QUASILINEAR PARABOLIC FUNCTIONAL EVOLUTION EQUATIONS

HERBERT AMANN
Institut für Mathematik, Universität Zürich, Winterthurerstr. 190, CH–8057 Zürich, Switzerland, email: herbert.amann@math.unizh.ch

Based on our recent work on quasilinear parabolic evolution equations and maximal regularity we prove a general result for quasilinear evolution equations with memory. It is then applied to the study of quasilinear parabolic differential equations in weak settings. We prove that they generate Lipschitz semiflows on natural history spaces. The new feature is that delays can occur in the highest order nonlinear terms. The general theorems are illustrated by a number of model problems.

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1. Introduction

In a recent paper [8] we have derived very general existence, uniqueness, and continuity theorems for abstract quasilinear evolution equations of the form

$$\dot{u} + A(u)u = F(u). \tag{1}$$

Here $A(u)$ is for each given u in an appropriate class of functions a bounded measurable function with values in a Banach space of bounded linear operators. Thus $\dot{v} + A(u)v = F(u)$ is for each suitable u a nonautonomous evolution equation on some Banach space. The new feature of our result is that the class of admissible functions, that is, the domain of definition of $A(\cdot)$ and $F(\cdot)$, is the same as the one where a solution of (1) is being sought for. More precisely, given Banach spaces $E_1 \overset{d}{\hookrightarrow} E_0$ and $1 < p < \infty$, we assume that A and F are defined on

$$L_p\big((0,T),E_1\big) \cap H_p^1\big((0,T),E_0\big) \tag{2}$$

and map this space into $L_\infty\big((0,T),\mathcal{L}(E_1,E_0)\big)$ and $L_r\big((0,T),E_0\big)$ for some $r > p$, respectively. Consequently, A and F will be nonlocal operators with respect to the time variable, in general. This distinguishes our work in [8] from all previous studies of nonlinear evolution equations where A and F always have been assumed to be local maps (see [7]).

The fact that we work on the function space (2) allows for great flexibility in applications. In particular, we can use the general results to treat evolution equations depending on the history of their solution (see [4], [6], and [9]).

It is the purpose of this paper to give a rigorous basis for such problems. More precisely, we develop a general existence, uniqueness, and continuity theory for functional evolution equations of the form

$$\dot{u} + A(u_t, u)u = F(u_t, u), \tag{3}$$

where, as usual in the theory of functional differential equations, $u_t(\theta) := u(t + \theta)$ for $t \geq 0$ and $-S \leq \theta \leq 0$. (This notation should not be confused with the partial derivative $\partial_t u$.) In particular, we show that in the autonomous case problems of this type generate semiflows on appropriate history spaces. So far only semilinear equations of the general form (3) have been considered where A is independent of u and u_t. For these problems there is a vast literature for which we refer to [29] and the references therein, for example.

The main results for (3) are given in Section 4. In the section following it we prove a rather general theorem for quasilinear parabolic differential equations with memory. The main new feature is that we can allow memory terms in the top order coefficients and that we derive the continuous dependence of the solution on its history. In the autonomous case this implies that (3) generates a semiflow on the history space, a fact which has, up to now, only been shown in semilinear problems.

Problems of this kind occur in several applications, for instance in climate models (see Section 2) or by regularizing ill posed problems in image processing (see [9]). For simplicity, we restrict ourselves to weak settings. However, it will be clear to the reader that the abstract results can also be applied to parabolic differential equations in strong settings (as in e.g., [6] and [9]).

In Section 2 we illustrate the power of our approach by applying the main result of Section 5 to some model problems. We restrict ourselves to simple cases to give the flavor of the techniques and do not strive for optimal results. In particular, we do not present sophisticated global existence results. Section 3 contains an existence and continuity theorem for parameter dependent quasilinear evolution equations. It is an easy consequence of the results in [7], but is put in a form suitable for the study of (3) in Section 4. In the last section we show how the results for the model cases of Section 2 follow from the basic result of Section 5.

2. Model problems

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, where $n \geq 2$. Assume that Γ_1 is a measurable subset of its boundary, Γ, denote by $\chi : \Gamma \to \{0,1\}$ the characteristic function of Γ_1, and put $\Gamma_0 := \Gamma \backslash \Gamma_1$. The pair (Ω, χ) is said to be (C^2) **regular** if Ω is a C^2 domain and χ is continuous. In this case Γ_0 and Γ_1 are both open (and closed) in Γ. In general, either Γ_0 or Γ_1 can be empty, of course. We write $\vec{\nu}$ for the outer unit normal on Γ (defined a.e. with respect to the $(n-1)$-dimensional Hausdorff measure).

In this section we consider the following evolution system

$$\left.\begin{aligned} \partial_t\big(\boldsymbol{e}(u)\big) + \nabla\cdot\vec{\boldsymbol{\jmath}}(u) &= \boldsymbol{f}(u) \qquad \text{on } \Omega\times(0,\infty), \\ \chi\vec{\nu}\cdot\vec{\boldsymbol{\jmath}}(u) + (1-\chi)\gamma u &= \chi\boldsymbol{g}(u) \qquad \text{on } \Sigma\times(0,\infty), \end{aligned}\right\} \tag{4}$$

γ being the trace operator and $\nabla\cdot$ denoting divergence. We are particularly interested in situations where (4) is history dependent. More precisely, we consider constitutive hypotheses of the following form

$$\left.\begin{aligned} &\bullet\quad \boldsymbol{e}(u) := \mu * u; \\ &\bullet\quad \vec{\boldsymbol{\jmath}}(u) := -\nu_0 * \big(a(\cdot,\sigma_0 * u)\nabla u\big) + \nu_1 * \big(b(\cdot,\sigma_1 * u)\nabla u\big); \\ &\bullet\quad \boldsymbol{f}(u) := \rho_0 * f(\cdot,\sigma_2 * u); \\ &\bullet\quad \boldsymbol{g}(u) := \rho_1 * g(\cdot,\sigma_3 * u), \end{aligned}\right\} \tag{5}$$

where μ, ν_j, ρ_j, and σ_j are bounded (possibly Banach space valued) Radon measures on $\mathbb{R}$, to be specified more precisely below. Throughout we suppose that

$$\left.\begin{aligned} &\bullet\quad a \in C^{0,1\text{-}}(\overline{\Omega}\times\mathbb{R}^m, \mathbb{R}^{n\times n}) \text{ such that} \\ &\qquad a(x,\xi) \text{ is symmetric and positive definit,} \\ &\qquad \text{uniformly for } x\in\overline{\Omega} \text{ and } \xi \text{ in bounded intervals;} \\ &\bullet\quad b \in C^{0,1\text{-}}(\overline{\Omega}\times\mathbb{R}^m, \mathbb{R}^{n\times n}); \\ &\bullet\quad f \in C^{0,1\text{-}}(\overline{\Omega}\times\mathbb{R}^m); \\ &\bullet\quad g \in C^{0,1\text{-}}(\Gamma\times\mathbb{R}^m) \end{aligned}\right\} \tag{6}$$

for some $m \in \mathbb{N}$. (For convenience, we put $[0,\infty] := [0,\infty) = \mathbb{R}^+$ and $[-\infty,0] := (-\infty,0]$.) Here and below, given metric spaces X and Y, an open subset O of $X\times Y$, and a Banach space F, we write $C^{0,1\text{-}}(O,F)$ for the set of all $f \in C(O,F)$ such that for each point (x,y) in O there exists a neighborhood $U \times V$ in O such that $f(\cdot,y): U \to F$ is Lipschitz continuous, uniformly with respect to $y \in V$. As usual, we omit the symbol F if $F = \mathbb{R}$.

In this section we also suppose that

$$\left.\begin{array}{l} \bullet\ \text{ either } p = 2, \\ \bullet\ \text{ or } n+2 < p < \infty \text{ and } (\Omega,\chi) \text{ is regular.} \end{array}\right\} \tag{7}$$

We set

$$H^1_{p,\chi} := \bigl\{\, v \in H^1_p := H^1_p(\Omega)\ ;\ (1-\chi)\gamma v = 0 \,\bigr\}, \quad H^{-1}_{p,\chi} := (H^1_{p',\chi})',$$

the dual being determined by means of the L_p duality pairing

$$\langle v, w\rangle := \int_\Omega v \cdot w\, dx, \qquad (v,w) \in L_{p'} \times L_p.$$

Note that

$$H^1_{p,\chi} \overset{d}{\hookrightarrow} L_p \overset{d}{\hookrightarrow} H^{-1}_{p,\chi},$$

where $\overset{d}{\hookrightarrow}$ denotes continuous and dense embedding. Also note that $H^{-1}_{p,\chi} = H^{-1}_p$ if $\chi = 0$, that is, $\Gamma = \Gamma_0$. In this case the second line of (4) reduces to the homogeneous Dirichlet boundary condition $\gamma u = 0$. We also put $H^j_\chi := H^j_{2,\chi}$ for $j = \pm 1$.

Furthermore,

$$W^{1-2/p}_{p,\chi} := \begin{cases} L_2, & \text{if } p = 2, \\ \bigl\{\, v \in W^{1-2/p}_p\ ;\ (1-\chi)\gamma v = 0 \,\bigr\} & \text{otherwise,} \end{cases}$$

where $W^s_p := W^s_p(\Omega)$ are the usual Sobolev-Slobodeckii spaces of order $s \in [0,1]$. Recall that, except for equivalent norms, $W^s_p = H^s_p$ for $s = 0,1$.

