1\text{-}}$ which possess the Volterra property. We also assume that

$$\left.\begin{array}{l} \bullet \ \text{there exists } \alpha \in (0,1] \text{ such that} \\ \quad \alpha\,|\xi|^2 \le \boldsymbol{a}_{jk}(u)\xi^j\xi^k \le |\xi|^2/\alpha \\ \quad \text{for } u \in \mathcal{H}^{1,1}_{p,q},\ \xi \in \mathbb{R}^n,\ \text{a.e. in } Q\,. \end{array}\right\} \tag{11}$$

For abbreviation, we define $R(\Gamma)$ by

$$R(\Gamma) := \begin{cases} W_q^{-1/q}(\Gamma)\,, & \text{if } (\Omega,\chi) \text{ is regular}\,, \\ L_2(\Gamma) & \text{otherwise}\,. \end{cases}$$

Finally, we suppose that

$$\left.\begin{array}{l} \bullet \ p < r \le \infty\,; \\ \bullet \ (\boldsymbol{f},\boldsymbol{g}) - (\boldsymbol{f},\boldsymbol{g})(0) \in \mathcal{C}^{1\text{-}}_{\mathrm{Volt}}\big(\mathcal{H}^{1,1}_{p,q}, L_r\big(J, H^{-1}_{q,\mathcal{B}} \times R(\Gamma)\big)\big)\,; \\ \bullet \ (\boldsymbol{f},\boldsymbol{g})(0) \in L_p\big(J, H^{-1}_{q,\mathcal{B}} \times R(\Gamma)\big)\,. \end{array}\right\} \tag{12}$$

With boundary value problem (8) we associate its Dirichlet form $\mathfrak{a}$, defined by

$$\mathfrak{a}(u)(v,w) := \big\langle \partial_j w, \boldsymbol{a}_{jk}(u)\partial_k v + \boldsymbol{a}_j(u)v \big\rangle + \big\langle w, \boldsymbol{c}_j(u)\partial_j v \big\rangle$$

for $u \in \mathcal{H}^{1,1}_{p,q}$ and $(v,w) \in H^1_{q,\mathcal{B}} \times H^1_{q',\mathcal{B}'}$.

Suppose that

$$\left.\begin{array}{l} \bullet \ 2/p + 1/q \notin \{1,2\}\,; \\ \bullet \ \text{either } p = q = 2 \text{ or } (\Omega,\chi) \text{ is regular}\,; \\ \bullet \ u^0 \in B^{1-2/p}_{q,p,\mathcal{B}}\,. \end{array}\right\} \tag{13}$$

By an $L_p(H^1_q)$ **solution** of (8) on J_T, where $0 < T \le T_0$, we mean a $u \in \mathcal{H}^{1,1}_{p,q,\mathrm{loc}}(J_T)$ satisfying $u(0) = u^0$ and

$$\langle v, \dot u\rangle + \mathfrak{a}(u)(u,v) = \big\langle v, \boldsymbol{f}(u)\big\rangle + \big\langle \chi\gamma v, \boldsymbol{g}(u)\big\rangle_\Gamma$$

for each $v \in H^1_{q',\mathcal{B}}$ and a.e. in J_T. An $L_2(H^1_2)$ solution is simply said to be a **weak solution**.

Theorem 2.1. *Let assumptions* (4) *and* (9)–(12) *be satisfied. Then problem* (8) *has a unique maximal $L_p(H^1_q)$ solution u^*. The maximal interval of existence, J^*, is open in J. If $u^* \in \mathcal{H}^{1,1}_{p,q}(J^*)$, then u^* is global, that is, $J^* = J$.*

Proof. Set $(E_1, E_0) := (H^1_{q,\mathcal{B}}, H^{-1}_{q,\mathcal{B}})$. Denote by $A(u) \in \mathcal{L}(E_1, E_0)$ the linear operator induced by the bilinear form $\mathfrak{a}(u)$ for $u \in \mathcal{W}^{1,1}_{p,q}$. It follows from assumptions (9)–(11), Proposition 1.1, and known results concerning maximal regularity for linear parabolic boundary value problems (cf. the Appendix in [7]) that

$$A \in \mathcal{C}^{1\text{-}}_{\mathrm{Volt}}\big(\mathcal{H}^{1,1}_{p,q}, \mathcal{MR}_p\big(J, (E_1, E_0)\big)\big) .$$

