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Operator-Valued Fourier Multipliers, Vector-Valued Besov Spaces, and Applications

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Abstract. It is shown that translation-invariant operators with operator-valued symbols act continuously on Besov spaces of Banach-space-valued distributions. This result is then used to extend and complement the known theory of vector-valued Besov spaces. In addition, its power is demonstrated by giving applications to a variety of problems from elliptic and parabolic differential and integrodifferential equations.

Introduction

Fourier multiplier theorems provide one of the most important tools in the study of partial differential and pseudodifferential equations. Among them Mikhlin's theorem, guaranteeing the continuity of pseudodifferential operators on L_p-spaces, plays a predominant rôle.

The simplest case of a pseudodifferential operator is provided by a translation-invariant operator which can always be written in the form

$$a(D) := \mathcal{F}^{-1} a \mathcal{F}$$

where a, the symbol of $a(D)$, is a sufficiently smooth function on $\mathbb{R}^n$, and $\mathcal{F}$ denotes the Fourier transform. We are particularly interested in the case where a takes its value in a Banach space E. Then we say that a belongs to the symbol class $S^m(\mathbb{R}^n, E)$, where $m \in \mathbb{R}$, if $a \in C^{n+1}\big(\mathbb{R}^n \backslash \{0\}, E\big)$ and there exists a constant c such that

$$|\partial^\alpha a(\xi)|_E \le c(1+|\xi|)^{m-|\alpha|} \ , \qquad \alpha \in \mathbb{N}^n \ , \quad |\alpha| \le n+1 \ .$$

It is a well-known consequence of Mikhlin's theorem that $a(D)$ is a bounded linear operator from $L_p(\mathbb{R}^n)$ into itself, where $1 < p < \infty$, if $a \in S^0(\mathbb{R}^n, \mathbb{C})$. In fact, in this

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'scalar' case the requirements for a can be considerably weakened (e.g., [Hör60], [Hör83, Section 7.9]).

In connection with an operator-theoretical approach to differential equations one is naturally led to study translation-invariant operators in $L_p(\mathbb{R}^n, E)$, that is, in L_p-spaces of vector-valued functions. Thanks to the work of BOURGAIN [Bou83], BURKHOLDER [Bur83], MCCONNELL [McC84], and ZIMMERMANN [Zim89] it is known that $a(D)$ maps $L_p(\mathbb{R}^n, E)$ continuously into itself for $1 < p < \infty$, provided a belongs to $S^0(\mathbb{R}^n, \mathbb{C})$ and E is a UMD space. The latter condition implies the reflexivity of E. Thus it restricts the class of admissible Banach spaces considerably. In particular, it rules out the use of Hölder spaces which are of great importance in the theory of nonlinear differential equations.

However, even if we are willing to restrict ourselves to the class of UMD spaces, the above generalization of Mikhlin's theorem is not sufficient for many purposes. In practice there occur naturally pseudodifferential operators with operator-valued symbols, that is, symbols in $S^m\big(\mathbb{R}^n, \mathcal{L}(E,F)\big)$, where $\mathcal{L}(E,F)$ is the Banach space of all bounded linear operators from E into the Banach space F. This is the case, for instance, in the systematic study of pseudodifferential operators on manifolds with singularities carried out, in particular, by B.-W. SCHULZE and his coworkers (e.g., [Sch91], [Sch94a]).

An inspection of that research shows that the authors always restrict themselves to a Hilbert space setting. This is necessary since they use in an essential way Plancherel's theorem which is known to be valid in Hilbert spaces only, that is, if E is a Hilbert space and $p = 2$.

For the study of nonlinear problems a Hilbert space L_2-setting is too narrow for many purposes. It would be very useful to be able to work in an $L_p(\mathbb{R}^n, E)$-setting with general p and an arbitrary Banach space. Unfortunately, this is impossible due to a result of G. PISIER. In fact, that author proved — but did not publish — more than fifteen years ago that, if Mikhlin's theorem holds on $L_p(\mathbb{R}, E)$ for $\mathcal{L}(E,E)$-valued symbols, then E is isomorphic to a Hilbert space (private communication; also see [LLLM96] for a proof). Thus there is no Mikhlin-type theorem on the scale of Bessel potential spaces $H_p^s(\mathbb{R}^n, E)$ for $1 < p < \infty$ and general E, since those spaces are isomorphic to $L_p(\mathbb{R}^n, E)$.

From the theory of function spaces it is known that, besides the scale of (scalar) Triebel-Lizorkin spaces, that contains the H_p^s-scale as a subscale, there is a second general scale of function spaces possessing similar properties, namely the scale of (scalar) Besov spaces (e.g., [Tri83]). This scale subsumes, in particular, the important Sobolev-Slobodeckii spaces of noninteger orders, as well as the scale of Hölder spaces. It is also known how to define vector-valued Besov spaces, and that many properties of the scalar spaces carry over to the vector-valued setting (cf. [Sch86] for the most complete results published so far).

In this paper we show that, given arbitrary Banach spaces E and F and any $a \in S^m\big(\mathbb{R}^n, \mathcal{L}(E,F)\big)$, the operator $a(D)$ maps the Besov space $B_{p,q}^{s+m}(\mathbb{R}^n, E)$ continuously into the Besov space $B_{p,q}^s(\mathbb{R}^n, F)$, where $s \in \mathbb{R}$ and $p, q \in [1, \infty]$ are arbitrary (see Section 5 for the definition of these spaces). Furthermore, similar assertions are true if these spaces are replaced by certain closed subspaces thereof that are of importance in applications. Thus in a vector-valued setting it is of advantage to use the scale of Besov spaces instead of the Bessel potential scale. Although general Besov spaces are somewhat complicated to describe, and not too easy to handle, there are rather precise embedding theorems relating them to the more familiar spaces $L_p(\mathbb{R}^n, E)$, $1 \leq p < \infty$, $BUC(\mathbb{R}^n, E)$, and $C_0(\mathbb{R}^n, E)$. Thus our operator-valued multiplier theorem can be used to derive continuity results of translation-invariant operators on those spaces as well, provided one is willing to sacrifice 'optimal regularity'.

In order to be able to define $a(D)$ in the operator-valued case on sufficiently general spaces of distributions we have to employ L. Schwartz' theory of vector-valued distributions and, in particular, a rather sophisticated version of his abstract kernel theorem (cf. [Sch57b], [Sch57a]). These results and some of their consequences that are important for our problems are collected in the first three sections without proofs. A detailed exposition including full proofs will be given in [Ama97].

The basic multiplier theorems are proven in Section 4, where they take a preliminary form since we introduce general Besov spaces in Section 5 only. In that section we collect the known results on vector-valued Besov spaces and use some simple versions of our multiplier results to derive useful representations for some Besov spaces and some of their important subspaces. In addition, we present without proofs some deeper new results which show that — using appropriate interpretation — vector-valued Besov spaces possess virtually the same properties as their scalar counterparts. This is true without any restriction on the Banach space E, except for the duality assertion where we have to require that E' possesses the Radon-Nikodym property, which is certainly not surprising. Since we do not use most of these deeper properties we do not give proofs but refer again to [Ama97]. It should be mentioned, however, that they rely in an essential manner on the multiplier theorems of Section 4. In Section 6 we prove the Besov-space-version of the operator-valued multiplier theorem, namely Theorem 6.2.

The remaining two sections are devoted to applications. In Section 7 we prove some general isomorphism theorems and resolvent estimates for translation-invariant operators on Besov spaces. These results imply, among other things, that those operators generate analytic semigroups on Besov spaces. Using suitable embedding theorems we then infer that they generate C^∞-semigroups on the classical function spaces.

Finally, in Section 8 we present a variety of applications of the preceding general results. Besides of showing, by means of simple model problems, how to obtain easily solvability results for degenerate elliptic and parabolic differential equations on cylinders, we derive some new maximal regularity theorems for general operator-valued convolution equations comprising, in particular, integrodifferential equations of parabolic type on the line and on the half-line. We also give almost trivial proofs of a maximal regularity theorem due to Da Prato and Grisvard [DG75] and of an existence result of Lunardi [Lun95] for bounded solutions of parabolic evolution equations. In addition, we extend these results to encompass wider classes of spaces. Lastly, we show how our results can be used to treat the Poisson equation on a general cone in $\mathbb{R}^n$, which is a simple model problem for elliptic boundary value problems on manifolds with singularities. It is hoped that the results of this paper open a way to attack these problems in a non-hilbertian framework.

Notations and Conventions In this paper all vector spaces are over $\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$. If in a given formula there occur explicitly (as, for example, in the Fourier transform) or implicitly (as, for example, in the resolvent set of a linear operator) complex numbers, it is always understood that this formula refers to the complexified spaces if $\mathbb{K} = \mathbb{R}$.

If Z is a nonempty subset of some vector space, we put $\dot{Z} := Z \setminus \{0\}$.

Let V and W be locally convex spaces (LCSs). Then $\mathcal{L}(V,W)$ is the LCS of all continuous linear operators equipped with the bounded convergence topology, and $\mathcal{L}(V) := \mathcal{L}(V,V)$. We denote by $\mathcal{L}\mathrm{is}(V,W)$ the set of all isomorphisms in $\mathcal{L}(V,W)$, and $\mathcal{L}\mathrm{aut}(V) := \mathcal{L}\mathrm{is}(V,V)$. We write $V \hookrightarrow W$ if V is a linear subspace of W and the canonical injection is continuous. We replace $\hookrightarrow$ by $\overset{d}{\hookrightarrow}$ if V is also dense in W.