Let I be an interval with nonempty interior $\mathring{I}$. Then

$$\mathcal{H}^1_{p,\chi}(I) := \mathcal{H}^{1,1}_{p,p,\chi}(\Omega \times I) := L_p(I, H^1_{p,\chi}) \cap H^1_p\bigl(\mathring{I}, H^{-1}_{p,\chi}\bigr)$$

and $\mathcal{H}^1_\chi := \mathcal{H}^1_{2,\chi}$. It will be shown below that

$$\mathcal{H}^1_{p,\chi}(I) \hookrightarrow C_0\bigl(\overline{I}, W^{1-2/p}_{p,\chi}\bigr), \tag{8}$$

where C_0 denotes the space of continuous functions vanishing at infinity.

For $0 < T \le \infty$ we put $J_T := [0,T)$ and $J_{-T} := (-T,0]$. Furthermore, we usually employ the same symbol for a function and its restriction to any of its subdomains, if no confusion seems likely.

Suppose that $0 < S \leq \infty$,

$$v \in C_0\bigl([-S,0], W^{1-2/p}_{p,\chi}\bigr), \tag{9}$$

and $0 < T \leq \infty$. By a **solution** (more precisely: an $\mathcal{H}^1_p$ **solution**) of (4) on $(0,T)$ **with history** v we mean a

$$u \in C\bigl([-S,T), W^{1-2/p}_{p,\chi}\bigr)$$

satisfying $u \,|\, J_{-S} = v$ and

$$u \in \mathcal{H}^1_{p,\chi}(J_\tau), \qquad 0 < \tau < T, \tag{10}$$

as well as, given any $w \in H^1_{p',\chi}$,

$$\partial_t \bigl\langle w, \boldsymbol{e}(u)\bigr\rangle + \bigl\langle \nabla w, \vec{\boldsymbol{\jmath}}(u)\bigr\rangle = \bigl\langle w, \boldsymbol{f}(u)\bigr\rangle + \bigl\langle \gamma w, \boldsymbol{g}(u)\bigr\rangle_\Gamma \qquad \text{on } (0,T) \tag{11}$$

in the sense of distributions, where $\langle\cdot,\cdot\rangle_\Gamma$ denotes the $L_p(\Gamma)$ duality pairing (with respect to the Hausdorff volume measure of Γ). In addition, all integrals occurring in (11) have to be well defined. Note that (10) and (11) imply that u is a weak solution in the usual sense if $p = 2$.

A solution u is **maximal** if there does not exist a solution being a proper extension of u. In this case J_T is the maximal existence interval for u.

Before considering some model problems we recall the concept of a semiflow. Let X be a metric space and suppose that $J(x)$ is for each $x \in X$ an open subinterval of $\mathbb{R}^+$ containing 0. Set

$$\mathcal{X} := \bigcup_{x \in X} J(x) \times \{x\}.$$

Then $\varphi : \mathcal{X} \to X$ is said to be a (local) semiflow on X if $\mathcal{X}$ is open in $\mathbb{R}^+ \times X$, $\varphi \in C(\mathcal{X}, X)$, $\varphi(0,x) = x$ for $x \in X$, and, given $(t,x) \in \mathcal{X}$ and $s \in J\bigl(\varphi(t,x)\bigr)$, it follows that $s + t \in J(x)$ and $\varphi\bigl(s, \varphi(t,x)\bigr) = \varphi(s+t,x)$. It is global if $\mathcal{X} = \mathbb{R}^+ \times X$. Furthermore, φ is a Lipschitz semiflow if, in addition, $\varphi \in C^{0,1\text{-}}(\mathcal{X}, X)$.

Given $T \in (0,\infty]$, a Banach space F, and $u \in C\bigl([-S,T], F\bigr)$, we recall that

$$u_t(\theta) := u(t-\theta), \qquad 0 \leq \theta \leq S, \quad 0 \leq t \leq T.$$

Note that $u_t \in C\bigl([-S,0], F\bigr)$ for $0 \leq t \leq T$.

Suppose that $\mathcal{V}$ is a Banach space such that $\mathcal{V} \hookrightarrow C\bigl([-S,0], W^{1-2/p}_{p,\chi}\bigr)$. Then we say that (4) is well posed in $\mathcal{H}^1_p$ and generates a semiflow on the history space $\mathcal{V}$ if there exists for each $v \in \mathcal{V}$ a unique maximal $\mathcal{H}^1_p$ solution, $u(v)$, of (4) and the map $(t,v) \mapsto u(v)_t$ is a semiflow on $\mathcal{V}$.

We start with a simple model problem of reaction-diffusion type:

$$\left.\begin{aligned} \partial_t u - \nabla \cdot \vec{\jmath}(u) &= \boldsymbol{f}(u) && \text{on } \Omega, \\ u &= 0 && \text{on } \Gamma_0, \\ \vec{\nu} \cdot \vec{\jmath}(u) &= \boldsymbol{g}(u) && \text{on } \Gamma_1, \end{aligned}\right\} \tag{12}$$

where

$$\boldsymbol{j}(u) := -a(\sigma_0 * u)\nabla u,$$

that is, we set $\nu_0 := \delta_0$, where δ_r is the Dirac measure supported in $r \in \mathbb{R}$, and $b := 0$. For notational simplicity, we usually do no longer indicate the x dependence of the nonlinearities.

First we suppose that $m = 1$ and the diffusion matrix depends on suitable space averages of u only. For this we assume that

$$K \in \mathcal{L}\bigl(L_2, C(\overline{\Omega})\bigr), \tag{13}$$

where $\mathcal{L}(E, F)$ is the Banach space of all continuous linear maps from the Banach space E into the Banach space F. We also set $\mathcal{L}(E) := \mathcal{L}(E, E)$.

We denote by $\mathcal{M}[0, S]$ the space of all real valued Radon measures of bounded variation on the interval $[0, S]$. We suppose that

$$\alpha \in \mathcal{M}[0, S] \tag{14}$$

and consider the nonlocal time-delayed quasilinear parabolic problem

$$\left.\begin{aligned} \partial_t u - \nabla \cdot \bigl(a(\alpha * Ku)\nabla u\bigr) &= f(\alpha * Ku) && \text{on } \Omega, \\ u &= 0 && \text{on } \Gamma_0, \\ \vec{\nu} \cdot a(\alpha * Ku)\nabla u &= g(\alpha * Ku) && \text{on } \Gamma_1. \end{aligned}\right\} \tag{15}$$

Here and below it is understood that the boundary conditions are taken in the sense of traces. In particular, the right hand side of the third equation of (15) reads more precisely as $g(\gamma\alpha * Ku)$. Observe that

$$(\alpha * Ku)(x, t) = \int_{[0,S]} (Ku)(x, t - \tau)\alpha\,(d\tau) \in C\bigl(\overline{\Omega} \times [0, T]\bigr)$$

for $(x, t) \in \overline{\Omega} \times J_T$ and $T > 0$, provided $u \in C_0\bigl([-S, T], L_2\bigr)$.

Theorem 2.1. *Let (13) and (14) be satisfied. Then (15) is well posed in $\mathcal{H}^1_\chi$ and generates a Lipschitz semiflow on the history space $C_0\bigl([-S, 0], L_2\bigr)$. It depends Lipschitz continuously on α and K. If the support of α is contained in $[s, S]$ for some $s \in (0, S]$, then this semiflow is global.*

The proof as well as the proofs of all the following theorems of this section are found in Section 6, where it will be made precise how (15) is a particular instant of (4). What is meant by a semiflow depending Lipschitz continuously on parameters is defined in Section 4.

Corollary 2.1. *If $r > 0$, then the nonlocal retarded problem*

$$\begin{aligned} \partial_t u - \nabla \cdot \big(a\big(Ku(t-r)\big)\nabla u\big) &= f\big(Ku(t-r)\big) && \text{on } \Omega, \\ u &= 0 && \text{on } \Gamma_0, \\ \vec{\nu} \cdot a\big(Ku(t-r)\big)\nabla u &= g\big(Ku(t-r)\big) && \text{on } \Gamma_1 \end{aligned}$$

is well posed in $\mathcal{H}^1_\chi$ and generates a global Lipschitz semiflow on the history space $C\big([-r,0], L_2\big)$. It depends Lipschitz continuously on K.

