QUASILINEAR PARABOLIC PROBLEMS VIA MAXIMAL REGULARITY

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Abstract. We use maximal L_p regularity to study quasilinear parabolic evolution equations. In contrast to all previous work we only assume that the nonlinearities are defined on the space in which the solution is sought for. It is shown that there exists a unique maximal solution depending continuously on all data, and criteria for global existence are given as well. These general results possess numerous applications, some of which will be discussed in separate publications.

Introduction

In this paper we develop a general existence, uniqueness, continuity, and differentiability theory for semilinear parabolic evolution equations of the form

$$\dot{u} + A(u)u = F(u) \text{ on } (0, \mathsf{T}), \quad u(0) = x, \tag{0.1}$$

where T is a given positive real number. This problem has already been treated by many authors, including ourselves (e.g., [1], [13], [14], [16], and the references therein). The main new feature of our present work, which distinguishes it from all previous investigations, is the fact that we use maximal L_p regularity in its full strength. This means that A and F are defined on the smallest possible space only, namely on that class of functions, more precisely: distributions, in which the solution of (0.1) is being sought for. Thus, in contrast to the earlier work, where the maps $A(\cdot)$ and $F(\cdot)$ are assumed to be defined on larger spaces carrying weaker topologies than the solution spaces, there is only one domain space over $(0, \mathsf{T})$ entering the investigation of (0.1). This fact allows for great flexibility in concrete applications, encompassing, in particular, problems being nonlocal in time, and leads to optimal results.

To be more precise we need some notation. Throughout, we assume that E_0 and E_1 are (real or complex) Banach spaces such that $E_1 \overset{d}{\hookrightarrow} E_0$, where $\hookrightarrow$ denotes continuous injection and the superscript d indicates that E_1 is also dense in E_0. (This notation is also used in the case of general topological spaces.) Henceforth, we always suppose that $0 < T \leq \mathsf{T}$ and put $J_T := [0, T)$ as well as $\mathsf{J} := J_{\mathsf{T}}$. We also use J to denote any one of the intervals J_T with $0 < T \leq \mathsf{T}$.

Throughout we suppose that $1 < p < \infty$. By $H_p^1(\mathring{J}, E_0)$ we denote the Sobolev space of all $u \in L_p(\mathring{J}, E_0)$ whose first order distributional derivative is in $L_p(\mathring{J}, E_0)$ as well, endowed with its usual norm. Then we put

$$\mathcal{H}_p^1(J, (E_1, E_0)) := L_p(J, E_1) \cap H_p^1(\mathring{J}, E_0).$$

We also set

$$E := (E_0, E_1)_{1/p', p},$$

$(\cdot, \cdot)_{\theta, r}$ denoting the real interpolation functor of exponent $\theta \in (0, 1)$ and (integrability) parameter $r \in [1, \infty]$. It is known that

$$\mathcal{H}_p^1(J, (E_1, E_0)) \hookrightarrow C(\overline{J}, E), \tag{0.2}$$

(cf. [2, Theorem III.4.10.2]).

As usual, if X and Y are Banach spaces (more generally, locally convex spaces), then we write $\mathcal{L}(\mathsf{X}, \mathsf{Y})$ for the space of all continuous linear maps from X into Y, and $\mathcal{L}(\mathsf{X}) := \mathcal{L}(\mathsf{X}, \mathsf{X})$. Recall that $\mathcal{L}(\mathsf{X}, \mathsf{Y})$ is a Banach space with the standard uniform operator norm, if X and Y are Banach spaces.

Given a map $A : J \to \mathcal{L}(E_1, E_0)$, we identify it with its point-wise extension

$$E_1^J \to E_0^J, \quad u \mapsto Au,$$

defined by

$$(Au)(t) := A(t)u(t), \qquad u \in E_1^J, \quad t \in J.$$

Then it follows that

$$\begin{aligned} L_\infty(J, \mathcal{L}(E_1, E_0)) &\hookrightarrow \mathcal{L}(L_p(J, E_1), L_p(J, E_0)) \\ &\hookrightarrow \mathcal{L}(\mathcal{H}_p^1(J, (E_1, E_0)), L_p(J, E_0)). \end{aligned} \tag{0.3}$$

We assume that

$$A(u) \in L_\infty(\mathsf{J}, \mathcal{L}(E_1, E_0)), \qquad u \in \mathcal{H}_p^1(\mathsf{J}, (E_1, E_0)).$$

Note that this means that the linear map

$$\mathcal{H}_p^1(\mathsf{J}, (E_1, E_0)) \to L_p(\mathsf{J}, E_0), \quad v \mapsto A(u)v$$

is for each $u \in \mathcal{H}_p^1\big(\mathsf{J},(E_1,E_0)\big)$ a local operator, whereas the nonlinear function $u \mapsto A(u)$ may be nonlocal.

We also assume that $A(u)$ has for each $u \in \mathcal{H}_p^1\big(\mathsf{J},(E_1,E_0)\big)$ the property of maximal L_p regularity, meaning that the linear evolution equation

$$\dot{v} + A(u)v = f \text{ on } (0,\mathsf{T}), \quad v(0) = 0,$$

has for each $f \in L_p(\mathsf{J},E_0)$ exactly one solution $v \in \mathcal{H}_p^1\big(\mathsf{J},(E_1,E_0)\big)$.

We further assume that

$$F(u) \in L_p(\mathsf{J},E_0), \qquad u \in \mathcal{H}_p^1\big(\mathsf{J},(E_1,E_0)\big),$$

and that (A,F) is suitably Lipschitz continuous in a sense made precise below and implying, in particular, that F is subordinate to A.

These assumptions imply that $\dot{u} + A(u)u - F(u)$ is for each u belonging to $\mathcal{H}_p^1\big(\mathsf{J},(E_1,E_0)\big)$ a well defined element of $L_p(\mathsf{J},E_0)$. In order to obtain a reasonable evolution problem we have to add one more assumption. Namely, we suppose that A and F are Volterra maps which means that the restriction of $\big(A(u),F(u)\big)$ to any J depends on $u\,|\,J$ only.

Given these assumptions, we show that (0.1) possesses a unique maximal solution and we give conditions for global existence. In addition, we prove that, under natural continuity hypotheses, the solution depends continuously — or even differentiably — on all data. This fact, which is in the present setting much more difficult to obtain than under standard hypotheses, is important for qualitative studies, in particular in connection with control problems.

Quasilinear parabolic evolution equations in the framework of maximal L_p regularity have already been studied by Clément and Li [9] (in a particular concrete setting) and by Prüss [15]. However, these authors assume that $A(\cdot)$ and $F(\cdot)$ are local maps from E into E_0 (in which case the Volterra property is automatic, of course). These assumptions impose serious restrictions in concrete applications which are not necessary by employing our approach. We also refer to Clément and Simonett [10] for a study of (0.1) using maximal regularity in continuous interpolation spaces, extending and improving on earlier results of Da Prato and Grisvard [11] and Angenent [7], as well as to the book by Lunardi [14] who bases her studies on maximal regularity in Hölder spaces (also see [2, Chapter III] for a detailed exposition of the various maximal regularity theories). In all those papers, as well as in many others devoted to concrete parabolic equation, it is always assumed that A and F are local operators mapping the corresponding trace space (or an appropriate superspace thereof) into E_0. Such a setting precludes the study of problems which are nonlocal in time.

This paper consists of two parts. In Part 1 we introduce precise hypotheses and present our general abstract results. It is the purpose of this paper to lay the abstract foundations for the local theory of quasilinear parabolic problems. Our abstract results possess numerous applications in various concrete settings. The general continuity and differentiability theorems are of particular importance in qualitative considerations and control theory. In order not to overburden this paper, applications will be presented in separate publications (also see [4], [6]). The second part contains the proofs of the abstract theorems.

Part 1. Main results

First we introduce precise assumptions before, in the second section, we present the existence and uniqueness theorem for problem (0.1). Section 3 is devoted to the continuity theorem. In the next section we describe the correct concept of differentiability in locally convex spaces and formulate the third main result of this paper, guaranteeing that the solution of (0.1) depends continuously differentiably on all data, provided, of course, the latter are suitably smooth. In Section 5 we give some simple extensions of our general results to systems.

For the sake of relatively simple statements, throughout this paper we use rather condensed notation which requires quite a bit of attention from the reader.

1. Assumptions

Since (E_0, E_1) is fixed during this part, we put

$$\mathcal{H}_p^1(J) := \mathcal{H}_p^1\big(J, (E_1, E_0)\big).$$

Suppose that

$$B \in \mathcal{L}\big(\mathcal{H}_p^1(J), L_p(J, E_0)\big).$$

Given $f \in L_p(J, E_0)$ and $x \in E$, by a **solution** of the linear Cauchy problem

$$\dot{u} + Bu = f \text{ on } \mathring{J}, \quad u(0) = x \tag{1.1$_x$}$$

we mean a **strong** (L_p) **solution**, that is, a function u belonging to $\mathcal{H}_p^1(J)$ and satisfying $(1.1)_x$. Recall that each $u \in H_p^1(\mathring{I}, E_0)$ is an absolutely continuous E_0 valued function, and its derivative $\dot{u}$ in the a.e. sense coincides with its distributional derivative ∂u, (e.g., [8, § 2]). Thus the differential equation in $(1.1)_x$ can be understood either in the sense of E_0 valued distributions on $\mathring{J}$ or in the a.e. sense. The initial condition is meaningful, due to (0.2).

The map B is said to possess (the property of) **maximal** (L_p) **regularity** on J (with respect to (E_1, E_0)) if $(1.1)_0$ has for each $f \in L_p(J)$ exactly one strong L_p solution on J. We denote by

$$\mathcal{MR}_p(J) := \mathcal{MR}_p\big(J, (E_1, E_0)\big)$$

the set of all $B \in L_\infty\big(J, \mathcal{L}(E_1, E_0)\big)$ possessing the property of maximal L_p regularity on J with respect to (E_1, E_0).

We write $\mathcal{MR}(E_1, E_0)$ for the set of all $C \in \mathcal{L}(E_1, E_0)$ such that the constant map $t \mapsto C$ belongs to $\mathcal{MR}_p\big(J, (E_1, E_0)\big)$. (This notation is justified since the property of maximal regularity for constant maps is independent of (bounded) intervals and of p; see [5, Remarks 6.1(d) and (e)]). It is known (see [12]) that each $C \in \mathcal{MR}(E_1, E_0)$ is the negative infinitesimal generator of a strongly continuous analytic semigroup on E_0. Furthermore, $A \in C\big(\overline{J}, \mathcal{L}(E_1, E_0)\big)$ belongs to $\mathcal{MR}_p\big(J, (E_1, E_0)\big)$ iff $A(t) \in \mathcal{MR}(E_1, E_0)$ for each $t \in \overline{J}$ (cf. Proposition 7.1 in [5]). This shows that the maximal regularity hypothesis restricts the class of evolution equations being studied here to (abstract) parabolic equations.

Of course, $\mathcal{MR}_p\big(J, \mathcal{L}(E_1, E_0)\big)$ and $\mathcal{MR}(E_1, E_0)$ are given the topologies induced by $L_\infty\big(J, \mathcal{L}(E_1, E_0)\big)$ and $\mathcal{L}(E_1, E_0)$, respectively. We assume throughout that $\mathcal{MR}(E_1, E_0) \neq \emptyset$.

Let X and Y be nonempty sets. A function $f : X^J \to Y^J$ is a **Volterra map** (or has the Volterra property) if, for each $T \in \mathring{J}$ and each pair $u, v \in X^J$ with $u \,|\, J_T = v \,|\, J_T$, it follows that $f(u) \,|\, J_T = f(v) \,|\, J_T$. For a given space $\mathfrak{F}$ of maps $X^J \to Y^J$ we denote by $\mathfrak{F}_{\mathrm{Volt}}$ the subset of all Volterra maps in $\mathfrak{F}$.

Let X and Y be metric spaces. Then $\mathcal{C}^{1\text{-}}(X, Y)$ is the space of all maps from X into Y which are bounded on bounded sets and uniformly Lipschitz continuous on such sets. If Y and Y_0 are Banach spaces such that $\mathsf{Y} \hookrightarrow \mathsf{Y}_0$, then we denote by $\mathcal{C}^{1\text{-}}(X; \mathsf{Y}, \mathsf{Y}_0)$ the set of all $f : X \to \mathsf{Y}_0$ such that $f - f(0) \in \mathcal{C}^{1\text{-}}(X, \mathsf{Y})$. Note that $\mathcal{C}^{1\text{-}}(X; \mathsf{Y}, \mathsf{Y}) = \mathcal{C}^{1\text{-}}(X, \mathsf{Y})$. If X is finite dimensional, then $\mathcal{C}^{1\text{-}}(\mathsf{X}, \mathsf{Y}) = C^{1\text{-}}(\mathsf{X}, \mathsf{Y})$, the space of all (locally) Lipschitz continuous maps from X into Y.

