Remarks on the Strong Solvability of the Navier-Stokes Equations

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Throughout this note $m \geq 3$ and either $\Omega = \mathbb{R}^m$, or Ω is a half-space of $\mathbb{R}^m$, or Ω is a smooth domain in $\mathbb{R}^m$ with a compact boundary $\partial\Omega$. We consider the Navier-Stokes equations

$$\begin{aligned} \nabla \cdot v &= 0 && \text{in } \Omega \ , \\ \partial_t v + (v \cdot \nabla) v - \nu \Delta v &= -\nabla p && \text{in } \Omega \ , \\ v &= 0 && \text{on } \partial\Omega \ , \\ v(\cdot, 0) &= v^0 && \text{in } \Omega \ . \end{aligned} \tag{1}$$

Of course, there is no boundary condition if $\Omega = \mathbb{R}^m$.

In a recent paper [1] we have investigated the strong solvability of (1) for initial data v^0 belonging to certain spaces of distributions (modulo gradients). In this note we explain some of our main results in a very particular and simple setting. As usual, we concentrate on the velocity field v since the pressure field p is determined up to a constant by v.

Function Spaces

We suppose that $1 < q < \infty$ and denote by H_q^k the Sobolev spaces $H_q^k(\Omega, \mathbb{R}^m)$ for $k \in \mathbb{N}$. We write $L_{q,\sigma}$ for the closure in $L_q = H_q^0$ of the space $\mathcal{D}_\sigma$ of all smooth solonoidal vector fields with compact supports in Ω. Moreover,

$$H_{q,0,\sigma}^2 := \begin{cases} \{\, u \in H_q^2 \ ; \ \nabla \cdot u = 0 \,\} \ , & \Omega = \mathbb{R}^m \ , \\ \{\, u \in H_q^2 \ ; \ \nabla \cdot u = 0, \ u | \partial\Omega = 0 \,\} \ , & \Omega \neq \mathbb{R}^m \ . \end{cases}$$

We write $B_{q,r}^s$ for the Besov spaces $B_{q,r}^s(\Omega, \mathbb{R}^m)$, $s \in \mathbb{R}$, $1 \leq r \leq \infty$, and refer to [13] for precise definitions. If $q > m$ then we put

$$B_{q',1,\sigma}^{1-m/q} := B_{q',1}^{1-m/q} \cap L_{q',\sigma}$$

and

$$B_{q,\infty,\sigma}^{-1+m/q} := (B_{q',1,\sigma}^{1-m/q})'$$

where $q' := q/(q-1)$ and the dual space is determined by means of the $L_{q,\sigma}$ duality pairing

$$(u, v) \mapsto \langle u, v \rangle := \int_\Omega u \cdot v \, dx \ , \qquad (u, v) \in L_{q',\sigma} \times L_{q,\sigma} \ .$$

Finally, we set

$$n_{q,\sigma}^{-1+m/q} := \begin{cases} L_{q,\sigma} & \text{if } q = m \ , \\ \text{closure of } L_{q,\sigma} \text{ in } B_{q,\infty,\sigma}^{-1+m/q} & \text{if } q > m \ . \end{cases}$$

If $q > m$ and $\Omega = \mathbb{R}^m$ then $n_{q,\sigma}^{-1+m/q}$ is simply the closure of $L_{q,\sigma}$, equivalently, of $\mathcal{D}_\sigma$, in $B_{q,\infty}^{-1+m/q}$. If $\partial\Omega \neq \emptyset$ then the situation is more complicated. To explain it we put

$$L_{q,\pi} := \left\{ v \in L_q \ ; \ \exists\, p \in L_{q,\mathrm{loc}}(\overline{\Omega}, \mathbb{R}^m) : v = \nabla p \right\} \ .$$

Then it is shown in [1] that $n_{q,\sigma}^{-1+m/q}$ is isometrically isomorphic to the closure of $L_q/L_{q,\pi}$ in $B_{q,\infty}^{-1+m/q}/(B_{q',1,\sigma}^{1-m/q})^\perp$ for $q > m$, where $(B_{q',1,\sigma}^{1-m/q})^\perp$ is the annihilator of the closed subspace $B_{q',1,\sigma}^{1-m/q}$ of $B_{q',1}^{1-m/q}$. Thus, loosely speaking, v belongs to $n_{q,\sigma}^{-1+m/q}$ for $q > m$ iff v is a distribution in $B_{q,\infty}^{-1+m/q}$ modulo gradients of functions in $L_{q,\mathrm{loc}}(\overline{\Omega}, \mathbb{R})$.

In [1] it is also shown that

$$L_{q,\sigma} \overset{d}{\hookrightarrow} n_{r,\sigma}^{-1+m/r} \overset{d}{\hookrightarrow} n_{s,\sigma}^{-1+m/s} \ , \qquad m < r < s < \infty \ , \tag{2}$$

where $\overset{d}{\hookrightarrow}$ denotes 'continuous and dense injection'.

Very Weak, Mild, and Strong Solutions

The solvability of (1) has been investigated by many authors under various hypotheses on v^0 and using several seemingly distinct concepts of weak solutions. (We refer to [1] for extensive discussions and references.) The following theorem shows that all these concepts coincide.