Proof. It suffices to choose $S := r$ and $\alpha := \delta_r$. □

To treat local reaction terms in a weak setting we replace the hypotheses on f and g in (6) by assuming, for simplicity, that $n \geq 3$, that $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is a Carathéodory function satisfying $f(\cdot, 0) \in L_{2n/(n+2)}$ and

$$|f(\cdot,\xi) - f(\cdot,\eta)| \leq c\big(1 + |\xi|^{2/n} + |\eta|^{2/n}\big)\, |\xi - \eta| \tag{16}$$

for $\xi, \eta \in \mathbb{R}^m$, and that $g_0 \in L_2(\Gamma)$. Observe that (16) is satisfied for the model nonlinearity

$$f(\cdot,\xi) = b\,|\xi|^{2/n}\,\xi + f_0 \tag{17}$$

with $m = 1$, $b \in L_\infty$, and $f_0 \in L_2$. Then we consider quasilinear parabolic problems with nonlocal time-delays in the diffusion matrix only, the reaction term being local, that is,

$$\left.\begin{aligned} \partial_t u - \nabla \cdot \big(a(\alpha * Ku)\nabla u\big) &= f(u) && \text{on } \Omega, \\ u &= 0 && \text{on } \Gamma_0, \\ \vec{\nu} \cdot a(\alpha * Ku)\nabla u &= g_0 && \text{on } \Gamma_1. \end{aligned}\right\} \tag{18}$$

Theorem 2.2. *Suppose that assumptions* (13), (14), *and* (16) *hold. Then* (18) *is well posed in $\mathcal{H}^1_\chi$ and generates a semiflow on the history space $C_0\big([-S,0], L_2\big)$ which depends Lipschitz continuously on α and K. It is global if the following additional conditions are satisfied:*

(i) $\operatorname{supp}(\alpha) \subset [s, S]$ *for some $s \in (0, S]$;*
(ii) *there exists a constant κ such that*

$$\big(f(\cdot,\xi) - f(\cdot,0)\big)\xi \leq \kappa(1 + |\xi|^2), \quad \xi \in \mathbb{R}.$$

Corollary 2.2. *If $r > 0$, then the model problem*

$$\begin{aligned} \partial_t u - \nabla \cdot \big(a\big(Ku(t-r)\big)\nabla u\big) &= b\,|u|^{2/n}\,u + f_0 && \text{on } \Omega, \\ u &= 0 && \text{on } \Gamma_0, \\ \vec{\nu} \cdot a\big(Ku(t-r)\big)\nabla u &= g_0 && \text{on } \Gamma_1 \end{aligned}$$

is well posed in $\mathcal{H}^1_\chi$ and generates a semiflow on $C_0\big([-r,0], L_2\big)$, depending Lipschitz continuously on K. It is global if $b \le 0$.

Proof. With (17) this follows by choosing $S := r$ and $\alpha := \delta_r$. □

There are many conceivable choices for K. For example, we could set

$$Ku := \langle k, u \rangle, \qquad u \in L_2,$$

for some fixed $k \in L_2$, so that Ku is constant on $\overline{\Omega}$. Nonlocal (non delayed) quasilinear parabolic initial boundary value problems, predominantly with this choice for K, have recently attracted some interest, in particular by M. Chipot and coworkers (cf. [6], [10]–[12], and the references therein).

Another important case is obtained by setting

$$Ku := k \star \widetilde{u}, \qquad u \in L_2,$$

where $k \in L_2(\mathbb{R}^n)$, $\widetilde{u}$ is the extension of u to $\mathbb{R}^n$ by zero in Ω^c, and $\star$ denotes convolution on $\mathbb{R}^n$. In particular, setting $k := \chi_{r\mathbb{B}^n}$, the characteristic function of the ball in $\mathbb{R}^n$ with center at 0 and radius r, it follows that

$$Ku(x) = \int_{\Omega(x,r)} u(y)\,dy, \qquad x \in \Omega,$$

where $\Omega(x,r) := (x + r\mathbb{B}^n) \cap \Omega$. Thus in this case the diffusion matrix (and f and g in the case of Theorem 2.1) depends on a suitably delayed space average of the solution over a neighborhood of x in Ω.

Next we consider model problems where the diffusion matrix depends on u in a local way with respect to the x variable, but not necessarily with respect to t. For this we suppose that $m = 2$ and consider the model problem

$$\left.\begin{aligned} \partial_t u - \nabla \cdot \big(a(u, \alpha * u)\nabla u\big) &= f(u, \alpha * u) && \text{on } \Omega, \\ u &= 0 && \text{on } \Gamma_0, \\ \vec{\nu} \cdot a(u, \alpha * u)\nabla u &= g(u, \alpha * u) && \text{on } \Gamma_1. \end{aligned}\right\} \qquad (19)$$

Then the following analogue to Theorem 2.1 is valid.

Theorem 2.3. *Suppose that* (Ω, χ) *is regular,* $n+2 < p < \infty$*, and* (14) *is satisfied. Then* (19) *is well posed in* $\mathcal{H}^1_{p,\chi}$ *and generates a Lipschitz semiflow on the history space* $C_0\big([-S,0], W^{1-2/p}_{p,\chi}\big)$ *depending Lipschitz continuously on* α*. If* $\mathrm{supp}(\alpha) \subset [s,S]$ *for some* $s \in (0,S]$ *and* $(a,f) = (a,f)(\alpha * u)$*, then this semiflow is global.*

Remarks 2.1. **(a)** For simplicity, we have omitted convection terms of the form $\vec{\boldsymbol{c}}(u) \cdot \nabla u$ and $\nabla \cdot \big(\vec{\boldsymbol{c}}(u)u\big)$, where $\vec{\boldsymbol{c}}(u)$ is a suitable nonlocal function of u. It will be clear from the general abstract results how this can be done.

(b) From Theorem 4.2 it will also be clear that we can obtain well posedness results for nonautonomous equations. Of course, in such a case the semiflow property is no longer valid.

(c) We can replace $\overline{\Omega}$ by a smooth submanifold of some Riemannian manifold, provided gradients, divergence, and normals are taken with respect to the corresponding Riemannian metric. □

Problems of the form (19) occur in applications, for example in certain climate models. For instance, in [20] and [21] G. Hetzer studies the quasilinear functional differential equation

$$c(\beta * u)\partial_t u - \nabla \cdot (k \nabla u) = R(t, u, \beta * u) \tag{20}$$

on the Euclidean unit sphere in $\mathbb{R}^3$, assuming that c is a bounded C^2 function being uniformly positive, $\beta \in C^2[0,T]$ for some $T > 0$, and k and R are sufficiently smooth functions with k being uniformly positive. By dividing (20) by $c(\beta * u)$ it is clear, due to Remarks 2.1, that this model fits into the framework of this paper.

In the theory of heat conduction in a rigid body the functions occurring in (4) have the following interpretation: u is the temperature, $\boldsymbol{e}(u)$ the interval energy density, $\vec{\boldsymbol{\jmath}}(u)$ the heat flux, $\boldsymbol{f}(u)$ and $\boldsymbol{g}(u)$, respectively, the density of external heat sources in Ω and on Γ, respectively. Considering bodies with memory one arrives at the following constitutive hypotheses:

$$\boldsymbol{e}(u) = u + h * u$$

and

$$\vec{\boldsymbol{\jmath}}(u) = -a(u, \alpha * u)\nabla u - k * \big(b(u, \alpha * u)\nabla u\big),$$

where we suppose that

$$h \in L_r\big(\mathbb{R}^+, C^1(\overline{\Omega})\big), \quad k \in L_r(\mathbb{R}^+, L_\infty), \quad \alpha \in \mathcal{M}\big(\mathbb{R}^+, C(\overline{\Omega})\big) \tag{21}$$

for some $r \in (1, p']$. Thus one is led to consider the problem

$$\partial_t(u + h * u) - \nabla \cdot \big(a(u, \alpha * u)\nabla u + k * \big(a(u, \alpha * u)\nabla u\big)\big) = f(u, \alpha * u) \quad (22)$$

in Ω, subject to the boundary conditions

$$\left.\begin{aligned} u &= 0 && \text{on } \Gamma_0, \\ \vec{\nu} \cdot a(u, \alpha * u)\nabla u + k * \big(a(u, \alpha * u)\nabla u\big) &= g(u, \alpha * u) && \text{on } \Gamma_1. \end{aligned}\right\} \quad (23)$$

Observe that, for example,

$$(h * u)(x, t) = \int_0^\infty h(x, \tau)u(x, t - \tau)\, d\tau, \qquad (x, t) \in \Omega \times \mathbb{R}^+,$$

where $u(t) = u(\cdot, t)$ etc.

Theorem 2.4. *Suppose that (Ω, χ) is regular, $n + 2 < p < \infty$, and (21) is satisfied. Then (22), (23) is well posed in $\mathcal{H}_p^1$ and generates a Lipschitz semiflow on the history space $\mathcal{H}_{p,\chi}^1(J_{-S})$, where $S \in (0, \infty]$ is such that*

$$\big(\mathrm{supp}(h) \cup \mathrm{supp}(k)\big) + \mathrm{supp}(\alpha) \subset [0, S].$$

If $h = 0$, then this is true for the history space $C_0\big([-S, 0], W_{p,\chi}^{1-2/p}\big)$.

Setting $h = 0$ and $\alpha = 0$ and assuming that k is real valued we obtain as a particular case the quasilinear Volterra integro differential equation

$$\partial_t u - \nabla \cdot \big(a(u)\nabla u\big) - \int_0^\infty h(\tau)\nabla \cdot \big(a(u(t - \tau))\nabla u(t - \tau)\big)\, d\tau = f(u).$$

Equations of this type, usually with zero Dirichlet boundary conditions, have been studied by many authors, even for more general fully nonlinear equations, by means of maximal regularity results in Hölder and Besov space settings (see [15], [26], and the references therein). Another approach to such equations is based on sophisticated results from the theory of abstract linear Volterra equations (see [27]). Using these techniques it is also possible to obtain existence results in the difficult singular case where the local second order operator $\nabla \cdot \big(a(u)\nabla u\big)$ is not present (cf. [17]–[19], [25], [27], [30], for example, and the references in those papers).

The only results known to the author for problems containing the term $\partial_t(h * a)$ concern linear and semilinear equations (e.g., [13], [14], [22], and [29]).