After these preparations we can introduce the following hypotheses:

$$\left.\begin{array}{l} \bullet\quad A \in \mathcal{C}^{1\text{-}}_{\mathrm{Volt}}\big(\mathcal{H}^1_p(\mathsf{J}), \mathcal{MR}_p(\mathsf{J})\big); \\ \bullet\quad p < q \le \infty; \\ \bullet\quad F \in \mathcal{C}^{1\text{-}}_{\mathrm{Volt}}\big(\mathcal{H}^1_p(\mathsf{J}); L_q(\mathsf{J}, E_0), L_p(\mathsf{J}, E_0)\big). \end{array}\right\} \tag{1.2}$$

2. Existence and uniqueness

Consider the quasilinear Cauchy problem (0.1). By a **solution** on J we mean a $u \in \mathcal{H}^1_{p,\mathrm{loc}}(J)$ such that $u\,|\,J_T$ belongs to $\mathcal{H}^1_p(J_T)$ for each $T \in \mathring{J}$ and is a solution of the linear Cauchy problem

$$\dot v + A(u)v = F(u) \text{ on } (0,T), \quad v(0) = x.$$

A solution is **maximal** if it cannot be extended to a solution on a strictly larger interval. In this case its domain is a **maximal interval of existence** for (0.1). A solution is **global** if it is defined on all of J.

Now we can formulate the following general existence and uniqueness result whose proof is given in Section 10.

Theorem 2.1. *Let assumption* (1.2) *be satisfied. Then* (0.1) *possesses a unique maximal solution u. The maximal interval of existence, $J_{\max}$, is open in J. If $u \in \mathcal{H}^1_p(J_{\max})$, then $J_{\max} = \mathsf{J}$.*

Recall that the maximal solution u belongs to $\mathcal{H}^1_{p,\mathrm{loc}}(J_{\max})$. The last part of Theorem 2.1 shows that L_p integrability of u and $\dot u$ on $J_{\max}$ guarantees already that u is global. The following corollary gives further sufficient conditions for global existence which are useful in concrete applications.

Corollary 2.2. *Let u be the unique maximal solution of* (0.1)*. If*

(i) $A(u) \in \mathcal{MR}_p(J_{\max})$,
(ii) $F(u) \in L_p(J_{\max}, E_0)$,

then $J_{\max} = \mathsf{J}$.

Proof. Assumptions (i) and (ii) guarantee that the linear Cauchy problem

$$\dot v + A(u)v = F(u) \text{ on } J_{\max}, \quad v(0) = x$$

has a unique solution $v \in \mathcal{H}^1_p(J_{\max})$ (cf. Lemma 6.1). The Volterra property of A and F and Lemma 6.2 imply that $u\,|\,J_T$ is for each $T \in \mathring{J}_{\max}$ the unique solution of

$$\dot w + A(u)w = F(u) \text{ on } J_T, \quad w(0) = x. \tag{2.1}$$

Due to the Volterra property, we also see that $v\,|\,J_T$ is a solution of (2.1) as well. Thus $v\,|\,J_T = u\,|\,J_T$ for $T \in \mathring{J}_{\max}$. Hence $u = v \in \mathcal{H}^1_p(J_{\max})$. □

3. Continuity

The unique maximal solution of (0.1), whose existence is guaranteed by Theorem 2.1, depends Lipschitz continuously on all data. For a precise formulation of this result we introduce concise notation.

We write B_{X} for the open unit ball in the Banach space X. Hence, using standard notation, $B_{\mathsf{X}}(x,r) := x + rB_{\mathsf{X}}$ is the open ball in X with center at x and radius $r > 0$. We also set $\mathcal{B}_T := B_{\mathcal{H}_p^1(J_T)}$.

Henceforth, we fix $q \in (p,\infty]$, put

$$\Phi(J) := \mathcal{C}^{1\text{-}}_{\mathrm{Volt}}\bigl(\mathcal{H}_p^1(J), \mathcal{MR}_p(J)\bigr) \times \mathcal{C}^{1\text{-}}_{\mathrm{Volt}}\bigl(\mathcal{H}_p^1(J); L_q(J,E_0), L_p(J,E_0)\bigr),$$

and denote a general point of $\Phi(J)$ by $\varphi = (A,F)$. We also set

$$\mathsf{Z}_r(J) := L_\infty\bigl(J, \mathcal{L}(E_1,E_0)\bigr) \times L_r(J,E_0), \qquad p \le r \le \infty,$$

and $\mathsf{Z}(J) := \mathsf{Z}_p(J)$. Given $\varphi \in \Phi(J_T)$ and $R > 0$, there exists $K > 0$ such that

$$\|\varphi(0)\|_{\mathsf{Z}(J_T)} \le K \tag{3.1}$$

and

$$\|\varphi(u) - \varphi(v)\|_{\mathsf{Z}_q(J_T)} \le K\,\|u - v\|_{\mathcal{H}_p^1(J_T)}, \qquad u,v \in R\mathcal{B}_T. \tag{3.2}$$

The set of all such φ is denoted by $\Phi(J_T;R,K)$. For $\varphi_0 \in \Phi(J_T)$ and $\varepsilon > 0$ we write $\mathcal{V}(J_T;\varphi_0,R,K,\varepsilon)$ for the set of all $\varphi \in \Phi(J_T;R,K)$ satisfying

$$\sup_{u \in R\mathcal{B}_T} \|\varphi(u) - \varphi_0(u)\|_{\mathsf{Z}(J_T)} \le \varepsilon.$$

Given $\varphi \in \Phi(\mathsf{J})$, we denote the unique maximal solution of (0.1) by $u(\varphi,x)$, write $J(\varphi,x)$ for the corresponding maximal interval of existence, and put

$$t^+(\varphi,x) := \sup J(\varphi,x).$$

Then $J(\varphi,x) = \bigl[0,t^+(\varphi,x)\bigr)$ by Theorem 2.1.

Now we can formulate the following general continuity result whose proof is given in Section 11.

Theorem 3.1. *Assume $(\varphi_0,x_0) \in \Phi(\mathsf{J}) \times E$. If $u(\varphi_0,x_0)$ belongs to $\mathcal{H}_p^1(\mathsf{J})$, then put $S := \mathsf{T}$. Otherwise fix any positive $S < t^+(\varphi_0,x_0)$. Set*

$$u_0 := u(\varphi_0,x_0)\,|\,J_S.$$

Then, given $R > \|u_0\|_{\mathcal{H}_p^1(J_S)}$, there exist positive constants K, ρ, and ε such that

$$J(\varphi,x) \supset [0,S], \qquad (\varphi,x) \in \mathcal{V}(J_S;\varphi_0,R,K,\varepsilon) \times B_E(x_0,\rho).$$

Moreover, there is a κ such that, setting $u_j := u(\varphi_j,x_j)$,

$$\|u_1 - u_2\|_{\mathcal{H}_p^1(J_S)} \le \kappa\bigl(\|(\varphi_1-\varphi_2)(u_1)\|_{\mathsf{Z}(J_S)} + \|x_1 - x_2\|_E\bigr) \tag{3.3}$$

for $(\varphi_1,x_1),(\varphi_2,x_2) \in \mathcal{V}(J_S;\varphi_0,R,K,\varepsilon) \times B_E(x_0,\rho)$.

Remark 3.2. Observe that $\Phi(J)$ is a subset of the Fréchet space

$$\mathcal{C}^{1\text{-}}\big(\mathcal{H}_p^1(J), L_\infty\big(J, \mathcal{L}(E_1, E_0)\big)\big) \times \mathcal{C}^{1\text{-}}\big(\mathcal{H}_p^1(J); L_q(J, E_0), L_p(J, E_0)\big), \qquad (3.4)$$

which is continuously embedded in the Fréchet space

$$\mathcal{C}\big(\mathcal{H}_p^1(J), L_\infty\big(J, \mathcal{L}(E_1, E_0)\big) \times L_p(J, E_0)\big) = \mathcal{C}\big(\mathcal{H}_p^1(J), \mathsf{Z}(J)\big), \qquad (3.5)$$

where $\mathcal{C}$ is the space of continuous functions which are bounded on bounded sets. Thus $\Phi(J)$ carries the $\mathcal{C}^{1\text{-}}$ topology induced by (3.4) as well as the weaker $\mathcal{C}$ topology induced by (3.5). Using these topologies, Theorem 3.1 shows that the solution map $(\varphi, x) \mapsto u(\varphi, x)$ is locally Lipschitz continuous in the following sense: For each $\varphi_0 \in \Phi$ and $x_0 \in E$ there exists a $\mathcal{C}^{1\text{-}}$ neighborhood $\mathcal{V}$ of φ_0 in Φ and a neighborhood W of x_0 in E such that

$$\mathcal{V} \times W \to \mathcal{H}_p^1(J_S), \quad (\varphi, x) \mapsto u(\varphi, x)$$

is well defined and uniformly Lipschitz continuous with respect to the $\mathcal{C}$ topology on $\mathcal{V}$. $\square$

Corollary 3.3. *The maximal existence time, that is, the map*

$$t^+ : \ \Phi(\mathsf{J}) \times E \to (0, \mathsf{T}],$$

is lower semicontinuous with respect to the $\mathcal{C}^{1\text{-}}$ *topology on* $\Phi(\mathsf{J})$.

4. Differentiabiliy

Given slightly stronger continuity requirements for φ, the solution map is even Gateaux differentiable. For this we need further explanation.

We denote by $\mathcal{L}^2(\mathsf{X}, \mathsf{Y})$ the Banach space of all continuous bilinear maps from X into Y, equipped with its usual norm.

Let $A : \mathcal{H}_p^1(J) \to \mathcal{L}\big(\mathcal{H}_p^1(J), L_p(J, E_0)\big)$ be differentiable at $u_0 \in \mathcal{H}_p^1(J)$. Then $DA(u_0)$, its derivative at u_0, is a bounded linear operator from $\mathcal{H}_p^1(J)$ into the space $\mathcal{L}\big(\mathcal{H}_p^1(J), L_p(J, E_0)\big)$. Hence

$$DA(u_0)[u, v] := \big(DA(u_0)u\big)v \in L_p(J, E_0), \qquad u, v \in \mathcal{H}_p^1(J).$$

Thus, by identifying $DA(u_0)$ with $(u, v) \mapsto DA(u_0)[u, v]$, it follows that

$$DA(u_0) \in \mathcal{L}^2\big(\mathcal{H}_p^1(J), L_p(J, E_0)\big).$$

If A is differentiable in a neighborhood U of u_0 in $\mathcal{H}_p^1(J)$, then it is continuously differentiable if

$$DA := \big(u \mapsto DA(u)\big) \in C\big(U, \mathcal{L}^2\big(\mathcal{H}_p^1(J), L_p(J, E_0)\big)\big).$$

Let W be an open subset of some locally convex space W. Then a function $f : W \to \mathsf{X}$ is G differentiable at $w \in W$ if there exists $Df(w) \in \mathcal{L}(\mathsf{W}, \mathsf{X})$ such that

$$Df(w)\omega = \lim_{s\to 0}\bigl(f(w+s\omega) - f(w)\bigr)/s, \qquad \omega \in \mathsf{W}.$$

It is continuously G differentiable in W if it is G differentiable at each $w \in W$ and

$$Df := \bigl(w \mapsto Df(w)\bigr) \in C\bigl(W, \mathcal{L}(\mathsf{W}, \mathsf{X})\bigr),$$

where $\mathcal{L}(\mathsf{W}, \mathsf{X})$ is given the topology of uniform convergence on bounded subsets of W. If W is a Banach space, then, as is well known, f is continuously G differentiable iff it is continuously differentiable in the usual (Fréchet) sense.

We denote by $\mathcal{C}^1\bigl(\mathsf{X}, \mathsf{Y}\bigr)$ the vector space of all $f : \mathsf{X} \to \mathsf{Y}$ such that f belongs to $\mathcal{C}(\mathsf{X}, \mathsf{Y})$ and Df to $\mathcal{C}\bigl(\mathsf{X}, \mathcal{L}(\mathsf{X}, \mathsf{Y})\bigr)$. It is a Fréchet space with the topology of uniform convergence on bounded sets of the functions and their derivatives. Note that

$$\mathcal{C}^1(\mathsf{X}, \mathsf{Y}) \hookrightarrow \mathcal{C}^{1\text{-}}(\mathsf{X}, \mathsf{Y}), \tag{4.1}$$

as follows from the mean value theorem.

Now, recalling (0.3), we can formulate the differentiability theorem for solutions of (0.1). Its proof is postponed to Section 12.

Theorem 4.1. *Suppose that* $(\varphi_0, x_0) \in \Phi(\mathsf{J}) \times E$. *If* $u(\varphi_0, x_0)$ *is in* $\mathcal{H}_p^1(\mathsf{J})$, *then put* $S = \mathsf{T}$. *Otherwise fix any positive* $S < t^+(\varphi_0, x_0)$. *Also suppose that* $\varphi_0 \in \mathcal{C}^1_{\mathrm{Volt}}\bigl(\mathcal{H}_p^1(J_S), \mathsf{Z}(J_S)\bigr)$, *set* $u_0 := u(\varphi_0, x_0)\,|\,J_S$, *and assume that*

$$B_0(u_0) := A_0(u_0) + DA_0(u_0)[\cdot, u_0] - DF_0(u_0) \in \mathcal{L}\bigl(\mathcal{H}_p^1(J_S), L_p(J_S, E_0)\bigr) \tag{4.2}$$

has the property of maximal L_p *regularity.*

Then there exists a neighborhood $\mathcal{U}$ *of* (φ_0, x_0) *in* $\mathcal{C}^1\bigl(\mathcal{H}_p^1(J_S), \mathsf{Z}(J_S)\bigr) \times E$ *such that problem* (0.1) *has for each* $(\varphi, x) \in \mathcal{U}$ *a unique solution* $u(\varphi, x)$ *in* $\mathcal{H}_p^1(J_S)$. *The solution map*

$$\mathcal{U} \to \mathcal{H}_p^1(J_S), \quad (\varphi, x) \mapsto u(\varphi, x) \tag{4.3}$$

is continuously G differentiable. Given $(\varphi, x) \in \mathcal{U}$ *and any* $(\widehat{\varphi}, \widehat{x})$ *belonging to* $\mathcal{C}^1\bigl(\mathcal{H}_p^1(J_S), \mathsf{Z}(J_S)\bigr) \times E$, *the G derivative of* (4.3) *at* (φ, x) *in the direction* $(\widehat{\varphi}, \widehat{x})$, *that is,* $Du(\varphi, x)(\widehat{\varphi}, \widehat{x})$, *is the unique solution in* $\mathcal{H}_p^1(J_S)$ *of the* *linearized Cauchy problem*

$$\begin{aligned}\dot v + A\bigl(u(\varphi,x)\bigr)v + DA\bigl(u(\varphi,x)\bigr)\,[v,u(\varphi,x)]\\ = \widehat F\bigl(u(\varphi,x)\bigr) + DF\bigl(u(\varphi,x)\bigr)v - \widehat A\bigl(u(\varphi,x)\bigr)u(\varphi,x)\end{aligned}$$

on $(0,S)$*, with* $v(0)=\widehat x$*.*

Remarks 4.2. (a) In general, $B(u_0)\notin L_\infty\bigl(J_S,\mathcal L(E_1,E_0)\bigr)$ so that the maximal regularity assumption (4.2) does not say that $B(u_0)\in\mathcal{MR}_p(J_S)$.

