Stability and Bifurcation in Viscous Incompressible Fluids

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1 The Problem

We consider the motion of a viscous incompressible fluid in a bounded smooth domain Ω of $\mathbb{R}^3$. It is governed by the laws of conservation of mass, momentum, and energy given by

$$\begin{aligned} \nabla \cdot \boldsymbol{v} &= 0 \ , \\ \partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} &= -\nabla p + \nabla \cdot \boldsymbol{S} + \boldsymbol{b} \ , \\ c(\partial_t \theta + \boldsymbol{v} \cdot \nabla \theta) &= -\nabla \cdot \boldsymbol{q} + \boldsymbol{S} : \boldsymbol{D} + r \ , \end{aligned} \tag{1.1}$$

respectively (e.g., [24], [27]). Here the velocity vector field $\boldsymbol{v}$, the pressure p, and the absolute temperature θ are the unknowns. The density has been normalized to 1, and $\boldsymbol{S}$ is the viscous part of the stress tensor, $\boldsymbol{b} = \boldsymbol{b}(x,t,\theta)$ is the body force, $c = c(\theta) > 0$ is the heat capacity, $\boldsymbol{q}$ is the heat flux vector,

$$\boldsymbol{D} := \boldsymbol{D}(\boldsymbol{v}) := \frac{1}{2}\left[\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^\top\right]$$

is the rate of deformation tensor, and $r = r(x,t,\theta)$ is the radiant heat. Moreover, $\boldsymbol{S} : \boldsymbol{D} := \mathrm{tr}(\boldsymbol{S}\boldsymbol{D})$, where $\mathrm{tr}(\cdot)$ denotes the trace.

Of course, equations (1.1) have to hold in $\Omega \times (0,\infty)$. As for boundary conditions, we suppose that $\Gamma := \partial\Omega = \Gamma_0 \cup \Gamma_1$, where Γ_0 and Γ_1 are disjoint and relatively open in Γ. Then we impose the conditions

$$\begin{aligned} \boldsymbol{v} &= \boldsymbol{v}_0 \qquad \text{on } \ \Gamma \times (0,\infty) \ , \\ \theta &= \theta_0 \qquad \text{on } \Gamma_0 \times (0,\infty) \ , \\ -\boldsymbol{q} \cdot \boldsymbol{\nu} &= h \qquad \text{on } \Gamma_1 \times (0,\infty) \ , \end{aligned} \tag{1.2}$$

where $\boldsymbol{v}_0 = \boldsymbol{v}_0(y,t)$, $\theta_0 = \theta_0(y,t)$, and $h = h(y,t,\theta)$ with $\boldsymbol{\nu}$ denoting the outer unit normal vector field. This means that we prescribe the usual adherence boundary condition for the velocity vector field, the temperature on Γ_0, and the inward heat flux through Γ_1 (by a possibly nonlinear function of θ). Lastly, we specify initial conditions

$$\boldsymbol{v} = \boldsymbol{v}^0(x) \ , \quad \theta = \theta^0(x) \ , \qquad x \in \Omega \ . \tag{1.3}$$

In order to obtain a meaningful problem we have to impose constitutive relations for $\boldsymbol{S}$ and $\boldsymbol{q}$. As for the stress tensor, we assume that it is a smooth function of $\boldsymbol{D}$ and θ only, and that it is symmetric in order to guarantee the conservation of angular momentum, that is, we assume that

$$\boldsymbol{S} = \boldsymbol{S}^\top = \boldsymbol{S}(\boldsymbol{D}, \theta) \ . \tag{1.4}$$

Thus we consider so-called **Stokesian fluids**.

The simplest class of fluids satisfying (1.4) is the class of **Newtonian fluids** for which $\boldsymbol{S} = 2\nu\boldsymbol{D}$ for some positive constant ν, the kinematic viscosity. In this case the first two equations in (1.1) reduce to the Navier-Stokes equations. In all other cases we are dealing with **non-Newtonian fluids**.

Of course, our system has to satisfy the principle of frame invariance. This implies that

$$\boldsymbol{S} = \alpha_1 \boldsymbol{D} + \alpha_2 \boldsymbol{D}^2 \ , \tag{1.5}$$

where the scalar functions α_j depend on the principal invariants $\mathrm{tr}(\boldsymbol{D})$, $|\boldsymbol{D}|^2 := \boldsymbol{D} : \boldsymbol{D}$, and $\det(\boldsymbol{D})$ of $\boldsymbol{D}$ and on the temperature θ only (e.g., [24], [27]). Since $\mathrm{tr}(\boldsymbol{D}) = \nabla \cdot \boldsymbol{v} = 0$, it follows that

$$\alpha_j = \alpha_j\big(|\boldsymbol{D}|^2, \det(\boldsymbol{D}), \theta\big) \ , \qquad j = 1, 2 \ . \tag{1.6}$$

As for regularity, we assume that

$$\alpha_j \in C^\infty(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+, \mathbb{R}) \ , \qquad j = 1, 2 \ ,$$

for simplicity, where $\mathbb{R}^+ := [0, \infty)$.

For the heat flux vector we impose the constitutive assumption that it depends smoothly on $\boldsymbol{D}$, θ, and $\nabla\theta$ only. Then it is a consequence of the principle of frame invariance that

$$\boldsymbol{q} = -\boldsymbol{Q}\nabla\theta \qquad \text{with} \quad \boldsymbol{Q} = \beta_0 + \beta_1 \boldsymbol{D} + \beta_2 \boldsymbol{D}^2 \ , \tag{1.7}$$

where the scalar functions β_j depend on the principal invariants of $\boldsymbol{D}$, the temperature θ, on $|\nabla\theta|^2$, $\nabla\theta \cdot \boldsymbol{D}\nabla\theta$, and $\nabla\theta \cdot \boldsymbol{D}^2\nabla\theta$ only. For simplicity, we assume that they are independent of $\nabla\theta$, that is, we assume that

$$\beta_j = \beta_j\big(|\boldsymbol{D}|^2, \det(\boldsymbol{D}), \theta\big) \tag{1.8}$$

with

$$\beta_j \in C^\infty(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+, \mathbb{R}) \ , \qquad j = 0, 1, 2 \ .$$

The simplest case in which assumptions (1.7) and (1.8) are satisfied occurs if $\beta_1 = \beta_2 = 0$ and $\beta_0 = \kappa$, where κ is a positive constant. Then (1.7) represents Fourier's law, $\boldsymbol{q} = -\kappa\nabla\theta$, and the third equation in (1.1) reduces to the convective heat equation. Note, however, that it contains the 'energy dissipation' term $\boldsymbol{S} : \boldsymbol{D}$ that accounts for heat production due to viscous motion. This term is omitted in almost all mathematical studies of heat conducting viscous fluids since it is of second order in $\nabla\boldsymbol{v}$. Below we shall see that it is precisely this term which carries important information for the asymptotic behavior of the system (1.1), (1.2).

If we consider Newtonian fluids, use Fourier's law, and drop the term $\boldsymbol{S} : \boldsymbol{D} = 2\nu\,|\boldsymbol{D}|^2$, then our system (1.1), (1.2) reduces to the well-known Boussinesq approximation for a compressible heat-conducting viscous fluid. In this case, that has been extensively studied (e.g., [13]), it is usually assumed, in addition, that $r = 0$ and h is an affine function of θ.

The initial boundary value problem (1.1)–(1.3) together with the constitutive assumptions (1.4) and (1.7), (1.8) is a highly complicated nonlinear system for the unknowns $(\boldsymbol{v}, p, \theta)$. Its complexity is illustrated by the fact that it contains the Navier-Stokes equations as a very particular subsystem. We are interested in the global strong solvability of this system. It is well-known that — in the isothermal case where the third equation in (1.1) and the corresponding boundary conditions are omitted — the Navier-Stokes equations are locally uniquely strongly solvable. But it is an open question whether these solutions exist globally, in general. On the other hand, it is also well-known that the Navier-Stokes equations possess unique global strong solutions for small data $(\boldsymbol{b}, \boldsymbol{v}_0, \boldsymbol{v}^0)$ (e.g., [15]).

In this paper we study the global strong solvability of the full system (1.1)–(1.4) and (1.7), (1.8) for 'small data'. Since we consider the non-isothermal case, we assume that there exists a uniform temperature distribution $\overline{\theta}$ at which the fluid is at rest if there are no external forces and initial disturbances. More precisely, we assume that there exist a positive constant $\overline{\theta}$ and functions

$$(\boldsymbol{b}_1, r_1) \in C^\infty(\overline{\Omega} \times \mathbb{R}^+, \mathbb{R}^3 \times \mathbb{R}) \ , \quad h_1 \in C^\infty(\Gamma_1 \times \mathbb{R}^+, \mathbb{R})$$

such that

$$(\boldsymbol{b}, r)(x, t, \theta) = (\boldsymbol{b}_0, r_0)(x, t) + (\boldsymbol{b}_1, r_1)(x, \theta)(\theta - \overline{\theta}) \tag{1.9}$$

for $(x,t,\theta) \in \overline{\Omega} \times \mathbb{R}^+ \times \mathbb{R}^+$, and

$$h(y,t,\theta) = h_0(y,t) + h_1(y,\theta)(\theta - \overline{\theta}) \tag{1.10}$$

for $(y,t,\theta) \in \Gamma_1 \times \mathbb{R}^+ \times \mathbb{R}^+$. The regularity properties of $\boldsymbol{b}_0$, r_0, and h_0 are described in the next section. It should be noted that in the autonomous case formulas (1.9) and (1.10) are consequences of the mean-value theorem.