The dual space of V is denoted by V' and is given the strong topology so that $V' = \mathcal{L}(V, \mathbb{K})$. Then $\langle \cdot, \cdot \rangle_V : V' \times V \to \mathbb{K}$ is the V'-V-duality pairing, that is, $\langle v', v \rangle_V$ is the value of $v' \in V'$ at $v \in V$.

Throughout this paper $E, E_0, E_1, E_2, \ldots$ denote Banach spaces whose norms are denoted by $|\cdot|$, if no confusion seems likely. Moreover,

$$E_1 \times E_2 \to E_0 \ , \quad (e_1, e_2) \mapsto e_1 \bullet e_2 \tag{0.1}$$

is a **multiplication**, that is, a continuous bilinear map of norm at most 1. Important examples of multiplications are:

(i) ordinary multiplication if E is a Banach algebra and $E_j = E$, $j = 0, 1, 2$;
(ii) multiplication with scalars: $\mathbb{K} \times E \to E$, $(\alpha, e) \mapsto \alpha e$;
(iii) the duality pairing $E' \times E \to \mathbb{K}$, $(e', e) \mapsto \langle e', e \rangle_E$;
(iv) the evaluation map $\mathcal{L}(E_1, E_0) \times E_1 \to E_0$, $(A, e) \mapsto Ae$;
(v) composition $\mathcal{L}(E_1, E_2) \times \mathcal{L}(E_0, E_1) \to \mathcal{L}(E_0, E_2)$, $(S, T) \to ST$.

In general, if $b \colon E_1 \times E_2 \to E_0$ is a nontrivial continuous bilinear map then

$$E_1 \times E_2 \to E_0 \ , \quad (e_1, e_2) \mapsto \|b\|^{-1} \, b(e_1, e_2)$$

is a multiplication. The trivial map $E_1 \times E_2$, $(e_1, e_2) \mapsto 0$ is a multiplication as well. Hence it is no restriction to presuppose the existence of a multiplication (0.1) on the spaces E_0, E_1, and E_2 (as long as we do not specify a particular one, of course).

1. Spaces of Vector-Valued Distributions

In this section we collect some basic facts about vector-valued distributions and introduce further notation. The results are 'elementary' in the sense that the usual 'scalar proofs' carry over to the vector-valued setting by using obvious modifications only.

Let X be a nonempty open subset of $\mathbb{R}^n$. We write $\mathcal{D}(X, E)$ for the space of all E-valued test functions on X, that is, the set of all $u \in C^\infty(X, E)$ having compact supports in X, endowed with the usual inductive limit topology. We also put $\mathcal{E}(X, E) := C^\infty(X, E)$, equipped with the Fréchet topology of compact convergence of all derivatives. Furthermore, $\mathcal{S}(\mathbb{R}^n, E)$ is the Schwartz space of rapidly decreasing E-valued smooth functions on $\mathbb{R}^n$ with the standard Fréchet topology. Lastly, $\mathcal{O}_M(\mathbb{R}^n, E)$ denotes the space of slowly increasing smooth E-valued functions on $\mathbb{R}^n$, that is, $\varphi \in \mathcal{O}_M(\mathbb{R}^n, E)$ if for each $\alpha \in \mathbb{N}^n$ there exist c_α and $m_\alpha \in \mathbb{N}$ such that

$$|\partial^\alpha \varphi(x)| \le c_\alpha (1 + |x|)^{m_\alpha} \ , \qquad x \in \mathbb{R}^n \ .$$

It is a LCS with the topology induced by the family of seminorms

$$\{\, \varphi \mapsto \|\varphi \partial^\alpha a\|_\infty \ ; \ \varphi \in \mathcal{S}(\mathbb{R}^n, \mathbb{K}),\ \alpha \in \mathbb{N}^n \,\} \ .$$

Of course, we set $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, where $\partial_j := \partial / \partial x_j$ for $1 \le j \le n$, and $\|\cdot\|_\infty$ is the supremum norm.

To simplify the writing we agree to put

$$\mathfrak{F}(X, E) := \mathfrak{F}(\mathbb{R}^n, E) \qquad \text{if } \mathfrak{F} \in \{\mathcal{S}, \mathcal{O}_M\} \ , \tag{1.1}$$

that is, if $\mathfrak{F}$ is one of the letters $\mathcal{S}$ and $\mathcal{O}_M$. Moreover,

$$\mathfrak{F}(X) := \mathfrak{F}(X, \mathbb{K}) \ , \qquad \mathfrak{F} \in \{\mathcal{D}, \mathcal{E}, \mathcal{S}, \mathcal{O}_M\} \ .$$

Then we set

$$\mathfrak{F}'(X, E) := \mathcal{L}\big(\mathfrak{F}(X), E\big) \ , \qquad \mathfrak{F} \in \{\mathcal{D}, \mathcal{E}, \mathcal{S}, \mathcal{O}_M\} \ ,$$

so that $\mathfrak{F}'(X, \mathbb{K}) = \mathfrak{F}(X)'$. Thus $\mathcal{D}'(X, E)$ is the space of E-valued distributions on X, and $\mathcal{S}'(\mathbb{R}^n, E)$ is the space of E-valued temperate distributions on $\mathbb{R}^n$. The support of an E-valued distribution is defined as in the scalar case. The scalar proof carries over to show that $\mathcal{E}'(X, E)$ is the set of all E-valued distributions having compact supports in X.

As usual, we identify $u \in L_{1,\mathrm{loc}}(X, E)$ with the E-valued distribution

$$\varphi \mapsto u(\varphi) := \int_X \varphi(x) u(x)\, dx \ , \qquad \varphi \in \mathcal{D}(X) \ .$$

Then $L_{1,\mathrm{loc}}(X, E)$ is the space of regular E-valued distributions on X which is given the obvious Fréchet topology. It follows that

$$\mathcal{D}(X, E) \overset{d}{\hookrightarrow} L_{1,\mathrm{loc}}(X, E) \overset{d}{\hookrightarrow} \mathcal{D}'(X, E) \ .$$

Moreover,

$$\mathcal{D}(X, E) \overset{d}{\hookrightarrow} \mathcal{E}(X, E) \overset{d}{\hookrightarrow} \mathcal{D}'(X, E) \ , \quad \mathcal{D}(X, E) \overset{d}{\hookrightarrow} \mathcal{E}'(X, E) \overset{d}{\hookrightarrow} \mathcal{D}'(X, E) \ ,$$

and

$$\mathcal{D}(\mathbb{R}^n, E) \overset{d}{\hookrightarrow} \mathcal{S}(\mathbb{R}^n, E) \overset{d}{\hookrightarrow} \mathcal{O}_M(\mathbb{R}^n, E) \overset{d}{\hookrightarrow} \mathcal{S}'(\mathbb{R}^n, E) \overset{d}{\hookrightarrow} \mathcal{D}(\mathbb{R}^n, E) \ .$$

For $u \in \mathcal{D}'(X, E)$ and $\alpha \in \mathbb{N}^n$ we define $\partial^\alpha u$ by

$$(\partial^\alpha u)(\varphi) := (-1)^{|\alpha|} u(\partial^\alpha \varphi) \ , \qquad \varphi \in \mathcal{D}(X) \ .$$

Then $\partial^\alpha \in \mathcal{L}\big(\mathcal{D}'(X, E)\big)$. Similarly, $(au)(\varphi) := u(a\varphi)$ for $a \in \mathcal{E}(X),\ u \in \mathcal{D}'(X, E)$, and $\varphi \in \mathcal{D}(X)$. Then 'point-wise multiplication'

$$\mathcal{E}(X) \times \mathcal{D}'(X, E) \to \mathcal{D}'(X, E) \ , \quad (a, u) \mapsto au$$

is a well-defined bilinear map which is hypocontinuous, that is, continuous in each variable, uniformly with respect to the other one restricted to bounded sets. It follows that $(u \mapsto au) \in \mathcal{L}\big(\mathcal{S}'(\mathbb{R}^n, E)\big)$ iff $a \in \mathcal{O}_M(\mathbb{R}^n)$.

Let $\mathcal{G}(X,E)$ be a LCS such that $\mathcal{G}(X,E) \hookrightarrow \mathcal{D}'(X,E)$. Then $\mathcal{G}(X,E)$ is said to be a space of E-valued distributions, an 'E-valued space of distributions on X', and we put $\mathcal{G}(X) := \mathcal{G}(X,\mathbb{K})$ if no confusion seems possible.