(b) If $(\varphi,x)\in\bigl(\Phi(\mathsf J)\times E\bigr)\cap\mathcal U$ then, by uniqueness, the solution $u(\varphi,x)$ of Theorem 4.1 coincides with the restriction to J_S of the corresponding maximal solution of (0.1) guaranteed by Theorem 2.1. Thus it is justified to use the symbol $u(\varphi,x)$ in either case.

(c) It should be observed that there are no additional requirements like the Volterra property, for example, for the elements in $\mathcal U\backslash\bigl\{(\varphi_0,x_0)\bigr\}$. □

Remark 4.3. Suppose that $\mathcal X$ is a nonempty subset of E and that we are only interested in solving (0.1) for $x\in\mathcal X$. Then it is not necessary that φ be defined on all of $\mathcal H_p^1(\mathsf J)$. To be more precise, set

$$\mathcal H_p^1(\mathsf J;\mathcal X):=\bigl\{\,v\in\mathcal H_p^1(\mathsf J)\ ;\ v(0)\in\mathcal X\,\bigr\},$$

endowed with the metric induced by the norm of $\mathcal H_p^1(\mathsf J)$. Assume, instead of (1.2), that

- $A\in\mathcal C^{1\text{-}}_{\mathrm{Volt}}\bigl(\mathcal H_p^1(\mathsf J;\mathcal X),\mathcal{MR}_p(\mathsf J)\bigr)$;
- $p<q\le\infty$;
- $F\in\mathcal C^{1\text{-}}_{\mathrm{Volt}}\bigl(\mathcal H_p^1(\mathsf J;\mathcal X);L_p(\mathsf J,E_0),L_q(\mathsf J,E_0)\bigr)$.

Then Theorems 2.1–4.1 as well as Corollaries 2.2 and 3.3 remain valid, with the obvious modifications, if x is restricted to belong to $\mathcal X$. In particular, the value 0 in (3.1) and the ball $R\mathcal B_T$ in (3.2) have to be replaced by an arbitrary, but fixed, $u_0\in\mathcal H_p^1(J_T;\mathcal X)$ and the ball $u_0+R\mathcal B_T$, respectively.

Proof. This follows from the proofs of those theorems. □

5. Systems

In this section we present simple but useful extensions of the preceding results to systems.

Let $E_{0,j}$ and $E_{1,j}$ be Banach spaces such that $E_{1,j}\overset{d}{\hookrightarrow}E_{0,j}$ for $j=1,\dots,N$. Set $E_k:=\prod_{j=1}^N E_{k,j}$ for $k=0,1$, endowed with the ℓ_2 norm. Then E_0 and E_1 are Banach spaces satisfying $E_1\overset{d}{\hookrightarrow}E_0$.

Suppose that $1 < p_j < \infty$ for $j = 1, 2, \ldots, N$ and put

$$\mathcal{H}^1_{\vec{p}}(J) := \mathcal{H}^1_{\vec{p}}\big(J, (E_1, E_0)\big) := \prod_{j=1}^N \mathcal{H}^1_{p_j}\big(J, (E_{1,j}, E_{0,j})\big),$$

$$L_{\vec{p}}(J, E_0) := \prod_{j=1}^N L_{p_j}(J, E_{0,j}), \quad E := \prod_{j=1}^N (E_{0,j}, E_{1,j})_{1/p'_j, p_j}.$$

Note that (0.2) implies

$$\mathcal{H}^1_{\vec{p}}\big(J, (E_1, E_0)\big) \hookrightarrow C\big(\overline{J}, E\big).$$

For $B \in \mathcal{L}\big(\mathcal{H}^1_{\vec{p}}(J), L_{\vec{p}}(J, E_0)\big)$ we define the property of **maximal $L_{\vec{p}}$ regularity** as in Section 1 by (formally) replacing everywhere the index p by $\vec{p}$. Then

$$\mathcal{MR}_{\vec{p}}(J) := \mathcal{MR}_{\vec{p}}\big(J, (E_1, E_0)\big)$$

is the set of all $B \in L_\infty\big(J, \mathcal{L}(E_1, E_0)\big)$ possessing the property of maximal $L_{\vec{p}}$ regularity on J.

We assume that

- $\mathcal{MR}(E_{1,j}, E_{0,j}) \neq \emptyset, \qquad j = 1, 2, \ldots, N.$

We also assume that

$$\left.\begin{aligned} &\bullet\quad A \in \mathcal{C}^{1\text{-}}_{\mathrm{Volt}}\big(\mathcal{H}^1_{\vec{p}}(\mathsf{J}), \mathcal{MR}_{\vec{p}}(\mathsf{J})\big);\\ &\bullet\quad \vec{p} < \vec{q} \leq \vec{\infty};\\ &\bullet\quad F \in \mathcal{C}^{1\text{-}}_{\mathrm{Volt}}\big(\mathcal{H}^1_{\vec{p}}(\mathsf{J}); L_{\vec{q}}(\mathsf{J}, E_0), L_{\vec{p}}(\mathsf{J}, E_0)\big), \end{aligned}\right\} \tag{5.1}$$

where $\vec{p} < \vec{q}$ means $p_j < q_j$ for $1 \leq j \leq N$, and $\vec{\infty} := (\infty, \ldots, \infty)$.

Theorem 5.1. *Let* (5.1) *be satisfied. Then Theorems 2.1, 3.1, and 4.1, as well as Corollaries 2.2 and 3.3 and Remark 4.3 hold (with the obvious modifications) in this case also.*

Proof. The reader will verify that the proofs of Sections 6–12 are easily adapted to give the assertion. □

Our next proposition gives a sufficient condition for $B \in L_\infty\big(J, \mathcal{L}(E_1, E_0)\big)$ to possess the property of maximal $L_{\vec{p}}$ regularity. For this we note that each such B possesses a unique $N \times N$ operator matrix representation, $B = [B_{jk}]$, where

$$B_{j,k} \in L_\infty\big(J, \mathcal{L}(E_{1,k}, E_{0,j})\big), \qquad 1 \leq j, k \leq N.$$

Clearly, B is 'upper triangular' if $B_{j,k} = 0$ for $1 \leq k < j \leq N$.

Proposition 5.2. *Suppose that* $B \in L_\infty\big(J, \mathcal{L}(E_1, E_0)\big)$ *is upper triangular and*

$$B_{j,j} \in \mathcal{MR}_{p_j}\big(J, (E_{1,j}, E_{0,j})\big), \qquad 1 \le j \le N.$$

Then $B \in \mathcal{MR}_{\vec{p}}\big(J, (E_1, E_0)\big)$.

Proof. Suppose that $f = (f_1, \ldots, f_N) \in L_{\vec{p}}(J, E_0)$. Then, by the upper triangularity, the Cauchy problem

$$\dot{u} + Bu = f \text{ in } (0,T), \quad u(0) = 0 \tag{5.2}$$

is equivalent to

$$\dot{u}_j + B_{j,j} u_j = f_j - \sum_{k=j+1}^{N} B_{j,k} u_k, \qquad 1 \le j \le N. \qquad (5.3)_j$$

From this, the maximal regularity property of $B_{j,j}$, and by solving $(5.3)_j$ from 'bottom to top', we see that (5.2) has a unique solution $u \in \mathcal{H}^1_{\vec{p}}(J)$. □

By combining this proposition with the perturbation theorems of [5] we easily obtain sufficient conditions for operators B with 'fully occupied' matrices $[B_{jk}]$ to possess the property of maximal $L_{\vec{p}}$ regularity, provided the subdiagonal entries $B_{j,k}$, $1 \le j < k \le N$, are suitably 'subordinate'. Details are left to the interested reader.

Part 2. Proof of the general theorems

In this part we prove the general theorems formulated in Part 1. Since we assume only that A and F are defined on $\mathcal{H}^1_p(\mathsf{J})$, and not on $C\big(\overline{\mathsf{J}}, E\big)$, the proofs are rather more complicated than in the 'classical' situation. First, we cannot carry out an iteration process (i.e., Banach's fixed point theorem) with the usual choice of the constant function $u_0(t) := x$ as starting point, since this function does not belong to $\mathcal{H}^1_p(\mathsf{J})$, in general, if $x \in E$. Second, since (A, F) is defined on the fixed space $\mathcal{H}^1_p(\mathsf{J})$, we have to prove suitable uniform extension theorems for $\mathcal{H}^1_p(J_T)$ functions. Lastly, the proof of the (global) continuity theorem is rather delicate also since the norms of the elements of $\mathcal{H}^1_p(J_T)$ do not converge to zero in a uniform fashion as T tends to zero.

In Section 6 we collect the basic facts on maximal regularity used below. Section 7 contains simple but most important extension theorems for $\mathcal{H}^1_p$ functions. Section 8 is of preparatory nature containing technical estimates needed in the proofs of the existence and continuity theorems given in

Sections 9–11. The last section is devoted to the proof of the differentiability theorem.

Throughout this part p is a fixed number in $(1,\infty)$ and, for abbreviation, we put

$$\mathcal{H}(J) := \mathcal{H}_p^1(J), \quad L(J) := L_p(J,E_0),$$

as well as

$$L_r(J) := L_r(J,E_0), \quad 1 \le r \le \infty, \qquad \mathcal{L} := \mathcal{L}(E_1,E_0).$$

We also set

$$\mathcal{MR}(J) := \mathcal{MR}_p(J), \quad \mathcal{MR} := \mathcal{MR}(E_1,E_0),$$

and recall that it is assumed that $\mathcal{MR} \neq \emptyset$.

Let X be a Banach space. Suppose that $0 < S < T$ and u maps J_S into X. Given $v:\ J_{T-S} \to \mathsf{X}$, we put

$$u \oplus_S v(t) := \begin{cases} u(t), & t \in J_S, \\ v(t-S), & t \in S + J_{T-S} = [S,T), \end{cases}$$

so that $u \oplus_S v$ maps J_T into X.

6. Maximal regularity

Let X and Y be Banach spaces. Then $\mathcal{L}\mathrm{is}(\mathsf{X},\mathsf{Y})$ is the set of all isomorphisms from X into Y. Recall that $\mathcal{L}\mathrm{is}(\mathsf{X},\mathsf{Y})$ is open in $\mathcal{L}(\mathsf{X},\mathsf{Y})$, and the inversion map $\mathcal{L}\mathrm{is}(\mathsf{X},\mathsf{Y}) \to \mathcal{L}\mathrm{is}(\mathsf{Y},\mathsf{X})$, $C \mapsto C^{-1}$ is smooth. We write γ_0 for the trace operator $u \mapsto u(0)$, so that $\gamma_0 \in \mathcal{L}\big(\mathcal{H}(J),E\big)$ by (0.2). We also set

$$\mathcal{H}_0(J) := \big\{\, u \in \mathcal{H}(J)\ ;\ \gamma_0 u = 0 \,\big\}.$$

Hence $\mathcal{H}_0(J)$ is a closed linear subspace of $\mathcal{H}(J)$, thus a Banach space.

The following lemma gives simple, but useful, characterizations of maximal regularity. Its proof is an almost obvious consequence of the open mapping theorem (cf. the proof of [3, Proposition 2.1]).

Lemma 6.1. *Suppose that $B \in \mathcal{L}\big(\mathcal{H}(J),L(J)\big)$. Then the following are equivalent.*

(i) *B has the property of maximal regularity.*

(ii) *$\partial + B \in \mathcal{L}\mathrm{is}\big(\mathcal{H}_0(J),L(J)\big)$.*

(iii) *$(\partial + B,\gamma_0) \in \mathcal{L}\mathrm{is}\big(\mathcal{H}(J),L(J)\times E\big)$.*

(iv) *The linear Cauchy problem*

$$\dot u + Bu = f \ \text{ on } J, \quad u(0) = x$$

has for each $(f,x) \in L(J)\times E$ a unique solution in $\mathcal{H}(J)$.

For $\kappa > 0$ we denote by $\mathcal{MR}(J,\kappa)$ the set of all $B \in \mathcal{MR}(J)$ satisfying

$$\|(\partial + B, \gamma_0)^{-1}\|_{\mathcal{L}(L(J)\times E, \mathcal{H}(J))} \le \kappa.$$

The proof of the following lemma is found in [5, Lemma 4.1]. Here and below, τ_s denotes left translation, that is, $\tau_s v(t) = v(t+s)$.