Denote by $\gamma' \in \mathcal{L}\big(W_q^{-1/q}(\Gamma), H^{-1}_{q,\mathcal{B}}\big)$ the dual operator of the trace map $\gamma \in \mathcal{L}\big(H^1_{q',\mathcal{B}'}, W^{1/q}_{q'}(\Gamma)\big)$, if (Ω, χ) is regular, and the dual in $\mathcal{L}\big(L_2(\Gamma), H^{-1}_{\mathcal{B}}\big)$ of $\gamma \in \mathcal{L}\big(H^1_{\mathcal{B}}, L_2(\Gamma)\big)$ otherwise. Then $f := \boldsymbol{f} + \gamma'\chi\boldsymbol{g}$ is well defined, and (12) implies $f(0) \in L_p(J, E_0)$ and

$$f - f(0) \in \mathcal{C}^{1\text{-}}_{\mathrm{Volt}}\big(\mathcal{H}^{1,1}_{p,q}, L_r(J, E_0)\big) .$$

Thus, thanks to (5) and (13), the hypotheses of Theorem 1.1 are satisfied. Clearly, u is an $L_p(H^1_q)$ solution of (8) on J_T iff $u \in \mathbb{W}^1_{p,\mathrm{loc}}\big(J_T, (E_1, E_0)\big)$ and u solves (1). Thus the assertion follows from Theorem 1.1. $\square$

Corollary 2.1. *Suppose that*

$$\boldsymbol{a}_{jk}(u^*) \in C\big(\overline{J^*}, C(\overline{\Omega})\big) \ \text{if} \ (p,q) \neq (2,2)$$

and

$$(\boldsymbol{f}, \boldsymbol{g})(u^*) \in L_p\big(J^*, H^{-1}_{q,\mathcal{B}} \times L_\sigma(\Gamma)\big) .$$

Then u^* is global.

Proof. Note that u^* is a solution of the linear problem

$$\dot{u} + A(u^*)u = f(u^*) \text{ in } J^* , \qquad u(0) = u^0 ,$$

where $A(u^*) \in \mathcal{MR}_p\big(J^*, (E_1, E_0)\big)$. Hence maximal regularity implies that $u^* \in \mathbb{W}^1_p\big(J^*, (E_1, E_0)\big)$. Thus the assertion is a consequence of the last assertion of Theorem 1.1. $\square$

We refrain from giving the precise formulation of the continuity assertion in the present setting. Instead we refer to [7]. In that paper the case of strongly coupled systems is treated as well.

3. Model problems

Let M be a σ-compact metric space endowed with a positive Radon measure. Then we denote by $\mathrm{Car}(M\times\mathbb{R}^k,\mathbb{R}^\ell)$ the set of all Carathéodory functions $f : M\times\mathbb{R}^k\to\mathbb{R}^\ell$. We write $f\in\mathrm{Car}^{1\text{-}}(M\times\mathbb{R}^k,\mathbb{R}^\ell)$ if f is a Carathéodory function such that $f(m,\cdot)\in C^{1\text{-}}(\mathbb{R}^k,\mathbb{R}^\ell)$ for a.a. $m\in M$. Moreover, given $\lambda\in[1,\infty)$, we denote by $\mathrm{Car}_{0,\lambda}(M\times\mathbb{R}^k,\mathbb{R}^\ell)$ the set of all $f\in\mathrm{Car}(M\times\mathbb{R}^k,\mathbb{R}^\ell)$ such that $f(\cdot,0)=0$ and there exists a constant κ with

$$|f(\cdot,\xi)-f(\cdot,\eta)|\le\kappa(1+|\xi|^{\lambda-1}+|\eta|^{\lambda-1})\,|\xi-\eta|\,,\qquad \xi,\eta\in\mathbb{R}^k\,.$$

Finally, we write $f\in C^{0,1\text{-}}(M\times\mathbb{R}^k,\mathbb{R}^\ell)$ if $f\in C(M\times\mathbb{R}^k,\mathbb{R}^\ell)$ and $f(m,\cdot)\in C^{1\text{-}}(\mathbb{R}^k,\mathbb{R}^\ell)$, uniformly with respect to $m\in M$.

To indicate the scope of Theorem 2.1 we consider now some model problems. We restrict ourselves to rather simple settings and leave it to the interested reader to deduce more general results from that theorem.