To be definite we also require that the specific heat be smooth, that is,

$$c \in C^\infty\big(\overline{\Omega}, (0,\infty)\big) \ .$$

All of our smoothness requirements are imposed for simplicity. It is not difficult to infer from our proofs that they can be considerably relaxed. We leave this point to the interested reader.

Lastly, we assume that

$$\alpha_1(0,0,\overline{\theta}) > 0 \ , \quad \beta_0(0,0,\overline{\theta}) > 0 \ , \tag{1.11}$$

but we do not put any further restriction on $\boldsymbol{S}$ or $\boldsymbol{Q}$. For abbreviation, we set

$$\nu := \alpha_1(0,0,\overline{\theta})/2 \ , \quad \kappa := \beta_0(0,0,\overline{\theta}) \ .$$

It should be noted that (1.11) is consistent with the second law of thermodynamics (e.g., [22]).

2 Global Existence and Stability

For a smooth manifold X, $s \in \mathbb{R}$, and $1 \le q \le \infty$ we denote by

$$W_q^s(X) := W_q^s(X,\mathbb{R}) \ , \quad \boldsymbol{W}_q^s(X) := W_q^s(X,\mathbb{R}^3)$$

the usual Sobolev-Slobodeckii spaces. If $X = \Omega$ then we omit Ω in this notation. Similar conventions apply to other spaces of functions and distributions, e.g., $\boldsymbol{C}^\rho := \boldsymbol{C}^\rho(\overline{\Omega},\mathbb{R}^3)$ for $\rho \in \mathbb{R}^+$, etc. We write τ_j for the trace operators on Γ_j, $j = 0,1$. Recall that

$$\tau_j \in \mathcal{L}\big(W_q^s, W_q^{s-1/q}(\Gamma_j)\big) \tag{2.1}$$

for $1 < q < \infty$ and $s > 1/q$. We use the same symbol in the vector-valued case. Of course, $\mathcal{L}(E,F)$ is the Banach space of all bounded linear operators from the Banach space E to the Banach space F, and $\mathcal{L}(E) := \mathcal{L}(E,E)$. We also fix an extension operator

$$\mathcal{R}_0 \in \mathcal{L}\big(W_q^{s-1/q}(\Gamma_0), W_q^s\big) \ , \qquad s > 1/q \ ,$$

satisfying

$$\tau_0 \mathcal{R}_0 = \mathrm{id} \quad \text{and} \quad \mathcal{R}_0 1 = 1 \ ,$$

where we identify 1 with the function (whose domain may vary from occurrence to occurrence) which is constantly equal to 1 (cf. [1, Theorem B.3]). We write τ for the trace operator on Γ and recall that (2.1) holds for τ if Γ_j is replaced by Γ and W by $\boldsymbol{W}$, respectively.

We put

$$\langle u, v \rangle := \int_\Omega u \cdot v \, dx \ , \quad \langle w, z \rangle_\partial := \int_\Gamma w \cdot z \ d\sigma \tag{2.2}$$

for $u, v : \overline{\Omega} \to \mathbb{R}^N$ and $w, z : \Gamma \to \mathbb{R}^N$, respectively, where it will be clear from the context which value of N has to be taken (usually $N = 1$ or $N = 3$), and where $d\sigma$ denotes the volume measure of $\partial\Omega$.

Throughout the remainder of this paper we fix $q \in (3, \infty)$ arbitrarily and use overdots to denote time-derivatives. We also put, letting $\gamma := \tau$ or $\gamma := \tau_0$,

$$W_{q,\gamma}^s := \begin{cases} \{u \in W_q^s \ ; \ \gamma u = 0\} \ , & 1/q < s < \infty \ , \\ W_q^s \ , & -2 + 1/q < s < 1/q \ , \\ (W_{q',\gamma}^{-s})' \ , & -\infty < s < -2 + 1/q \ , \end{cases} \tag{2.3}$$

where $q' := q/(q-1)$ and the dual space is taken with respect to the standard L_q-duality pairing $\langle \cdot, \cdot \rangle$, defined in (2.2). Of course, W in (2.3) can be replaced by $\boldsymbol{W}$.

Let J be a nontrivial subinterval of $\mathbb{R}^+$ containing 0. By a **solution** of (1.1)–(1.3) on J we mean a function

$$(\boldsymbol{v}, p, \theta) \in C(J, \boldsymbol{W}_q^2 \times W_q^1 \times W_q^1)$$

with

$$(\boldsymbol{v}, \theta - \mathcal{R}_0 \theta_0) \in C^1(J, \boldsymbol{L}_q \times W_{q,\tau_0}^{-1})$$

satsifying the normalization condition

$$\int_\Omega p(\cdot, t) \, dx = 0 \ , \qquad t \in J \ ,$$

the first two equations in (1.1) and (1.2) in the strong sense, and the third equation in (1.1) and (1.2) in the weak L_q-sense, as well as (1.3). By 'weak

L_q-sense' we mean that θ satisfies the identity

$$\begin{aligned}\langle\psi,\dot\theta+\boldsymbol{v}\cdot\nabla\theta\rangle &= \bigl\langle\nabla\psi,\tfrac{1}{c}\boldsymbol{q}\bigr\rangle+\bigl\langle\psi,\bigl(\nabla\tfrac{1}{c}\bigr)\cdot\boldsymbol{q}+\tfrac{1}{c}\boldsymbol{S}:\boldsymbol{D}+\tfrac{1}{c}r\bigr\rangle\\ &\quad+\bigl\langle\tau_1\psi,\tau_1\bigl(\tfrac{1}{c}\bigr)h\bigr\rangle_\partial\end{aligned}\tag{2.4}$$

for $\psi\in W^1_{q',\tau_0}$ and $t\in J$. A solution of (1.1)–(1.3) is **global** if $J=\mathbb{R}^+$.

We denote by σ_0 the least eigenvalue of the Stokes eigenvalue problem

$$\left.\begin{aligned}\nabla\cdot\boldsymbol{v}&=\boldsymbol{0}\\ -\nu\Delta\boldsymbol{v}+\nabla p&=\sigma\boldsymbol{v}\end{aligned}\right\}\quad\text{in }\Omega\ ,$$
$$\boldsymbol{v}=\boldsymbol{0}\qquad\text{on }\Gamma\ .$$

It is well-known that $\sigma_0>0$. We also denote by μ_0 the least eigenvalue of the linear elliptic eigenvalue problem

$$\begin{aligned}-\kappa\Delta w-\overline{r}_1w&=\mu w &&\text{in }\Omega\ ,\\ w&=0 &&\text{on }\Gamma_0\ ,\\ \kappa\partial_\nu w-\overline{h}_1w&=0 &&\text{on }\Gamma_1\ ,\end{aligned}\tag{2.5}$$

where $\overline{r}_1:=r_1(\cdot,\overline{\theta})$ and $\overline{h}_1:=h_1(\cdot,\overline{\theta})$.

2.1 Remark Recall that

$$\mu_0=\min\bigl\{\,I(w)\ ;\ w\in W^1_{2,\gamma_0},\ \textstyle\int_\Omega w^2\,dx=1\,\bigr\}\ ,\tag{2.6}$$

where

$$I(w):=\int_\Omega\bigl[\kappa\,|\nabla w|^2-\overline{r}_1w^2\bigr]\,dx-\int_{\Gamma_1}\overline{h}_1\tau_1w^2\,d\sigma\ .$$

Hence $\overline{r}_1\le 0$, $\overline{h}_1\le 0$, and

$$\mathrm{vol}(\Gamma_0)+\|\overline{r}_1\|_C+\|\overline{h}_1\|_{C(\Gamma_1)}>0$$

is a sufficient condition for $\mu_0>0$, thanks to Poincaré's inequality if $\Gamma_0\ne\emptyset$ and $(\overline{r}_1,\overline{h}_1)=(0,0)$. On the other hand, if $\Gamma_0=\emptyset$ then it follows from (2.6) that

$$I(1):=-\int_\Omega\overline{r}_1\,dx-\int_\Gamma\overline{h}_1\,d\sigma<0$$

guarantees that $\mu_0<0$. ∎

Given a Banach space E and $\alpha \in (0,1)$, we denote by $BUC^\alpha(\mathbb{R}^+, E)$ the Banach space of all bounded and uniformly α-Hölder continuous functions from $\mathbb{R}^+$ to E, endowed with the usual norm, which we denote by $\|\cdot\|_{C^\alpha(\mathbb{R}^+,E)}$, or even by $\|\cdot\|_{C^\alpha}$, if no confusion seems possible. Moreover, we write $e^{\omega t} f$ for the function $t \mapsto e^{\omega t} f(t)$.