Suppose that $0 < T \leq \infty$ and $q \geq m$. By a **very weak q-solution** of (1) on $[0,T)$ we mean a function

$$v \in C\big([0,T), L_{q,\sigma}\big) \tag{3}$$

satisfying

$$\int_0^T \left\{ \big\langle (\partial_t + \nu\Delta)w, v \big\rangle + \langle \nabla w, v \otimes v \rangle \right\} dt = \big\langle w(0), v^0 \big\rangle$$

for all

$$w \in L_1\big((0,T), H_{q',0,\sigma}^2\big) \cap W_1^1\big((0,T), L_{q',\sigma}\big)$$

vanishing near T.

Denoting by $P : L_q \to L_{q,\sigma}$ the Helmholtz projector we recall that the Stokes operator $S := S_q$ in $L_{q,\sigma}$ is defined by $S := -\nu P\Delta \,|\, H_{q,0,\sigma}^2$. If (3) is satisfied then v is said to be a **mild solution** in $L_{q,\sigma}$ of (0.1) on $[0,T)$ if

$$v(t) = e^{-tS}v^0 - \int_0^t e^{-(t-\tau)S} P(v \otimes v)(\tau)\, d\tau \ , \qquad 0 \leq t < T \ ,$$

in $L_{q,\sigma}$.

Finally, by a **strong** q**-solution** of (0.1) on $[0, T)$ we mean a function

$$v \in C\big([0,T), n_{q,\sigma}^{-1+m/q}\big) \cap C\big((0,T), H_{q,0,\sigma}^{2}\big) \cap C^1\big((0,T), L_{q,\sigma}\big)$$

satisfying $v(0) = v^0$ and

$$\dot{v} + Sv = -P(v \cdot \nabla)v \ , \qquad 0 < t < T \ . \tag{4}$$

Note that (2) implies

$$C\big([0,T), L_{q,\sigma}\big) \hookrightarrow C\big([0,T), n_{q,\sigma}^{-1+m/q}\big) \ .$$

Theorem 1 *Suppose that $q \geq m$ and $v^0 \in L_{q,\sigma}$. Then the following are equivalent:*

(i) *v is a very weak q-solution on $[0, T)$.*

(ii) *v is a mild solution in $L_{q,\sigma}$ on $[0, T)$.*

(iii) *v is a strong q-solution in $C\big([0,T), L_{q,\sigma}\big)$.*

It should be remarked that the only related result known so far is due to Fabes, Jones, and Rivière [4]. These authors essentially proved the equivalence of (i) and (ii) in the case where $\Omega = \mathbb{R}^m$.

Uniqueness

Our next theorem guarantees uniqueness of very weak solutions.

Theorem 2 *If $q \geq m$ and $v^0 \in L_{q,\sigma}$ then* (1) *possesses at most one very weak q-solution on $[0, T)$.*

If $\Omega = \mathbb{R}^3$ then it has recently been shown by Monniaux [10] that there exists at most one mild m-solution. If $\partial\Omega \neq \emptyset$ then Lions and Masmoudi [9] have sketched a different proof for uniqueness of mild m-solutions. Our proof in [1] is rather simple, relying on maximal L_q-regularity if $q = m$.

Existence

Now we turn to existence and present the following general result.

Theorem 3 *Suppose that $q \geq m$ and $v^0 \in n_{q,\sigma}^{-1+m/q}$.*

(i) *The Navier-Stokes equations possess a unique maximal strong q-solution satisfying*

$$t^{(1-m/q)/2} v_q(t) \to 0 \ \text{in} \ L_q \quad \text{as} \ t \to 0$$

if $q > m$.

(ii) *If $v^0 \in L_{q,\sigma}$ then*

$$v_q \in C\big([0, t_q^+), L_{q,\sigma}\big) \ ,$$

where t_q^+ is the maximal existence time.

(iii) *For each $T > 0$ there exists $R > 0$ such that $t_q^+ > T$ whenever v^0 satisfies*

$$\|v^0\|_{n_{q,\sigma}^{-1+m/q}} \leq R \ .$$

(iv) $v \in C\big(\overline{\Omega} \times (0, t_q^+), \mathbb{R}^m\big)$.

This theorem extends and simplifies corresponding existence results due to Kato [6], Giga and Miyakawa [5], Kobayashi and Muramatu [7], and others (see [1] for extensive references and the relation of our work to previous results). In particular, it should be noted that v_q is the unique strong q-solution in $C\big([0, t_q^+), L_{q,\sigma}\big)$ if $v^0 \in L_{q,\sigma}$, as follows from Theorems 1 and 2.

Theorem 3 and (2) imply that, given $r > q$, problem (1) has a unique maximal strong r-solution v_r on the maximal interval of existence $[0, t_r^+)$. In [1] it is shown that $t_r^+ \geq t_q^+$ and $v_r \supset v_q$. Denote by $n_{\infty,\sigma}^{-1}$ the inductive limit of the spaces $n_{r,\sigma}^{-1+m/r}$, $r \geq m$, and set $t^+ := \sup\{\, t_r^+ \ ; \ r \geq q \,\}$. Then it follows that there is a unique function

$$v : [0, t^+) \to n_{\infty,\sigma}^{-1}$$

such that $v \,|\, [0, t_r^+) = v_r$ for $q \leq r < \infty$. This function is the unique **maximal strong solution** of (1).