Another model case is the retarded quasilinear parabolic problem

$$\left.\begin{aligned} \partial_t \boldsymbol{e}(u)(t) - \nabla \cdot \vec{\boldsymbol{\jmath}}(u)(t) &= f\big(u(t), u(t - r)\big) && \text{on } \Omega, \\ u &= 0 && \text{on } \Gamma_0, \\ \vec{\nu} \cdot \vec{\boldsymbol{\jmath}}(u)(t) &= g\big(u(t), u(t - r)\big) && \text{on } \Gamma_1, \end{aligned}\right\} \quad (24)$$

where $r > 0$,

$$\boldsymbol{e}(u)(t) := u(t) + hu(t-r) \tag{25}$$

and

$$\vec{\jmath}(u)(t) := -a\big(u(t), u(t-r)\big)\nabla u(t) - kb\big(u(t), u(t-r)\big)\nabla u(t-r) \tag{26}$$

with

$$h, k \in C^1(\overline{\Omega}). \tag{27}$$

Theorem 2.5. *Suppose that* (Ω, χ) *is regular and* $n+2 < p < \infty$. *Also suppose that* (25)–(27) *are satisfied for some* $r > 0$. *Then* (24) *is well posed in* $\mathcal{H}_p^1$ *and generates a Lipschitz semiflow on the history space* $\mathcal{H}_{p,\chi}^1(J_{-r})$. *If* $h = 0$, *then this is true for the history space* $C\big([-r, 0], W_{p,\chi}^{1-2/p}\big)$. *These semiflows are global, if* a, b, f, *and* g *depend on* $u(t-r)$ *only and not on* $u(t)$.

A very particular instant of problem (24) is the retarded semilinear parabolic equation

$$\partial_t\big(u(t) + \alpha u(t-r)\big) - a\Delta u(t) - b\Delta u(t-r) = f\big(u(t-r)\big) \tag{28}$$

in $\Omega \times (0, \infty)$, where α, a, and b are constants with $a > 0$, subject to the boundary conditions

$$\left.\begin{aligned} u &= 0 && \text{on } \Gamma_0, \\ a\partial_\nu u(t) + b\partial_\nu u(t-r) &= g\big(u(t-r)\big) && \text{on } \Gamma_1. \end{aligned}\right\} \tag{29}$$

As usual, ∂_ν is the normal derivative on Γ. It follows from Theorem 2.5 that, given any history $v \in \mathcal{H}_{p,\chi}^1(J_{-r})$, problem (28), (29) possesses a unique global $\mathcal{H}_p^1$ solution $u(v)$, and the map

$$\mathbb{R}^+ \times \mathcal{H}_{p,\chi}^1(J_{-r}) \hookrightarrow \mathcal{H}_{p,\chi}^1(J_{-r}), \quad (t, v) \mapsto u(v)_t$$

is a well defined Lipschitz semiflow on $\mathcal{H}_{p,\chi}^1(J_{-r})$, provided (Ω, χ) is regular and $p > n+2$. Furthermore, if $\alpha = 0$, then this remains true if we replace $\mathcal{H}_{p,\chi}^1(J_{-r})$ by the history space $C\big([-r, 0], W_{p,\chi}^{1-2/p}\big)$.

3. Parameter dependent evolution equations

Let E_0 and E_1 be Banach spaces such that $E_1 \overset{d}{\hookrightarrow} E_0$. We fix $p \in (1, \infty)$ and, given a subinterval I of $\mathbb{R}$ with nonempty interior, we put

$$\mathcal{H}_p^1\big(I, (E_1, E_0)\big) := L_p(I, E_1) \cap H_p^1(\mathring{I}, E_1).$$

It follows that

$$\mathcal{H}_p^1\big(I,(E_1,E_0)\big) \hookrightarrow C_0\big(\overline{I},E\big), \tag{30}$$

where

$$E := (E_0,E_1)_{1/p',p}$$

with $(\cdot,\cdot)_{\theta,p}$ being the real interpolation functor of exponent $\theta \in (0,1)$ and parameter p (cf. Theorem III.4.10.2 of [2]).

Suppose that $a := \inf I > -\infty$ with $a \in I$ and

$$A \in L_\infty\big(I,\mathcal{L}(E_1,E_0)\big).$$

Then A is said to have (the property of) **maximal L_p regularity** (on I with respect to (E_1,E_0)) if the Cauchy problem

$$\dot{u} + Au = f \text{ on } \mathring{I}, \quad u(a) = 0$$

has for each $f \in L_p(I,E_0)$ a unique solution $\mathcal{H}_p^1\big(I,(E_1,E_0)\big)$ and if, in addition, given $\tau, T \in I$ with $\tau < T$, the homogeneous problem

$$\dot{v} + Av = 0 \text{ on } (\tau,T), \quad v(\tau) = 0 \tag{31}$$

possesses in $\mathcal{H}_p^1\big([\tau,T),(E_1,E_0)\big)$ the trivial solution only. The proof of Lemma 4.1 in [5] shows that assumption (31) is equivalent to: A has maximal L_p regularity on every nontrivial bounded subinterval of I being closed on the left. (In Lemma 4.1 of [5] hypothesis (31) is missing.)

We fix a positive number T and set $\mathsf{J} := J_{\mathsf{T}}$. Then we denote by

$$\mathcal{MR}_p(\mathsf{J}) := \mathcal{MR}_p\big(\mathsf{J},(E_1,E_0)\big)$$

the set of all $A \in L_\infty\big(\mathsf{J},\mathcal{L}(E_1,E_0)\big)$ possessing maximal L_p regularity, endowed with the topology induced by $L_\infty\big(\mathsf{J},\mathcal{L}(E_1,E_0)\big)$. We write $\mathcal{MR}_p(E_1,E_0)$ for the subset of $\mathcal{MR}_p(\mathsf{J})$ consisting of all constant maps $t \mapsto A$ therein and assume that

$$\mathcal{MR}_p(E_1,E_0) \neq \emptyset. \tag{32}$$

Let X and Y be nonempty sets and J a subinterval of $\mathbb{R}^+$ containing 0. A function $f : X^J \to Y^J$ is a **Volterra map** if, for each $T \in \mathring{J}$ and each pair $u, v \in X^J$ with $u\,|\,J_T = v\,|\,J_T$, it follows that $f(u)\,|\,J_T = f(v)\,|\,J_T$. Let X and Y be metric spaces. Then $\mathcal{C}^{1\text{-}}(X,Y)$ is the space of all maps from X into Y which are bounded on bounded sets and uniformly Lipschitz continuous on such sets. If Y is a Banach space, then $\mathcal{C}^{1\text{-}}(X,Y)$ is endowed with the Fréchet topology of uniform convergence on bounded sets of the functions and their first order difference quotients on such sets. Note that

$\mathcal{C}^{1\text{-}}(X,Y)$ equals $C^{1\text{-}}(X,Y)$, the space of all (locally) Lipschitz continuous maps from X into Y, provided X is finite dimensional.

For abbreviation we put

$$\mathcal{H}_p^1(J_T) := \mathcal{H}_p^1\big(J_T,(E_1,E_0)\big), \quad \mathcal{H} := \mathcal{H}_p^1(\mathsf{J})$$

for $0 < T \le \mathsf{T}$ and assume that

$$\bullet \quad \Xi \text{ is a Banach space and } \alpha \in \mathcal{L}(\Xi, E). \tag{33}$$

We denote by $\gamma_0 \in \mathcal{L}(\mathcal{H}, E)$ the trace map for $t = 0$, that is, $\gamma_0(u) = u(0)$ for $u \in \mathcal{H}$, and set

$$D := \big\{\, (\xi,u) \in \Xi \times \mathcal{H} \ ;\ \alpha(\xi) = \gamma_0(u) \,\big\}. \tag{34}$$

Note that D is the kernel of

$$\big((\xi,u) \mapsto \alpha(\xi) - \gamma_0(u)\big) \in \mathcal{L}(\Xi \times \mathcal{H}, E).$$

Thus it is a closed linear subspace of $\Xi \times \mathcal{H}$, hence a Banach space.

For $\xi \in \Xi$ we put

$$\mathcal{H}_{\alpha(\xi)} := \big\{\, u \in \mathcal{H} \ ;\ \gamma_0(u) = \alpha(\xi) \,\big\}$$

and assume that

$$\left.\begin{array}{l} \bullet \quad A \in \mathcal{C}^{1\text{-}}\big(D, \mathcal{MR}_p\big(\mathsf{J},(E_1,E_0)\big)\big); \\ \bullet \quad F \in \mathcal{C}^{1\text{-}}\big(D, L_p(\mathsf{J},E_0)\big) \text{ for some } r \in (p,\infty]; \\ \bullet \quad (A,F)(\xi,\cdot) \text{ is for each } \xi \in \Xi \text{ a Volterra map on } \mathcal{H}_{\alpha(\xi)}. \end{array}\right\} \tag{35}$$

We consider the parameter dependent quasilinear evolution problem

$$\dot u + A(\xi,u)u = F(\xi,u) \text{ on } (0,\mathsf{T}), \quad u(0) = \alpha(\xi) \tag{36}$$

for $\xi \in \Xi$.