First we assume that

$$\left.\begin{array}{l}\bullet\quad \text{assumption (4) holds;}\\ \bullet\quad a=a^\top\in\mathrm{Car}^{1\text{-}}(\Omega\times\mathbb{R},\mathbb{R}^{n\times n})\,;\\ \bullet\quad \text{there exists } \alpha\in(0,1] \text{ such that}\\ \qquad\qquad \alpha\,|\xi|^2\le\xi\cdot a(x,\eta)\xi\le|\xi|^2/\alpha\\ \qquad \text{for a.a. } x\in\Omega \text{ and all } (\xi,\eta)\in\mathbb{R}^n\times\mathbb{R}\,.\end{array}\right\}\tag{14}$$

We also assume that

$$\left.\begin{array}{l}\bullet\quad \Omega' \text{ is a measurable subset of } \Omega\,;\\ \bullet\quad 1\le\lambda<1+4/n \text{ and } f\in\mathrm{Car}_{0,\lambda}(Q\times\mathbb{R})\,;\\ \bullet\quad g\in\mathrm{Car}_{0,1}(\Sigma\times\mathbb{R})\,;\\ \bullet\quad (f_0,g_0)\in L_2(Q)\times L_2(\Sigma)\,.\end{array}\right\}\tag{15}$$

Then we consider the nonlocal parabolic problem

$$\left.\begin{aligned}\partial_t u-\nabla\cdot\Bigl(a\bigl(\cdot,\textstyle\int_{\Omega'}u(x,\cdot)\,dx\bigr)\nabla u\Bigr)&=f(\cdot,u)+f_0 &&\text{in } Q\,,\\ u&=0 &&\text{on } \Sigma_0\,,\\ \nu\cdot a\bigl(\cdot,\textstyle\int_{\Omega'}u(x,\cdot)\,dx\bigr)\nabla u&=g(\cdot,u)+g_0 &&\text{on } \Sigma_1\,,\\ u(\cdot,0)&=u^0 &&\text{on } \Omega\,.\end{aligned}\right\}\tag{16}$$

Theorem 3.1. *Problem* (16) *has for each $u^0\in L_2(\Omega)$ a unique maximal weak solution u^*. If f^0, g^0, and u^0 are positive, then so is u^*. If f and g are decreasing in the last variable, then u^* is global.*

Proof. Taking into account (7) and appropriate embedding theorems of the form

$$\mathcal{H}^{1,1}(I) \hookrightarrow L_r\big(J, L_s(\Omega)\big)\,,$$

which are deduced from Theorem 3 in [8], we infer from (14) and (15) that conditions (9), (11), and (12) are satisfied for $p = q = 2$, where

$$\boldsymbol{a}(u) := a\Big(\cdot, \int_{\Omega'} u(x,\cdot)\,dx\Big)$$

and $(\boldsymbol{f},\boldsymbol{g})(u) := \big(f(\cdot,\cdot,u), g(\cdot,\cdot,\gamma u)\big)$. Hence the first assertion is a consequence of Theorem 2.1. The second one follows from a weak form of the maximum principle. The weak maximum principle also allows us to show that $\boldsymbol{f}(u^*) \in L_p(J^*, H_{\mathcal{B}}^{-1})$ and $\boldsymbol{g}(u^*) \in L_p\big(J^*, L_\sigma(\Gamma)\big)$, provided f and g are decreasing in the last variable. Thus Corollary 2.1 implies the global existence assertion. For details we refer to [7]. $\square$

Of course, we can consider situations where f and g are nonlocal as well and where different entries of the diffusion matrix contain different nonlocal terms.

It should be noted that the model nonlinearities

$$f(\cdot,\cdot,\xi) := a_0\,|\xi|^{\lambda-1}\,\xi\,, \quad g(\cdot,\cdot,\xi) := b\,|\xi|\,, \qquad \xi \in \mathbb{R}\,,$$

with $a_0 \in L_\infty(Q)$ and $b \in L_\infty(\Sigma)$, satisfy (15).

We mention that an application of the results in [4], based on hypothesis (3), would require $\lambda = 1$.

Problems of this type have been intensively studied by M. Chipot and coworkers (cf. [9]–[20]). More precisely, in those papers the differential equations in Ω are either of the form

$$\partial_t u - a\big(\langle v, u\rangle\big)\Delta u + a_0 u = f_0\,,$$

where $v \in L_2(\Omega)$, and the boundary conditions are linear and homogeneous, or the problems are semilinear with nonlocal lower order terms. (The Laplace operator can be replaced by a general second order elliptic operator.) It is crucial that $a\big(\langle v, u(\cdot,t)\rangle\big)$ is a pure function of t, that is, independent of $x \in \Omega$. The proofs, except the ones in [20], rely on Schauder's fixed point theorem and are completely different from our approach. (In all those papers the asymptotic behavior of solutions is investigated as well, a problem we do not touch here.)