If $m \in \mathbb{N}$ and $1 \le p \le \infty$ then $W_p^m(X,E)$ is the Sobolev space of all E-valued distributions on X such that $\partial^\alpha u \in L_p(X,E)$ for $|\alpha| \le m$. It is a Banach space with its usual norm which is denoted by $\|\cdot\|_{m,p}$. If $1 \le p < \infty$ and $\theta \in (0,1)$, we put

$$[u]_{\theta,p}^p := \int_{X\times X} \frac{|u(x)-u(y)|^p}{|x-y|^{n+\theta p}}\, d(x,y) \ .$$

Then, given $p \in [1,\infty)$ and $s \in \mathbb{R}^+ \backslash \mathbb{N}$, the Slobodeckii space $W_p^s(X,E)$ consists of all $u \in W_p^{[s]}(X,E)$ such that $[\partial^\alpha u]_{s-[s],p} < \infty$ for $|\alpha| = m$, where $[s]$ is the integer part of s. It is a Banach space with the norm

$$u \mapsto \|u\|_{s,p} := \Bigl(\|u\|_{[s],p}^p + \sum_{|\alpha|=[s]} [\partial^\alpha u]_{s-[s],p}^p\Bigr)^{1/p} \ .$$

It follows that

$$\mathcal{D}(X,E) \hookrightarrow W_p^s(X,E) \stackrel{d}{\hookrightarrow} \mathcal{D}'(X,E)$$

and

$$\mathcal{S}(\mathbb{R}^n,E) \stackrel{d}{\hookrightarrow} W_p^s(\mathbb{R}^n,E) \stackrel{d}{\hookrightarrow} \mathcal{S}'(\mathbb{R}^n,E)$$

for $s \in \mathbb{R}^+$ and $1 \le p < \infty$.

We denote by $BUC^m(X,E)$, $m \in \mathbb{N}$, the Banach space of all $u \in C(\mathbb{R}^n,E)$ such that $\partial^\alpha u$ is bounded and uniformly continuous on X for $|\alpha| \le m$, equipped with the norm $\|\cdot\|_{m,\infty}$. If $s \in \mathbb{R}^+ \backslash \mathbb{N}$ then $BUC^s(X,E)$ is the Hölder space of all u belonging to $BUC^{[s]}(X,E)$ such that $\partial^\alpha u$ is uniformly Hölder continuous of exponent $s-[s]$ for $|\alpha| = [s]$. It is a Banach space with the norm

$$u \mapsto \|u\|_{s,\infty} := \|u\|_{[s],\infty} + \max_{|\alpha|=[s]} [\partial^\alpha u]_{s-[s]} \ ,$$

where

$$[u]_\theta := [u]_{\theta,X} := \sup_{\substack{x,y\in X\\ x\ne y}} \frac{|u(x)-u(y)|}{|x-y|^\theta} \ .$$

Clearly,

$$BUC^s(X,E) \hookrightarrow W_\infty^{[s]}(X,E) \hookrightarrow \mathcal{D}'(X,E) \ , \qquad s \in \mathbb{R}^+ \ .$$

We also introduce the 'little Hölder spaces' $buc^s(X,E)$ as follows: if $s \in \mathbb{N}$ then

$$buc^s(X,E) := BUC^s(X,E) \ ,$$

and if $s \in \mathbb{R}^+ \backslash \mathbb{N}$ then it is the closed subspace of $BUC^s(\mathbb{R}^n,E)$ consisting of all u such that $\max_{|\alpha|=[s]} [\partial^\alpha u]_{s-[s],Y} \to 0$ as $\operatorname{diam}(Y) \to 0$.

Lastly, we denote by $C_0^s(X,E)$ the closure of $\mathcal{D}(X,E)$ in $BUC^s(X,E)$. Then

$$\mathcal{S}(\mathbb{R}^n,E) \overset{d}{\hookrightarrow} C_0^s(\mathbb{R}^n,E) \hookrightarrow buc^s(\mathbb{R}^n,E) \hookrightarrow BUC^s(\mathbb{R}^n,E) \hookrightarrow \mathcal{S}'(\mathbb{R}^n,E) \tag{1.2}$$

for $s \in \mathbb{R}^+$. It follows that $u \in C_0^m(X,E)$ for $m \in \mathbb{N}$ iff $u \in C^m(X,E)$ and, given any $\varepsilon > 0$, there exists a compact subset K of X such that $|\partial^\alpha u(x)| < \varepsilon$ for $x \in X \backslash K$ and $|\alpha| \le m$, that is, $\partial^\alpha u$ 'vanishes at infinity' for $|\alpha| \le m$.

Obviously, $W_p^0 = L_p$ and $\|\cdot\|_{0,p} = \|\cdot\|_p$, and we omit the superscript s in BUC^s and C_0^s if it equals 0.

We denote by $\mathcal{F}$ the Fourier transform, and set $\widehat{\varphi} := \mathcal{F}\varphi$ for $\varphi \in L_1(\mathbb{R}^n,E)$. The Riemann-Lebesgue lemma asserts that

$$\mathcal{F} \in \mathcal{L}\big(L_1(\mathbb{R}^n,E), C_0(\mathbb{R}^n,E)\big) \ .$$

For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n,E)$ we put $\widehat{u}(\varphi) := u(\widehat{\varphi})$ and $\mathcal{F}u := \widehat{u}$. Then it follows that

$$\mathcal{F} \in \mathcal{L}\mathrm{aut}\big(\mathcal{S}(\mathbb{R}^n,E)\big) \cap \mathcal{L}\mathrm{aut}\big(\mathcal{S}'(\mathbb{R}^n,E)\big) \tag{1.3}$$

and

$$\mathcal{F}^{-1}u = (2\pi)^{-n}\check{\widehat{u}} = (2\pi)^{-n}\widehat{\check{u}} \ , \qquad u \in \mathcal{S}'(\mathbb{R}^n,E) \ , \tag{1.4}$$

where $\check{\varphi}(x) := \varphi(-x)$ and $\check{u}(\varphi) := u(\check{\varphi})$ for $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n,E)$.

Setting $D_j := -i\partial_j$ for $1 \le j \le n$ we obtain

$$\widehat{D^\alpha u} = \xi^\alpha \widehat{u} \ , \quad \widehat{\xi^\alpha u} = (-1)^{|\alpha|} D^\alpha \widehat{u} \ , \qquad u \in \mathcal{S}'(\mathbb{R}^n,E) \ , \quad \alpha \in \mathbb{N}^n \ . \tag{1.5}$$

We define dilation σ_t for $t > 0$ by

$$\sigma_t\varphi(x) := \varphi(tx) \ , \quad (\sigma_t u)(\varphi) := t^{-n} u(\sigma_{1/t}\varphi)$$

for $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n,E)$. Then $\{\, \sigma_t \ ; \ t > 0 \,\}$ is a group of automorphisms of $\mathcal{S}(\mathbb{R}^n,E)$ and of $\mathcal{S}'(\mathbb{R}^n,E)$, and $(\sigma_t)^{-1} = \sigma_{1/t}$. Moreover,

$$\mathcal{F} \circ \sigma_t = t^{-n} \sigma_{1/t} \circ \mathcal{F} \ , \quad \partial^\alpha \circ \sigma_t = t^{|\alpha|} \sigma_t \circ \partial^\alpha \ , \qquad \alpha \in \mathbb{N}^n \ , \quad t > 0 \ . \tag{1.6}$$

It is also easily verified that $L_p(\mathbb{R}^n,E)$ is invariant under the group $\{\, \sigma_t \ ; \ t > 0 \,\}$ of dilations and that

$$\|\sigma_t u\|_p = t^{-n/p} \|u\|_p \ , \qquad u \in L_p(\mathbb{R}^n,E) \ , \quad 1 \le p \le \infty \ , \quad t > 0 \ . \tag{1.7}$$

Suppose that $u \in \mathcal{D}'(\mathbb{R},E)$ and $\mathrm{supp}(u) \subset \mathbb{R}^+$. Also suppose that there exists $\omega \in \mathbb{R}$ such that $e^{-\langle \xi, \cdot \rangle} u \in \mathcal{S}'(\mathbb{R},E)$ for each $\xi > \omega$, where $\langle \xi, x \rangle = \xi x$, of course. Then

$$\zeta = \xi + i\eta \mapsto \widetilde{u}(\zeta) := \big(e^{-\langle \cdot, \xi \rangle} u\big)\widehat{\ }(\eta) \ , \tag{1.8}$$

the Laplace transform of u, is a well-defined analytic function on $[\mathrm{Re}\,\zeta > \omega]$ which is polynomially bounded in the sense that there are constants c and k such that

$$|\widetilde{u}(\zeta)| \le c(1 + |\zeta|)^k \ , \qquad \mathrm{Re}\,\zeta > \omega \ .$$

Conversely, if $\widetilde{u}$ is a polynomially bounded analytic function on $[\mathrm{Re}\,\zeta > \omega]$ for some $\omega \in \mathbb{R}$ then it is the Laplace transformation of a distribution $u \in \mathcal{D}'(\mathbb{R},E)$ whose support is contained in $\mathbb{R}^+$. This is a special case of a more general Paley-Wiener theorem. It is easily seen that $e^{-\langle \xi, \cdot \rangle} u \in \mathcal{S}'(\mathbb{R}, E)$ for each $\xi > 0$ if $u \in \mathcal{S}'(\mathbb{R}, E)$ with $\mathrm{supp}(u) \subset \mathbb{R}^+$.