Lemma 6.2. *There exists $\widehat{\kappa} \ge 1$ such that, given any $\kappa > 0$ and B belonging to $\mathcal{MR}(J_T,\kappa)$,*

$$\tau_s B \in \mathcal{MR}(J_{S-s}, \widehat{\kappa}\kappa), \qquad 0 \le s < S \le T.$$

The crucial part of this assertion is the fact that the norm of the inverse of $(\partial + B, \gamma_0)$ can be controlled under restrictions and translations.

7. Extensions

For $x \in E$ we set

$$\mathbb{M}_T(x) := \{\, u \in \mathcal{H}(J_T)\ ;\ u(0) = x \,\}.$$

For completeness we include a proof for the following simple lemma.

Lemma 7.1. *Suppose that $u \in \mathcal{H}(J_S)$ and $v \in \mathbb{M}_{T-S}\big(u(S)\big)$. Then $u \oplus_S v$ belongs to $\mathcal{H}(J_T)$, and $\partial(u \oplus_S v) = \partial u \oplus_S \partial v$. Moreover,*

$$\|u_1 \oplus_S v_1 - u_2 \oplus_S v_2\|_{\mathcal{H}(J_T)} \le 2\big(\|u_1 - u_2\|_{\mathcal{H}(J_S)} + \|v_1 - v_2\|_{\mathcal{H}(J_{T-S})}\big)$$

for $u_j \in \mathcal{H}(J_S)$ and $v_j \in \mathbb{M}_{T-S}\big(u_j(S)\big)$, $j = 1,2$.

Proof. Since $u \in H_p^1(J_S, E_0)$ and $v \in H_p^1(J_{T-S}, E_0)$, these functions are absolutely continuous. Thus, given $\varphi \in C^\infty(0,\infty)$ having its support in $\mathring{J}_T$,

$$\begin{aligned} -\int_0^T \partial\varphi(u \oplus_S v)\,dt &= -\int_0^S \dot\varphi u\,dt - \int_0^{T-S} \tau_S\dot\varphi v\,dt \\ &= \int_0^S \varphi\dot u\,dt + \int_0^{T-S} \tau_S\varphi\dot v\,dt = \int_0^T \varphi(\partial u \oplus_S \partial v)\,dt \end{aligned}$$

in E_0, due to $u(S) = v(0)$. Hence $\partial(u \oplus_S v) = \partial u_S \oplus \partial v_S \in L(J_T)$. This shows that $u \oplus_S v \in H_p^1(J_T, E_0)$. Consequently, $u \oplus_S v \in \mathcal{H}(J_T)$. The rest is obvious. □

In the following basic extension lemma it is important to observe that the norms of the extension operators are uniformly bounded with respect to T.

Lemma 7.2. *There exist a positive constant κ and, for each $T \in (0,\mathsf{T}]$, a map*

$$\mathrm{ext}_T \in \mathcal{L}\big(\mathcal{H}(J_T),\mathcal{H}(\mathsf{J})\big)$$

such that

$$\|\mathrm{ext}_T\, u\|_{\mathcal{H}(\mathsf{J})} \le \kappa\big(\|u\|_{\mathcal{H}(J_T)} + \|u(0)\|_E\big)$$

and $\mathrm{ext}_T\, u \supset u$ *for* $u \in \mathcal{H}(J_T)$.

Proof. Fix $B \in \mathcal{MR}$ and set

$$\kappa := \|(\partial + B,\gamma_0)^{-1}\|_{\mathcal{L}(L(\mathsf{J})\times E,\mathcal{H}(\mathsf{J}))}\,(1+\|B\|_{\mathcal{L}}).$$

For $u \in \mathcal{H}(J_T)$ put

$$\mathrm{ext}_T\, u := (\partial + B,\gamma_0)^{-1}\big(((\partial+B)u)^{\sim}, u(0)\big) \in \mathcal{H}(\mathsf{J}),$$

where $((\partial+B)u)^{\sim} \in L(\mathsf{J})$ is the extension by zero of $(\partial+B)u \in L(J_T)$. Then ext_T has the desired properties. $\square$

8. Preliminary estimates

In this section we present estimates which are needed for the derivation of the existence and continuity theorems. For this we set

$$\eta(t) := t^{1/p-1/q}, \qquad t \ge 0.$$

For $u_0 \in \mathcal{H}(\mathsf{J})$ we put

$$\mathbb{N}_S(u_0;r,\rho) := \big\{\, u \in \mathcal{H}(J_S)\ ;\ \|u-u_0\|_{\mathcal{H}(J_S)} \le r,\ \|u(0)-u_0(0)\|_E \le \rho \,\big\}$$

for $r,\rho > 0$ and $0 < S \le \mathsf{T}$.

Lemma 8.1. *Suppose that $(\varphi_0,u_0) \in \Phi(\mathsf{J}) \times \mathcal{H}(\mathsf{J})$. Then, for any R strictly bigger than $\|u_0\|_{\mathcal{H}(\mathsf{J})}$, there exist positive constants K, κ, r, ρ, and ε such that, given $\varphi \in \mathcal{V}(\mathsf{J};\varphi_0,R,K,\varepsilon)$,*

$$\tau_s A(u) \in \mathcal{MR}(J_\sigma,\kappa)$$

and

$$\|\tau_s\varphi(u)\|_{Z(J_\sigma)} \le \kappa$$

as well as

$$\big\|\tau_s\big(A(u)-A(v)\big)\big\|_{L_\infty(J_\sigma,\mathcal{L})} \le \kappa\big(\|u-v\|_{\mathcal{H}(J_{s+\sigma})} + \|u(0)-v(0)\|_E\big)$$

and

$$\big\|\tau_s\big(F(u)-F(v)\big)\big\|_{L(J_\sigma)} \le \kappa\eta(\sigma)\big(\|u-v\|_{\mathcal{H}(J_{s+\sigma})} + \|u(0)-v(0)\|_E\big)$$

for $0 \le s < S \le \mathsf{T}$, $0 < \sigma \le S-s$, *and* $u,v \in \mathbb{N}_S(u_0;r,\rho)$.

Proof. For $0 < S \le \mathsf{T}$ and $u \in \mathcal{H}(J_S)$ we put

$$\varepsilon_S(u) := u_0 + \mathrm{ext}_S(u - u_0) \in \mathbb{M}_{\mathsf{T}}\bigl(u(0)\bigr)$$

if $S < \mathsf{T}$, and $\varepsilon_S(u) := u$ otherwise. By Lemma 7.2 there exists a constant κ_0 such that

$$\|\varepsilon_S(u) - u_0\|_{\mathcal{H}(\mathsf{J})} \le \kappa_0\bigl(\|u - u_0\|_{\mathcal{H}(J_S)} + \|u(0) - u_0(0)\|_E\bigr). \tag{8.1}$$

Fix $R > \|u_0\|_{\mathcal{H}(\mathsf{J})}$ and $\overline{r}, \overline{\rho} > 0$ such that $\kappa_0(\overline{r} + \overline{\rho}) < R - \|u_0\|_{\mathcal{H}(\mathsf{J})}$. Then it follows from (8.1) that

$$\varepsilon_S(u) \in R\mathcal{B}_{\mathsf{T}}, \qquad u \in \mathbb{N}_S(u_0; r, \rho), \quad 0 < r \le \overline{r}, \quad 0 < \rho \le \overline{\rho}, \quad 0 < S \le \mathsf{T}. \tag{8.2}$$

Choose $K > 0$ such that $\varphi_0 \in \mathcal{V}(\mathsf{J}; R, K)$. Then (8.1) and (8.2) imply

$$\bigl\|A_0(\varepsilon_S(u)) - A_0(u_0)\bigr\|_{L_\infty(\mathsf{J},\mathcal{L})} \le K\,\|\varepsilon_S(u) - u_0\|_{\mathcal{H}(\mathsf{J})} \le \kappa_0 K(r + \rho) \tag{8.3}$$

for $u \in \mathbb{N}_S(u_0; r, \rho)$, $0 < r \le \overline{r}$, $0 < \rho \le \overline{\rho}$, and $0 < S \le \mathsf{T}$. Since $A_0(u_0)$ belongs to $\mathcal{MR}(\mathsf{J})$, there exists $\kappa_1 > 0$ such that

$$\bigl\|\bigl(\partial + A_0(u_0), \gamma_0\bigr)^{-1}\bigr\|_{\mathcal{L}(L(\mathsf{J})\times E, \mathcal{H}(\mathsf{J}))} \le \kappa_1.$$

Now we fix $r \in (0, \overline{r}\,]$ and $\rho \in (0, \overline{\rho}\,]$ such that $\kappa_0 K(r + \rho) < 1/4\kappa_1$ and set $\varepsilon := 1/4\kappa_1$. Then we infer from

$$A\bigl(\varepsilon_S(u)\bigr) - A_0(u_0) = A\bigl(\varepsilon_S(u)\bigr) - A_0\bigl(\varepsilon_S(u)\bigr) + A_0\bigl(\varepsilon_S(u)\bigr) - A_0(u_0)$$

and (8.3) that

$$\bigl\|A\bigl(\varepsilon_S(u)\bigr) - A_0(u_0)\bigr\|_{L_\infty(\mathsf{J},\mathcal{L})} \le \bigl\|A\bigl(\varepsilon_S(u)\bigr) - A_0\bigl(\varepsilon_S(u)\bigr)\bigr\|_{L_\infty(\mathsf{J},\mathcal{L})} + 1/4\kappa_1.$$

Thus (8.2) and the choice of ε imply

$$\bigl\|A\bigl(\varepsilon_S(u)\bigr) - A_0(u_0)\bigr\|_{L_\infty(\mathsf{J},\mathcal{L})} \le 1/2\kappa_1$$

for $\varphi \in \mathcal{V}_\varepsilon := \mathcal{V}(\mathsf{J}; \varphi_0, R, K, \varepsilon)$, $u \in \mathbb{N}_S(u_0; r, \rho)$, and $0 < S \le \mathsf{T}$. Now we deduce from the obvious fact that the injection map $L_\infty(\mathsf{J}, \mathcal{L}) \hookrightarrow \mathcal{L}\bigl(\mathcal{H}(\mathsf{J}), L(\mathsf{J})\bigr)$ has norm 1 and from a well known perturbation theorem for bounded invertibility, which is based on a Neumann series argument, that

$$\bigl(\partial + A\bigl(\varepsilon_S(u)\bigr), \gamma_0\bigr) \in \mathcal{L}\mathrm{is}\bigl(\mathcal{H}(\mathsf{J}), L(\mathsf{J}) \times E\bigr)$$

and that the inverse of this map is bounded by $2\kappa_1$ for φ, u, and S as above. The Volterra property guarantees that

$$\tau_s(A)\bigl(\varepsilon_S(u)\bigr)\,\big|\,J_\sigma = \tau_s A(u)\,|\,J_\sigma, \qquad 0 \le s < S, \quad 0 < \sigma \le S - s.$$

Thus it follows from Lemma 6.2 and the above that

$$\tau_s A(u)\,|\,J_\sigma \in \mathcal{MR}(J_\sigma, 2\kappa_1\widehat{\kappa}) \tag{8.4}$$

for $u \in \mathbb{N}_S(u_0; r, \rho)$, $0 \le s < S \le \mathsf{T}$, and $0 < \sigma \le S - s$.

From (8.2), (8.4), and Lemma 7.2 we deduce that, given $\varphi \in \mathcal{V}_\varepsilon$,

$$
\begin{aligned}
&\big\|\tau_s\big(A(u) - A(v)\big)\big\|_{L_\infty(J_\sigma,\mathcal{L})} \\
&\quad = \big\|\tau_s\big(A\big(\varepsilon_{s+\sigma}(u)\big) - A\big(\varepsilon_{s+\sigma}(v)\big)\big)\big\|_{L_\infty(J_\sigma,\mathcal{L})} \\
&\quad \le \big\|\big(A\big(\varepsilon_{s+\sigma}(u)\big) - A\big(\varepsilon_{s+\sigma}(v)\big)\big\|_{L_\infty(\mathsf{J},\mathcal{L})} \\
&\quad \le K\,\|\varepsilon_{s+\sigma}(u) - \varepsilon_{s+\sigma}(v)\|_{\mathcal{H}(\mathsf{J})} \\
&\quad \le \kappa_0 K\big(\|u - v\|_{\mathcal{H}(J_{s+\sigma})} + \|u(0) - v(0)\|_E\big)
\end{aligned}
\tag{8.5}
$$

for $u, v \in \mathbb{N}_S(u_0; r, \rho)$, $0 \le s < S \le \mathsf{T}$, and $0 < \sigma \le S - s$, due to the fact that $w\,|\,J_{s+\sigma} \in \mathbb{N}_{s+\sigma}(u_0; r, \rho)$ for $w \in \mathbb{N}_S(u_0; r, \rho)$. Similarly, using Hölder's inequality in addition, we find

$$
\begin{aligned}
&\big\|\tau_s\big(F(u) - F(v)\big)\big\|_{L(J_\sigma)} \\
&\quad \le \eta(\sigma)\,\big\|\tau_s\big(F(u) - F(v)\big)\big\|_{L_q(J_\sigma)} \\
&\quad \le \eta(\sigma)\,\big\|F\big(\varepsilon_{s+\sigma}(u)\big) - F\big(\varepsilon_{s+\sigma}(v)\big)\big\|_{L_q(\mathsf{J})} \\
&\quad \le K\eta(\sigma)\,\|\varepsilon_{s+\sigma}(u) - \varepsilon_{s+\sigma}(v)\|_{\mathcal{H}(\mathsf{J})} \\
&\quad \le \kappa_0 K\eta(\sigma)\big(\|u - v\|_{\mathcal{H}(J_{s+\sigma})} + \|u(0) - v(0)\|_E\big)
\end{aligned}
\tag{8.6}
$$

for u, v, s, σ, and S as above.