Now we strengthen the regularity assumptions on Ω and a by assuming that

$$\left.\begin{array}{l}\bullet\ \ (\Omega,\chi) \text{ is regular};\\ \bullet\ \ a = a^\top \in C^{0,1\text{-}}(\overline{\Omega}\times\mathbb{R},\mathbb{R}^{n\times n});\\ \bullet\ \ \text{the uniform ellipticity condition } (14)_3 \text{ is satisfied}.\end{array}\right\} \tag{17}$$

In addition, we suppose that

$$\left.\begin{array}{l}\bullet\ \ b \in \mathrm{Car}^{1\text{-}}(\Omega\times\mathbb{R},\mathbb{R}^{n\times n});\\ \bullet\ \ k \in L_\rho(\mathbb{R}^+,\mathbb{R}) \text{ for some } \rho > 1;\\ \bullet\ \ \mu \text{ is a bounded Radon measure on } \mathbb{R}^+;\\ \bullet\ \ f \in \mathrm{Car}^{1\text{-}}(Q\times\mathbb{R},\mathbb{R}) \text{ with } f(\cdot,0)=0;\\ \bullet\ \ g \in \mathrm{Car}^{1\text{-}}(\Sigma\times\mathbb{R},\mathbb{R}) \text{ with } g(\cdot,0)=0.\end{array}\right\} \tag{18}$$

Then we consider the quasilinear parabolic problem with memory:

$$\left.\begin{aligned}\partial_t u - \nabla\cdot\big(a(\cdot,\mu*u)\nabla u\big)&\\ + k*\big(\nabla\cdot b(\cdot,u)\nabla u\big) &= f(\cdot,u)+f_0 &&\text{in } Q,\\ u &= 0 &&\text{on } \Sigma_0,\\ \nu\cdot a(\cdot,\mu*u)\nabla u &= g(\cdot,u)+g_0 &&\text{on } \Sigma_1,\\ u &= \overline{u} &&\text{on } \Omega\times(-\mathbb{R}^+),\end{aligned}\right\} \tag{19}$$

where, given $u \in \mathcal{W}^{1,1}_{p,q}(J_T)$, convolution $*$ is understood in the following sense:

$$\mu*u(t) := \int_{\mathbb{R}^+} \widetilde{u}(t-\tau)\,\mu(d\tau), \qquad t\in J_T, \tag{20}$$

with $\widetilde{u}\,|\,J_T := u$ and $\widetilde{u}\,|\,(-\mathbb{R}^+) := \overline{u}$. Similarly,

$$k*\big(\nabla\cdot b(\cdot,u)\nabla u\big) := \int_{\mathbb{R}^+} k(\tau)\nabla\cdot\Big(b\big(\cdot,\widetilde{u}(t-\tau)\big)\nabla\widetilde{u}(t-\tau)\Big)\,d\tau.$$

In particular, we can choose for μ the Dirac measure supported at $\{r\}$ for some $r>0$. Then

$$a\big(\cdot,\mu*u(t)\big) = a\big(\cdot,u(t-r)\big)$$

so that we obtain a quasilinear retarded problem with memory (if $k\neq 0$). If $r=0$, then the leading second order term is local. Also note that we may choose $b := a$. Moreover, it suffices if $\overline{u}$ is prescribed on the interval $(-S,0]$ only if $\mathrm{supp}(\mu)\cup\mathrm{supp}(k)\subset[0,S)$.

Theorem 3.2. *Let assumptions* (17) *and* (18) *be satisfied and suppose that*

$$2/p + n/q < 1. \tag{21}$$

Then problem (19) *has for each* $(f_0, g_0) \in L_p\big(J, H^{-1}_{q,\mathcal{B}} \times W_q^{-1/q}(\Gamma)\big)$ *and each*

$$\overline{u} \in \mathbb{W}^1_p\left(-\mathbb{R}^+, (H^1_{q,\mathcal{B}}, H^{-1}_{q,\mathcal{B}})\right)$$

a unique maximal $L_p(H^1_q)$ *solution* u^*, *that is,*

$$u^* \in \mathbb{W}^1_{p,\mathrm{loc}}\big((-\infty, T^*), (H^1_{q,\mathcal{B}}, H^{-1}_{q,\mathcal{B}})\big)\,.$$

If $\operatorname{supp}\mu \subset [r, S)$ *for some* $r \in (0, S)$, *then the unique maximal* $L_p(H^1_q)$ *solution of*

$$\left.\begin{aligned} \partial_t u - \nabla\cdot\big(a(\cdot, \mu * u)\nabla u\big) &= f_0 && \text{in } Q\,, \\ u &= 0 && \text{on } \Sigma_0\,, \\ \nu\cdot a(\cdot, \mu * u)\nabla u &= g_0 && \text{on } \Sigma_1\,, \\ u &= \overline{u} && \text{on } \Omega\times(-S, 0]\,, \end{aligned}\right\} \tag{22}$$

is global.