2. Point-Wise Multiplications

Suppose that $a \in \mathcal{E}(X, E_1)$. Then point-wise multiplication (with respect to multiplication (0.1)) with a regular E_2-valued distribution u is defined by

(2.1) $$a \bullet u(x) := a(x) \bullet u(x) \ , \qquad \text{a.a. } x \in X \ .$$

Clearly, $a \bullet u$ is a regular E_0-valued distribution, and this definition is consistent with the definition of multiplication of a smooth function with a distribution if $E_1 = \mathbb{K}$ and $\bullet$ denotes multiplication with scalars. However, if u is a general E_2-valued distribution, it is far from being obvious how to define a 'point-wise multiplication' that reduces to (2.1) if u is regular. In order to do this we have to rely on the general theory of the topological tensor products and a rather general version of Schwartz' kernel theorem as given in [Sch57b] (also see [Sch57a]). Whereas Schwartz considers distributions with values in general LCSs, we restrict ourselves to the Banach-space-valued case. This simplifies the theory to some extent. For a complete presentation in this framework, as well as for further results, we refer to [Ama97, Chapter VI] where, in particular, proofs for the statements of this and the next section can be found (also see [DL90] and [Fat83] for some parts of the theory in 'one variable').

Suppose that $\mathfrak{F} \in \{\mathcal{D}, \mathcal{D}', \mathcal{E}, \mathcal{E}', \mathcal{S}, \mathcal{S}', \mathcal{O}_M\}$ and recall convention (1.1). Then we set $\varphi \otimes e := \varphi e$ for $\varphi \in \mathfrak{F}(X)$ and $e \in E$. Consequently, the tensor product

$$\mathfrak{F}(\mathbb{R}^n) \otimes E := \mathrm{span}\big\{ \varphi \otimes e \ ; \ \varphi \in \mathfrak{F}(\mathbb{R}^n), \ e \in E \big\}$$

has a meaning, where the span is taken in $\mathfrak{F}(\mathbb{R}^n, E)$. It follows that $\mathcal{D}(X) \otimes E$ is dense in $\mathfrak{F}(X, E)$. Given $a := \varphi \otimes e_1$ and $u := \psi \otimes e_2$, where $\varphi, \psi \in \mathcal{D}(X)$ and $e_1, e_2 \in E$, it is natural to put $a \bullet u := \varphi\psi \otimes (e_1 \bullet e_2)$. Then one would like to extend this definition to all $(a, u) \in \mathcal{E}(X, E_1) \times \mathcal{D}'(X, E_2)$ by density and continuity. This is indeed possible as the following fundamental result shows.

Theorem 2.1. *There exists a unique hypocontinuous bilinear map*

$$\mathcal{E}(X, E_1) \times \mathcal{D}'(X, E_2) \to \mathcal{D}'(X, E_0) \ , \quad (a, u) \mapsto a \bullet u \ ,$$

called **point-wise multiplication induced by** (0.1)*, such that*

$$(\varphi \otimes e_1) \bullet (\psi \otimes e_2) = \varphi\psi \otimes (e_1 \bullet e_2)$$

for $a := \varphi \otimes e_1 \in \mathcal{D}(X) \otimes E_1$ and $u := \psi \otimes e_2 \in \mathcal{D}(X) \otimes E_2$. It restricts to a hypocontinuous bilinear map

$$\mathfrak{F}_1(X, E_1) \times \mathfrak{F}_2(X, E_2) \to \mathfrak{F}_0(X, E_0) \ ,$$

where $(\mathfrak{F}_1, \mathfrak{F}_2; \mathfrak{F}_0)$ stands for any one of the triplets $(\mathcal{E}, \mathcal{D}; \mathcal{D})$, $(\mathcal{E}, \mathcal{E}; \mathcal{E})$, $(\mathcal{O}_M, \mathcal{S}; \mathcal{S})$, $(\mathcal{O}_M, \mathcal{O}_M; \mathcal{O}_M)$, $(\mathcal{E}, \mathcal{E}'; \mathcal{E}')$, or $(\mathcal{O}_M, \mathcal{S}'; \mathcal{S}')$.

Remarks 2.2. (a) Since $E_2 \times E_1 \to E_0$, $(e_2, e_1) \mapsto e_1 \bullet e_2$ is a multiplication as well, the assertions of Theorem 2.1 are 'symmetric' with respect to E_1 and E_2, that is, the rôles of E_1 and E_2 can be interchanged. This fact will often be employed in the following, usually without further mention.

(b) Point-wise multiplication induced by (0.1), as defined in Theorem 2.1, coincides on regular distributions with point-wise multiplication defined in (2.1).

(c) *Leibniz' rule* is valid: if p is a polynomial in n indeterminates then

$$p(\partial)(a \bullet u) = \sum_{\beta} \frac{1}{\beta!} (\partial^\beta a) \bullet p^{(\beta)}(\partial) u$$

for $a \in \mathcal{E}(X, E_1)$ and $u \in \mathcal{D}'(X, E_2)$, where $p^{(\beta)} := \partial^\beta p$. In particular:

$$\partial^\alpha (a \bullet u) = \sum_{\beta \leq \alpha} \tbinom{\alpha}{\beta} (\partial^\beta a) \bullet \partial^{\alpha - \beta} u \ , \qquad a \in \mathcal{E}(X, E_1) \ , \quad u \in \mathcal{D}'(X, E_2) \ .$$

(d) If $(a, u) \in \mathcal{E}(X, E_1) \times \mathcal{D}'(X, E_2)$ then $\operatorname{supp}(a \bullet u) \subset \operatorname{supp}(a) \cap \operatorname{supp}(u)$.

(e) Suppose that there are multiplications

$$\begin{tikzcd}
E_1 \times E_2 \arrow[d] & & E_2 \times E_3 \arrow[d] \\
E_0 \times E_3 \arrow[dr] & & E_1 \times E_4 \arrow[dl] \\
& E_5 &
\end{tikzcd}$$

all denoted by $\bullet$, which are associative, that is, $(e_1 \bullet e_2) \bullet e_3 = e_1 \bullet (e_2 \bullet e_3)$ for $e_j \in E_j$, $j = 1, 2, 3$. Then point-wise multiplication is associative as well, when defined, that is,

$$(u_1 \bullet u_2) \bullet u_3 = u_1 \bullet (u_2 \bullet u_3) \ , \qquad u_j \in \mathfrak{F}_j(X, E_j) \ , \quad j = 1, 2, 3 \ ,$$

where $(\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3) = \big\{ (\mathcal{E}, \mathcal{E}, \mathcal{D}'), (\mathcal{O}_M, \mathcal{O}_M, \mathcal{S}') \big\}$. □

Suppose that $\mathfrak{F} \in \{\mathcal{D}, \mathcal{E}, \mathcal{S}\}$. Then it is reasonable to expect that there is a 'duality pairing'

$$\mathfrak{F}'(X, E') \times \mathfrak{F}(X, E) \to \mathbb{K}$$

that is a natural extension of the $\mathfrak{F}'(X)$-$\mathfrak{F}(X)$-duality pairing. It is a corollary to the following more general theorem that this is true indeed. In this theorem we put $\mathfrak{F}'' := \mathfrak{F}$ for $\mathfrak{F} \in \{\mathcal{D}, \mathcal{E}, \mathcal{S}\}$. This is consistent with $\mathfrak{F}(X)'' = \mathfrak{F}(X)$ as follows from the reflexivity of $\mathfrak{F}(X)$ for $\mathfrak{F} \in \{\mathcal{D}, \mathcal{E}, \mathcal{S}\}$.

Theorem 2.3. *Suppose that* $\mathfrak{F} \in \{\mathcal{D}, \mathcal{E}, \mathcal{S}, \mathcal{D}', \mathcal{E}', \mathcal{S}'\}$. *Then there exists a unique hypocontinuous bilinear map*

$$\mathfrak{F}'(X, E_1) \times \mathfrak{F}(X, E_2) \to E_0 \ , \quad (u', u) \mapsto \langle u' \bullet u \rangle_{\mathfrak{F}} \ ,$$

the **scalar product induced by multiplication** (0.1)*, such that*

$$\big\langle(\varphi\otimes e_1)\bullet(\psi\otimes e_2)\big\rangle_{\mathfrak{F}} = \langle\varphi,\psi\rangle_{\mathcal{D}}(e_1\bullet e_2)$$

for $\varphi,\psi\in\mathcal{D}(X)$ *and* $e_j\in E_j,\ \ j=1,2$.

Remarks 2.4. (a) Suppose that $u\in L_{1,\mathrm{loc}}(X,E_1)$ and $v\in\mathcal{D}(X,E_2)$. Then

$$\langle u\bullet v\rangle_{\mathcal{D}} = \int_X u(x)\bullet v(x)\,dx\ .$$

(b) Parseval's formula is valid, that is,

$$\langle u\bullet\varphi\rangle_{\mathcal{S}} = (2\pi)^{-n}\langle\widehat{u}\bullet\check{\widehat{\varphi}}\rangle_{\mathcal{S}} = \langle\mathcal{F}u,\mathcal{F}^{-1}u\rangle_{\mathcal{S}}$$

for $u\in\mathcal{S}'(\mathbb{R}^n,E_1)$ and $\varphi\in\mathcal{S}(\mathbb{R}^n,E_2)$. □

Corollary 2.5. *Let* $\mathfrak{F}\in\{\mathcal{D},\mathcal{E},\mathcal{S}\}$*. Then there exists a unique hypocontinuous bilinear map*

$$\mathfrak{F}'(X,E')\times\mathfrak{F}(X,E)\to\mathbb{K}\ ,\quad (u',u)\mapsto\langle u',u\rangle_{\mathfrak{F}(X,E)}\ ,$$

the **duality pairing between** $\mathfrak{F}'(X,E')$ **and** $\mathfrak{F}(X,E)$*, such that*

$$\langle u',u\rangle_{\mathfrak{F}(X,E)} = \int_X \big\langle u'(x),u(x)\big\rangle_E\,dx$$

for $u'\in\mathcal{D}(X,E')$ *and* $u\in\mathcal{D}(X,E)$.