Note that

$$\|u\|_{\mathcal{H}(J_{s+\sigma})} + \|u(0)\|_E \le \|u_0\|_{\mathcal{H}(\mathsf{J})} + \|u_0(0)\|_E + r + \rho$$

for $u \in \mathbb{N}_S(u_0; r, \rho)$ and $s + \sigma \le S$. Hence (8.5) and (8.6) imply the existence of a constant κ_2 such that

$$\|\tau_s\varphi(u)\|_{\mathsf{Z}(J_\sigma)} \le \|\tau_s\varphi(0)\|_{\mathsf{Z}(J_\sigma)} + \kappa_2, \qquad \varphi \in \mathcal{V}_\varepsilon.$$

Since it follows from $\varphi \in \mathcal{V}(\mathsf{J}; R, K)$ that $\|\varphi(0)\|_{\mathsf{Z}(\mathsf{J})} \le K$, we see that

$$\|\tau_s\varphi(u)\|_{\mathsf{Z}(J_\sigma)} \le K + \kappa_2, \qquad \varphi \in \mathcal{V}_\varepsilon. \tag{8.7}$$

Now, setting $\kappa := \max\{2\kappa_1\widehat{\kappa}, \kappa_0 K, K + \kappa_2\}$, estimates (8.4)–(8.7) give the assertion. □

9. Local existence

For abbreviation, we put

$$\mathbb{M}_S(x, r) := \big\{\, u \in \mathbb{M}_S(x) \ ; \ \|u - e_x\|_{\mathcal{H}(J_S)} \le r \,\big\}$$

for $0 < S \leq \mathsf{T}$, $x \in E$ and $r > 0$. Observe that $\mathbb{M}_S(x,r)$ is a closed subset of the Banach space $\mathcal{H}(J_S)$, hence a complete metric space. We also set $Z_S := Z(J_S)$ for $0 < S \leq \mathsf{T}$.

On the basis of the preceding preparations we can now prove the following fundamental local existence, uniqueness, and continuity theorem for the quasilinear Cauchy problem

$$\dot{u} + A(u)u = F(u) \text{ on } \mathsf{J}, \quad u(0) = x. \tag{9.1$_{(\varphi,x)}$}$$

Proposition 9.1. *Suppose that $(\varphi_0, x_0) \in \Phi(\mathsf{J}) \times E$. Then there exist positive constants R, K, T, r, ε, and ρ such that $(9.1)_{(\varphi,x)}$ has for each x belonging to $B_E(x_0,\rho)$ and $\varphi \in \mathcal{V}(\mathsf{J};\varphi_0,R,K,\varepsilon)$ a unique solution $u(\varphi,x)$ in the set $\mathbb{M}_T(x,r)$. Moreover, there exists a positive constant κ such that, setting $u_j := u(\varphi_j, x_j)$,*

$$\|u_1 - u_2\| \leq \kappa\big(\|(\varphi_1 - \varphi_2)(u_1)\|_{\mathsf{Z}_T} + \|x_1 - x_2\|_E\big)$$

for $x_1, x_2 \in B_E(x_0,\rho)$ and $\varphi_1, \varphi_2 \in \mathcal{V}(\mathsf{J};\varphi_0,R,K,\varepsilon)$.

Proof. (1) We fix $C \in \mathcal{H}(E_1,E_0)$ and put

$$e_x(t) := e^{-tC}x, \qquad 0 < t \leq \mathsf{T}, \quad x \in E.$$

Note that $(x \mapsto e_x) \in \mathcal{L}\big(E, \mathcal{H}(\mathsf{J})\big)$ and that there exists $\widetilde{\kappa} \geq 1$ such that

$$\|e_x\|_{\mathcal{H}(J_T)} \leq \widetilde{\kappa}\,\|x\|_E, \qquad 0 < T \leq \mathsf{T}, \quad x \in E \tag{9.2}$$

(cf. [2, Proposition III.4.10.3]).

Set $u_0 := e_{x_0}$ and fix $R > \|u_0\|_{\mathcal{H}(\mathsf{J})}$. From Lemma 8.1 we know that there exist positive constants K, ε, and $\overline{r}, \overline{\rho} \in (0,1]$, as well as $\overline{\kappa} \geq K \vee \widetilde{\kappa} \vee 1$ such that, given $\varphi \in \mathcal{V} := \mathcal{V}(J,\varphi_0,R,K,\varepsilon)$, it follows that

$$A(u) \in \mathcal{MR}(J_S, \overline{\kappa}) \tag{9.3}$$

and

$$\|\varphi(u)\|_{\mathsf{Z}_S} \leq \overline{\kappa} \tag{9.4}$$

as well as

$$\|A(u) - A(v)\|_{L_\infty(J_S,\mathcal{L})} \leq \overline{\kappa}\big(\|u - v\|_{\mathcal{H}(J_S)} + \|u(0) - v(0)\|_E\big) \tag{9.5}$$

and

$$\|F(u) - F(v)\|_{L(J_S)} \leq \overline{\kappa}\eta(\sigma)\big(\|u - v\|_{\mathcal{H}(J_S)} + \|u(0) - v(0)\|_E\big) \tag{9.6}$$

for $u, v \in \mathbb{N}_S(r,\rho) := \mathbb{N}_S(u^0; r, s)$, $0 < r \leq \overline{r}$, $0 < \rho \leq \overline{\rho}$, and $0 < S \leq \mathsf{T}$.

In the rest of this proof we always presuppose that

$$\varphi \in \mathcal{V}, \quad r \in (0,\overline{r}\,], \quad \rho \in (0,\overline{\rho}\,], \quad S \in (0,\mathsf{T}].$$

Note that from $v = (v - u_0) + u_0$ it follows that

$$\|v\|_{\mathcal{H}(J_S)} \leq r + \|u_0\|_{\mathcal{H}(J_S)}, \qquad v \in \mathbb{N}_S(r, \rho). \tag{9.7}$$

Similarly, $e_{v(0)} - v = e_{v(0)} - e_{x_0} + u_0 - v$ implies

$$\|e_{v(0)} - v\|_{\mathcal{H}(J_S)} \leq \overline{\kappa}\rho + r, \qquad v \in \mathbb{N}_S(r, \rho), \tag{9.8}$$

due to (9.2) and $\overline{\kappa} \geq \widetilde{\kappa}$.

(2) For $v \in \mathbb{N}_S(r, \rho)$ we put

$$G(v) := \big(A(e_{v(0)}) - A(v)\big)v + F(v) \in L(J_S).$$

Observe that (9.5), (9.7), and (9.8) imply

$$\big\|\big(A(e_{v(0)}) - A(v)\big)v\big\|_{L(J_S)} \leq \overline{\kappa}\big(r + \overline{\kappa}\rho\big)\big(r + \|u_0\|_{\mathcal{H}(J_S)}\big).$$

Similarly, from $F(v) = \big(F(v) - F(u_0)\big) + F(u_0)$ and (9.6) we infer, due to $\overline{r}, \overline{\rho} \leq 1$, that

$$\|F(v)\|_{L(J_S)} \leq 2\overline{\kappa}\eta(S) + \|F(u_0)\|_{L(J_S)}.$$

Thus, setting

$$\alpha(S) := 2\overline{\kappa}\eta(S) + \overline{\kappa}(1 + \overline{\kappa})\,\|u_0\|_{\mathcal{H}(J_S)} + \|F(u_0)\|_{L(J_S)},$$

we see that

$$\|G(v)\|_{L(J_S)} \leq \overline{\kappa}(r + \overline{\kappa}\rho)r + \alpha(S). \tag{9.9}$$

Since, for $u, v \in \mathbb{N}_S(r, \rho)$ with $u(0) = v(0)$,

$$G(u) - G(v) = \big(A(e_{u(0)}) - A(u)\big)(u - v) + \big(A(v) - A(u)\big)v + F(u) - F(v),$$

we find, similarly, that

$$\|G(u) - G(v)\|_{L(J_S)} \leq \overline{\kappa}\big(2r + \overline{\kappa}\rho + \alpha(S)\big)\,\|u - v\|_{\mathcal{H}(J_S)}. \tag{9.10}$$

(3) For $x \in B_E(x_0, \rho)$ and $v \in \mathbb{M}_S(x, r)$ we denote by $U_S := U_S(v, x)$ the unique solution in $\mathcal{H}(J_S)$ of the linear Cauchy problem

$$\dot{u} + A(e_x)u = G(v) \text{ on } J_S, \quad u(0) = x.$$

Due to Lemma 6.1(iii), it is well defined. Thus, by the definition of u_0,

$$(U_S - u_0)^{\cdot} + A(e_x)(U_S - u_0) = \big(C - A(e_x)\big)u_0 + G(v) \tag{9.11}$$

on J_S, and $(U_S - u_0)(0) = x - x_0$.

Set $\beta(S) := (\|C\|_{\mathcal{L}} + \overline{\kappa})\,\|u_0\|_{\mathcal{H}(J_S)} + \alpha(S)$. Using (9.2), we obtain

$$\|e_x\|_{\mathcal{H}(\mathsf{J})} \leq \|u_0\|_{\mathcal{H}(\mathsf{J})} + \overline{\kappa}\rho.$$

By making ρ smaller, if necessary, we can assume that $e_x \in R\mathcal{B}_T$ for x belonging to $B_E(x_0, \rho)$. Consequently, $\|A(e_x)\|_{L_\infty(J,\mathcal{L})} \leq \overline{\kappa}$, due to (9.4). Hence it follows from (9.4) and (9.9) that $\overline{\kappa}(r+\overline{\kappa}\rho)r+\beta(S)$ is a bound for the $L(J_S)$ norm of the right-hand side of (9.11). Hence we infer from (9.3) that

$$\|U_S(v,x)-u_0\|_{\mathcal{H}(J_S)} \le \overline{\kappa}^2(r+\overline{\kappa}\rho)r+\overline{\kappa}\beta(S)+\overline{\kappa}\rho.$$

Similarly, (9.10) implies

$$\|U_S(v,x)-U_S(w,x)\|_{\mathcal{H}(J_S)} \le \overline{\kappa}^2\big(2r+\overline{\kappa}\rho+\beta(S)\big)\,\|v-w\|_{\mathcal{H}(J_S)}$$

for $x\in B_E(x_0,\rho)$ and $v,w\in\mathbb{M}_S(x,r)$.

(4) Now we fix r and ρ such that $8\overline{\kappa}^2 r\le 1$, $8\overline{\kappa}^3\rho\le 1$, and $8\overline{\kappa}\rho\le r$. We also fix $T\in(0,\mathsf{T}]$ with $8\overline{\kappa}^2\beta(T)\le r$, which is possible due to $\beta(S)\downarrow 0$ as $S\downarrow 0$. Then it follows from step (3) that

$$\|U_S(v,x)-u_0\|_{\mathcal{H}(J_S)}\le r/2$$

and

$$\|U_S(v,x)-U_S(w,x)\|_{\mathcal{H}(J_S)}\le \|v-w\|_{\mathcal{H}(J_S)}/2$$

for $x\in B_E(x_0,\rho)$ and $v,w\in\mathbb{M}_S(x,r)$, and for $0<S\le T$. This shows that $U(\cdot,x)$ maps the complete metric space $\mathbb{M}_S(x,r)$ into itself and is a strict contraction. Hence Banach's fixed point theorem guarantees that $U_S(\cdot,x)$ has a unique fixed point, $u_S(x)$, in $\mathbb{M}_S(x,r)$. Clearly, u is a fixed point of $U_S(\cdot,x)$ iff it is a solution of $(9.1)_{(\varphi,x)}$ on J_S belonging to $\mathbb{M}_S(x,r)$. From this we infer that

$$u_{S_1}(x)\supset u_{S_2}(x),\qquad 0<S_2<S_1\le T.$$

This proves the first assertion.