Proof. Define $\widetilde{u}$ as in (20). Then, given $u \in \mathcal{W}^{1,1}_{p,q}(J_T)$ for $0 < T \le T_0$ and putting $I_T := (-\mathbb{R}^+) \cup J_T$, one verifies that $\widetilde{u} \in \mathcal{W}^{1,1}_{p,q}(I_T)$. It follows from (7), (21), and a Sobolev type embedding theorem that

$$\mathcal{W}^{1,1}_{p,q}(I_T) \hookrightarrow C_0\big(\overline{I}_T, C(\overline{\Omega})\big). \tag{23}$$

Thus, setting $\boldsymbol{a}(u) := a(\cdot, \mu * u)$, well known properties of convolutions and assumption (17) imply the validity of conditions (9) and (11). Similarly, we deduce from (18) that, setting

$$\boldsymbol{f}(u) := -k * \nabla\cdot\big(b(\cdot, u)\nabla u\big) + f(\cdot, u) + f_0$$

and $\boldsymbol{g}(u) := g(\cdot, u) + g_0$, that $(\boldsymbol{f}, \boldsymbol{g})$ satisfies (12). Thus the first assertion follows from Theorem 2.1. If $(\boldsymbol{f}, \boldsymbol{g}) = (f_0, g_0)$, that is, in case of problem (22), it is an easy consequence of (23) that Corollary 2.1 is applicable to give global existence. □

It should be remarked that problems like (19) cannot be treated at all by theorems invoking hypotheses of type (3).

There is a large literature on parabolic equations involving delays and memory terms. However, most of it concerns semilinear equations (e.g., [21], [22]). Very little seems to be known about an L_p theory for quasilinear equations with memory terms in the top order part (see [23] and the references therein, and [24]). In fact, we do not know of any result for quasilinear equations in which (nondistributed) delay terms occur within the diffusion matrix.

4. Parabolic boundary value problems: strong settings

Throughout this section we suppose that

$$\bullet \quad (\Omega,\chi) \text{ is regular}. \tag{24}$$

We set

$$\mathcal{B}u := \chi\partial_\nu u + (1-\chi)\gamma u$$

and

$$H^2_{q,\mathcal{B}} := \bigl\{\, u \in H^2_q(\Omega)\ ;\ \mathcal{B}u = 0 \,\bigr\},$$

as well as

$$B^s_{q,p,\mathcal{B}} := \begin{cases} \bigl\{\, u \in B^s_{q,p}(\Omega)\ ;\ \mathcal{B}u = 0 \,\bigr\}, & 1+1/q < s < 2, \\ \bigl\{\, u \in B^s_{q,p}(\Omega)\ ;\ (1-\chi)\gamma u = 0 \,\bigr\}, & 1/q < s < 1+1/q, \\ \quad B^s_{q,p}, & 0 < s < 1/q \end{cases}$$

for $1 < p, q < \infty$. It is known that

$$(L_q, H^2_{q,\mathcal{B}})_{1/p',p} \doteq B^{2-2/p}_{q,p,\mathcal{B}}, \qquad \frac{2}{p} + \frac{1}{q} \ne \{1,2\}, \tag{25}$$

and $B^{1/2}_{2,2,\mathcal{B}} = H^1_{\mathcal{B}}$.

We also put

$$\mathcal{H}^{1,2}_{p,q} := \mathbb{W}^1_p\bigl(I, (H^2_{q,\mathcal{B}}, L_q)\bigr), \quad \mathcal{H}^{1,2}_{p,q} := \mathcal{H}^{1,2}_{p,q}(J).$$

Then we assume that

$$\left.\begin{array}{l} \bullet\quad 1 < p, q < \infty,\ 2/p + 1/q \notin \{1,2\}; \\ \bullet\quad \boldsymbol{a}_{jk} = \boldsymbol{a}_{kj} \in \mathcal{C}^{1\text{-}}_{\mathrm{Volt}}\bigl(\mathcal{H}^{1,2}_{p,q}, C\bigl(\overline{J}, C(\overline{\Omega})\bigr)\bigr),\ 1 \le j,k \le n; \\ \bullet\quad \text{condition (11) is satisfied for } u \in \mathcal{H}^{1,2}_{p,q}, \end{array}\right\} \tag{26}$$

and put

$$\boldsymbol{\mathcal{A}}(u)v := -\boldsymbol{a}_{jk}(u)\partial_j\partial_k v.$$