Clearly, v satisfies (4) on $(0, t^+)$ and $v(t) \to v^0$ in $n_{q,\sigma}^{-1+m/q}$ as $t \to 0$. Moreover, v is smooth on $\overline{\Omega} \times (0, t^+)$.

Global Existence

Although Theorem 3 guarantees the existence of a unique maximal strong solution on an arbitrarily large interval for small initial data, it does not imply that v is a global solution, that is, $t^+ = \infty$. This fact can be established, however, provided Ω is bounded and v^0 is small in $n_{\infty,\sigma}^{-1}$. Here and below $\|\cdot\|_q$ is the norm in L_q.

Theorem 4 *Suppose that Ω is bounded, $q \geq m$, and $v^0 \in n_{q,\sigma}^{-1+m/q}$. Given $r \geq q$, there exists $R > 0$ such that $t^+ = \infty$, provided*

$$\|v^0\|_{n_{r,\sigma}^{-1+m/r}} \leq R \ . \tag{5}$$

Furthermore, there exists $\omega > 0$ such that

$$\|v(t)\|_r \leq c e^{-\omega t} \ , \qquad t \geq 0 \ .$$

Suppose that $v^0 \in L_{r,\sigma}$. Then it follows from the fact that $n_{r,\sigma}^{-1+m/r}$ is isometrically isomorphic to a closed subspace of $B_{r,\infty}^{-1+m/r} / (B_{r',1,\sigma}^{1-m/r})^{\perp}$ that condition (5) is satisfied, provided

$$\|v^0\|_{B_{r,\infty}^{-1+m/r}} \leq R \ .$$

This shows that Theorem 4 is the analogue for bounded domains of a recent result which has been proven by Cannone [2] if $\Omega = \mathbb{R}^3$ and by Cannone, Planchon, and Schonbek [3] if Ω is a half-space of $\mathbb{R}^3$. The proofs of these authors rely heavily on the fact that there are rather explicit representations of the Stokes semigroup as well as of the Helmholtz projector in the situations they consider. This is not the case for bounded domains. Thus our approach is rather different and is based on semigroup, inter- and extrapolation theories.

Leray-Hopf Weak Solutions

If $v^0 \in L_{2,\sigma}$ then u is said to be a **weak solution** on $[0,T)$ of (1), provided

$$u \in L_\infty\big((0,T), L_{2,\sigma})\big) \cap L_2\big((0,T), H_2^1\big)$$

and

$$\int_0^T \big\{-\langle\dot\varphi, u\rangle + \nu\langle\nabla\varphi, \nabla u\rangle + \big\langle\varphi, (u\cdot\nabla)u\big\rangle\big\}\,dt = \big\langle\varphi(0), v^0\big\rangle$$

for all $\varphi \in \mathcal{D}\big([0,T), \mathcal{D}_\sigma\big)$. The function $u\colon \mathbb{R}^+ \to L_{2,\sigma}$ is a **Leray-Hopf weak solution** of (1) if $u\,|\,[0,T)$ is a weak solution on $[0,T)$ for every $T > 0$ and if u satisfies the energy inequality

$$\|u(t)\|_2^2 + 2\nu\int_0^t \|\nabla u(\tau)\|_2^2\,d\tau \le \|v^0\|_2^2\ , \qquad t > 0\ .$$

Recall that it is well-known that there exists at least one Leray-Hopf weak solution of (1).

The following theorem establishes the relations between the maximal strong solution v and Leray-Hopf weak solutions.

Theorem 5 *Suppose that $v^0 \in L_{2,\sigma} \cap L_{q,\sigma}$ for some $q \ge m$.*

(i) *The unique maximal strong solution v of (1) is a weak solution on $[0,T)$ for every $T < t^+$ and belongs to $C\big([0,t^+), L_2\big)$.*

(ii) *If u is any Leray-Hopf weak solution then $u \supset v$. In particular, u is smooth and unique on $(0,t^+)$.*

This theorem guarantees local uniqueness and smoothness of Leray-Hopf weak solutions without further restrictions. In particular, if v exists globally then there is a unique Leray-Hopf weak solution and it is smooth for $t > 0$. This is in contrast to known uniqueness theorems of Serrin [11], Fabes, Jones, and Rivière [4], Sohr and von Wahl [12], Kozono and Sohr [8], and others, which are conditional in the sense that they require the solutions to belong to more restricted classes.

The proofs of the above theorems are given in [1] together with many additional details. In particular, there are precise descriptions of the function spaces related to Navier-Stokes equations and being useful in precise regularity statements. These descriptions are also essential for the derivation of precise mapping properties of the convection term, a result which is basic for establishing the sharp results given above. In addition, we consider more general domains, the case where $m/3 < q < m$, and non-vanishing exterior forces.

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