Theorem 3.1. *Let assumptions* (33) *and* (35) *be satisfied and suppose that $\xi \in \Xi$. Then:*

(i) (Existence and Uniqueness) *There exist a maximal open subinterval $J^* := J(\xi)$ of J containing 0 and a unique $u^* := u(\xi) : J^* \to E_0$ such that $u^* \,|\, J_T$ belongs to $\mathcal{H}_p^1(J_T)$ and satisfies*

$$\dot u^* + A(\xi,u^*)u^* = F(\xi,u^*) \text{ on } (0,T), \quad u^*(0) = \alpha(\xi)$$

for $0 < T < T^* := \sup J^*$.

(ii) (Global existence) *If $J^* \ne \mathsf{J}$, then $u^* \notin \mathcal{H}_p^1(J^*)$.*

(iii) (Continuous dependence on ξ) *If $u^* \in \mathcal{H}$, then put $T_0 := \mathsf{T}$. Otherwise, fix any positive $T_0 < T^*$. Then there exist $r, \kappa > 0$ such that, given any $\xi_j \in \Xi$ satisfying*

$$\|\xi_j - \xi\|_\Xi < r, \qquad j = 1, 2,$$

it follows that $u(\xi_j) \in \mathcal{H}_p^1(J_{T_0})$ and

$$\|u(\xi_1) - u(\xi_2)\|_{\mathcal{H}_p^1(J_{T_0})} \le \kappa \, \|\xi_1 - \xi_2\|_\Xi.$$

(iv) (Continuous dependence on A and F) *Let T_0 be defined as above and let $\big((A_j, F_j)\big)$ be a sequence such that (A_j, F_j) satisfies* (35) *for each $j \in \mathbb{N}$ and $(A_j, F_j) \to (A, F)$ in*

$$\mathcal{C}^{1\text{-}}\big(D, L_\infty\big(\mathsf{J}, \mathcal{L}(E_1, E_0)\big) \times L_p(\mathsf{J}, E_0)\big).$$

Denote by $u_j(\xi)$ the solution of (36) *with (A, F) replaced by (A_j, F_j). Then $u_j(\xi) \to u(\xi)$ in $\mathcal{H}_p^1(J_{T_0})$.*

Proof. (i) and (ii) follow from Theorem 2.1 and Remark 4.3 in [8]. Assertions (iii) and (iv) are easily deduced from the proof of Theorem 3.1 therein by modifying appropriately the situation considered in Remark 4.3 of [8]. (In [8] assumption (31) has to be added to the definition of maximal L_p regularity since Lemma 4.1 of [5] is used in the proofs.) □

Remark 3.1. Let Π be a Banach space and suppose that

- $A \in \mathcal{C}^{1\text{-}}\big(\Pi \times \mathcal{H}, \mathcal{MR}_p(\mathsf{J}, (E_1, E_0))\big)$;
- $F \in \mathcal{C}^{1\text{-}}\big(\Pi \times \mathcal{H}, L_r(\mathsf{J}, E_0)\big)$ for some $r \in (p, \infty]$;
- $(A, F)(\pi, \cdot)$ is for each $\pi \in \Pi$ a Volterra map.

Then the quasilinear parameter dependent initial value problem

$$\dot{u} + A(\pi, u)u = F(\pi, u) \quad \text{on } (0, T), \qquad u_0 = e$$

has for each $e \in E$ a unique maximal solution in the sense specified in (i) of Theorem 3.1. Furthermore, assertions (ii)–(iv) are also valid.

Proof. This follows from the preceding theorem by setting $\Xi := \Pi \times E$ and $\alpha(\xi) := e$ for $\xi = (\pi, e) \in \Xi$. □

4. Functional evolution equations

Now we fix $S \in (0,\infty]$ and suppose that

$$\mathcal{V} \in \bigl\{ \mathcal{H}_p^1\bigl((-S,0),(E_1,E_0)\bigr), C_0\bigl([-S,0],E\bigr) \bigr\}. \tag{37}$$

We put

$$\mathcal{D} := \bigl\{ \, (v,w) \in \mathcal{V} \times \mathcal{H} \ ; \ v(0) = w(0) \, \bigr\},$$

fix a (parameter) Banach space Π, and suppose that

$$\left.\begin{array}{l} \bullet \quad A \in \mathcal{C}^{1\text{-}}\bigl(\Pi \times \mathcal{D}, \mathcal{MR}_p(\mathsf{J},(E_1,E_0))\bigr); \\ \bullet \quad F \in \mathcal{C}^{1\text{-}}\bigl(\Pi \times \mathcal{D}, L_r(\mathsf{J},E_0)\bigr) \text{ for some } r \in (p,\infty]; \\ \bullet \quad (A,F)(\pi,v,\cdot) \text{ is for each } (\pi,v) \in \Pi \times \mathcal{V} \\ \quad\ \ \text{a Volterra map on } \mathcal{H}_{v(0)}. \end{array}\right\} \tag{38}$$

Then, given $\pi \in \Pi$ and $v \in \mathcal{V}$, we consider the following parameter dependent quasilinear functional differential equation

$$\dot{u} + A(\pi,u_t,u)u = F(\pi,u_t,u) \quad \text{on } (0,\mathsf{T}), \qquad u_0 = v. \tag{39}$$

By an $\mathcal{H}_p^1$ **solution** u of (39) on J_T, where $0 < T \le \mathsf{T}$, we mean a $u : [-S,T) \to E_0$ satisfying $u \,|\, [-S,0) = v$ and $u \,|\, J_\tau \in \mathcal{H}_p^1(J_\tau)$ as well as

$$\dot{u}(t) + A(\pi,u_t,u)(t)u(t) = F(\pi,u_t,u)(t), \qquad 0 < t < \tau,$$

for $0 < \tau < T$. It is maximal if there does not exist an $\mathcal{H}_p^1$ solution being a proper extension of u. In this case J_T is called maximal existence interval for u.

The following general existence, uniqueness, and continuity theorem is the first main result of this paper.

Theorem 4.1. *Let assumptions* (37) *and* (38) *be satisfied. Then:*

(i) (Existence and Uniqueness) *Problem* (39) *has for each* $(\pi,v) \in \Pi \times \mathcal{V}$ *a unique maximal* $\mathcal{H}_p^1$ *solution* $u(\pi,v)$.

(ii) (Global existence) *Denote by* $J(\pi,v)$ *the maximal existence interval of* $u(\pi,v)$. *If* $J(\pi,v) \ne \mathsf{J}$, *then* $u(\pi,v) \notin \mathcal{H}_p^1\bigl(J(\pi,v)\bigr)$.

(iii) (Continuous dependence on π and v) *If* $u(\pi,v) \in \mathcal{H}_p^1(\mathsf{J})$, *then set* $T_0 := \mathsf{T}$. *Otherwise, fix any* $T_0 \in \mathring{J}(\pi,v)$. *Then there exist* $r,\kappa > 0$ *such that, given* $(\pi_j,v_j) \in \Pi \times \mathcal{V}$ *satisfying*

$$\|\pi_j - \pi\|_\Pi + \|v_j - v\|_V < r, \qquad j = 1,2,$$

it follows that $u(\pi,v_j) \in \mathcal{H}_p^1(J_{T_0})$ *and*

$$\|u(\pi_1,v_1) - u(\pi_2,v_2)\|_{\mathcal{H}(J_{T_0})} \le \kappa\bigl(\|\pi_1 - \pi_2\|_\Pi + \|v_1 - v_2\|_V\bigr).$$

(iv) (Continuous dependence on A and F) *Let T_0 be defined as in (iii) and let $\big((A_j,F_j)\big)$ be a sequence such that (A_j,F_j) satisfies* (38) *for each $j\in\mathbb{N}$ and $(A_j,F_j)\to(A,F)$ in*

$$\mathcal{C}^{1\text{-}}\big(\Pi\times\mathcal{D},L_\infty(\mathsf{J},\mathcal{L}(E_1,E_0))\times L_p(\mathsf{J},E_0)\big).$$

Denote by $u_j(\pi,v)$ the maximal solution of (39) *with (A,F) replaced by (A_j,F_j). Then $u_j(\pi,v)\to u(\pi,v)$ in $\mathcal{H}^1_p(J_{T_0})$.*

Proof. For $(v,w)\in\mathcal{D}$ we set

$$v\oplus w(t):=\begin{cases} v(t), & -S<t\le 0,\\ w(t), & 0<t<\mathsf{T}.\end{cases}$$

If $\mathcal{V}=C_0\big([-S,0],E\big)$, then it is obvious that

$$\big((v,w)\mapsto(v\oplus w)_t\big)\in\mathcal{C}^{1\text{-}}(\mathcal{D},\mathcal{V}),\qquad 0\le t<\mathsf{T}. \tag{40}$$

In the other case this follows from Lemma 7.1 in [8].

Set $\Xi:=\Pi\times\mathcal{V}$ and $\alpha(\xi):=v(0)$ for $\xi=(\pi,v)\in\Xi$. Then (33) is satisfied and $D=\Pi\times\mathcal{D}$.