After these preparations we can formulate the first main result of this paper, namely the following global existence, uniqueness, and stability result, where $\overline{c} := c(\overline{\theta})$. For abbreviation we put

$$F := \boldsymbol{L}_q \times W_{q,\tau_0}^{-1} \times W_q^{-1}(\Gamma_1) \times \boldsymbol{W}_q^{2-1/q}(\Gamma) \times W_q^{1-1/q}(\Gamma_0)$$

and we fix $\rho \in (0,1)$.

2.2 Theorem *Suppose that* $\mu_0 > 0$ *and fix* $\omega \in \big[0, \sigma_0 \wedge (\mu_0/\overline{c})\big)$. *Then there exist positive constants* K_0 *and* K_1 *such that the following is true: for each set of data*

$$\big((\boldsymbol{b}_0, r_0, h_0, \boldsymbol{v}_0, \theta_0), (\boldsymbol{v}^0, \theta^0)\big) \in BUC^\rho(\mathbb{R}^+, F) \times \boldsymbol{W}_q^2 \times W_q^1$$

satisfying the compatibility conditions

$$\nabla \cdot \boldsymbol{v}^0 = 0 \ , \quad \boldsymbol{v}^0 \,|\, \Gamma = \boldsymbol{v}_0(\cdot, 0) \ , \quad \theta^0 \,|\, \Gamma_0 = \theta_0(\cdot, 0) \ ,$$

and

$$\int_\Gamma \boldsymbol{v}_0(\cdot, t) \cdot \boldsymbol{\nu} \, d\sigma = 0 \ , \qquad t \geq 0 \ ,$$

as well as the smallness condition

$$\|\boldsymbol{v}^0\|_{\boldsymbol{W}_q^2} + \|\theta^0 - \overline{\theta}\|_{W_q^1} + \|e^{\omega t}(\boldsymbol{b}_0, r_0, h_0, \boldsymbol{v}_0, \theta_0 - \overline{\theta})\|_{C^\rho(\mathbb{R}^+,F)} \leq K_0 \ ,$$

there exists a unique global solution $(\boldsymbol{v}, p, \theta)$ *of* (1.1)–(1.3), *and*

$$\|(\boldsymbol{v}, p, \theta - \overline{\theta})(t)\|_{\boldsymbol{W}_q^2 \times W_q^1 \times W_q^1} + \|(\boldsymbol{v}, \theta - \mathcal{R}_0\theta_0)^{\cdot}(t)\|_{\boldsymbol{L}_q \times W_{q,\tau_0}^{-1}} \leq K_1 e^{-\omega t}$$

for $t \geq 0$.

The proof of this theorem, that extends and sharpens the main result of [5], is postponed to Section 4.

2.3 Remarks **(a)** We have chosen the weak formulation (2.4) for the 'temperature equation' in order to treat the nonlinear boundary condition in

(1.2) with relative ease. It can be shown by means of standard arguments that we obtain in fact smooth classical solutions, provided all data are smooth and suitable compatibility conditions are satisfied.

(b) Consider the very special case $(\boldsymbol{b}_0, r_0, h_0, \boldsymbol{v}_0, \theta_0) = (\boldsymbol{0}, 0, 0, \boldsymbol{0}, \overline{\theta})$. Then problem (1.1)–(1.3) possesses a rest state, namely

$$(\boldsymbol{v}, p, \theta) = (\boldsymbol{0}, 0, \overline{\theta}) =: \overline{z} \ ,$$

thanks to (1.5), (1.7), and (1.9), (1.10). Thus Theorem 2.2 says that the rest state $\overline{z}$ is stable (in the specified topologies) with respect to small perturbations of all data. In fact, the rest state $\overline{z}$ is even exponentially stable if the time-dependent perturbations decay exponentially in time, provided, of course, $\mu_0 > 0$. ■

Next we consider the isothermal case. Thus we assume that

$$\left.\begin{aligned} \nabla \cdot \boldsymbol{v} &= 0 \\ \partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} &= -\nabla p + \nabla \cdot \boldsymbol{S} + \boldsymbol{b}_0 \end{aligned}\right\} \quad \text{in } \Omega \times (0,\infty) \ , \qquad \begin{aligned} \boldsymbol{v} &= \boldsymbol{v}_0 \quad &&\text{on } \Gamma \times (0,\infty) \ , \\ \boldsymbol{v}(\cdot, 0) &= \boldsymbol{v}^0 &&\text{on } \Omega \ , \end{aligned} \tag{2.7}$$

is decoupled from the equation for θ. As for this system, the following global existence, uniqueness, and stability theorem is valid:

2.4 Theorem *Suppose that $\boldsymbol{S} = \boldsymbol{S}(\boldsymbol{D})$ is independent of θ and fix $\omega \in [0, \sigma_0)$. Then there exist positive constants K_0 and K_1 such that the following is true: for each*

$$\big((\boldsymbol{b}_0, \boldsymbol{v}_0), \boldsymbol{v}^0\big) \in BUC^\rho\big(\mathbb{R}^+, \boldsymbol{L}_q \times \boldsymbol{W}_q^{2-1/q}(\Gamma)\big) \times \boldsymbol{W}_q^2 \ ,$$

where $0 < \rho < 1$, satisfying the compatibility conditions

$$\nabla \cdot \boldsymbol{v}^0 = 0 \ , \quad \boldsymbol{v}^0 \,|\, \Gamma = \boldsymbol{v}_0(\cdot, 0) \ , \quad \int_\Gamma \boldsymbol{v}_0(\cdot, t) \cdot \boldsymbol{\nu} \, d\sigma = 0 \ , \quad t \geq 0 \ ,$$

as well as the smallness condition

$$\|\boldsymbol{v}^0\|_{\boldsymbol{W}_q^2} + \|e^{\omega t}(\boldsymbol{b}_0, \boldsymbol{v}_0)\|_{C^\rho} \leq K_0 \ ,$$

there exists a unique strong solution

$$(\boldsymbol{v}, p) \in C(\mathbb{R}^+, \boldsymbol{W}_q^2 \times W_q^1) \ , \qquad \boldsymbol{v} \in C^1(\mathbb{R}^+, \boldsymbol{L}_{q,\sigma}) \ ,$$

of (2.7). Furthermore,

$$\|\boldsymbol{v}(t)\|_{\boldsymbol{W}_q^2} + \|\dot{\boldsymbol{v}}(t)\|_{\boldsymbol{L}_q} + \|p(t)\|_{W_q^1} \leq K_1 e^{-\omega t}$$

for $t \geq 0$.

Proof Choose $\Gamma_1 := \emptyset$ and denote by ω_0 the smallest eigenvalue of $-\Delta$ on Ω under Dirichlet boundary condition. Fix a positive constant δ such that $\delta > \overline{c}\sigma_0 - \kappa\omega_0$ and define r_1 by $r_1(\cdot, \theta) := -\delta$. Using this choice in (2.5) it follows that $\mu_0 > \sigma_0 \overline{c}$. Hence the assertion follows from Theorem 2.2 since (2.7) is a now subsystem of (1.1)–(1.3). ∎

Theorem 2.3 is an extension of the main result of [4], where the no-slip condition $\boldsymbol{v} = \boldsymbol{0}$ on Γ has been treated only. It should be observed that Theorem 2.3 applies to general isothermal Stokesian flows (1.5), (1.6) under the sole assumption that $\alpha_1(0,0) > 0$.

The general class of Stokesian fluids considered above contains the class of **fluids with nonlinear viscosity** characterized by $\alpha_2 = 0$. This class, in turn, encompasses the particularly important subclass of **generalized Newtonian fluids** satisfying $\alpha_1 = \alpha_1(|\boldsymbol{D}|^2)$, that is, the (generalized) viscosity depends on the second principal invariant of $\boldsymbol{D}$ only. These classes seem to describe adequately the behavior of certain polymeric fluids and low molecular weight biological liquids (such as blood, for example), and are very popular in chemical engineering, colloidal mechanics, rheology, and glaciology (cf. [7], [23], and, in particular, the references cited in [19]). Many of the concrete models belong to the **power class** for which

$$\alpha_1(s) = 2\nu + \nu_0 s^{(r-2)/2} , \qquad s \in \mathbb{R}^+ , \tag{2.8}$$

where ν_0 and r are nonnegative constants. If $\nu_0 > 0$ and $r > 2$ then the fluid is 'shear thickening', whereas it is 'shear thinning' if $r < 2$.