Lastly, we assume that

$$\left.\begin{array}{l} \bullet\quad \boldsymbol{f} - \boldsymbol{f}(0) \in \mathcal{C}^{1\text{-}}_{\mathrm{Volt}}\bigl(\mathcal{H}^{1,2}_{p,q}, L_r(J, L_q)\bigr) \text{ for some } r \in (p,\infty]; \\ \bullet\quad \boldsymbol{f}(0) \in L_p(J, L_q); \\ \bullet\quad u^0 \in B^{2-2/p}_{q,p,\mathcal{B}}. \end{array}\right\} \tag{27}$$

Then we consider the parabolic initial boundary value problem

$$\left.\begin{aligned} \partial_t u + \boldsymbol{\mathcal{A}}(u)u &= \boldsymbol{f}(u) && \text{in } Q\,, \\ \mathcal{B}u &= 0 && \text{on } \Gamma\,, \\ u(\cdot,0) &= u^0 && \text{on } \Omega\,. \end{aligned}\right\} \tag{28}$$

By an $L_p(H_q^2)$ **solution** of it on J_T, where $0 < T \le T_0$, we mean a function $u \in \mathcal{W}^{1,2}_{p,q,\mathrm{loc}}(J_T)$ satisfying (28) pointwise a.e.

Theorem 4.1. *Let assumptions* (24), (26), *and* (27) *be satisfied. Then problem* (28) *has a unique maximal $L_p(H_q^2)$ solution u^*. The maximal interval of existence, J^*, is open in J. If $u^* \in \mathcal{H}^{1,2}_{p,q}(J^*)$, then u^* is global, that is, $J = J^*$.*

Proof. Set $(E_1, E_0) := (H^2_{q,\mathcal{B}}, L_q)$ and define $A(v)$ by $A(v)u := \mathcal{A}(v)u$ for $u, v \in \mathcal{H}^{1,2}_{p,q}$. It follows from assumption (26), Theorem 8.2 in [25], and Proposition 1.1(i) that

$$A \in \mathcal{C}^{1\text{-}}_{\mathrm{Volt}}\big(\mathcal{W}^{1,2}_{p,q}, \mathcal{MR}_p\big(J, (E_1, E_0)\big)\big)\,.$$

Due to (25) and assumption (27) the assertion now follows from Theorem 1.1. □

In order to illustrate the power of this theorem we consider two simple model cases. For this we suppose that

$$\left.\begin{array}{l} \bullet\quad a_{jk} = a_{kj} \in C^{0,1\text{-}}(\overline{\Omega} \times \mathbb{R}, \mathbb{R})\,; \\ \bullet\quad \text{the uniform ellipticity condition } (14)_3 \text{ is satisfied} \\ \qquad \text{for } a := [a_{jk}]\,. \end{array}\right\} \tag{29}$$

We consider a nonlocal problem where the diffusion matrix and the nonlinearity depend on the gradient of the solution.

Theorem 4.2. *Let assumptions* (24) *and* (29) *be satisfied. Suppose that Ω' and Ω'' are measurable subsets of Ω, that $1 \le \lambda < n/(n-2)$, and that $f_1 \in \mathrm{Car}_{0,\lambda}(Q \times \mathbb{R}, \mathbb{R})$ and $f_2 \in \mathrm{Car}^{1\text{-}}(Q \times \mathbb{R}, \mathbb{R})$. Define the map $f : Q \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by*

$$f(x,t,\xi,\eta) := f_1(x,t,\xi) + f_2(x,t,\eta)\,, \qquad (x,t) \in Q\,, \quad \xi, \eta \in \mathbb{R}. \tag{30}$$

Then, given $f_0 \in L_2(Q)$ *and* $u^0 \in H^1_{\mathcal{B}}$, *there exists a unique maximal* $L_2(H^2)$ *solution of*

$$\begin{aligned} \partial_t u - a_{jk}\bigl(\cdot, \textstyle\int_{\Omega'} |\nabla u|^2\,dx\bigr)\partial_j\partial_k u &= f\bigl(\cdot,\cdot,u,\textstyle\int_{\Omega''} |\nabla u|^2\,dx\bigr) + f_0 && \text{in } Q\,,\\ u &= 0 && \text{on } \Sigma_0\,,\\ \partial_\nu u &= 0 && \text{on } \Sigma_1\,,\\ u(\cdot,0) &= u^0 && \text{on } \Omega\,. \end{aligned}$$