(5) Set $\mathcal{U}:=\mathcal{V}\times B_E(x_0,\rho)$ and suppose that $(\varphi_j,x_j)\in\mathcal{U}$ for $j=1,2$. Let u_j be the unique solution of $(9.1)_{(\varphi_j,x_j)}$ on J_T belonging to $\mathbb{M}_T(x_j,r)$. Then

$$(u_1-u_2)^{\cdot}+A_1(u_1)(u_1-u_2)=\big(A_2(u_2)-A_1(u_1)\big)u_2+F_1(u_1)-F_2(u_2) \tag{9.12}$$

on J_T, and $(u_1-u_2)(0)=x_1-x_2$. Note that

$$A_2(u_2)-A_1(u_1)=A_2(u_2)-A_2(u_1)+A_2(u_1)-A_1(u_1)$$

implies, due to (9.5) and (9.7),

$$\begin{aligned}\big\|\big(A_2(u_2)-A_1(u_1)\big)u_2\big\|_{L(J_T)}&\\ &\le\overline{\kappa}\big(r+\|u_0\|_{\mathcal{H}(J_T)}\big)\big(\|u_1-u_2\|_{\mathcal{H}(J_T)}+\|x_1-x_2\|_E\big)\\ &\quad+\kappa^*\,\|(A_1-A_2)(u_1)\|_{L_\infty(J_T,\mathcal{L})}\end{aligned}$$

with $\kappa^*:=1+\overline{\kappa}\,\|x_0\|_E$, where $\|u_0\|_{\mathcal{H}(J_T)}$ has been estimated by (9.2). Similarly,

$$F_1(u_1)-F_2(u_2)=F_1(u_1)-F_2(u_1)+F_2(u_1)-F_2(u_2),$$

and (9.6) imply

$$\begin{aligned}&\|F_1(u_1)-F_2(u_2)\|_{L(J_T)}\\&\qquad\le\overline{\kappa}\eta(T)\big(\|u_1-u_2\|_{\mathcal{H}(J_T)}+\|x_1-x_2\|_E\big)+\|(F_1-F_2)(u_1)\|_{L(J_T)}.\end{aligned}$$

Thus, due to the choice of r and ρ, the $L(J_T)$ norm of the right-hand side of (9.12) is estimated from above by

$$\frac{1}{2\overline{\kappa}}\,\|u_1-u_2\|_{\mathcal{H}(J_T)}+\widehat{\kappa}\big(\|(\varphi_1-\varphi_2)(u_1)\|_{\mathsf{Z}_T}+\|x_1-x_2\|_E\big),$$

where $\widehat{\kappa}:=\overline{\kappa}\big(\kappa^*+\eta(T)\big)$. Now, setting $\kappa:=2\overline{\kappa}\big(\widehat{\kappa}+1\big)$, the last assertion follows from (9.3). □

10. Global existence and uniqueness

By a more or less obvious extension procedure we can now prove our main existence and uniqueness result.

Proof of Theorem 2.1. (1) Fix $(\varphi,x)\in\Phi(\mathsf{J})\times E$. Proposition 9.1 guarantees the existence of T_0 in $(0,\mathsf{T}]$ and $r_0>0$ such that $(9.1)_{(\varphi,x)}$ has on J_{T_0} a unique solution belonging to $\mathbb{M}_{T_0}(x,r_0)$.

(2) Suppose that $0<S<\mathsf{T}$ and $w\in\mathcal{H}(J_S)$ is a solution of $(9.1)_{(\varphi,x)}$ on J_S. For $u\in\mathbb{M}_{\mathsf{T}-S}\big(w(S)\big)$ put

$$\varphi_{S,w}(u):=\tau_S\varphi(w\oplus_S u)\in\mathsf{Z}_{\mathsf{T}-S}.$$

Also set $u_0:=w\oplus_S e_{w(S)}\in\mathcal{H}(\mathsf{J})$. Fix positive constants R and K with $u_0\in R\mathcal{B}_{\mathsf{T}}$ and $\varphi\in\Phi(R,K)$. It follows from Lemma 8.1 that there exist $\kappa\ge1$ and $\overline{r}>0$ such that

$$\|\varphi_{S,w}(u)\|_{\mathsf{Z}_\sigma}\le\kappa,$$

and

$$\|A_{S,w}(u)-A_{S,w}(v)\|_{L_\infty(J_\sigma,\mathcal{L})}\le\kappa\,\|u-v\|_{\mathcal{H}(J_\sigma)}$$

as well as

$$\|F_{S,w}(u)-F_{S,w}(v)\|_{L(J_\sigma)}\le\kappa\eta(\sigma)\,\|u-v\|_{\mathcal{H}(J_\sigma)}$$

and

$$A_{S,w}(u)\in\mathcal{MR}(J_\sigma,\kappa)$$

for $0<\sigma\le\mathsf{T}-S$ and $u,v\in\mathbb{M}_\sigma\big(w(S),\overline{r}\big)$.

Now obvious modifications of steps (2)–(4) of the proof of Proposition 9.1 (replacing $\mathbb{N}_S(r,\rho)$ by $\mathbb{M}_\sigma\big(w(S),r\big)$) imply the existence of $r>0$ and S_1 belonging to $(S,\mathsf{T}]$ such that the Cauchy problem

$$\dot u+\tau_S A(w\oplus_S u)u=\tau_S F(w\oplus_S u)\text{ on }J_{S_1-S},\quad u(0)=w(S)$$

possesses a unique solution $u_1 \in \mathbb{M}_{S_1-S}\big(w(S), r\big)$. By Lemma 7.1 it is easily verified that $w \oplus_S u_1$ is a solution of $(9.1)_{(\varphi,x)}$ on J_{S_1} belonging to $\mathcal{H}(J_{S_1})$.

(3) Let $0 < S_1 \leq S_2 \leq \mathsf{T}$ and let $u_j \in \mathcal{H}(J_{S_j})$ be a solution of $(9.1)_{(\varphi,x)}$ on J_{S_j}. Set

$$\tau := \max\big\{\, t \in [0, S_1] \ ;\ u_1(s) = u_2(s) \text{ for } 0 \leq s \leq t \,\big\}.$$

Suppose that $\tau < S_1$. Then the preceding step guarantees the existence of σ_0 in the interval $(0, S_1 - \tau]$ and $r > 0$ and, for each $\sigma \in (0, \sigma_0]$, of a unique $v \in \mathbb{M}_\sigma\big(u_1(\tau), r\big)$ such that $u_1 \oplus_\tau v$ is a solution of $(9.1)_{(\varphi,x)}$ on $J_{\tau+\sigma}$. Choose $\sigma \in (0, \sigma_0]$ such that $u_j(\cdot + \tau) \in \mathbb{M}_\sigma\big(u_1(\tau), r\big)$ for $j = 1, 2$. Then uniqueness implies

$$u_1 \,|\, J_{\tau+\sigma} = u_1 \oplus_\tau v = u_2 \,|\, J_{\tau+\sigma}.$$

Since this contradicts the definition of τ, it follows that $u_1 \subset u_2$.

(4) Denote by t^+ the supremum of all $S \in (0, \mathsf{T}]$ such that $(9.1)_{(\varphi,x)}$ has on J_S a solution $u_S \in \mathcal{H}(J_S)$. Step (1) implies that t^+ is well defined, and from step (3) it follows that u_S is uniquely determined. Thus $u(\cdot, x) \in \mathcal{H}_{\mathrm{loc}}\big(0, t^+\big)$ can be defined by $u(\cdot, x) \,|\, J_S := u_S$ for $0 < S < t^+$. Then $u(\cdot, x)$ is the unique solution of $(9.1)_{(\varphi,x)}$ on $\big[0, t^+\big) =: J_{\max}$.

Suppose that $u(\cdot, x) \in \mathcal{H}\big(J_{\max}\big)$. Then $\overline{x} := u\big(t^+, x\big)$ is well defined in E, due to (0.2). If $t^+ < \mathsf{T}$, then step (2) shows that we can extend $u(\cdot, x)$ to a solution of $(9.1)_{(\varphi,x)}$ on J_S belonging to $\mathcal{H}(J_S)$ for some $S > t^+$. Since this contradicts the definition of t^+, it follows that $J_{\max} = \mathsf{J}$. This proves the theorem. $\square$

11. Proof of the continuity theorem

In this section we give a proof of the continuity theorem which guarantees that the solution of (9.1) depends Lipschitz continuously on (φ, x).

Proof of Theorem 3.1. Let S be fixed as prescribed.

(1) Fix $R > \|u_0\|_{\mathcal{H}(J_S)}$. By replacing J in Lemma 8.1 by J_S we find constants $K, r_0, \rho_0, \varepsilon_0 > 0$ and $\kappa \geq 1$ such that

$$\big\|\tau_s\big(A(u) - A(v)\big)\big\|_{L_\infty(J_\sigma, \mathcal{L})} \leq \kappa\big(\|u - v\|_{\mathcal{H}(J_{s+\sigma})} + \|u(0) - v(0)\|_E\big) \tag{11.1}$$

and

$$\big\|\tau_s\big(F(u) - F(v)\big)\big\|_{L(J_\sigma)} \leq \kappa\eta(\sigma)\big(\|u - v\|_{\mathcal{H}(J_{s+\sigma})} + \|u(0) - v(0)\|_E\big), \tag{11.2}$$

as well as

$$\tau_s A(u) \in \mathcal{MR}(J_\sigma, \kappa) \tag{11.3}$$

for

$$0 \le s < S, \quad 0 < \sigma \le S - s, \quad u, v \in \mathbb{N}_{s+\sigma}(u_0; r_0, \rho_0),$$
$$\varphi \in \mathcal{V}_{\varepsilon_0} := \mathcal{V}(J_S; \varphi_0, R, K, \varepsilon_0).$$

(2) Fix $r, \rho_0 \in (0,1]$ such that

$$3r \le r_0, \quad 16\kappa^2 r \le 1, \quad 16\kappa^2 \rho_0 \le r. \tag{11.4}$$

We claim that

$$\left.\begin{aligned}
&\text{if } u(\varphi, x) \in \overline{\mathcal{B}}_s(u_0, r/2)\\
&\text{for some } s \in (0, S) \text{ and } (\varphi, x) \in \Phi(J_S) \times E,\\
&\text{then } u(\varphi, x) \oplus_s v \in \overline{\mathcal{B}}_{s+\sigma}(u_0, r_0) \text{ for } 0 < \sigma \le S - s \text{ and}\\
&v \in Q_\sigma(s, \varphi, x) := \mathbb{M}_\sigma\big(u(\varphi, x)(s)\big) \cap \overline{\mathcal{B}}_\sigma(\tau_s u_0, r).
\end{aligned}\right\} \tag{11.5}$$

Indeed, using $u_0 = u_0 \oplus_s \tau_s u_0$, we deduce from Lemma 7.1 that

$$\begin{aligned}
\|u(\varphi, x) \oplus_s v - u_0\|_{\mathcal{H}(J_{s+\sigma})} &\le 2\big(\|u(\varphi, x) - u_0\|_{\mathcal{H}(J_s)} + \|v - \tau_s u_0\|_{\mathcal{H}(J_\sigma)}\big)\\
&\le 3r.
\end{aligned}$$

Hence (11.4) implies the assertion.

(3) Suppose that

$$s \in (0, S) \text{ and } (\varphi, x) \in \mathcal{V}_{\varepsilon_0} \times E \text{ satisfy } u(\varphi, x) \in \mathbb{N}_s(u_0; r/2, \rho_0). \tag{11.6}$$

Set

$$\varphi_{x,s}(v) := \tau_s\big[\varphi\big(u(\varphi, x) \oplus_s v\big)\big], \qquad v \in Q_\sigma(s, \varphi, x), \quad 0 < \sigma \le S - s.$$

Then (11.1) and (11.5) guarantee that

$$\|A_{x,s}(v) - A_{x,s}(w)\|_{L_\infty(J_\sigma, \mathcal{L})} \le \kappa \, \|v - w\|_{\mathcal{H}(J_\sigma)} \tag{11.7}$$

for

$$v, w \in Q_\sigma(s, \varphi, x) \quad \text{and} \quad 0 < \sigma \le S - s. \tag{11.8}$$

Put $\overline{A} := A_0(u_0)$ and observe that

$$\begin{aligned}
\tau_s \overline{A} - A_{x,s}(v) &= \tau_s\big(A_0(u_0) - A\big(u(\varphi, x) \oplus_s v\big)\big)\\
&= \tau_s\big(A_0(u_0) - A(u_0)\big) + \tau_s\big(A(u_0) - A\big(u(\varphi, x) \oplus_s v\big)\big).
\end{aligned}$$

Hence we deduce from (11.1), (11.4), (11.5), and (11.7) that

$$\begin{aligned}
&\|\tau_s \overline{A} - A_{x,s}(v)\|_{L_\infty(J_\sigma, \mathcal{L})}\\
&\qquad \le 2\kappa\big(\|u_0 - u(\varphi, x)\|_{\mathcal{H}(J_s)} + \|\tau_s u_0 - v\|_{\mathcal{H}(J_\sigma)}\big) + \kappa \, \|x - x_0\|_E\\
&\qquad\quad + \big\|\tau_s\big((A - A_0)(u_0)\big)\big\|_{L_\infty(J_\sigma, \mathcal{L})}\\
&\qquad \le 3\kappa r + \kappa \rho_0 + \big\|\tau_s\big((A - A_0)(u_0)\big)\big\|_{L_\infty(J_\sigma, \mathcal{L})}
\end{aligned} \tag{11.9}$$

if σ and v satisfy (11.8). Similarly, it follows from (11.2) that

$$\begin{aligned}\|(F_0)_{x_0,s}(\tau_s u_0) - F_{x,s}(v)\|_{L(J_\sigma)} \\ \leq 2\kappa\eta(\sigma)\big(\|u_0 - u(\varphi,x)\|_{\mathcal{H}(J_s)} + \|\tau_s u_0 - v\|_{\mathcal{H}(J_\sigma)}\big) \\ + \kappa\eta(\sigma)\,\|x - x_0\|_E + \big\|\tau_s\big((F-F_0)(u_0)\big)\big\|_{L(J_\sigma)} \\ \leq 3\kappa\eta(\sigma)r + \kappa\eta(\sigma)\rho_0 + \big\|\tau_s\big((F-F_0)(u_0)\big)\big\|_{L(J_\sigma)},\end{aligned} \tag{11.10}$$

provided σ and v satisfy (11.8).