Generalized Newtonian fluids attracted already considerable mathematical interest (c.f. [16], [15], [14], [8], and [18]). In all these papers rather stringent growth, monotonicity, and coercivity assumptions for $\alpha_1(\cdot)$ are imposed. Then by means of the theory of monotone operators and Galerkin type approximations the existence of weak solutions (in the usual L_2-case) on any given finite interval J is shown for the isothermal problem (2.7). Under further restrictions uniqueness results are established as well. (To be more precise, in [18] the no-slip boundary condition is replaced by a not very realistic space-periodic boundary condition.) The results of [18] are partly extended in [19] to a rather restricted class of fluids with nonlinear viscosity. The latter paper also contains considerations concerning global exponential stability. However, since the existence of the corresponding (weak) solution has not been proven, these considerations are formal.

All of the above works deal with the isothermal case. Aside from the writer's results in [5] there seems to exist only one paper, namely [20], containing an existence theorem for non-Newtonian fluids in the nonisothermal case. In that paper the authors consider the Boussinesq approximation, omitting the energy dissipation term $\boldsymbol{S} : \boldsymbol{D}$, with $\boldsymbol{b}_1$ a constant vector and $(\boldsymbol{b}_0, r) = (\boldsymbol{0}, 0)$, under space-periodic boundary conditions. Moreover, they study generalized Newtonian fluids under the same stringent growth and coercivity assumptions as in [18] and prove the existence of a unique weak solution in some cases. In addition, they establish the existence of a global attractor and give an estimate for its fractal dimension.

3 Instability and Bifurcation

In order to have an easy possibility of referring to the data we introduce some abbreviation. Note that

$$\boldsymbol{W}_{q,\boldsymbol{\nu}}^{2-1/q}(\Gamma) := \left\{ \boldsymbol{v} \in \boldsymbol{W}_q^{2-1/q}(\Gamma) \ ; \ \textstyle\int_\Gamma \boldsymbol{v} \cdot \boldsymbol{\nu} \, d\sigma = 0 \right\}$$

is a closed linear subspace of $\boldsymbol{W}_q^{2-1/q}(\Gamma)$, hence a Banach space. Thus

$$F_0 := \boldsymbol{L}_q \times W_{q,r_0}^{-1} \times W_q^{-1}(\Gamma_1) \times \boldsymbol{W}_{q,\boldsymbol{\nu}}^{2-1/q}(\Gamma) \times W_q^{1-1/q}(\Gamma_0)$$

is a Banach space as well, a closed linear subspace of F.

Now we discuss the solvability of problem (1.1)–(1.3) if $\mu_0 \leq 0$. The proofs of the following theorems are also postponed to Section 4.

3.1 Theorem *Suppose that $\mu_0 \leq 0$, fix $T > 0$, and put $J := [0, T]$. Then there exists a positive constant K such that problem* (1.1)–(1.3) *possesses for each set of data*

$$\big((\boldsymbol{b}_0, r_0, h_0, \boldsymbol{v}_0, \theta_0), \boldsymbol{v}^0, \theta^0\big) \in C^\rho(J, F_0) \times \boldsymbol{W}_q^2 \times W_q^1 \ ,$$

where $0 < \rho < 1$, satisfying

$$\|(\boldsymbol{b}_0, r_0, h_0, \boldsymbol{v}_0, \theta_0)\|_{C^\rho(J,F)} + \|\boldsymbol{v}^0\|_{\boldsymbol{W}_q^2} + \|\theta^0 - \overline{\theta}\|_{W_q^1} \leq K$$

and the compatibility conditions

$$\nabla \cdot \boldsymbol{v}^0 = 0 \ , \quad \boldsymbol{v}^0 \,|\, \Gamma = \boldsymbol{v}_0(\cdot, 0) \ , \quad \theta^0 \,|\, \Gamma_0 = \theta_0(\cdot, 0)$$

a unique solution on J.

Although T can be arbitrarily large, Theorem 3.1 is not a global existence theorem since K depends on T and may shrink to zero as T tends to infinity. In particular, in contrast to Theorem 2.2 we cannot prove that the stationary state $\overline{z}$ is stable (in the sense of Lapunov). In fact, the contrary is true. To be more precise, let us consider the situation where disturbances from the rest state occur via initial perturbations only. In other words, consider the following autonomous special case of (1.1)–(1.3):

$$\left.\begin{aligned} \nabla\cdot\boldsymbol{v} &= 0 \\ \partial_t\boldsymbol{v}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} &= -\nabla p+\nabla\cdot\boldsymbol{S}+\boldsymbol{b}_1(\cdot,\theta)(\theta-\overline{\theta}) \\ c(\partial_t\theta+\boldsymbol{v}\cdot\nabla\theta) &= -\nabla\cdot\boldsymbol{q}+\boldsymbol{S}:\boldsymbol{D}+r_1(\cdot,\theta)(\theta-\overline{\theta}) \end{aligned}\right\}\ \text{in}\quad \Omega\times(0,\infty)\ ,$$

$$\begin{aligned} \boldsymbol{v} &= \boldsymbol{0} && \text{on}\quad \Gamma\times(0,\infty)\ , \\ \theta &= \overline{\theta} && \text{on}\ \Gamma_0\times(0,\infty)\ , \\ -\boldsymbol{q}\cdot\boldsymbol{\nu} &= h_1(\cdot,\theta)(\theta-\overline{\theta}) && \text{on}\ \Gamma_1\times(0,\infty)\ , \\ \boldsymbol{v}(\cdot,0) &= \boldsymbol{v}^0\ ,\quad \theta(\cdot,0)=\theta^0 && \text{on}\ \Omega\ . \end{aligned} \tag{3.1}$$

Then the rest state $\overline{z}$ of (3.1) is **stable** (in the sense of Lapunov) if for each $\varepsilon>0$ there exists $\delta>0$ such that problem (3.1) possesses for each

$$(\boldsymbol{v}^0,\theta^0)\in\boldsymbol{W}_q^2\times W_q^1 \qquad \text{with}\quad \|\boldsymbol{v}^0\|_{\boldsymbol{W}_q^2}+\|\theta^0-\overline{\theta}\|_{W_q^1}<\delta\ ,$$

satisfying the compatibility conditions

$$\nabla\cdot\boldsymbol{v}^0=0\ ,\quad \boldsymbol{v}^0\,|\,\Gamma=\boldsymbol{0}\ ,\quad \theta^0\,|\,\Gamma_0=\overline{\theta}\ ,$$

a unique global solution $(\boldsymbol{v},p,\theta)$ and

$$\|\boldsymbol{v}(t)\|_{\boldsymbol{W}_q^2}+\|p(t)\|_{W_q^1}+\|(\theta-\overline{\theta})(t)\|_{W_q^1}<\varepsilon\ ,\qquad t\in\mathbb{R}^+\ .$$

The rest state is **unstable** (in the sense of Lapunov) if it is not stable,

3.2 Theorem *Suppose that $\mu_0<0$. Then the rest state $\overline{z}$ of* (3.1) *is unstable.*

It should be noted that the stability of $\overline{z}$ is completely determined by the sign of the least eigenvalue μ_0 of problem (2.5). Indeed, if $\mu_0>0$ then Theorem 2.2 shows that $\overline{z}$ is stable, in fact, even exponentially stable, whereas it is unstable by Theorem 3.2 if $\mu_0<0$.

Now we turn our consideration to a neighborhood of $\mu_0=0$ where the stationary state $\overline{z}$ of (3.1) looses its stability. For the sake of easy calculations we assume that $\Gamma_1=\Gamma$. We also assume, for definiteness, that

$$I(1)=-\int_\Omega r_1(\cdot,\overline{\theta})\,dx-\int_\Gamma h_1(\cdot,\overline{\theta})\,d\sigma<0 \tag{3.2}$$

and consider the parameter-dependent problem

$$\left.\begin{aligned} \nabla\cdot\boldsymbol{v}&=0\\ \partial_t\boldsymbol{v}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}&=-\nabla p+\nabla\cdot\boldsymbol{S}+\boldsymbol{b}_1(\cdot,\theta)(\theta-\overline{\theta})\\ c(\partial_t\theta+\boldsymbol{v}\cdot\nabla\theta)&=-\nabla\cdot\boldsymbol{q}+\boldsymbol{S}:\boldsymbol{D}+\varepsilon r_1(\cdot,\theta)(\theta-\overline{\theta}) \end{aligned}\right\}\ \text{in } \Omega\times(0,\infty)\ , \tag{3.3$_\varepsilon$}$$

$$\boldsymbol{v}=\boldsymbol{0}\ ,\quad -\boldsymbol{q}\cdot\boldsymbol{\nu}=\varepsilon h_1(\cdot,\theta)(\theta-\overline{\theta}) \qquad \text{on } \Gamma\times(0,\infty)\ ,$$
$$\boldsymbol{v}(\cdot,0)=\boldsymbol{v}^0\ ,\quad \theta(\cdot,0)=\theta^0 \qquad \text{on } \Omega\ .$$

for $\varepsilon\in\mathbb{R}$. Note that problem $(3.3)_\varepsilon$ possesses for $\varepsilon\in\mathbb{R}$ the 'line of trivial stationary states'

$$Z_0:=\big\{\,(\varepsilon,\overline{z})\ ;\ \varepsilon\in\mathbb{R}\big\}\subset\mathbb{R}\times[\boldsymbol{W}_{q,\tau}^2\times W_q^1\times W_q^1]\ .$$

Moreover, thanks to (3.2), Remark 2.1, and Theorem 3.2, the trivial stationary state $\overline{z}$ of $(3.3)_\varepsilon$ is unstable if $\varepsilon>0$.