Proof. Let $E_1:=E',\ E_2:=E$, and $E_0:=\mathbb{K}$, and put $e_1\bullet e_2:=\langle e_1,e_2\rangle_E$. Then the assertion follows from Theorem 2.3 and Remark 2.4(a). □

Remark 2.6. It is also true that

$$\langle u',u\rangle_{\mathcal{D}(X,E)} = \int_X\big\langle u'(x),u(x)\big\rangle_E\,dx\ ,\qquad (u',u)\in L_{1,\mathrm{loc}}(X,E')\times\mathcal{D}(X,E)\ ,$$

and

$$\langle u',u\rangle_{\mathcal{S}(\mathbb{R}^n,E)} = \int_{\mathbb{R}^n}\big\langle u'(x),u(x)\big\rangle_E\,dx\ ,\qquad (u',u)\in L_p(\mathbb{R}^n,E')\times\mathcal{S}(\mathbb{R}^n,E)\ ,$$

where $1\le p<\infty$. □

3. Convolutions

For $u_j\in\mathcal{D}(\mathbb{R}^n,E_j),\ \ j=1,2$, we can define the convolution induced by multiplication (0.1) by

$$\text{(3.1)}\qquad u_1 *_\bullet u_2(x) := \int_{\mathbb{R}^n} u_1(x-y)\bullet u_2(y)\,dy = \int_{\mathbb{R}^n} u_1(y)\bullet u_2(x-y)\,dy$$

for $x\in\mathbb{R}^n$.

The following theorem extends this definition to vector-valued distributions.

Theorem 3.1. *Suppose that either* $u_1 \in \mathcal{D}'(\mathbb{R}^n, E_1)$ *and* $u_2 \in \mathcal{E}'(\mathbb{R}^n, E_2)$, *or that* $u_1 \in \mathcal{S}(\mathbb{R}^n, E_1)$ *and* $u_2 \in \mathcal{S}'(\mathbb{R}^n, E_2)$. *Then there is a unique distribution in* $\mathcal{D}'(\mathbb{R}^n, E_0)$ *or in* $\mathcal{O}_M(\mathbb{R}^n, E_0)$, *respectively, the* **convolution of** u_1 **and** u_2 **with respect to multiplication** (0.1), *denoted by* $u_1 *_\bullet u_2$, *such that*

$$(\varphi_1 \otimes e_1) *_\bullet (\varphi_2 \otimes e_2) = (\varphi_1 * \varphi_2) \otimes (e_1 \bullet e_2) \ , \qquad \varphi_1, \varphi_2 \in \mathcal{D}(\mathbb{R}^n) \ , \quad e_j \in E_j \ ,$$

for $j = 1, 2$, *and such that the 'convolution maps'* $(u_1, u_2) \mapsto u_1 *_\bullet u_2$ *are bilinear and hypocontinuous*:

$$\mathcal{D}'(\mathbb{R}^n, E_1) \times \mathcal{E}'(\mathbb{R}^n, E_2) \to \mathcal{D}'(\mathbb{R}^n, E_0)$$

and

$$\mathcal{S}(\mathbb{R}^n, E_1) \times \mathcal{S}'(\mathbb{R}^n, E_2) \to \mathcal{O}_M(\mathbb{R}^n, E_0) \ ,$$

respectively.

Similarly as in Remark 2.2.(a) we see that the rôles of E_1 and E_2 can be interchanged. This is often used without further mention.

Suppose that $m \geq 2$. Then the distributions $u_j \in \mathcal{D}'(\mathbb{R}^n, E_j)$, $1 \leq j \leq m$, are said to satisfy condition (Σ) if the map

$$\prod_{j=1}^{m} \operatorname{supp}(u_j) \to \mathbb{R}^n \ , \quad (x_1, \ldots, x_m) \mapsto x_1 + \cdots + x_m$$

is proper, that is, preimages of compact sets are compact. This is the case, for example, if $m - 1$ of them have compact supports, or if there exists a proper closed convex cone Γ of $\mathbb{R}^n$ containing $\operatorname{supp}(u_j)$ for $1 \leq j \leq m$.

Let $u_j \in \mathcal{D}'(\mathbb{R}^n, E_j)$, $j = 1, 2$, satisfy condition (Σ). Then for each bounded open subset X of $\mathbb{R}^n$ there exists $\rho > 0$ such that

$$\bigl(x_j \in \operatorname{supp}(u_j),\ x_1 + x_2 \in X\bigr) \Rightarrow \bigl(|x_j| \leq \rho,\ j = 1, 2\bigr) \ .$$

Fix $\psi_j \in \mathcal{D}(\mathbb{R}^n)$, $j = 1, 2$, with $\psi_j \,|\, (\rho\mathbb{B}^n) = 1$, where $\mathbb{B}^n$ is the open unit-ball in $\mathbb{R}^n$. Then $\psi_j u_j \in \mathcal{E}'(\mathbb{R}^n, E_j)$ so that $(\psi_1 u_1) *_\bullet (\psi_2 u_2)$ is well-defined. It can be shown that there exists a unique $u_0 \in \mathcal{D}'(\mathbb{R}^n, E_0)$ that is independent of X and of the choice of ψ_j, such that $u_0 \,|\, X = (\psi_1 u_1) *_\bullet (\psi_2 u_2) \,|\, X$. Moreover, u_0 coincides with $u_1 *_\bullet u_2$ whenever that convolution is well-defined. For this reason we put also in this case $u_1 *_\bullet u_2 := u_0$ and call $u_1 *_\bullet u_2$ again **convolution of** u_1 **and** u_2 **with respect to multiplication** (0.1).

Remarks 3.2. Unless explicit restrictions are given, we suppose that either u_j belong to $\mathcal{D}'(\mathbb{R}^n, E_j)$ for $j = 1, 2$ and satisfy condition (Σ), or that $u_1 \in \mathcal{S}'(\mathbb{R}^n, E_1)$ and $u_2 \in \mathcal{S}(\mathbb{R}^n, E_2)$.

(a) $u_1 *_\bullet u_2(\varphi) = u_1 * (u_2 * \check{\varphi})(0) = \bigl\langle u_1 \bullet (\check{u}_2 * \varphi) \bigr\rangle_{\mathcal{D}}$ for $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

(b) If $u_j \in L_{1,\mathrm{loc}}(\mathbb{R}^n, E_j)$ satisfy condition (Σ) then $u_1 *_\bullet u_2 \in L_{1,\mathrm{loc}}(\mathbb{R}^n, E_0)$ and $u_1 *_\bullet u_2(x)$ is given by (3.1).

(c) *Associativity* Let the associativity hypotheses of Remark 2.2(e) be satisfied and suppose that either $u_j \in \mathcal{D}'(\mathbb{R}^n, E_j)$, $j = 1,2,3$, satisfy condition (Σ), or $u_1 \in \mathcal{S}'(\mathbb{R}^n, E_1)$ and $u_k \in \mathcal{S}(\mathbb{R}^n, E_k)$, $k = 2,3$. Then

$$u_1 *_\bullet (u_2 *_\bullet u_3) = (u_1 *_\bullet u_2) *_\bullet u_3 \ .$$

(d) *Commutativity* Suppose that $E_1 = E_2 =: E$ and multiplication (0.1) is symmetric. Then $u_1 *_\bullet u_2 = u_2 *_\bullet u_1$.

(e) *Support Theorem* Suppose that $u_j \in \mathcal{D}'(\mathbb{R}^n, E_j)$, $j = 1,2$, satisfy condition (Σ). Then $\mathrm{supp}(u_1 *_\bullet u_2) \subset \mathrm{supp}(u_1) + \mathrm{supp}(u_2)$.

(f) $\partial^{\alpha+\beta}(u_1 *_\bullet u_2) = \partial^\alpha u_1 *_\bullet \partial^\beta u_2$, $\alpha, \beta \in \mathbb{N}^n$.

(g) Let τ_a denote translation by $a \in \mathbb{R}^n$ (defined as usually). Then

$$\tau_a(u_1 *_\bullet u_2) = (\tau_a u_1) *_\bullet u_2 = u_1 *_\bullet \tau_a u_2 \ , \qquad a \in \mathbb{R}^n \ ,$$

and $(u_1 *_\bullet u_2)\check{\ } = \check{u}_1 *_\bullet \check{u}_2$.

(h) Suppose that $a \in E_1$ and $u \in \mathcal{D}'(\mathbb{R}^n, E_2)$. Then

$$\partial^\alpha \big[(\delta \otimes a) *_\bullet u\big] = (\partial^\alpha \delta \otimes a) *_\bullet u = a \bullet \partial^\alpha u = \partial^\alpha (a \bullet u)$$

for $\alpha \in \mathbb{N}^n$, where δ is the Dirac distribution. □

Given a nonempty subset K of $\mathbb{R}^n$, we put

$$\mathcal{D}'_K(\mathbb{R}^n, E) := \big\{\, u \in \mathcal{D}'(\mathbb{R}^n, E) \ ; \ \mathrm{supp}(u) \subset \overline{K} \,\big\}$$

and $\mathcal{D}'_+(E) := \mathcal{D}'_{\mathbb{R}^+}(\mathbb{R}, E)$. We observe that $\mathcal{D}'_K(\mathbb{R}^n, E)$ is a closed linear subspace of $\mathcal{D}'(\mathbb{R}^n, E)$.