(4) Let (11.6) and (11.8) be satisfied. Put

$$H_s(\varphi,x,v) := \big(\tau_s\overline{A} - A_{x,s}(v)\big)v + F_{s,x}(v) \in L(J_\sigma).$$

Since

$$\begin{aligned}\|v\|_{\mathcal{H}(J_\sigma)} &\leq \|v - \tau_s u_0\|_{\mathcal{H}(J_\sigma)} + \|\tau_s u_0\|_{\mathcal{H}(J_\sigma)} \\ &\leq r + \|\tau_s u_0\|_{\mathcal{H}(J_\sigma)}\end{aligned} \tag{11.11}$$

we infer from (11.5), (11.9), and (11.10) that

$$\begin{aligned}\|H_s(\varphi,x,v) - (F_0)_{x_0,s}(\tau_s u_0)\|_{L(J_\sigma)} \\ \leq r3\kappa\big(r + \|\tau_s u_0\|_{\mathcal{H}(J_\sigma)} + \eta(\sigma)\big) \\ + \kappa\rho_0\big(1+\eta(\sigma)\big) + \beta\,\|(\varphi-\varphi_0)(u_0)\|_{\mathsf{Z}_S},\end{aligned} \tag{11.12}$$

where $\beta := 1 + \|u_0\|_{\mathcal{H}(\mathsf{J})}$. Similarly, from

$$\begin{aligned}&H_s(\varphi,x,v) - H_s(\varphi,x,w) \\ &\quad = \big(\tau_s\overline{A} - A_{x,s}(v)\big)(v-w) + \big(A_{x,s}(w) - A_{x,s}(v)\big)w + F_{x,s}(v) - F_{x,s}(w),\end{aligned}$$

(11.1), (11.2), and (11.9) we deduce that

$$\begin{aligned}\|H_s(\varphi,x,v) - H_s(\varphi,x,w)\|_{L(J_\sigma)} \\ \leq \kappa\big(4r + \rho_0 + \|\tau_s u_0\|_{\mathcal{H}(J_\sigma)} + \eta(\sigma) \\ + \|(\varphi-\varphi_0)(u_0)\|_{\mathsf{Z}_S}\big)\,\|v-w\|_{\mathcal{H}(J_\sigma)}.\end{aligned} \tag{11.13}$$

(5) Suppose that (11.6) and (11.8) hold. Denote by $V(v) := V(s,\varphi,x,v)$ the unique solution in $\mathcal{H}(J_\sigma)$ of the linear Cauchy problem

$$\dot{u} + \tau_s\overline{A}u = H_s(\varphi,x,v) \text{ on } J_\sigma, \quad u(0) = u(\varphi,x)(s).$$

Then $w := V(v) - \tau_s u_0 \in \mathcal{H}(J_\sigma)$ satisfies

$$\dot{w} + \tau_s\overline{A}w = H_s(\varphi,x,v) - (F_0)_{x_0,s}(\tau_s u_0) \text{ on } J_\sigma,$$

and $w(0) = u(\varphi,x)(s) - u_0(s)$. Thus (11.3) and Lemma 6.1(iii) imply

$$\|w\|_{\mathcal{H}(J_\sigma)} \leq \kappa\big(\|H_s(\varphi,x,v) - (F_0)_{x_0,s}(\tau_s u_0)\|_{L(J_\sigma)} + \|u(\varphi,x)(s) - u_0(s)\|_E\big).$$

Hence we deduce from (11.12) that

$$\begin{aligned}\|V(v)-\tau_s u_0\|_{\mathcal{H}(J_\sigma)} \le{}& r3\kappa^2\big(r+\alpha(s,\sigma)\big)+\kappa^2\rho_0\\ &+\kappa\big(\beta\,\|(\varphi-\varphi_0)(u_0)\|_{\mathsf{Z}_S}+\|u(\varphi,x)(s)-u_0(s)\|_E\big),\end{aligned}$$

where

$$\alpha(s,\sigma):=\|\tau_s u_0\|_{\mathcal{H}(J_\sigma)}+2\eta(\sigma).$$

Similarly, since

$$\big(V(v)-V(w)\big)^{\boldsymbol{\cdot}}+\tau_s\overline{A}\big(V(v)-V(w)\big)=H_s(\varphi,x,v)-H_s(\varphi,x,w)$$

on J_σ, and $\big(V(v)-V(w)\big)(0)=0$, we obtain from (11.13) that

$$\|V(v)-V(w)\|_{\mathcal{H}(J_\sigma)}\le\kappa^2\big(4r+\rho_0+\alpha(s,\sigma)+\|(\varphi-\varphi_0)(u_0)\|_{\mathsf{Z}_S}\big)\,\|v-w\|_{\mathcal{H}(J_\sigma)}.$$

Now we fix positive numbers $\overline{\varepsilon}\le\varepsilon_0$, $\overline{\rho}\le 1$, and $\overline{\sigma}=\overline{\sigma}(s)$ such that

$$16\beta\kappa^2\overline{\varepsilon}\le r,\quad 4\kappa\overline{\rho}\le r,\quad 12\kappa^2\alpha(s,\overline{\sigma})\le 1.$$

Then, recalling (11.4), we see that

$$\|V(v)-\tau_s u_0\|_{\mathcal{H}(J_{\sigma(s)})}\le r$$

and

$$\|V(v)-V(w)\|_{\mathcal{H}(J_{\sigma(s)})}\le\|v-w\|_{\mathcal{H}(J_{\sigma(s)})}/2$$

for $0<\sigma(s)\le\overline{\sigma}\wedge(S-s)$, provided

$$\varphi\in\mathcal{V}_{\overline{\varepsilon}},\quad u(\varphi,x)\in\mathbb{N}_s(u_0;r/2,\rho_0),\quad \|u(\varphi,x)(s)-u_0(s)\|_E\le\overline{\rho}. \tag{11.14}$$

Thus $V(\varphi,x,\cdot)$ maps the complete metric space $Q_{\sigma(s)}(s,\varphi,x)$ into itself and is a strict contraction. Hence it has a unique fixed point, $v(s,\varphi,x)$, in $Q_{\sigma(s)}(s,\varphi,x)$.

(6) Suppose that $0<s<S$ and $(\varphi,x)\in\mathcal{V}_{\overline{\varepsilon}}\times E$ satisfy (11.14), and fix $\sigma(s)$ with $0<\sigma(s)\le\overline{\sigma}\wedge(S-s)$. Note that $v(s,\varphi,x)$ is a solution of

$$\dot{u}+A_{s,x}(u)u=F_{s,x}(u)\text{ on }J_{\sigma(s)},\quad u(0)=u(\varphi,x)(s).$$

Thus we infer from Lemma 7.1 that $u(\varphi,x)\oplus_s v(s,\varphi,x)$ is a solution of $(9.1)_{(\varphi,x)}$ on $J_{s+\sigma(s)}$. Hence Theorem 2.1 implies

$$u(\varphi,x)(s)\oplus_s v(s,\varphi,x)=u(\varphi,x)\,|\,J_{s+\sigma(s)}.$$

This shows that $J(\varphi,x)\supset\big[0,s+\sigma(s)\big]$, provided (11.14) is true.

(7) Now we claim that there exists $\overline{\tau}>0$ such that

$$12\kappa^2\alpha(s,\tau)\le 1,\qquad 0<\tau\le\overline{\tau}\wedge(S-s),\quad s\in[0,S). \tag{11.15}$$

To prove this, it suffices to show that for every $\xi > 0$ there exists $\delta > 0$ such that, denoting by χ_D the characteristic function of a subset D of J_S,

$$\|u_0\chi_D\|^p_{L_p(J_S)} + \|\dot{u}_0\chi_D\|^p_{L_p(J_S)} < \xi$$

for every measurable set $D \subset J_S$ with measure, $|D|$, less than δ. Suppose this to be false. Then there exists for each $j \in \mathbb{N}$ a measurable subset $D_j \subset J_S$ satisfying $|D_j| < 2^{-j}$ and

$$\|u_0\chi_{D_j}\|^p_{L_p(J_S)} + \|\dot{u}_0\chi_{D_j}\|^p_{L_p(J_S)} \geq \xi. \tag{11.16}$$

Set $C_n := \bigcup_{j\geq n} D_j$ and $C := \bigcap_{n\geq 0} C_n$. Then C_n and C are measurable subsets of J_S satisfying $|C| \leq |C_n| \leq \sum_{j\geq n} 2^{-j} = 2^{-n+1}$ for $n \in \mathbb{N}$. Hence $|C| = 0$ and $\chi_{C_n} \downarrow 0$ a.e. as $n \to \infty$. Since $\|u_0\chi_{D_j}\|_{L_p(J_S)} \leq \|u_0\chi_{C_j}\|_{L_p(J_S)}$ for $j \in \mathbb{N}$, it follows from Lebesgue's theorem that the left-hand side of (11.16) converges to zero as $j \to \infty$, which is impossible. This proves (11.15).

(8) We set $\tau(s) := \overline{\tau} \wedge (S - s)$ and suppose that

$$\left.\begin{aligned} &0 < s < S,\ (\varphi_j, x_j) \in \mathcal{V}_{\overline{\varepsilon}} \times B_E(x_0, \rho_0),\\ &u(\varphi_j, x_j) \in \mathcal{B}_s(u_0, r/2),\\ &\|u(\varphi_j, x_j)(s) - u_0(s)\|_E \leq \overline{\rho},\ j = 1, 2. \end{aligned}\right\} \tag{11.17}$$

Then we put $u_j := u(\varphi_j, x_j)$ and $v_j := \tau_s u_j$. It follows from steps (5)–(7) that $v_j \in Q_{\tau(s)}(s, \varphi_j, x_j)$ and that

$$\begin{aligned} (v_1 - v_2)^{\cdot} &+ \tau_s A_1(u_1 \oplus_s v_1)(v_1 - v_2)\\ &= \tau_s\bigl[A_2(u_2 \oplus_s v_2) - A_1(u_1 \oplus_s v_1)\bigr]v_2\\ &\quad + \tau_s\bigl[F_1(u_1 \oplus_s v_1) - F_2(u_2 \oplus_s v_2)\bigr] \end{aligned} \tag{11.18}$$

on $J_{\tau(s)}$ with $(v_1 - v_2)(0) = (u_1 - u_2)(s)$. Since v_2 satisfies (11.11), we see, similarly as in step (4), that the $L(J_{\tau(s)})$ norm of the right-hand side is estimated from above by

$$\begin{aligned} &2\kappa\bigl(r + \alpha(s, \tau(s))\bigr)\bigl(\|u_1 - u_2\|_{\mathcal{H}(J_s)} + \|v_1 - v_2\|_{\mathcal{H}(J_{\tau(s)})} + \|x_1 - x_2\|_E\bigr)\\ &\quad + \beta\,\|(\varphi_1 - \varphi_2)(u_1)\|_{\mathsf{Z}_{s+\tau(s)}}. \end{aligned}$$

Due to $2\kappa\bigl(r + \alpha(s, \tau(s))\bigr) \leq 1/2\kappa$, by (11.4) and (11.15), it follows from (11.18) and (11.3) that

$$\begin{aligned} &\|\tau_s(u_1 - u_2)\|_{\mathcal{H}(J_{\tau(s)})}\\ &\qquad \leq \|u_1 - u_2\|_{\mathcal{H}(J_s)} + \|x_1 - x_2\|_E\\ &\qquad\quad + 2\kappa\bigl(\beta\,\|(\varphi_1 - \varphi_2)(u_1)\|_{\mathsf{Z}_{s+\tau(s)}} + \|u_1(s) - u_2(s)\|_E\bigr). \end{aligned} \tag{11.19}$$

(9) Assume that $0 < s < S$ and that there exist constants $\varepsilon(s) \in (0, \overline{\varepsilon}]$, $\rho(s) \in (0, \overline{\rho} \wedge \rho_0]$, and $\mu(s) > 0$ such that, setting $u_j := u(\varphi_j, x_j)$,

$$\|u_1 - u_2\|_{\mathcal{H}(J_s)} \le \mu(s)\bigl(\|(\varphi_1 - \varphi_2)(u_1)\|_{\mathsf{Z}_s} + \|x_1 - x_2\|_E\bigr) \qquad (11.20)_s$$

for $(\varphi_j, x_j) \in \mathcal{V}_{\varepsilon(s)} \times B_E\bigl(x_0, \rho(s)\bigr)$ and $j = 1, 2$.

Let $\lambda(s)$ be the norm of the trace map $\mathcal{H}(J_s) \to E$, $u \mapsto u(s)$. Then it follows from (11.20) that

$$\|u(\varphi, x)(s) - u_0(s)\|_E \le \lambda(s)\mu(s)\bigl(\|(\varphi - \varphi_0)(u_1)\|_{\mathsf{Z}_s} + \rho(s)\bigr)$$

for $(\varphi, x) \in \mathcal{V}_{\varepsilon(s)} \times B_E\bigl(x_0, \rho(s)\bigr)$. From this we deduce that there are constants $\varepsilon_1(s) \in \bigl(0, \varepsilon(s)\bigr]$ and $\rho_1(s) \in \bigl(0, \rho(s)\bigr]$ such that

$$u(\varphi, x) \in \mathcal{B}_s(u_0, r/2) \qquad (11.21)$$

and

$$\|u(\varphi, x)(s) - u_0(s)\|_E \le \overline{\rho}$$

for $(\varphi, x) \in \mathcal{V}_{\varepsilon_1(s)} \times B_E\bigl(x_0, \rho_1(s)\bigr)$. Thus (11.17) implies that (11.19) holds for

$$(\varphi_j, x_j) \in \mathcal{V}_{\varepsilon_1(s)} \times B_E\bigl(x_0, \rho_1(s)\bigr).$$

Now, setting

$$(\varepsilon, \rho)\bigl(s + \tau(s)\bigr) := (\varepsilon_1, \rho_1)(s), \quad \mu\bigl(s + \tau(s)\bigr) := 4\bigl[\beta\kappa + \bigl(1 + \kappa\lambda(s)\bigr)\mu(s)\bigr],$$

it follows from $(11.20)_s$ and Lemma 7.1 that

$$\begin{aligned} &\|u_1 - u_2\|_{\mathcal{H}(J_{s+\tau(s)})} \\ &\qquad \le \mu\bigl(s + \tau(s)\bigr)\bigl(\|(\varphi_1 - \varphi_2)(u_1)\|_{\mathsf{Z}_{s+\tau(s)}} + \|x_1 - x_2\|_E\bigr) \end{aligned} \qquad (11.22)_s$$

for $(\varphi_j, x_j) \in \mathcal{V}_{\varepsilon(s+\tau(s))} \times B_E\bigl(x_0, \rho(s + \tau(s))\bigr)$ and $j = 1, 2$.