The following theorem gives more precise information near $\varepsilon=0$. Below we denote the intersection of $(-\varepsilon_0,\varepsilon_0)\times[\boldsymbol{W}_{q,\tau}^2\times W_q^1\times W_q^1]$ and Z_0 again by Z_0. We also put $\overline{\boldsymbol{b}}_1:=\boldsymbol{b}_1(\cdot,\overline{\theta})$.

3.3 Theorem *There exist a constant $\varepsilon_0>0$, a neighborhood W of $\overline{z}$ in $\boldsymbol{W}_{q,\tau}^2\times W_q^1\times W_q^1$, and a function*

$$\big[\varepsilon\mapsto z(\varepsilon):=\big(\boldsymbol{v}(\varepsilon),p(\varepsilon),\theta(\varepsilon)\big)\big]\in C^\infty\big((-\varepsilon_0,\varepsilon_0),\boldsymbol{W}_{q,\tau}^2\times W_q^1\times W_q^1\big)$$

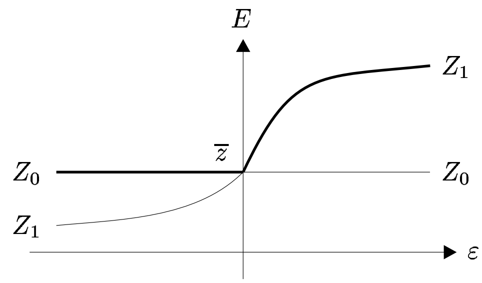
satisfying $z(0)=\overline{z}$ such that the set of stationary states of $(3.3)_\varepsilon$ contained in $(-\varepsilon_0,\varepsilon_0)\times W$ consists precisely of the line of trivial stationary states Z_0 and the curve

$$Z_1:=\big\{\,\big(\varepsilon,z(\varepsilon)\big)\ ;\ -\varepsilon_0<\varepsilon<\varepsilon_0\,\big\}\ .$$

Moreover, Z_0 and Z_1 intersect transversally in $(0,\overline{z})$ and the trivial stationary state $\overline{z}$ of $(3.3)_\varepsilon$ is stable for $-\varepsilon_0<\varepsilon<0$ and unstable for $\varepsilon>0$.

Suppose, in addition, that $P\overline{\boldsymbol{b}}_1\neq0$. Then the nontrivial stationary state $z(\varepsilon)$ of $(3.3)_\varepsilon$ is stable for $0<\varepsilon<\varepsilon_0$ and unstable for $-\varepsilon_0<\varepsilon<0$.

This theorem shows that there occurs bifurcation of stationary states at $\varepsilon=0$ and that it is transcritical if $P\overline{\boldsymbol{b}}_1\neq0$. Thus, if this condition is satisfied and if we put $E:=\boldsymbol{W}_{q,\tau}^2\times W_q^1\times W_q^1$, the set of stationary states of $(3.3)_\varepsilon$, $\varepsilon \in \mathbb{R}$, has in a neighborhood of $\varepsilon = 0$ the form



where the stable stationary states lie on the heavy and the unstable ones on the light branches of Z_0 and Z_1.

3.4 Remarks **(a)** First suppose that $r_1 = 0$ and

$$\int_\Gamma h_1(\cdot, \overline{\theta})\, d\sigma > 0 \; . \tag{3.4}$$

Since

$$h_1(\cdot,\theta)(\theta - \overline{\theta}) = h_1(\cdot,\overline{\theta})(\theta - \overline{\theta}) + o\big((\theta - \overline{\theta})^2\big) \; ,$$

it follows from (3.4) that — on the average and for small deviations of θ from the rest state temperature $\overline{\theta}$ — the influx of heat through the boundary Γ is positive [resp. negative] if $\theta > \overline{\theta}$ [resp. $\theta < \overline{\theta}$] and $\varepsilon > 0$. Thus small deviations of θ from $\overline{\theta}$ are expected to lead on the average to an increase of the temperature difference $\theta - \overline{\theta}$, which affects also the second equation in $(3.3)_\varepsilon$ to produce deviations of $\boldsymbol{v}$ from $\mathbf{0}$. Hence (3.4) and $\varepsilon > 0$ are expected to have a destabilizing effect. On the other hand, if $\varepsilon < 0$ then, on the average, heat is extracted through Γ if $\theta > \overline{\theta}$, and injected if $\theta < \overline{\theta}$. Thus it can be expected that (3.4) and $\varepsilon < 0$ have a stabilizing effect trying to bring small temperature deviations — and, consequently, small velocity deviations — back to the stationary state $\overline{z}$. Similar heuristic arguments apply if $r_1 \neq 0$ and (3.2) is satisfied. Of course, there are also the other nonlinear terms in the differential equation, in particular the term $\boldsymbol{S} : \boldsymbol{D}$, which has a destabilizing effect near $\boldsymbol{v} = \mathbf{0}$. But Theorem 3.3 shows that this heuristic argument is basically correct.

(b) As already observed earlier, Theorem 2.2 implies that the rest state $\overline{z}$ is locally (or conditionally) asymptotically stable if $\mu_0 > 0$. There arises, of course, the question whether it is also globally (or unconditionally) stable, that is, whether problem (3.1) possesses for arbitrarily large initial values a global solution that tends towards $\overline{z}$ as $t \to \infty$. It is a consequence of Theorem 3.3 that, *in general, the trivial stationary state $\overline{z}$ is not globally asymptotically stable.* Indeed, Theorem 3.3 guarantees the existence of nonlinearities for which the trivial steady state $\overline{z}$ is locally asymptotically stable and such that there exists a nontrivial steady state as well. Thus $\overline{z}$ cannot be globally asymptotically stable in such a situation.

(c) It will be seen from the proof of Theorem 3.3 that the fact that transcritical bifurcation occurs depends crucially on the presence of the term $\boldsymbol{S} : \boldsymbol{D}$ in the temperature equation. If this term is dropped — as in the standard Boussinesq approximation for Newtonian fluids — the nature of bifurcation, that is, the direction of the curve Z_1 of nontrivial stationary states at $(0, \overline{z})$ cannot be decided, in general.

As noted in (b), the fact that transcritical bifurcation occurs implies that 'the onset of instability' happens 'earlier' (that is, in our situation for a value $\mu_0 > 0$) than predicted by a linear stability analysis (where it occurs at $\mu_0 = 0$). In the case of the Boussinesq approximation for Newtonian fluids much work has been done determining this 'onset of instability', usually by establishing suitable energy estimates. In particular, there have been derived criteria that are necessary and sufficient for the onset of instability, that is, criteria guaranteeing that (in our situation) the trivial rest state is globally asymptotically stable whenever $\mu_0 > 0$ (cf. [13], [11], [26], [29] and the references cited in this work). Theorem 3.3 implies that *necessary and sufficient conditions for instability cannot be expected to hold in general* for heat-conducting incompressible viscous fluids — even in the Newtonian case — *if the complete energy conservation law is taken into consideration,* that is, if the dissipation function $\boldsymbol{S} : \boldsymbol{D}$ is not omitted. ■

4 Proofs

We begin by introducing some notation. Let J be a nontrivial subinterval of $\mathbb{R}^+$ containing 0. Given a Banach space E, we denote by $BUC(J, E)$ the Banach space of all bounded and uniformly continuous functions $u : J \to E$, endowed with the supremum norm $\|\cdot\|_\infty$. Moreover, given $\alpha \in (0,1)$ and $\varepsilon > 0$ with $2\varepsilon \in J$,

$$[u]^*_{\alpha,[\varepsilon,2\varepsilon]} := \sup_{\substack{\varepsilon<s<t<2\varepsilon \\ t-s<1}} \frac{\|u(s)-u(t)\|}{|s-t|^\alpha}$$

and

$$[\![u]\!]_\alpha := \sup_{\varepsilon\in \dot{J}} (1\wedge\varepsilon)^\alpha [u]^*_{\alpha,[\varepsilon,2\varepsilon]} \ , \qquad u \in BUC(J,E) \ .$$