Example 3.3. Suppose that $u_j \in \mathcal{D}'_+(E_j)$ are regular distributions. Then the convolution $u_1 *_\bullet u_2$ belongs to $\mathcal{D}'_+(E_0)$ and is a regular distribution as well, and

$$u_1 *_\bullet u_2(t) = \int_0^t u_1(t-s) \bullet u_2(s)\, ds \ , \qquad \text{a.a. } t \in \mathbb{R}^+ \ .$$

Proof. This is an easy consequence of Remarks 3.2(b) and (e). □

In the following theorem we collect some of the properties of convolutions in a particularly important setting.

Theorem 3.4. *Suppose that $\mathfrak{F} \in \{\mathcal{D}, \mathcal{S}, \mathcal{E}', \mathcal{D}'_\Gamma\}$, where Γ is a proper closed convex cone in $\mathbb{R}^n$. Then convolution is a well-defined hypocontinuous bilinear map*

$$\mathfrak{F}(\mathbb{R}^n, E_1) \times \mathfrak{F}(\mathbb{R}^n, E_2) \to \mathfrak{F}(\mathbb{R}^n, E_0) \ , \quad (u_1, u_2) \mapsto u_1 *_\bullet u_2$$

possessing the associativity and commutativity properties of Remarks 3.2(c) *and* (d), *respectively. If $(E, \bullet)$ is a [commutative] Banach algebra then $\big(\mathfrak{F}(\mathbb{R}^n, E), *_\bullet\big)$ is also a [commutative] algebra, a* **convolution algebra**. *If $(E, \bullet)$ has a unit e_0 and $\mathfrak{F} \in \{\mathcal{E}', \mathcal{D}'_\Gamma\}$, then $\big(\mathfrak{F}(\mathbb{R}^n, E), *_\bullet\big)$ has a unit as well, namely $\delta \otimes e_0$.*

The next theorem is of particular importance for the remainder of this paper. It guarantees the existence of convolutions if support restrictions are replaced by suitable integrability or boundedness conditions. Its proof does not rely on any deep theory of vector-valued distributions but is literally the same as in the scalar case.

Theorem 3.5. *Suppose that* $(\mathfrak{F}_1, \mathfrak{F}_2; \mathfrak{F}_0)$ *is any one of the triplets*

$$(BUC, L_1; BUC), (C_0, L_1; C_0), (L_p, L_1; L_p), (L_\infty, L_1; BUC), (L_q, L_{q'}; C_0) \ ,$$

where $1 \le p < \infty$ *and* $1 < q < \infty$. *Then convolution with respect to multiplication* (0.1) *extends from* $\mathfrak{F}_1(\mathbb{R}^n, E_1) \times \mathcal{D}(\mathbb{R}^n, E_2)$ *to a multiplication*

$$\mathfrak{F}_1(\mathbb{R}^n, E_1) \times \mathfrak{F}_2(\mathbb{R}^n, E_2) \to \mathfrak{F}_0(\mathbb{R}^n, E_0) \ .$$

It is given by (3.1).

Assuming the hypotheses of Theorem 3.5 we see that the estimate

$$\|u_1 *_\bullet u_2\|_{\mathfrak{F}_0(\mathbb{R}^n, E_0)} \le \|u_1\|_{\mathfrak{F}_1(\mathbb{R}^n, E_1)} \, \|u_2\|_{\mathfrak{F}_2(\mathbb{R}^n, E_2)}$$

is valid for $u_j \in \mathfrak{F}_j(\mathbb{R}^n, E_j)$, $\ j = 1, 2$. This is **Young's inequality** for convolutions (in the vector-valued setting).

Lastly, it is most important that the convolution theorem carries over to the vector-valued situation.

Theorem 3.6. *If* $u_1 \in \mathcal{S}'(\mathbb{R}^n, E_1)$ *and* $u_2 \in \mathcal{S}(\mathbb{R}^n, E_2)$ *then* $(u_1 *_\bullet u_2)\hat{} = \widehat{u}_1 \bullet \widehat{u}_2$.

Remark 3.7. Given $a \in \mathcal{O}_M(\mathbb{R}^n, E_1)$, we put

$$a(D)u := \mathcal{F}^{-1} a_\bullet \mathcal{F} u := \mathcal{F}^{-1}(a \bullet \widehat{u}) \ , \qquad u \in \mathcal{S}'(\mathbb{R}^n, E_2) \ .$$

Then (1.3) and Theorem 2.1 guarantee that

$$a(D) \in \mathcal{L}\big(\mathcal{S}(\mathbb{R}^n, E_2), \mathcal{S}(\mathbb{R}^n, E_0)\big) \cap \mathcal{L}\big(\mathcal{S}'(\mathbb{R}^n, E_2), \mathcal{S}'(\mathbb{R}^n, E_0)\big) \ . \tag{3.2}$$

Moreover, $a(D)u = \mathcal{F}^{-1}(a) *_\bullet u$ for $u \in \mathcal{S}(\mathbb{R}^n, E_2)$, thanks to Theorem 3.6. In the following, we put

$$a(D)u := \mathcal{F}^{-1} a_\bullet \mathcal{F} u := \mathcal{F}^{-1}(a) *_\bullet u$$

whenever $a \in \mathcal{S}'(\mathbb{R}^n, E_1)$ and $u \in \mathcal{S}'(\mathbb{R}^n, E_2)$ are such that the convolution product on the right-hand side is well-defined. Then $a(D)$ is called **translation-invariant (pseudodifferential) operator** with **symbol** a (related to multiplication (0.1)). □

Throughout the remainder of this paper we simply write au and $a * u$ for $a \bullet u$ and $a *_\bullet u$, since it will always be clear from the context which particular multiplication we are using in a given formula.

4. Fourier Multipliers

Suppose that $\mathfrak{F} \in \{ BUC, C_0, L_p \ ; \ 1 \leq p < \infty \}$. Then $\mathfrak{F}(\mathbb{R}^n, E) \hookrightarrow \mathcal{S}'(\mathbb{R}^n, E)$ and we can ask for conditions on $a \in \mathcal{S}'(\mathbb{R}^n, E_1)$ such that $a(D)$ is well-defined and belongs to $\mathcal{L}\big(\mathfrak{F}(\mathbb{R}^n, E_2), \mathfrak{F}(\mathbb{R}^n, E_0)\big)$. In this case a is said to be a **Fourier multiplier for** $\mathfrak{F}$. On the basis of Theorem 3.5 we can give an easy criterion for this to happen.

We put

$$\mathcal{F}L_1(\mathbb{R}^n, E) := \Big(\{ u \in \mathcal{S}'(\mathbb{R}^n, E) \ ; \ \mathcal{F}^{-1}u \in L_1(\mathbb{R}^n, E) \}, \ \|\cdot\|_{\mathcal{F}L_1} \Big) \ ,$$

where

$$\|u\|_{\mathcal{F}L_1} := \|\mathcal{F}^{-1}u\|_1 \ . \tag{4.1}$$

It is clear that $\mathcal{F}L_1(\mathbb{R}^n, E)$ is a Banach space, and the Riemann-Lebesgue lemma, the density of $\mathcal{S}$ in L_1, and (1.3) imply that

$$\mathcal{S}(\mathbb{R}^n, E) \overset{d}{\hookrightarrow} \mathcal{F}L_1(\mathbb{R}^n, E) \overset{d}{\hookrightarrow} C_0(\mathbb{R}^n, E) \ . \tag{4.2}$$

Moreover, if E is a Hilbert space and $k > n/2$ then

$$W_2^k(\mathbb{R}^n, E) \overset{d}{\hookrightarrow} \mathcal{F}L_1(\mathbb{R}^n, E) \ . \tag{4.3}$$

In fact, the standard scalar proof carries over to the vector-valued situation — thanks to the validity of Plancherel's theorem in $L_2(\mathbb{R}^n, E)$, if E is a Hilbert space — to give the estimate

$$\|u\|_{\mathcal{F}L_1} \leq c\,\|u\|_2^{1-\theta} \Big(\max_{|\alpha|=k} \|\partial^\alpha u\|_2 \Big)^\theta \leq c\,\|u\|_{2,k} \ , \qquad u \in W_2^k(\mathbb{R}^n, E) \ ,$$

where $\theta := n/(2k)$ (cf. [BL76, Lemma 6.5.1]).

The reason for introducing the space $\mathcal{F}L_1$ is the following simple multiplier result.

Theorem 4.1. *Suppose that* $\mathfrak{F} \in \{ BUC, C_0, L_p \ ; \ 1 \leq p < \infty \}$. *Then*

$$\big[a \mapsto a(D)\big] \in \mathcal{L}\Big(\mathcal{F}L_1(\mathbb{R}^n, E_1), \mathcal{L}\big(\mathfrak{F}(\mathbb{R}^n, E_2), \mathfrak{F}(\mathbb{R}^n, E_0)\big)\Big) \ .$$

Moreover,

$$\big[a \mapsto a(D)\big] \in \mathcal{L}\Big(\mathcal{F}L_1(\mathbb{R}^n, E_1), \mathcal{L}\big(L_\infty(\mathbb{R}^n, E_2), BUC(\mathbb{R}^n, E_0)\big)\Big) \ .$$

The norms of these linear maps are bounded by 1.