Proof. Define $\boldsymbol{\mathcal{A}}(u)$ and $\boldsymbol{f}(u)$ by

$$\boldsymbol{\mathcal{A}}(u) := -a_{jk}\Bigl(\cdot, \int_{\Omega'} |\nabla u|^2\,dx\Bigr)\partial_j\partial_k$$

and

$$\boldsymbol{f}(u) := f\Bigl(\cdot,\cdot,u,\int_{\Omega''} |\nabla u|^2\,dx\Bigr) + f_0$$

for $u \in \mathcal{H}^{1,2}$. It follows from (2), (25), and Sobolev's embedding theorem that

$$\mathcal{H}^{1,2} \hookrightarrow C\bigl(\overline{J}, H^1\bigr) \hookrightarrow C\bigl(\overline{J}, L_{2^*}\bigr)\,,$$

where $2^* := 2n/(n-2)$ if $n > 2$, and $2^* \in (2,\infty)$ if $n = 2$. Using these facts, it is easy to verify that $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{f}$ satisfy the assumptions specified in (26) and (27), respectively. Hence Theorem 4.1 implies the assertion. □

In the following particularly simple case it is easy to prove global existence.

Theorem 4.3. *Suppose that* $a \in C^{1\text{-}}(\mathbb{R},\mathbb{R})$ *and there exists* $\alpha \in (0,1]$ *such that* $\alpha \le a(\xi) \le 1/\alpha$ *for* $\xi \in \mathbb{R}$. *Also suppose that* $f_1, f_2 \in \mathrm{Car}^{1\text{-}}(\Omega \times \mathbb{R},\mathbb{R})$ *are uniformly bounded, and define* f *by* (30). *Then, given* $f_0 \in L_2(\Omega)$ *and* $u^0 \in H^1_{\mathcal{B}}$, *there exists a unique global* $L_2(H^2)$ *solution of*

$$\left.\begin{aligned} \partial_t u - a\bigl(\textstyle\int_{\Omega'} |\nabla u|^2\,dx\bigr)\Delta u &= f\bigl(\cdot,u,\textstyle\int_{\Omega''} |\nabla u|^2\,dx\bigr) + f_0 && \text{in } Q\,,\\ u &= 0 && \text{on } \Sigma_0\,,\\ \partial_\nu u &= 0 && \text{on } \Sigma_1\,,\\ u(\cdot,u^0) &= u^0 && \text{on } \Omega\,. \end{aligned}\right\} \tag{31}$$

Proof. Theorem 4.2 guarantees that this boundary value problem possesses a unique maximal solution $u^* \in \mathcal{H}^{1,1}_{\mathrm{loc}}(J^*)$, where $J^* = [0,T^*)$ with $T^* \in (0,T]$. Set

$$a^*(t) := a\Bigl(\int_{\Omega'} |\nabla u(x,t)|^2\,dx\Bigr)$$

and

$$g^*(t) := \Bigl(f\Bigl(\cdot, u^*(\cdot,t), \int_{\Omega''} |\nabla u^*(x,t)|^2\,dx\Bigr) + f_0\Bigr)\Big/ a^*(t)$$

for $t \in J^*$. Also put

$$\alpha(\tau) := \int_0^\tau \frac{ds}{a^*(s)}\,, \qquad \tau \in J^*\,,$$

and $\tau^* := t^*$ if $\alpha(t^*) \le t^*$, whereas $\tau^* := \alpha^{-1}(t^*)$ otherwise. In addition, set $g(t) := g^*(t)$ for $0 \le t < t^*$, and $g(t) := 0$ if $t^* < t < \tau^*$. Observe that $g \in L_2\bigl((0,\tau^*), L_2\bigr)$. Hence, by maximal regularity, the linear boundary value problem

$$\left.\begin{aligned} \partial_t w - \Delta w &= g \qquad && \text{in } \Omega \times (0,\tau^*)\,, \\ \mathcal{B}w &= 0 && \text{on } \Gamma \times (0,\tau^*)\,, \\ w(\cdot,0) &= u^0 && \text{on } \Omega \end{aligned}\right\} \tag{32}$$

has a unique solution $w \in \mathcal{H}^{1,2}\bigl([0,\tau^*)\bigr)$. Put $v(\tau) := u\bigl(\alpha(\tau)\bigr)$ for $\tau \in J_{\tau^*}$. Then one verifies that v is an $L_2(H^2)$ solution of (32) on $[0,\tau^*)$. Thus, by uniqueness, $w \supset v$. Consequently, $v \in \mathcal{H}^{1,2}\bigl([0,\tau^*)\bigr)$. From this we deduce that $u = v \circ \alpha^{-1} \in \mathcal{H}^{1,2}(J^*)$. Now $J^* = J$ follows from the last part of Theorem 4.1. $\square$

This theorem generalizes the existence and uniqueness result of [26], where (32) is studied — again by different methods — in the case where $f = 0$ and $\Gamma_1 = \emptyset$.