Then $BUC_\alpha^\alpha(J,E)$ is the Banach space of all $u \in BUC(J,E)$ satisfying $[\![u]\!]_\alpha < \infty$ and

$$\lim_{\varepsilon\to 0} \varepsilon^\alpha [u]^*_{\alpha,[\varepsilon,2\varepsilon]} = 0 \ ,$$

equipped with the norm

$$\|\cdot\|_{C_\alpha^\alpha} := \|\cdot\|_\infty + [\![\cdot]\!]_\alpha \ .$$

The Banach space $BUC_\alpha^{1+\alpha}(J,E)$ consists of all $u \in BUC_\alpha^\alpha(J,E)$ such that $\partial u := \dot{u} \in BUC_\alpha^\alpha(J,E)$, endowed with the norm

$$u \mapsto \|u\|_{C_\alpha^{1+\alpha}} := \|u\|_{C_\alpha^\alpha} + \|\dot{u}\|_{C_\alpha^\alpha} \ .$$

Lastly, given $\omega \in \mathbb{R}^+$ and $\beta \in \{\alpha, 1+\alpha\}$,

$$e^{-\omega} BUC_\alpha^\beta(J,E) := \big\{\, u \in BUC(J,E)\ ;\ e^{\omega t}u \in BUC_\alpha^\beta(J,E)\,\big\} \ .$$

These are Banach spaces as well with the norms

$$u \mapsto \|u\|_{e^{-\omega}C_\alpha^\beta} := \|e^{\omega t}u\|_{C_\alpha^\beta} \ .$$

It is obvious that

$$BUC^\alpha(J,E) \hookrightarrow BUC_\alpha^\alpha(J,E) \ . \tag{4.1}$$

Let (E_0,E_1) be a densely injected Banach couple, that is, E_0 and E_1 are Banach spaces such that $E_1 \hookrightarrow E_0$ and E_1 is dense in E_0. We denote by $\mathcal{H}(E_1,E_0)$ the set of all operators $B \in \mathcal{L}(E_1,E_0)$ such that $-B$, considered as a linear operator in E_0 with domain E_1, is the infinitesimal generator of a strongly continuous analytic semigroup on E_0. We write $s(B)$ for the upper spectral bound of $B \in \mathcal{H}(E_1,E_0)$, defined by

$$s(B) := \sup\big\{\, \mathrm{Re}\,\lambda\ ;\ \lambda \in \sigma(B)\,\big\} \ ,$$

where $\sigma(B)$ is the spectrum of B.

We denote by $\boldsymbol{L}_{q,\sigma}$ the closure in $\boldsymbol{L}_q$ of the set of all smooth solenoidal vector fields with compact supports in Ω, and we put $\boldsymbol{W}^2_{q,\sigma} := \boldsymbol{W}^2_q \cap \boldsymbol{L}_{q,\sigma}$ and $\boldsymbol{W}^2_{q,\tau,\sigma} := \boldsymbol{W}^2_{q,\tau} \cap \boldsymbol{L}_{q,\sigma}$. Then $P \in \mathcal{L}(\boldsymbol{L}_q, \boldsymbol{L}_{q,\sigma})$ is the Helmholtz projector associated with the Helmholtz-Weyl decomposition of $\boldsymbol{L}_q$ (e.g., [9], [10, Theorem III.1.2]). By applying P to the second equation in (1.1) we obtain the reduced system

$$\begin{aligned}
\left.\begin{aligned}\partial_t \boldsymbol{v} + P(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} &= P\nabla\cdot\boldsymbol{S} + P\boldsymbol{b}\\ c(\partial_t\theta + \boldsymbol{v}\cdot\nabla\theta) &= -\nabla\cdot\boldsymbol{q} + \boldsymbol{S}:\boldsymbol{D} + r\end{aligned}\right\} \quad & \text{in } \ \Omega\times(0,\infty)\ ,\\
\boldsymbol{v} = \boldsymbol{v}_0 \qquad & \text{on } \ \Gamma\times(0,\infty)\ ,\\
\theta = \theta_0 \qquad & \text{on } \Gamma_0\times(0,\infty)\ ,\\
-\boldsymbol{q}\cdot\boldsymbol{\nu} = h \qquad & \text{on } \Gamma_1\times(0,\infty)\ ,\\
\boldsymbol{v}(\cdot,0) = \boldsymbol{v}^0\ ,\quad \theta(\cdot,0) = \theta^0 \qquad & \text{on } \ \Omega\ ,
\end{aligned} \tag{4.2}$$

where we are looking for solutions $(\boldsymbol{v}, \theta)$ satisfying $\boldsymbol{v}(t) \in \boldsymbol{W}^2_{q,\sigma}$ for $t > 0$. Having solved (4.2) we can recover the 'pressure field' $p(t)$ as ususal by solving a suitable Neumann boundary value problem (c.f. [9], [10]). Hence we can restrict our considerations to the 'reduced problem' (4.2) and leave the details of the proofs of the asserted properties of p to the interested reader.

Finally, we put

$$E_0 := \boldsymbol{L}_{q,\sigma} \times W^{-1}_{q,\tau_0} , \quad E_1 := \boldsymbol{W}^2_{q,\tau,\sigma} \times W^1_{q,\tau_0} .$$

Then (E_0, E_1) is a densely injected Banach couple and

$$E_1 \hookrightarrow \boldsymbol{C}^1 \times C \tag{4.3}$$

since $q > 3$, thanks to Sobolev's embedding theorem.

Proof of Theorem 2.2 We fix $\omega \in \big[0, \sigma_0 \wedge (\mu_0/\overline{c})\big)$ and $\rho \in (0,1)$ and put

$$\mathbb{E}_0 := e^{-\omega} BUC^\rho_\rho(\mathbb{R}^+, E_0)$$

and

$$\mathbb{E}_1 := e^{-\omega} BUC^\rho_\rho(\mathbb{R}^+, E_1) \cap e^{-\omega} BUC^{1+\rho}_\rho(\mathbb{R}^+, E_0) .$$

We also set

$$\mathbb{F}_0 := e^{-\omega} BUC^\rho_\rho(\mathbb{R}^+, F_0) , \quad \mathbb{E} := \mathbb{E}_0 \times E_1 ,$$

and denote by $\mathbb{F}$ the set of all

$$\big((\boldsymbol{b}_0, r_0, h_0, \boldsymbol{v}_0, \theta_0), \boldsymbol{v}^0, \theta^0\big) \in \mathbb{F}_0 \times E_1$$

satsifying the comaptibility conditions

$$\tau \boldsymbol{v}^0 = \boldsymbol{v}_0(\cdot, 0) , \quad \tau_0 \theta^0 = \theta_0(\cdot, 0) .$$

Note that $\mathbb{F}$ is a closed linear subspace of $\mathbb{F}_0 \times E_1$, hence a Banach space.

Next we introduce a new 'variable' $u := (u_1, u_2) \in \mathbb{E}_1$ by

$$\boldsymbol{v} = u_1 + \mathcal{R}\boldsymbol{v}_0 , \quad \theta = u_2 + \mathcal{R}_0\theta_0 = u_2 + \overline{\theta} + \mathcal{R}_0(\theta_0 - \overline{\theta}) \tag{4.4}$$

and 'parameters'

$$\eta := (\boldsymbol{b}_0, r_0, h_0, \boldsymbol{v}_0, \theta_0 - \overline{\theta}) , \quad u^0 := \big(\boldsymbol{v}^0 - \mathcal{R}\boldsymbol{v}_0(\cdot, 0), \theta^0 - \mathcal{R}_0\theta_0(\cdot, 0)\big) \in E_1$$

for

$$\xi := \big((\boldsymbol{b}_0, r_0, h_0, \boldsymbol{v}_0, \theta_0 - \overline{\theta}), \boldsymbol{v}^0, \theta^0 - \overline{\theta}\big) \in \mathbb{F} \ .$$

Rewriting (4.2) in these variables we obtain a problem of the form

$$\dot{u} = F(u, \eta) \ , \quad t > 0 \ , \qquad u(0) = u^0 \ .$$

Thanks to (4.3) it is an exercise to verify that $F \in C^\infty(\mathbb{E}_1 \times \mathbb{F}, \mathbb{E}_0)$. We put

$$\gamma := \tau_1 \,|\, W^1_{q,\tau_0} \in \mathcal{L}\big(W^1_{q,\tau_0}, W^{1-1/q}_q(\Gamma_1)\big)$$

and denote by

$$\gamma^\sharp \in \mathcal{L}\big(W^{-1/q}_q(\Gamma_1), W^{-1}_{q,\tau_0}\big)$$

the dual of $\tau_1 \,|\, W^1_{q',\tau_0}$. Then one finds that

$$\partial_1 F(0,0) = \begin{bmatrix} \nu P \Delta & P \overline{\boldsymbol{b}}_1 \\ 0 & [\kappa \Delta + \overline{r}_1 + \gamma^\sharp \overline{h}_1 \gamma] / \overline{c} \end{bmatrix} \in \mathcal{L}(E_1, E_0) \ . \tag{4.5}$$