For $(\xi,u)=(\pi,v,u)\in D$ define $\mathcal{A}(\xi,u)$ and $\mathcal{F}(\xi,u)$ by

$$(\mathcal{A},\mathcal{F})(\xi,u)(t):=(A,F)\big(\pi,(v\oplus u)_t,u\big)(t),\qquad t\in\mathsf{J}.$$

It follows from (40), $\big((v,u)\to u\big)\in\mathcal{L}(\mathcal{D},\mathcal{H})$, and (38) that $\mathcal{A}$ and $\mathcal{F}$ satisfy (35). Thus Theorem 3.1 implies the assertions. □

Let X and Y be metric spaces and put $Z:=X\times Y$. Suppose that $J(z)$ is for each $z\in Z$ an open subinterval of $\mathbb{R}^+$ containing 0. Set

$$\mathcal{Z}:=\bigcup_{z\in Z}J(z)\times\{z\}.$$

Then $\varphi:\mathcal{Z}\to X$ is a parameter dependent Lipschitz semiflow on X, provided

$$\varphi(\cdot,\cdot,y):\ \mathcal{Z}_y:=\big\{\,(t,x)\in\mathbb{R}^+\times X\ ;\ (t,x,y)\in\mathcal{Z}\,\big\}\to X$$

is for each $y\in Y$ a Lipschitz semiflow on X. It depends Lipschitz continuously on the parameters $y\in Y$ if $\big((t,z)\mapsto\varphi(t,z)\big)\in\mathcal{D}^{0,1\text{-}}(\mathcal{Z},X)$.

Suppose that (38) is satisfied for every $\mathsf{T}>0$. Then the map (A,F) is said to be **autonomous** if, given $s,t\in\mathbb{R}^+$ and $u\in\mathcal{H}^1_p(J_{s+t})$, it follows that, setting $v^s(\tau):=u(s+\tau)$ for $0\le\tau\le t$, that

$$(A,F)(\pi,u_{t+s},u)(t+s)=(A,F)\big(\pi,(v^s)_t,v^s\big)(t).$$

Note that this is true, in particular, if $(A, F)(\pi, v, \cdot)$ is a local map.

Let (38) be satisfied for every $\mathsf{T} > 0$. Then we consider the quasilinear functional differential equation

$$\dot{u} + A(\pi, u_t, u)u = F(\pi, u_t, u) \text{ on } (0, \infty), \quad u_0 = v. \tag{41}$$

Clearly, u is an $\mathcal{H}_p^1$ solution if it is an $\mathcal{H}_p^1$ solution of (39) for every $\mathsf{T} > 0$.

The following theorem is the second main abstract theorem of this paper.

Theorem 4.2. *Let* (37) *be true and suppose that* (38) *holds for every* $\mathsf{T} > 0$. *Then*

(i) *Problem* (41) *has for each* $(\pi, v) \in \Pi \times \mathcal{V}$ *a unique maximal* $\mathcal{H}_p^1$ *solution,* $u(\pi, v)$.

(ii) *If* $u(\pi, v) \in \mathcal{H}_p^1\big(J_T \cap J(\pi, v)\big)$ *for every* $T > 0$, *where* $J(\pi, v)$ *is the maximal existence interval for* $u(\pi, v)$, *then* $u(\pi, v)$ *exists globally, that is,* $J(\pi, v) = \mathbb{R}^+$.

(iii) *For each* $T \in \mathring{J}(\pi, v)$ *there are* $r, \kappa > 0$ *such that*

$$\|u(\pi_1, v_1) - u(\pi_2, v_2)\|_{\mathcal{H}_p^1(J_T)} \le \kappa\big(\|\pi_1 - \pi_2\|_\Pi + \|v_1 - v_2\|_\mathcal{V}\big),$$

whenever $(\pi_j, v_j) \in \Pi \times \mathcal{V}$ *satisfy*

$$\|\pi_j - \pi\|_\Pi + \|v_j - v\|_\mathcal{V} < r, \qquad j = 1, 2.$$

(iv) *If* (A, F) *is autonomous then the map* $(t, v, \pi) \mapsto u(\pi, v)_t$ *defines a Lipschitz semiflow on* $\mathcal{V}$ *depending Lipschitz continuously on* $\pi \in \Pi$.

Proof. (i)–(iii) are obviously implied by (i)–(iii) of Theorem 4.1.

(iv) We fix $\pi \in \Pi$ and omit it from the notation since it does not play a role in the following argument. Given $v \in \mathcal{V}$ and $t, s \in \mathbb{R}^+$ with $t + s \in J(v)$, set $w(t) := u(v)(t + s)$. Then the fact that (A, F) is autonomous implies that

$$\dot{w} + A(\pi, w_t, w)w = F(\pi, w_t, w) \text{ on } \mathring{J}(v) - s$$

and $w_0 = u(v)_s$. Note that $u(v)_s \in \mathcal{V}$ and $w \,|\, J_\tau \in \mathcal{H}_p^1(J_\tau)$ for $\tau \in \mathring{J}(v) - s$. Hence we infer from (i) that $J\big(u(v)_s\big) \supset J(v) - s$ and $w = u\big(u(v)_s\big)$ on $J(v) - s$. On the other hand, set

$$\widetilde{w}(t) := \begin{cases} u(v)(t), & -S \le t < s, \\ u\big(u(v)_s\big)(t - s), & s \le t \in J\big(u(v)_s\big) + s. \end{cases}$$

Then using Lemma 7.1 of [8] one verifies that $\widetilde{w}$ is an $\mathcal{H}_p^1$ solution of (41) on $J\big(u(v)_s\big)+s$. Thus, by uniqueness, $J(v)\supset J\big(u(v)_s\big)+s$. This implies that $J(v)=J\big(u(v)_s\big)+s$ and

$$u(v)_{s+t}=u\big(u(v)_s\big)_t$$

for $s\in J(v)$ and $t\in J\big(u(v)_s\big)$. Now the assertion is a consequence of (iii), the strong continuity of the translation group on $C_0(\mathbb{R},E)$ and $\mathcal{H}_p^1(\mathbb{R})$, and, in the case where $\mathcal{V}=C_0\big([-S,0],E\big)$, of (30). □

It is easy to derive from Theorem 4.1(iv) the continuous dependence of $u(\pi,v)$ on A and F. We leave this to the reader.

5. Parabolic boundary value problems

Let I be a nonempty closed interval and F a Banach space. We denote by $\mathcal{M}(I,F)$ the Banach space of all bounded F valued Radon measures on I (see Section 2.2 in [3] for a brief introduction to the theory of vector valued measures and the corresponding integration). We identify $\mathcal{M}(I,F)$ with the closed linear subspace of $\mathcal{M}(\mathbb{R},F)$ consisting of all bounded F valued Radon measures being supported in I. We also identify $L_1(I,F)$ with the closed linear subspace of $\mathcal{M}(I,F)$ consisting of all measures being absolutely continuous with respect to Lebesgue's measure dt. Thus, in particular, we identify $f\in L_1(I,F)$ with its trivial extension (by zero on I^c) in $L_1(\mathbb{R},F)$.

Let F_0, F_1, and F_2 be Banach spaces and $F_1\times F_2\to F_0$ and assume that $(x,y)\mapsto x\bullet y$ is a multiplication, that is, a continuous bilinear form of norm at most 1. In particular, given Banach spaces E and F, we can choose $F_1:=\mathcal{L}(E,F)$, $F_2:=E$, $F_0:=F$, and $A\bullet e:=Ae$ for $A\in\mathcal{L}(E,F)$ and $e\in E$.

We put $\infty-\infty:=\infty$. Given $0\le R\le S\le\infty$ with $S>0$, $0<T\le\infty$, $u\in C_0\big([-S,T],F_1\big)$, and $\mu\in\mathcal{M}\big([0,S-R],F_2\big)$, the convolution integral

$$u*\mu(t):=\int u(t-\tau)\bullet\mu\,(d\tau) \tag{42}$$

is well defined for $-R\le t\le T$. It is not difficult to see that $(u,\mu)\mapsto u*\mu$ defines a multiplication

$$C_0\big([-S,T],F_1\big)\times\mathcal{M}\big([0,S-R],F_2\big)\to BUC\big([-R,T],F_0\big). \tag{43}$$

It also follows from Young's inequality that the map $(v,w)\mapsto v*w$ is a well defined multiplication

$$L_\xi\big((0,S-R),F_1\big)\times L_\eta\big((-S,T),F_2\big)\to L_\zeta\big((-R,T),F_0\big), \tag{44}$$

provided $\xi, \eta, \zeta \in [1, \infty]$ satisfy

$$1/\xi + 1/\eta = 1 + 1/\zeta. \tag{45}$$

For abbreviation, we set

$$X := \begin{cases} L_2, & \text{if } p = 2, \\ C(\overline{\Omega}) & \text{otherwise.} \end{cases}$$

In order to specify the measures appearing in (5) we fix R and S as above and suppose throughout that $1 < s \leq p'$. Then we introduce the following Banach spaces:

$$\begin{aligned} &\mathsf{H}_0 := L_s\bigl(J_R, \mathcal{L}(H^{-1}_{p,\chi})\bigr); && \mathsf{H} := L_s\bigl(J_R, \mathcal{L}(L_p)\bigr); \\ &\mathsf{P} := \mathcal{M}\bigl(\overline{J}_R, \mathcal{L}(C(\overline{\Omega}), L_\xi)\bigr); && \mathsf{P}_\Gamma := \mathcal{M}\bigl(\overline{J}_R, \mathcal{L}(C(\Gamma), L_\eta(\Gamma))\bigr); \\ &\Sigma := \mathcal{M}\bigl(\overline{J}_{S-R}, \mathcal{L}(X, C(\overline{\Omega}, \mathbb{R}^m))\bigr); && \Sigma_\Gamma := \mathcal{M}\bigl(\overline{J}_{S-R}, \mathcal{L}(X, C(\Gamma, \mathbb{R}^m))\bigr), \end{aligned}$$

where $pn/(n+p) \leq \xi \leq \infty$ with $\xi > 1$, and $p(n-1)/n \leq \eta \leq \infty$ with $\eta > 1$, and where we agree to set $L_s(J_R, F) := \{0\}$ if $R = 0$. We put

$$\Pi := \mathsf{H}_0 \times \mathsf{H} \times \mathsf{H} \times \mathsf{P} \times \mathsf{P}_\Gamma \times \Sigma \times \Sigma \times \Sigma \times \Sigma_\Gamma$$

and denote the general point of this Banach space by

$$\pi = (h, h_0, h_1, \rho_0, \rho_1, \sigma_0, \sigma_1, \sigma_2, \sigma_3).$$

Given $\pi \in \Pi$, we set

$$\mu := \delta_0 + h\,dt, \quad \nu_0 := \delta_0 + h_0\,dt, \quad \nu_1 := h_1\,dt. \tag{46}$$

Now we can formulate and prove the third main result of this paper, the following existence, uniqueness, and continuity theorem for problem (4).