(10) By Proposition 9.1 we can find constants $s_0 \in (0, S)$, $\varepsilon(s_0) \in (0, \overline{\varepsilon}]$, $\rho(s_0) \in (0, \overline{\rho} \wedge \rho_0]$, and $\mu(s_0) > 0$ such that $(11.20)_{s_0}$ holds. Hence step (9) shows that $(11.22)_{s_0}$ is also true. Now we obtain from step (9) by a finite induction argument that there are constants $\varepsilon \in (0, \overline{\varepsilon}]$, $\rho \in (0, \overline{\rho} \wedge \rho_0]$, and $\kappa > 0$ such that (3.3) is true. $\square$

12. Proof of the differentiability theorem

Although Theorem 4.1 is essentially a consequence of the implicit function theorem, we cannot refer to standard results but have to give a direct proof, due to the fact that our setting involves locally convex spaces and Gateaux differentiability only.

Proof of Theorem 4.1. (1) We set $\Xi := \mathcal{C}^1\big(\mathcal{H}(J_S), \mathsf{Z}_S\big) \times E$ and denote its general point by $\xi = (\varphi, x)$. Then we define the map

$$f : \Xi \times \mathcal{H}(J_S) \to L(J_S) \times E$$

by

$$f(\xi, u) := \big(\partial u + A(u)u - F(u), u(0) - x\big), \qquad \xi = (\varphi, x) \in \Xi, \quad u \in \mathcal{H}(J_S),$$

where, as usual, $\varphi = (A, F)$. It follows from (0.2), (4.1), and the definition of the topology of $\mathcal{C}^1\big(\mathcal{H}(J_S), \mathsf{Z}_S\big)$ that f is continuous. Note that $f(\xi_0, u_0) = 0$.

(2) The map $f(\cdot, u) : \Xi \to L(J_S) \times E$ is for each $u \in \mathcal{H}(J_S)$ Gateaux differentiable, and

$$D_1 f(\xi, u)\widehat{\xi} = \big(\widehat{A}(u)u - \widehat{F}(u), -\widehat{x}\big), \qquad \xi = (\varphi, x), \quad \widehat{\xi} = (\widehat{\varphi}, \widehat{x}) \in \Xi.$$

Note that the Gateaux derivative $\xi \mapsto D_1 f(\xi, u)$ is constant. Thus it follows that

$$D_1 f \in \mathcal{C}\big(\Xi \times \mathcal{H}(J_S), \mathcal{L}\big(\Xi, L(J_S) \times E\big)\big).$$

(3) The map $f(\xi, \cdot) : \mathcal{H}(J_S) \to L(J_S) \times E$ is differentiable for fixed $\xi \in \Xi$, and

$$D_2 f(\xi, u) = \big(\partial + A(u) + DA(u)[\cdot, u] - DF(u), \gamma_0\big).$$

From this we see that

$$D_2 f \in \mathcal{C}\big(\Xi \times \mathcal{H}(J_S), \mathcal{L}\big(\mathcal{H}(J_S), L(J_S) \times E\big)\big).$$

Observe that

$$D_2 f(\xi_0, u_0) = \big(\partial + B_0(u_0), \gamma_0\big) \in \mathcal{L}\mathrm{is}\big(\mathcal{H}(J_S), L(J_S) \times E\big)\big),$$

as follows from the assumed maximal regularity and Lemma 6.1.

(4) Set $g(\xi, u) := u - \big[D_2 f(\xi_0, u_0)\big]^{-1} f(\xi, u)$. Then $f(\xi, u) = 0$ is equivalent to $g(\xi, u) = u$. Thus $g(\xi_0, u_0) = u_0$.

We deduce from step (3) that $g(\xi, \cdot)$ is continuously differentiable with

$$D_2 g(\xi, u) = 1 - \big[D_2 f(\xi_0, u_0)\big]^{-1} D_2 f(\xi, u) \in \mathcal{L}\big(\mathcal{H}(J_S)\big).$$

Thus there exist $r > 0$ and a neighborhood $\mathcal{U}_0$ of ξ_0 in Ξ such that

$$\|D_2 g(\xi, u)\| \le 1/2, \quad \|g(\xi, u) - u_0\| \le r, \qquad \xi \in \mathcal{U}_0, \quad u \in \overline{\mathcal{B}}_S(u_0, r).$$

Consequently,

$$\|g(\xi, u) - g(\xi, v)\| \le \sup_{0<t<1} \big\|D_2 g\big(\xi, v + t(u - v)\big)\big\| \, \|u - v\| \le \|u - v\|/2 \quad (12.1)$$

for $\xi \in \mathcal{U}_0$ and $u, v \in \overline{\mathcal{B}}_S(u_0, r)$. Hence Banach's fixed point theorem guarantees that $g(\xi, \cdot)$ has for each $\xi \in \mathcal{U}_0$ a unique fixed point, $u(\xi)$, in $\overline{\mathcal{B}}_S(u_0, r)$. Moreover,

$$\begin{aligned} u(\xi) - u(\xi') &= g\big(\xi, u(\xi)\big) - g\big(\xi', u(\xi')\big) \\ &= g\big(\xi, u(\xi)\big) - g\big(\xi, u(\xi')\big) + g\big(\xi, u(\xi')\big) - g\big(\xi', u(\xi')\big) \end{aligned}$$

implies, due to (12.1), that

$$\|u(\xi) - u(\xi')\| \le 2 \left\| g\big(\xi, u(\xi')\big) - g\big(\xi', u(\xi')\big) \right\|, \qquad \xi, \xi' \in \mathcal{U}_0.$$

This proves the first assertion and the continuity of the solution map.

(5) Set

$$\mathsf{p}_R(\varphi, x) := \sup_{v \in R\mathcal{B}_S} \|\varphi(v)\|_{Z_S} + \|x\|_E.$$

Then there exist $R, \rho > 0$ such that

$$\mathcal{U}_{R,2\rho} := \big\{ \xi \in \Xi \ ; \ \mathsf{p}_R(\xi - \xi_0) < 2\rho \big\} \subset \mathcal{U}_0.$$

Also put $\mathcal{U} := \mathcal{U}_{R,\rho}$. For $\widehat{\xi} \in \Xi$ fix a positive $\varepsilon = \varepsilon(\widehat{\xi}\,)$ satisfying $\varepsilon \mathsf{p}_R(\widehat{\xi}\,) \le \rho$. Then $\xi + t\widehat{\xi} \in \mathcal{U}_0$ for $(\xi, t) \in \mathcal{U} \times (-\varepsilon, \varepsilon)$. Hence

$$\begin{aligned} 0 &= f\big(\xi + t\widehat{\xi}, u(\xi + t\widehat{\xi}\,)\big) - f\big(\xi, u(\xi)\big) \\ &= f\big(\xi + t\widehat{\xi}, u(\xi + t\widehat{\xi}\,)\big) - f\big(\xi, u(\xi + t\widehat{\xi}\,)\big) + f\big(\xi, u(\xi + t\widehat{\xi}\,)\big) - f\big(\xi, u(\xi)\big). \end{aligned}$$

Thus we infer from steps (2) and (3) and the mean value theorem, due to the fact that $D_1 f(\cdot, v)$ is constant for $v \in \mathcal{H}(J_S)$, that

$$\begin{aligned} \int_0^1 D_2 f\big(\xi, u(\xi) + s\big(u(\xi + t\widehat{\xi}\,) - u(\xi)\big)\big)\, ds\, \big(u(\xi + t\widehat{\xi}\,) - u(\xi)\big) \\ = -t D_1 f\big(\xi, u(\xi + t\widehat{\xi}\,)\big)\widehat{\xi}. \end{aligned} \tag{12.2}$$

Set

$$C(\xi, t) := \int_0^1 D_2 f\big(\xi, u(\xi) + s\big(u(\xi + t\widehat{\xi}\,) - u(\xi)\big)\big)\, ds.$$

Then the continuity of $D_2 f$ and of $u(\cdot)$ imply that

$$C(\cdot, \cdot) \in C\big(\mathcal{U} \times (-\varepsilon, \varepsilon), \mathcal{L}\big(\mathcal{H}(J_S), L(J_S) \times E\big)\big).$$

Moreover,

$$C(\xi_0, 0) = D_2 f(\xi_0, u_0) \in \mathcal{L}\mathrm{is}\big(\mathcal{H}(J_S), L(J_S) \times E\big).$$

Thus the openness of $\mathcal{L}$is and the continuity of the inversion map $C \mapsto C^{-1}$ imply that we can assume that $\big[C(\xi,t)\big]^{-1}$ exists for $(\xi,t) \in \mathcal{U} \times (-\varepsilon,\varepsilon)$, and that

$$C^{-1} \in C\big(\mathcal{U} \times (-\varepsilon,\varepsilon), \mathcal{L}\big(L(J_S) \times E, \mathcal{H}(J_S)\big)\big).$$

Consequently, we infer from (12.2) that

$$\big(u(\xi + t\widehat{\xi}\,) - u(\xi)\big)/t = -\big[C(\xi,t)\big]^{-1} D_1 f\big(\xi, u(\xi + t\widehat{\xi}\,)\big)\widehat{\xi}$$

for $(\xi,t) \in \mathcal{U} \times (-\varepsilon,\varepsilon)$. Hence, by the continuity of u, C^{-1}, and $D_1 f$, it follows that u is G differentiable in $\mathcal{U}$ and

$$Du(\xi)\widehat{\xi} = -\big[D_2 f\big(\xi, u(\xi)\big)\big]^{-1} D_1 f\big(\xi, u(\xi)\big)\widehat{\xi}.$$

Now the assertions follow. $\square$

References

[1] H. Amann. *Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems,* in "Function spaces, differential operators and nonlinear analysis (Friedrichroda, 1992)," volume 133 of *Teubner-Texte Math.*, pages 9–126. Teubner, Stuttgart, 1993.

[2] H. Amann. "Linear and Quasilinear Parabolic Problems, Volume I: Abstract Linear Theory," Birkhäuser, Basel, 1995.

[3] H. Amann. *Nonautonomous parabolic equations involving measures,* Zap. Nauchn. Sem. St.-Petersburg. Otdel. Mat. Inst. Steklov. (POMI) Vol. 306 (Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funktsii. 34) (2003), 16–52, 229.

[4] H. Amann. *Maximal regularity and quasilinear evolution equations,* Oberwolfach Reports Vol. 1(3), (2004), 1957–1961.

[5] H. Amann. *Maximal regularity for nonautonomous evolution equations,* Adv. Nonl. Studies Vol. 4 (2004), 417–430.

[6] H. Amann. *Maximal regularity and quasilinear parabolic boundary value problems,* in C.-C. Chen, M. Chipot, C.-S. Lin, editors, "Recent Advances in Elliptic and Parabolic Problems," pages 1–17. Proc. Int. Conf., Hsinchu, Taiwan, 16.–20. Feb. 2004, World Scientific, 2005.

[7] S.B. Angenent. *Nonlinear analytic semiflows,* Royal Soc. Edinburgh Vol. 115A (1990), 91–107.

[8] V. Barbu. "Nonlinear Semigroups and Differential Equations in Banach Spaces," Noordhoff, Leyden, 1976.

[9] Ph. Clément and S. Li. *Abstract parabolic quasilinear equations and application to a groundwater flow problem,* Adv. Math. Sci. Appl. Vol. 3(Special Issue) (1993/94), 17–32.

[10] Ph. Clément and G. Simonett. *Maximal regularity in continuous interpolation spaces and quasilinear parabolic equations,* J. Evol. Equ., Vol. 1(1) (2001), 39–67.

[11] G. Da Prato and P. Grisvard. *Equations d'évolutions abstraites non linéaires de type parabolique,* Ann. Mat. Pura Appl. (4), Vol. 120 (1979), 329–396.

[12] G. Dore. *L^p regularity for abstract differential equations,* in "Functional analysis and related topics, 1991 (Kyoto)," volume 1540 of *Lecture Notes in Math.*, pages 25–38. Springer, Berlin, 1993.

[13] A. Friedman. "Partial Differential Equations," Holt, Rinehart & Winston, New York, 1969.

[14] A. Lunardi. "Analytic Semigroups and Optimal Regularity in Parabolic Problems," Birkhäuser, Basel, 1995.

[15] J. Prüss. *Maximal regularity for evolution equations in L_p-spaces,* Conf. Semin. Mat. Univ. Bari, (285) (2002), 1–39 (2003).

[16] P.E. Sobolevskii. *Equations of parabolic type in a Banach space,* Amer. Math. Soc. Transl., Ser. 2 Vol. 49 (1966), 1–62.