The operator $-\nu P \Delta$ is the Stokes operator. Hence

$$-\nu P \Delta \in \mathcal{H}(\boldsymbol{W}^2_{q,\gamma,\sigma}, \boldsymbol{L}_{q,\sigma})$$

thanks to results of Solonnikov [25], von Wahl [28], Giga [12], and Miyakawa [21]. The negative of the operator in the right lower corner of the above matrix belongs to $\mathcal{H}(W^1_{q,\tau_0}, W^{-1}_{q,\tau_0})$, as follows from the author's results on extrapolation (cf. [3, Theorem 8.5] and [6, Chapter V and Volume 2 (to appear)]). Hence we infer from [6, Theorem I.1.6.1] and the fact that the multiplication operator $P\overline{\boldsymbol{b}}_1$ belongs to $\mathcal{L}(W^1_{q,\tau_0}, \boldsymbol{L}_{q,\sigma})$ that

$$-\partial_1 F(0,0) \in \mathcal{H}(E_1, E_0) \ . \tag{4.6}$$

Moreover, it is an easy consequence of the triangular structure of the matrix in (4.5) that

$$s\big(\partial_1 F(0,0)\big) = -\big[\sigma_0 \wedge (\mu_0 / \overline{c})\big] < 0 \ . \tag{4.7}$$

We put $\gamma_0 u := u(0)$ for $u \in C(\mathbb{R}^+, E_1)$ and

$$\Phi(u, \xi) := \big(\partial u - F(u, \eta), \gamma_0 u - u^0\big) \ , \qquad u \in \mathbb{E}_1 \ , \quad \xi = (\eta, u^0) \in \mathbb{F} \ ,$$

where $\partial u := \dot{u}$. Then $\Phi \in C^\infty(\mathbb{E}_1 \times \mathbb{F}, \mathbb{E}_0 \times E_1)$ with $\Phi(0,0) = (0,0)$ and

$$\partial_1 \Phi(0,0) = \big(\partial - \partial_1 F(0,0), \gamma_0\big) \ .$$

Owing to (4.6) and (4.7) it follows from [6, Theorem III.2.5.5] that $\partial_1 \Phi(0,0)$ is an isomorphism from $\mathbb{E}_1$ onto $\mathbb{E}_0 \times E_1$. Hence the implicit function theorem guarantees the existence of a neighborhood $U \times V$ of $(0,0)$ in $\mathbb{E}_1 \times \mathbb{F}$

and of a map $\varphi \in C^\infty(V,U)$ such that, given any $\xi \in V$, the equation $\Phi(u,\xi)=0$ has a unique solution $u=\varphi(\xi)$ in U, that is, the assertions

$$\big(\ (u,\xi)\in U\times V\ ,\quad \Phi(u,\xi)=0\ \big)\qquad\text{and}\qquad \big(\ \xi\in V\ ,\quad \Phi\big(\varphi(\xi),\xi\big)=0\ \big)$$

are equivalent. Now, defining $(\boldsymbol{v},\theta)$ by (4.4) with $u=\varphi(\xi)$, it is clear that $(\boldsymbol{v},\theta)$ is a solution of (4.2). From this and (4.1) we easily infer the assertions of Theorem 2.2, except for uniqueness since, by definition, our class of solutions belongs to a space larger than $\mathbb{E}_1$. However, uniqueness in the class of solutions defined in Section 2 can be obtained by adapting the interpolation-extrapolation technique employed in [4, Section 3], that is, the uniqueness proof given there, to the present situation. ■

Proof of Theorem 3.1 Put $\omega:=0$ and replace $\mathbb{R}^+$ in the preceding proof by J. Since $\mu_0\le 0$, condition (4.7) is no longer satisfied. Hence we have to replace [6, Theorem III.2.5.5] by [6, Theorem III.2.5.6]. With these modifications the above proof implies the assertion. ■

4.1 Remarks **(a)** It is clear that it suffices for the validity of the above proofs that $\Phi\in C^1(\mathbb{E}_1\times\mathbb{F},\mathbb{E}_0\times E_1)$. This observation leads to a considerable relaxation of the regularity assumptions. We leave it to the reader to find the minimal hypotheses.

(b) The fact that $u=\varphi(\xi)$ is a smooth function of ξ implies, of course, that the solution $(\boldsymbol{v},p,\theta)$ of (1.1)–(1.3) depends continuously on all data in the topologies specified in Theorems 2.2 and 3.1, respectively.

(c) The above proofs show that the solution $(\boldsymbol{v},p,\theta)$ of (1.1)–(1.3), whose existence is guaranteed by Theorems 2.2 and 3.1, even satisfies

$$(\boldsymbol{v},p,\theta)\in C^\rho(\dot{J},\boldsymbol{W}_q^2\times W_q^1\times W_q^1)\ ,$$

where $\dot{J}:=J\backslash\{0\}$ and $J=\mathbb{R}^+$ if $\mu_0>0$.

(d) Since the implicit function theorem relies on the contraction mapping theorem, it is, in principle, possible to obtain concrete estimates for the sizes of the neighborhoods of the stationary state $\overline{z}$ in which the existence of a solution can be guaranteed, that is, for the constants K_0, K_1, and K. However, as usual, these estimates are much too pessimistic to be of practical relevance. ■

Proof of Theorem 3.2 The reduced system (4.2) belonging to (3.1) is of the form

$$\dot{u} = G(u) \ , \qquad t > 0 \ , \qquad u(0) = u^0 \ , \tag{4.8}$$

since $\eta = 0$ in this case. Here $G \in C^\infty(E_1, E_0)$ and $G(0) = 0$. Thus (4.8) is an autonomous ordinary differential equation in the Banach space E_0 and $-\partial G(0) \in \mathcal{H}(E_1, E_0)$, where $\partial G(0)$ is given by the operator matrix in (4.5). Since Ω is bounded, E_1 is compactly embedded in E_0. This implies that $\partial G(0)$, considered as an unbounded linear operator in E_0, has a compact resolvent. Thus, since $\mu_0 < 0$, it follows that $\partial G(0)$ possesses at least one but at most finitely many eigenvalues with positive real part. Hence the hypotheses of [17, Theorem 9.1.3] are satisfied and the assertion follows. ∎

Proof of Theorem 3.3 The reduced system (4.2) belonging to $(3.3)_\varepsilon$ is of the form

$$\dot{u} = H(\varepsilon, u) \ , \quad t > 0 \ , \qquad u(0) = u^0 \ ,$$

where $H \in C^\infty(\mathbb{R} \times E_1, E_0)$ and $H(\varepsilon, 0) = 0$ for $\varepsilon \in \mathbb{R}$. Put

$$A(\varepsilon) := \partial_2 F(\varepsilon, 0) \in \mathcal{L}(E_1, E_0) \ , \qquad \varepsilon \in \mathbb{R} \ ,$$

and note that

$$A := A(0) = \begin{bmatrix} \nu P \Delta & P\overline{\boldsymbol{b}}_1 \\ 0 & (\kappa/\overline{c})\Delta \end{bmatrix} \ .$$

From this we infer that

$$\ker A = \mathbb{R}e \qquad \text{with} \quad e := (\boldsymbol{e}_1, e_2) \in E_1 = \boldsymbol{W}^2_{q,\tau,\sigma} \times W^1_q \ ,$$

where $\boldsymbol{e}_1$ is the unique solution of the Stokes equation $-\nu P \Delta \boldsymbol{v} = P\overline{\boldsymbol{b}}_1$ and $e_2 = 1$. Since 0 is a simple isolated eigenvalue of $-\Delta$ under Neumann boundary conditions, it follows that 0 is a simple isolated eigenvalue of A, considered as a linear operator in E_0.

Consider the linear operator

$$A^\sharp := \begin{bmatrix} \nu P \Delta & 0 \\ P\overline{\boldsymbol{b}}_1 \cdot & (\kappa/\overline{c})\Delta \end{bmatrix} \in \mathcal{L}(E_1^\sharp, E_0^\sharp) \ ,$$

where $E_1^\sharp := \boldsymbol{W}^2_{q',\tau,\sigma} \times W^1_{q'}$ and $E_0^\sharp := \boldsymbol{L}_{q',\sigma} \times (W^1_q)'$. Then interpolation-extrapolation theory (cf. [6, Theorem V.1.5.12]) implies $A^\sharp \supset A'$, where A' is the dual of the unbounded linear operator A in E_0. From this we deduce that $\ker A' = \ker A^\sharp = \mathbb{R}e'$, where $e' = (\boldsymbol{0}, 1)$.

Since

$$\partial_1\partial_2 H(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & [\overline{r}_1 + \gamma^\sharp \overline{h}_1 \gamma]/\overline{c} \end{bmatrix} \in \mathcal{L}(E_1, E_0) \ ,$$

it follows that

$$\langle e', \partial_1\partial_2 H(0,0)e \rangle = (1/\overline{c})\Big[\int_\Omega \overline{r}_1 \, dx + \int_\Gamma \overline{h}_1 \, d\sigma\Big] = -I(1)/\overline{c} > 0 \ ,$$

thanks to (3.2). Thus the transversality criterion for bifurcation from a simple eigenvalue is satisfied. Consequently, bifurcation theory (e.g., [2, Theorem 26.13] or [17, Theorem 9.1.10]) tells us that, in some neighborhood $(-\varepsilon_0, \varepsilon_0) \times U$ of $(0,0)$ in $\mathbb{R} \times E_1$, the solution set of $H(\varepsilon, u) = 0$ consists precisely of the line of trivial solutions $\Gamma_0 := (-\varepsilon_0, \varepsilon_0) \times \{0\}$ and a smooth curve Γ_1 intersecting Γ_0 transversally in $(0,0)$.