Theorem 5.1. *Suppose that assumptions* (6) *and* (7) *are satisfied. Fix* $\pi \in \Pi$ *and define* μ, ν_0, *and* ν_1 *by* (46), *and* (4) *by* (5). *If* $R = \infty$, *then assume in addition that, for* $j = 0, 1$,

$$\left.\begin{array}{l} \textit{either } \rho_j \textit{ is compactly supported} \\ \textit{or it is absolutely continuous with respect to } dt. \end{array}\right\} \tag{47}$$

Finally, suppose that $\mathcal{V}$ *equals either* $\mathcal{H}^1_{p,\chi}(J_{-S})$ *or* $C_0\bigl(\overline{J}_{-S}, W^{1-2/p}_{p,\chi}\bigr)$ *with*

$$V = \mathcal{H}^1_{p,\chi}(J_{-S}), \quad \textit{if } h \neq 0. \tag{48}$$

Then:

(i) *Problem* (4) *has for each history* $v \in \mathcal{V}$ *a unique maximal* $\mathcal{H}^1_p$ *solution,* $u(v)$.

(ii) *If* $u(v) \in \mathcal{H}_p^1\big(J_T \cap J(v)\big)$ *for each* $T > 0$*, where* $J(v)$ *is the maximal existence interval of* $u(v)$*, then* $u(v)$ *exists globally, that is,* $J(v) = \mathbb{R}^+$.

(iii) *The map* $(t, v) \mapsto u(v)_t$ *is a Lipschitz semiflow on* $\mathcal{V}$ *depending Lipschitz continuously on* $\pi \in \Pi$ *(subject to condition* (47)*, of course).*

Proof. (1) Set $(E_0, E_1) := (H_{p,\chi}^{-1}, H_{p,\chi}^1)$. Then $E_1 \overset{d}{\hookrightarrow} E_0$, and, except for equivalent norms, $E = W_{p,\chi}^{1-2/p}$. Indeed, if $p = 2$, this follows from Proposition 2.1 and Theorem 15.1 in Chapter I of [24] (also see Section 1.15.10 in [28]). If $p > n + 2$, then it is implied by Theorem 7.2 of [1], for example. (Note that this proves (8).)

(2) Fix $\mathsf{T} > 0$. The Sobolev embedding $W_{p,\chi}^{1-2/p} \hookrightarrow C(\overline{\Omega})$ if $p > n + 2$ and (8) imply, together with the definition of $v \oplus w$, that

$$\big((v, w) \mapsto v \oplus w\big) \in \mathcal{C}^{1\text{-}}\big(\mathcal{D}, C_0([-S, \mathsf{T}], X)\big).$$

Hence we infer from (43) that the map

$$\big(\sigma, (v, w)\big) \mapsto \sigma * (v \oplus w) \tag{49}$$

belongs to $\mathcal{C}^{1\text{-}}\big(\Sigma \times \mathcal{D}, BUC([-R, \mathsf{T}], C(\overline{\Omega}, \mathbb{R}^m))\big)$. Set

$$\big(\widetilde{a}, \widetilde{b}, \widetilde{f}\,\big)(\sigma, v, w) := (a, b, f)\big(\cdot, \sigma * (v \oplus w)\big).$$

Then (6) and the asserted continuity properties of (49) imply that

$$\widetilde{a}, \widetilde{b} \in \mathcal{C}^{1\text{-}}\big(\Sigma \times \mathcal{D}, BUC([-R, \mathsf{T}], C(\overline{\Omega}, \mathbb{R}^{n\times n}))\big) \tag{50}$$

and

$$\widetilde{f} \in \mathcal{C}^{1\text{-}}\big(\Sigma \times \mathcal{D}, BUC([-R, \mathsf{T}], C(\overline{\Omega}))\big). \tag{51}$$

Similarly,

$$\begin{aligned}\widetilde{g} &:= \big((\sigma, (v, w)) \mapsto g(\cdot, \sigma * (v \oplus w))\big)\\ &\qquad \in \mathcal{C}^{1\text{-}}\big(\Sigma_\Gamma \times \mathcal{D}, BUC([-R, \mathsf{T}], C(\Gamma))\big).\end{aligned} \tag{52}$$

(3) For $\pi \in \Pi$ and $(v, u) \in \mathcal{D}$ define $A(\pi, v, u)$ by

$$\big\langle \varphi, A(\pi, v, u)w \big\rangle := \big\langle \nabla\varphi, \widetilde{a}(\sigma_0, v, u)\nabla w \big\rangle$$

for $(\varphi, w) \in H_{p',\chi}^1 \times H_{p,\chi}^1$. Then it follows from (50) that

$$A \in \mathcal{C}^{1\text{-}}\big(\Pi \times \mathcal{D}, C(\overline{\mathsf{J}}, \mathcal{L}(E_1, E_0))\big). \tag{53}$$

Observe that $\widetilde{a}(\sigma_0, v, u)(x, t)$ is symmetric and uniformly positive semidefinit on $\overline{\Omega} \times \overline{\mathsf{J}}$. Thus, if $p = 2$, well known results on the weak solvability of linear parabolic equations, essentially due to J.-L. Lions [23] (also see Theorem 2 in Chapter XVIII of [16], Chapter 23 in [31], or Theorem 11.7 in [12]), guarantee that

$$A(\pi, v, u) \in \mathcal{MR}_p\big(\mathsf{J}, (E_1, E_0)\big). \tag{54}$$

If $p > n+2$, this will be shown elsewhere. In particular, (32) is satisfied.

(4) For $\pi \in \Pi$ and $(v, u) \in \mathcal{D}$ we define $F_0(\pi, v, u)$ by

$$\begin{aligned}\big\langle w, F_0(\pi, v, u)\big\rangle :=& \big\langle \nabla w, \big(-h_0 * \widetilde{a}(\sigma_0, v, u) + h_1 * \widetilde{b}(\sigma_1, v, u)\big)\nabla u\big\rangle \\ &+ \big\langle w, \rho_0 * \widetilde{f}(\sigma_2, v, u)\big\rangle + \big\langle \gamma w, \rho_1 * \widetilde{g}(\sigma_3, v, u)\big\rangle_\Gamma\end{aligned} \tag{55}$$

for $w \in H^1_{p,\chi}$. Using $\nabla u \in L_p(\mathsf{J}, L_p)$ and (50) we see that

$$\big((\sigma_0, (v, u)) \mapsto \widetilde{a}(\sigma_0, v, u)\nabla u\big) \in \mathcal{C}^{1\text{-}}\big(\Sigma \times \mathcal{D}, L_p(\mathsf{J}, L_p)\big).$$

Thus, setting $1/r := 1/p + 1/s - 1 \in [0, 1/p)$, we infer from (44), (45) that

$$\big((h_0, \sigma_0, (v, u)) \mapsto -h_0 * \widetilde{a}(\sigma_0, v, u)\nabla u\big)$$

belongs to $\mathcal{C}^{1\text{-}}\big(\mathsf{H} \times \Sigma \times \mathcal{D}, L_r(\mathsf{J}, L_p)\big)$. The same is true, if $\widetilde{a}$, h_0, and σ_0 are replaced by $\widetilde{b}$, h_1, and σ_1, respectively. From (43), (47), and (51) we deduce that

$$\big((\rho_0, \sigma_2, (v, u)) \mapsto \rho_0 * \widetilde{f}(\sigma_2, v, u)\big) \in \mathcal{C}^{1\text{-}}\big(\mathsf{P} \times \Sigma \times \mathcal{D}, L_\infty(\mathsf{J}, L_\xi)\big).$$

Note that $H^1_{p',\chi} \hookrightarrow L_{\xi'}$ by Sobolev's embedding theorem. Similarly, (43), (47), and (52) imply

$$\big((\rho_1, \sigma_3, (v, u)) \mapsto \rho_1 * \widetilde{g}(\sigma_3, v, u)\nabla u\big) \in \mathcal{C}^{1\text{-}}\big(\mathsf{P}_\Gamma \times \Sigma_\Gamma \times \mathcal{D}, L_\infty(\mathsf{J}, L_\eta(\Gamma))\big).$$

Furthermore, the trace theorem implies

$$\gamma \in \mathcal{L}\big(H^1_{p',\chi}, L_{\eta'}(\Gamma)\big).$$

From these considerations and the boundedness of J it follows that

$$F_0 \in \mathcal{C}^{1\text{-}}\big(\Pi \times \mathcal{D}, L_r(\mathsf{J}, E_0)\big). \tag{56}$$