It should be clear from the considerations in the preceding section that, by assuming that $2/p + n/q < 1$, we can treat differential equations of the form

$$\partial_t u - a_{jk}(\cdot, u, \mu * \nabla u)\partial_j \partial_k u = f(\cdot,\cdot,u,\nabla u)\,,$$

for example, subject to homogeneous boundary conditions. We leave this to the interested reader.

References

1. H. Amann. *Linear and Quasilinear Parabolic Problems, Volume I: Abstract Linear Theory*. Birkhäuser, Basel, 1995.
2. H. Amann. Maximal regularity for nonautonomous evolution equations. *Adv. Nonl. Studies*, (2004).
3. Ph. Clément, S. Li. Abstract parabolic quasilinear equations and application to a groundwater flow problem. *Adv. Math. Sci. Appl.*, **3**(Special Issue) (1993/94), 17–32.

4. J. Prüss. Maximal regularity for evolution equations in L_p-spaces. *Conf. Semin. Mat. Univ. Bari*, (285) (2002), 1–39 (2003).
5. H. Amann. Maximal regularity and local, nonlocal, and retarded quasilinear parabolic problems, 2004. In preparation.
6. J. Prüss, R. Schnaubelt. Solvability and maximal regularity of parabolic evolution equations with coefficients continuous in time. *J. Math. Anal. Appl.*, **256**(2) (2001), 405–430.
7. H. Amann. Local, nonlocal, and retarded quasilinear parabolic problems, 2004. In preparation.
8. H. Amann. Linear parabolic problems involving measures. *Rev. R. Acad. Cien. Serie A. Mat. (RACSAM)*, **95** (2001), 85–119.
9. N.H. Chang, M. Chipot. On some model diffusion problems with a nonlocal lower order term. *Chinese Ann. Math. Ser. B*, **24**(2) (2003), 147–166.
10. M. Chipot. *Elements of nonlinear analysis*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, 2000.
11. M. Chipot, B. Lovat. Some remarks on nonlocal elliptic and parabolic problems. *Nonlinear Anal.*, **30**(7) (1997), 4619–4627.
12. M. Chipot, B. Lovat. On the asymptotic behaviour of some nonlocal problems. *Positivity*, **3**(1) (1999), 65–81.
13. M. Chipot, B. Lovat. Existence and uniqueness results for a class of nonlocal elliptic and parabolic problems. *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, **8**(1) (2001), 35–51.
14. M. Chipot, L. Molinet. Asymptotic behaviour of some nonlocal diffusion problems. *Appl. Anal.*, **80**(3-4) (2001), 279–315.
15. M. Chipot, A. Rougirel. On some class of problems with nonlocal source and boundary flux. *Adv. Differential Equations*, **6**(9) (2001), 1025–1048.
16. N.-H. Chang, M. Chipot. On some mixed boundary value problems with nonlocal diffusion. *Adv. Math. Sci. Appl.*, **14** (2004), 1–24.
17. A. Rougirel. Blow-up and convergence results for a one-dimensional non-local parabolic problem. *Z. Anal. Anwendungen*, **20**(1) (2001), 93–113.
18. A. Rougirel. A result of blowup driven by a nonlocal source term. *Nonlinear Anal.*, **47**(1) (2001), 113–122.
19. A. Rougirel. Blow-up rate for parabolic problems with nonlocal source and boundary flux. *Electron. J. Differential Equations*, (2003), No. 98, 18 pp. (electronic).
20. M. Siegwart. *Nonlocal Quasilinear Parabolic Problems*. Doctoral dissertation, Universität Zürich, Switzerland, 2003.
21. J. Wu. *Theory and applications of partial functional-differential equations*, volume 119 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1996.
22. T. Caraballo. Nonlinear partial functional differential equations: existence and stability. *J. Math. Anal. Appl.*, **262**(1) (2001), 87–111.
23. J. Prüss. Quasilinear parabolic Volterra equations in spaces of integrable functions. In *Semigroup theory and evolution equations (Delft, 1989)*, volume 135 of *Lecture Notes in Pure and Appl. Math.*, pages 401–420. Dekker, New

York, 1991.
24. R. Zacher. *Quasilinear parabolic problems with nonlinear boundary conditions*. Doctoral dissertation, Universität Halle-Wittenberg, Germany, 2003.
25. R. Denk, M. Hieber, J. Prüss. $\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type. *Mem. Amer. Math. Soc.*, **166**(788) (2003), viii+114.
26. M. Chipot, V. Valente, G. Vergara Caffarelli. Remarks on a nonlocal problem involving the Dirichlet energy. *Rend. Sem. Mat. Univ. Padova*, **110** (2003), 199–220.