It is not difficult to check, thanks to the simple form of e and e', that

$$\langle e', \partial_2^2 H(0,0)[e]^2 \rangle = (2\nu/\overline{c}) \int_\Omega |\boldsymbol{D}(\boldsymbol{e}_1)|^2 \, dx \ . \tag{4.9}$$

Since

$$P\overline{\boldsymbol{b}}_1 = -\nu P \Delta \boldsymbol{e}_1 = -2\nu P \nabla \cdot \boldsymbol{D}(\boldsymbol{e}_1) \ ,$$

it follows that $\boldsymbol{D}(\boldsymbol{e}_1) \neq \boldsymbol{0}$ if $P\overline{\boldsymbol{b}}_1 \neq 0$. Thus

$$\langle e', \partial_2^2 H(0,0)[e]^2 \rangle > 0 \qquad \text{if} \quad P\overline{\boldsymbol{b}}_1 \neq 0 \ ,$$

which guarantees transcritical bifurcation (cf. [2, Proposition 27.5]. Although that proposition has been formulated in [2] in a finite-dimensional setting, it is easily verified that its proof carries over to the infinite-dimensional situation considered here; also see [17, Section 9.1.4]).

Lastly, the asserted stability properties follow easily from standard stability considerations in bifurcation theory (e.g., [2], [17]) and the fact that $-\partial_2 H(\varepsilon, u) \in \mathcal{H}(E_1, E_0)$ for (ε, u) in a sufficiently small neighborhood of $(0,0)$, as follows from [6, Theorem I.1.3.1]. ∎

5 Final Remarks

It seems worthwhile to add some comments on the scope of the methods used in this paper. As mentioned earlier, we have made some simplifying assumptions in order to facilitate the calculations.

First we have studied the motion of the fluid in the neighborhood of the rest state $(\mathbf{0}, 0, \overline{\theta})$ where $\overline{\theta}$ is a constant temperature distribution. However, the proofs of Theorems 2.2, 2.4, and 3.1 remain valid (with obvious modifications) if it is only assumed that

$$\overline{\zeta} := (\overline{\boldsymbol{v}}, \overline{p}, \overline{\theta}) \in \boldsymbol{W}^2_{q,\sigma} \times W^1_q \times W^1_q$$

is a stationary state of (1.1)–(1.3) (satisfying the appropriate compatibility conditions, of course) such that

$$\partial_1 F(0,0) \in \mathcal{H}(E_1, E_0) \ ,$$

where $\partial_1 F(0,0)$ is induced in the obvious way by the linearization of the reduced problem (4.2) at $\overline{\zeta}$. Of course, then the stability of the rest state $\overline{\zeta}$ is governed by the spectrum of $\partial_1 F(0,0)$.

Second, in order to prove the bifurcation result we have considered the parametrized problem $(3.3)_\varepsilon$. As discussed in Remark 3.4(a) the parameter ε measures the strength of the heat production in Ω and on Γ, respectively. Of course, other parameters can be used as well. In general, however, it will be difficult to verify the transversality condition or to get information on the direction, hence the stability, of the bifurcating branch of nontrivial steady states.

Lastly, it should be observed that for the standard Boussinesq approximation for Newtonian (or non-Newtonian) fluids, in which the term $\boldsymbol{S} : \boldsymbol{D}$ is omitted, the left-hand side of (4.9) as well as all higher derivatives of $H(0,0)$ vanish. Hence, as already pointed out in Remark 3.49c), we cannot obtain any information on the direction or the stability of the bifurcating curve in this case.

References

[1] H. Amann. Dynamic theory of quasilinear parabolic equations – II. Reaction-diffusion systems. *Diff. Int. Equ.*, **3** (1990), 13–75.

[2] H. Amann. *Ordinary Differential Equations. An Introduction to Nonlinear Analysis.* W. de Gruyter, Berlin, 1990.

[3] H. Amann. Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems. In H.J. Schmeisser, H. Triebel, editors, *Function Spaces, Differential Operators and Nonlinear Analysis*, pages 9–126. Teubner, Stuttgart, Leipzig, 1993.

[4] H. Amann. Stability of the rest state of a viscous incompressible fluid. *Arch. Rat. Mech. Anal.*, **126** (1994), 231–242.

[5] H. Amann. Heat-conducting incompressible viscous fluids. In A. Sequeira, editor, *Navier-Stokes Equations and Related Non-Linear Problems*, pages 231–243. Plenum Press, New York, 1995.

[6] H. Amann. *Linear and Quasilinear Parabolic Problems.* Birkhäuser, Basel, 1995.

[7] R.B. Bird, R.C. Armstrong, O. Hassager. *Dynamics of Polymeric Liquids,* Vol. 1: *Fluid Mechanics.* Wiley, New York, 1987.

[8] D. Cioranescu. Quelques examples de fluides Newtonien generalisés. In J.F. Rodrigues, A. Sequeira, editors, *Mathematical Topics in Fluid Mechanics*, pages 1–31. Pitman Research Notes in Math., #274, 1992.

[9] D. Fujiwara, H. Morimoto. An L_r-theorem of the Helmholtz decomposition of vector fields. *J. Fac. Sci. Univ. Tokyo, Sect. IA Math.*, **24** (1977), 685–700.

[10] G.P. Galdi. *An Introduction to the Mathematical Theory of the Navier-Stokes Equations.* Springer Verlag, New York, 1994.

[11] G.P. Galdi, M. Padula. A new approach to energy theory in the stability of fluid motion. *Arch. Rat. Mech. Anal.*, **110** (1990), 187–286.

[12] Y. Giga. Analyticity of the semigroup generated by the Stokes operator in L_r spaces. *Math. Z.*, **178** (1981), 297–329.

[13] D.D. Joseph. *Stability of Fluid Motions.* Springer Verlag, Berlin, 1976.

[14] S. Kaniel. On the initial value problem for an incompressible fluid with nonlinear viscosity. *J. Math. Mech.*, **19** (1970), 681–707.

[15] O.A. Ladyženskaja. *Mathematical Theory of Viscous Incompressible Flow.* Gordon & Breach, New York, 1969.

[16] O.A. Ladyženskaya. New equations for the description of motion of viscous incompressible fluids and solvability in the large of boundary value problems for them. *Proc. Steklov Inst. Math.*, **102** (1967), 95–118.

[17] A. Lunardi. *Analytic Semigroups and Optimal Regularity in Parabolic Equations.* Birkhäuser, Basel, 1995.

[18] J. Málek, J. Nečas, M. Růžička. On the non-Newtonian incompressible fluids. *Math. Meth. & Models in Appl. Sciences* (M^3AS), **3** (1993), 35–63.

[19] J. Málek, K.R. Rajagopal, M. Růžička. Existence and regularity of solutions and the stability of the rest state for fluids with shear dependent viscosity. 1994, Preprint.

[20] J. Málek, M. Růžička, G. Thäter. Fractal dimension, attractors and Boussinesq approximation in three dimensions. 1993, Preprint.

[21] T. Miyakawa. On the initial value problem for the Navier-Stokes equations in L^p spaces. *Hiroshima Math. J.*, **11** (1981), 9–20.

[22] K.R. Rajagopal. Mechanics of non-Newtonian fluids. In G.P. Galdi, J. Nečas, editors, *Recent Developments in Theoretical Fluid Dynamics*, pages 129–162. Pitman Research Notes in Math., #291, 1993.

[23] W.R. Schowalter. *Mechanics of Non-Newtonian Fluids.* Pergamon Press, Oxford, 1978.

[24] J. Serrin. Mathematical principles of classical fluid dynamics. *Handbuch der Physik,* Springer Verlag, **VIII/1** (1959), 125–263.

[25] V.A. Solonnikov. Estimates for solutions of nonstationary Navier-Stokes equations. *J. Soviet Math.*, **8** (1977), 467–529.

[26] B. Straughan. *The Energy Method, Stability, and Nonlinear Convection.* Springer Verlag, New York, 1992.

[27] C. Truesdell, W. Noll. The Non-Linear Field Theories of Mechanics. *Handbuch der Physik, Springer Verlag*, **III/3** (1965).

[28] W. von Wahl. *The Equations of Navier-Stokes and Abstract Parabolic Equations.* Vieweg & Sohn, Braunschweig, 1985.

[29] W. von Wahl. Necessary and sufficient conditions for the stability of flows in incompressible viscous fluids. *Arch. Rat. Mech. Anal.*, **126** (1994), 103–129.

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