(5) Now suppose that $h \neq 0$ so that (48) is satisfied. Since

$$\big((v, u) \mapsto v \oplus u\big) \in \mathcal{C}^{1\text{-}}\big(\mathcal{D}, \mathcal{H}^1_{p,\chi}(J_{-S} \cup \mathsf{J})\big)$$

(see (40)), it follows from

$$\mathcal{H}^1_{p,\chi}(J_{-S} \cup \mathsf{J}, H^{-1}_{p,\chi}) \hookrightarrow L_p(J_{-S} \cup \mathsf{J}) \hookrightarrow L_p(\mathbb{R}, H^{-1}_{p,\chi}) \tag{57}$$

and (44) that

$$\big((h, (v, u)) \mapsto \widetilde{h} * (v \oplus u)^{\sim}\big) \in \mathcal{C}^{1\text{-}}\big(\mathsf{H}_0 \times \mathcal{D}, L_r(\mathbb{R}, H^{-1}_{p,\chi})\big), \tag{58}$$

where $\sim$ denotes extension by zero, due to $\widetilde{h} \in L_s\big(\mathbb{R}, \mathcal{L}(H^{-1}_{p,\chi})\big)$. Similarly, using Lemma 7.1 of [8] we see that

$$\big((h,(v,u)) \mapsto \widetilde{h} * (\dot{v} \oplus \dot{u})^{\sim}\big) \in \mathcal{C}^{1\text{-}}\big(\mathsf{H}_0 \times \mathcal{D}, L_r(\mathbb{R}, H^{-1}_{p,\chi})\big). \tag{59}$$

Since convolution and distributional derivatives commute, it is not difficult to see that

$$-\int_{\mathbb{R}} \dot{\varphi}\big\langle w, \widetilde{h} * (v \oplus u)^{\sim}\big\rangle\, dt = \int_{\mathbb{R}} \varphi\big\langle w, h * (\dot{v} \oplus \dot{u})\big\rangle\, dt \tag{60}$$

for each smooth φ having compact support in $\mathring{\mathsf{J}}$ and each $w \in H^1_{p',\chi}$ (cf. Lemma 7.1 in [8]).

Given $\big(\pi,(v,w)\big) \in \Pi \times \mathcal{D}$, set

$$F_1(\pi,v,w) := -\partial_t\big(h * (v \oplus w)\big) := -\big[\partial_t\big(\widetilde{h} * (v \oplus w)^{\sim}\big)\big] \,\big|\, \mathring{\mathsf{J}}.$$

Then we infer from (59) and (60) that

$$F_1 \in \mathcal{C}^{1\text{-}}\big(\Pi \times \mathcal{D}, L_r(\mathsf{J}, E_0)\big). \tag{61}$$

(6) Put $F := F_0$ if $h = 0$, and $F := F_0 + F_1$ otherwise. Then assumption (38) is satisfied, due to (53), (54), (56), and (61), since the Volterra property is obvious.

Finally, set

$$(\mathcal{A}, \mathcal{F})(\pi, u_t) := (A, F)(\pi, u \,|\, J_{-S}, u \,|\, \mathsf{J})$$

for $u : J_{-S} \cup \mathsf{J} \to E_0$ with $(u \,|\, J_{-S}, u \,|\, \mathsf{J}) \in \mathcal{D}$. Then, given $(\pi, v) \in \Pi \times \mathcal{V}$, one verifies that u is an $\mathcal{H}^1_p$ solution on J_T of (4) with history $v \in \mathcal{V}$ iff u is an $\mathcal{H}^1_p$ solution on J_T of the parameter dependent functional differential equation

$$\dot{u} + \mathcal{A}(\pi, u_t)u = \mathcal{F}(\pi, u_t), \quad u_0 = v.$$

Hence the assertion follows from Theorem 4.2. □

6. Proofs for the model problems

It is now not difficult to prove the theorems of Section 2 by observing that the corresponding model problems are particular instances of (4), (5).

Proof of Theorem 2.1. (1) Set $m := 1$, $p := 2$, $h := 0$, $h_0 := h_1 := 0$, $\rho_0 := j\delta_0$ with $j : C(\overline{\Omega}) \hookrightarrow L_2$, $\rho_1 := j_\Gamma \delta_0$ with $j_\Gamma : C(\Gamma) \hookrightarrow L_2(\Gamma)$, $\sigma_0 := \sigma_1 := \sigma_2 := K\alpha$, and $\sigma_3 := \gamma K\alpha$. Then

$$\pi := (h, h_0, h_1, \rho_0, \rho_1, \sigma_0, \sigma_1, \sigma_2, \sigma_3) \in \Pi$$

with $\xi := \eta := 2$ and $R := 0$. Hence everything but the last assertion follows from Theorem 5.1.

(2) Suppose that $\mathrm{supp}(\alpha) \subset [s, S]$ for some $s \in (0, S]$. Then

$$\alpha * Ku(t) = \int_{[s,S]} Ku(t-\tau)\,\mu(d\tau) + \int_{[s,S]} Kv(t-\tau)\,\mu(d\tau), \qquad 0 \le t \le s.$$

Thus on the interval $[0, s]$ the diffusion matrix and the nonlinearities are known functions so that (15) reduces on J_s to a linear equation which has a unique $\mathcal{H}_p^1$ solution on J_s with initial value $v(0)$. Next we consider (15) on the interval $[s, 2s]$. Here we are now also faced with a linear problem with initial value $u(s) \in E$ having a unique solution. By iterating this argument we see that (15) is globally solvable since we can 'piece together' the solutions on the intervals $\big(k\tau, (k+1)\tau\big)$ by means of Lemma 7.1 of [8]. $\square$

It should be observed that the argument of the second part of this proof is the 'method of steps', well known in the theory of retarded differential equations (e.g., [29]).

Problem (18) does not fit completely into the framework of Theorem 5.1 since f is not continuous. However, easy modifications of the proof of the latter theorem give the stated results.

Proof of Theorem 2.2. (1) It follows from (16) and known properties of Nemytskii operators that

$$\big(u \mapsto f(\cdot, u)\big) \in \mathcal{C}^{1\text{-}}\big(C([0,T], L_2), C([0,T], L_\xi)\big)$$

where $\xi := 2n/(n+2)$. Since $H_\chi^1 \hookrightarrow L_{\xi'}$ we see that $F(u)$ is well defined for $u \in C\big([0,T], L_2\big)$ by

$$\big\langle v, F(u)\big\rangle := \big\langle v, f(\cdot, u)\big\rangle + \langle \gamma v, g_0\rangle_\Gamma, \qquad v \in H_\chi^1,$$

and that

$$F \in \mathcal{C}^{1\text{-}}\big(C([0,T], L_2), C([0,T], H_\chi^{-1})\big).$$

Thus (30) implies

$$F \in \mathcal{C}^{1\text{-}}\big(\mathcal{H}_\chi^1(J_T), L_\infty(J_T, H_\chi^{-1})\big)$$

for every $T > 0$. With this definition of F the proof of Theorem 5.1 remains valid. Thus all but the last assertion follow from that theorem.

(2) If the additional assumptions are satisfied we apply again the method of steps. However, in this case we have to solve at each step a semilinear

equation since the diffusion matrix is known but the right hand side is still a function of u on the corresponding interval.

Using condition (ii) and well known arguments for weak solutions of semilinear parabolic equations we easily deduce that $\|u(\cdot,t)\|_{L_2} \le c$ for $0 \le t < \tau$, where u is the maximal solution of the semilinear problem on J_s and $\tau \in (0,s]$ is its maximal existence time. Consequently, $F(u) \in L_\infty(J_\tau, H_\chi^{-1})$. Now maximal regularity implies $u \in \mathcal{H}_\chi^1(J_\tau)$. Hence we infer from Theorem 3.1(ii), for example, applied to the semilinear problem, that u exists on J_s and belongs to $\mathcal{H}_\chi^1(J_s)$. Thus $u(s) \in L_2$ and the method of steps can be carried through. □

Proof of Theorem 2.3. Here we put $m := 2$ and define σ_0 by $\sigma_0 := [\delta_0 \otimes I, \alpha \otimes I]$ with I being the identity in $\mathcal{L}\big(C(\overline{\Omega})\big)$, that is,

$$\langle \sigma_0, \varphi \rangle = \Big(\varphi(0,\cdot), \int_{\mathbb{R}} \varphi(t,\cdot)\, \alpha(dt) \Big), \qquad \varphi \in C_0\big(\mathbb{R}, C(\overline{\Omega})\big).$$

Now the assertions follow by the arguments of the proof of Theorem 2.1. □

It is now clear how Theorem 5.1 can be applied to prove Theorems 2.3 and 2.4. Theorem 2.5 is again obtained by the method of steps. At each step there has to be solved a quasilinear problem to which Theorem 2.1 of [8] can be applied. For this we have to observe that the translation group acts strongly continuously on $H_p^1(\mathbb{R}, H_{p,\chi}^{-1})$ and that it commutes with differentiation. Thus $\partial_t\big(hv(t-\cdot)\big)$ is well defined in $L_p(J_r, H_{p,\chi}^{-1})$. If the solution exists globally on J_r then we can go on to the next step. Otherwise, we have arrived at the maximal solution. This is true for every step which can be carried out. Details are left to the reader.

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