Proof. This is an immediate consequence of Theorem 3.5 and Remark 3.7. □

In the remainder of this section we establish sufficient conditions for a distribution to belong to $\mathcal{F}L_1$. For this we fix a radial function ψ satisfying the following conditions:

$$\psi \in \mathcal{D}(\mathbb{R}^n) \ , \quad \psi \,|\, \mathbb{B}^n = 1 \ , \quad \operatorname{supp}(\psi) \subset 2\mathbb{B}^n =: \Omega_0 \ . \tag{4.4}$$

Then we put $\widetilde{\psi} := \psi - \sigma_2\psi = \psi - \psi(2\,\cdot\,)$ and

$$\psi_0 := \psi \ , \quad \psi_k := \sigma_{2^{-k}}\widetilde{\psi} = \psi(2^{-k}\,\cdot\,) - \psi(2^{-k+1}\,\cdot\,) \ , \qquad k \in \dot{\mathbb{N}} \ , \tag{4.5}$$

and

$$\eta_j := \sigma_{2^{-j}}\widetilde{\psi} = \psi(2^{-j}\,\cdot\,) - \psi(2^{-j+1}\,\cdot\,) \ , \qquad j \in \mathbb{Z} \ . \tag{4.6}$$

We also let

$$\Omega_k := [2^{k-1} \le |\xi| \le 2^{k+1}] \ , \qquad k \in \dot{\mathbb{N}} \ , \tag{4.7}$$

and

$$\Sigma_j := [2^{j-1} \le |\xi| \le 2^{j+1}] \ , \qquad j \in \mathbb{Z} \ . \tag{4.8}$$

Then it follows that

$$\psi_k, \eta_j \in \mathcal{D}(\mathbb{R}^n) \ , \quad \operatorname{supp}(\psi_k) \subset \Omega_k \ , \quad \operatorname{supp}(\eta_j) \subset \Sigma_j \tag{4.9}$$

for $k \in \mathbb{N}$ and $j \in \mathbb{Z}$. Moreover,

$$\sum_{k=0}^{m} \psi_k(\xi) = \psi(2^{-m}\xi) \ , \qquad \xi \in \mathbb{R}^n \ , \quad m \in \mathbb{N} \ , \tag{4.10}$$

and

$$\sum_{j=-m'}^{m} \eta_j(\xi) = \psi(2^{-m}\xi) - \psi(2^{m'+1}\xi) \ , \qquad \xi \in \mathbb{R}^n \ , \quad m, m' \in \mathbb{N} \ . \tag{4.11}$$

Consequently,

$$\sum_{k=0}^{\infty} \psi_k(\xi) = 1 \ , \qquad \xi \in \mathbb{R}^n \ , \tag{4.12}$$

and

$$\sum_{j=-\infty}^{\infty} \eta_j(\xi) = 1 \ , \qquad \xi \in (\mathbb{R}^n)^{\bullet} \ , \tag{4.13}$$

where for each $\xi \in (\mathbb{R}^n)^{\bullet}$ at most two terms in the above series are different from zero. Thus $(\psi_k) := (\psi_k)_{k\in\mathbb{N}}$ and $(\eta_j) := (\eta_j)_{j\in\mathbb{Z}}$ are resolutions of the identity on $\mathbb{R}^n$ and on $(\mathbb{R}^n)^{\bullet}$, respectively, the **dyadic resolutions of the identity induced by** ψ.

Letting $\psi_{-1} := 0$, put

$$\chi_k := \psi_{k-1} + \psi_k + \psi_{k+1} \ , \qquad k \in \mathbb{N} \ . \tag{4.14}$$

Then it is an easy consequence of (1.6) and (4.4)–(4.9) that

$$\psi_k = \psi_k \chi_k \ , \qquad k \in \mathbb{N} \ , \tag{4.15}$$

and

$$2^{k|\alpha|}\, |\partial^\alpha \psi_k| \le c(\alpha, \psi)\chi_{\Omega_k} \ , \qquad k \in \mathbb{N} \ , \quad \alpha \in \mathbb{N}^n \ , \tag{4.16}$$

where χ_X is the characteristic function of the set X.

Given any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}^n$,

$$\|\varphi \partial^\alpha (\sigma_{2^{-m}} \psi - \mathbf{1})\|_\infty \to 0 \qquad \text{as } m \to \infty \ ,$$

where $\mathbf{1} := \chi_{\mathbb{R}^n}$. Consequently, $\sigma_{2^{-m}} \psi \to \mathbf{1}$ in $\mathcal{O}_M(\mathbb{R}^n)$ as $m \to \infty$, so that we infer from (4.10) that

$$\sum_{k=0}^\infty \psi_k = \mathbf{1} \qquad \text{in } \mathcal{O}_M(\mathbb{R}^n) \ .$$

Thus, given $u \in \mathcal{S}'(\mathbb{R}^n, E)$, it follows from Theorem 2.1 that $\sum_{k=0}^\infty \psi_k \widehat{u} = \widehat{u}$ in the topology of $\mathcal{S}'(\mathbb{R}^n, E)$. Hence we obtain from (1.3) that

$$\sum_{k=0}^\infty \psi_k(D) u = u \quad \text{in } \mathcal{S}'(\mathbb{R}^n, E) \ , \qquad u \in \mathcal{S}'(\mathbb{R}^n, E) \ . \tag{4.17}$$

In the following, we denote by $|\xi|^{|\alpha|} u$ the function $\xi \mapsto |\xi|^\alpha u(\xi)$, that is, $|\xi|^{|\alpha|}$ is interpreted as a multiplication operator, without fearing confusion.

Lemma 4.2. *Suppose that $a \in \mathcal{D}'\big((\mathbb{R}^n)^\bullet, E\big)$ and all its derivatives of order less than or equal to $n+1$ are regular distributions.*

(i) *Given $j \in \mathbb{Z}$, suppose that*

$$\mu_j := \max_{|\alpha| \le n+1} \big\| \, |\xi|^{|\alpha|} \, \partial^\alpha a \big\|_{\infty, \Sigma_j} < \infty \ .$$

Then $\eta_j a \in \mathcal{F}L_1(\mathbb{R}^n, E)$ and

$$\|\eta_j a\|_{\mathcal{F}L_1} \le c \mu_j \ ,$$

where $c = c(n, \psi)$ is independent of a and j.

(ii) *If $a \in W_\infty^{n+1}(\Omega_0, E)$, then $\psi a \in \mathcal{F}L_1(\mathbb{R}^n, E)$ and*

$$\|\psi a\|_{\mathcal{F}L_1} \le c \, \|a\|_{n+1, \infty, \Omega_0} \ ,$$

where c is independent of a.

Proof. (i) Since $\eta_j a = (\sigma_{2^{-j}} \widetilde{\psi}) a = \sigma_{2^{-j}} (\widetilde{\psi} \sigma_{2^j} a)$, it follows from (1.6) that

$$\mathcal{F}^{-1}(\eta_j a) = 2^{jn} \sigma_{2^j} \mathcal{F}^{-1}(\widetilde{\psi} \sigma_{2^j} a) \ .$$

Now we obtain from (1.3) and (4.1) that

$$\|\eta_j a\|_{\mathcal{F}L_1} = \|\widetilde{\psi} \sigma_{2^j} a\|_{\mathcal{F}L_1} \ . \tag{4.18}$$

Leibniz' rule and (1.6) imply

$$\partial^\alpha (\widetilde{\psi} \sigma_{2^j} a) = \sum_{\beta \le \alpha} \tbinom{\alpha}{\beta} 2^{j|\beta|} (\sigma_{2^j} \partial^\beta a) \partial^{\alpha - \beta} \widetilde{\psi} \ .$$

Thus, since $\operatorname{supp}(\partial^{\alpha-\beta}\widetilde{\psi}) \subset \Sigma_0$, it follows that

$$|\partial^\alpha(\widetilde{\psi}\sigma_{2^j}a)(\xi)| \le c \sum_{\beta\le\alpha} 2^{j|\beta|}\,|\partial^\beta a(2^j\xi)|\,\chi_{\Sigma_0}(\xi) \le c\mu_j\chi_{\Sigma_0}(\xi)$$

for $|\alpha| \le n+1$ and a.a. $\xi \in \mathbb{R}^n$. Hence $\partial^\alpha(\widetilde{\psi}\sigma_{2^j}a) \in L_1(\mathbb{R}^n, E)$ and

$$\|\partial^\alpha(\widetilde{\psi}\sigma_{2^j}a)\|_1 \le c\mu_j\ , \qquad |\alpha| \le n+1\ .$$

Consequently, by (1.4), (1.5), and the Riemann-Lebesgue lemma,

$$x^\alpha \mathcal{F}^{-1}(\widetilde{\psi}\sigma_{2^j}a) = \mathcal{F}^{-1}\big(D^\alpha(\widetilde{\psi}\sigma_{2^j}a)\big) \in C_0(\mathbb{R}^n, E)$$

and

$$|x^\alpha \mathcal{F}^{-1}(\widetilde{\psi}\sigma_{2^j}a)(x)| \le c\mu_j\ , \qquad x \in \mathbb{R}^n\ , \quad |\alpha| \le n+1\ . \tag{4.19}$$

Since, by the multinomial theorem,

$$|x|^k = \big((x_1^2+\cdots+x_n^2)^k\big)^{1/2} \le c(k,n)\sum_{|\alpha|=k}|x^\alpha|\ , \qquad k \in \mathbb{N}\ , \tag{4.20}$$

it follows from (4.19) that

$$|\mathcal{F}^{-1}(\widetilde{\psi}\sigma_{2^j}a)(x)| \le c\mu_j\,|x|^{-n-1}\ , \qquad x \in (\mathbb{R}^n)^\bullet\ . \tag{4.21}$$

Hence, thanks to (4.19) and (4.21),

$$\|\widetilde{\psi}\sigma_{2^j}a\|_{\mathcal{F}L_1} = \|\mathcal{F}^{-1}(\widetilde{\psi}\sigma_{2^j}a)\|_1 \le c\mu_j\Big[\int_{|x|\le 1} 1\,dx + \int_{|x|\ge 1}|x|^{-n-1}\,dx\Big] = c\mu_j\ .$$

Now the assertion is a consequence of (4.18).

(ii) From Leibniz' rule we infer that

$$|\partial^\alpha(\psi a)| \le c\sum_{\beta\le\alpha}|\partial^\beta a|\,\chi_{\Omega_0} \le c\,\|a\|_{n+1,\infty,\Omega_0}\,\chi_{\Omega_0}$$

for $|\alpha| \le n+1$. Now the arguments of (i) apply if we replace $\widetilde{\psi}$ by ψ and set $j := 0$ and $\mu_0 := \|a\|_{n+1,\infty,\Omega_0}$. □

Theorem 4.3. *Suppose that $a \in L_\infty(\mathbb{R}^n, E)$ and all derivatives of order at most $n+1$ are regular distributions on $(\mathbb{R}^n)^\bullet$. Put*

$$\mu_j := \max_{|\alpha|\le n+1}\big\|\,|\xi|^{|\alpha|}\,\partial^\alpha a\big\|_{\infty,\Sigma_j}\ , \qquad j \in \mathbb{Z}\ ,$$

and

$$\lambda_0 := \max_{|\alpha|\le n+1}\|\partial^\alpha a\|_{\infty,\Omega_0}\ .$$

Also suppose that

$$\kappa := \Big(\sum_{j=-\infty}^{\infty}\mu_j\Big) \wedge \Big(\lambda_0 + \sum_{j=1}^{\infty}\mu_j\Big) < \infty\ .$$

Then $a \in \mathcal{F}L_1(\mathbb{R}^n, E)$ and $\|a\|_{\mathcal{F}L_1} \le c\kappa$, where c is independent of κ and a.

Proof. Given $\varphi \in \mathcal{S}(\mathbb{R}^n)$, it follows from (4.11) and Lebesgue's theorem that

$$\Big(\sum_{j=-k'}^{k} \eta_j a\Big)(\varphi) = \int_{\mathbb{R}^n} a\big(\psi(2^{-k}\cdot) - \psi(2^{k'+1}\cdot)\big)\varphi\, dx \to a(\varphi)$$

as $k, k' \to \infty$. Thus the Banach-Steinhaus theorem and the fact that $\mathcal{S}(\mathbb{R}^n)$ is a Montel space imply

$$\sum_{j=-\infty}^{\infty} \eta_j a = a \qquad \text{in } \mathcal{S}'(\mathbb{R}^n, E)\ . \tag{4.22}$$

Similarly,

$$\sum_{k=0}^{\infty} \psi_k a = a \qquad \text{in } \mathcal{S}'(\mathbb{R}^n, E)\ . \tag{4.23}$$

Lemma 4.2 gives

$$\sum_{j=-k'}^{k} \|\eta_j a\|_{\mathcal{F}L_1} \le c \sum_{j=-\infty}^{\infty} \mu_j$$

and, thanks to $\psi_j = \eta_j$ for $j \in \mathbb{N}$,

$$\sum_{j=0}^{k} \|\psi_k a\|_{\mathcal{F}L_1} \le \lambda_0 + \sum_{j=0}^{\infty} \mu_j$$

for $k, k' \in \mathbb{N}$. Hence the series on the left-hand sides of (4.22) and (4.23) converge in the Banach space $\mathcal{F}L_1(\mathbb{R}^n, E)$ if the right-hand sides in the corresponding estimates are finite. Now the assertion follows from the continuous injection of $\mathcal{F}L_1(\mathbb{R}^n, E)$ in $\mathcal{S}'(\mathbb{R}^n, E)$ and from (4.22) and (4.23), respectively. □

Corollary 4.4. *Suppose that $a \in W_\infty^{n+1}(\mathbb{R}^n, E)$ and there exists $\varepsilon > 0$ such that*

$$\|a\|_{n+1,\infty} + \max_{|\alpha|\le n+1} \Big\| \,|\xi|^{|\alpha|+\varepsilon}\, \partial^\alpha a \Big\|_\infty \le \mu < \infty\ .$$

Then $a \in \mathcal{F}L_1(\mathbb{R}^n, E)$ and the estimate $\|a\|_{\mathcal{F}L_1} \le c\mu$ holds, where $c := c(\varepsilon, n)$ is independent of μ and a.

Proof. Note that $\lambda_0 \le \|a\|_{n+1,\infty} \le \mu$ and

$$\max_{|\alpha|\le n+1} |\xi|^\alpha\, |\partial^\alpha a(\xi)| \le \mu\, |\xi|^{-\varepsilon}\ , \qquad \text{a.a. } \xi \in (\mathbb{B}^n)^c\ .$$

Hence $\mu_j \le \mu 2^{-(j-1)\varepsilon}$ for $j \in \dot{\mathbb{N}}$ so that $\kappa \le \mu\big[1 + (1 - 2^{-\varepsilon})^{-1}\big]$. □

For $m \in \mathbb{R}$ we say that a belongs to the space $S^m(\mathbb{R}^n, E)$ of E-valued **symbols of degree** m on $\mathbb{R}^n$ if $a \in C^{n+1}\big((\mathbb{R}^n)^\bullet, E\big)$ and there exists a constant c such that

$$|\partial^\alpha a(\xi)| \le c(1 + |\xi|)^{m-|\alpha|}\ , \qquad \xi \in (\mathbb{R}^n)^\bullet\ , \qquad |\alpha| \le n+1\ . \tag{4.24}$$

We also put $\Lambda(\xi) := (1+|\xi|^2)^{1/2}$ for $\xi \in \mathbb{R}^n$ and

$$\|a\|_{S^m} := \max_{|\alpha|\le n+1} \|\Lambda^{|\alpha|-m}\partial^\alpha a\|_\infty$$

for $a \in S^m(\mathbb{R}^n, E)$. Then $\|\cdot\|_{S^m}$ is a norm and $S^m(\mathbb{R}^n, E)$ is a Banach space. These symbol spaces will be important in the following sections. In the moment, we restrict ourselves to the case $m = 0$ and prove the following simple but important multiplier result.

Proposition 4.5. *Suppose that $\psi \in \mathcal{D}(\mathbb{R}^n)$ with $\psi|\mathbb{B}^n = 1$ and $\operatorname{supp}(\psi) \subset 2\mathbb{B}^n$, and let (ψ_k) be the dyadic resolution of the identity on $\mathbb{R}^n$ induced by ψ. Also suppose that*

$$\mathfrak{F} \in \{\, BUC, C_0, L_p \;;\; 1 \le p < \infty \,\} .$$

Then

$$(\psi_k a)(D) \in \mathcal{L}\big(\mathfrak{F}(\mathbb{R}^n, E_2), \mathfrak{F}(\mathbb{R}^n, E_0)\big)$$

and

$$(\psi_k a)(D) \in \mathcal{L}\big(L_\infty(\mathbb{R}^n, E_2), BUC(\mathbb{R}^n, E_0)\big)$$

for $a \in S^0(\mathbb{R}^n, E_1)$, and

$$\sup_{k\in\mathbb{N}} \|(\psi_k a)(D)\| \le c\,\|a\|_{S^0} \ , \qquad a \in S^0(\mathbb{R}^n, E_1) \ ,$$

where c is independent of a.

Proof. If $a \in S^0(\mathbb{R}^n, E)$ then $\|a\|_{n+1,\infty} \le \|a\|_{S^0}$ and $\mu_j \le \|a\|_{S^0}$ for $j \in \mathbb{N}$. Hence the assertion follows from Lemma 4.2 and Theorem 4.1. $\square$

Multiplier theorems of the type of Theorem 4.3 and its corollary are well-known in the scalar case (e.g., [BL76, Exercise 6.8.3] or [Hie91, Lemma 3.3]). In this case the proof is usually based on (4.3) so that the order of differentiation can be reduced from $n+1$ to $[n/2]+1$. This requires, however, the use of Plancherel's theorem which is valid only if E is a Hilbert space. We emphasize that in Corollary 4.4 and Proposition 4.5 we do not have to impose restrictions on the Banach space E.

After having finished this paper we learned that L. Weis announced an analogue to Proposition 4.5, which he obtained in collaboration with M. Jung, provided the resolution $(\psi_k)_{k\in\mathbb{N}}$ of the identity on $\mathbb{R}^n$ is replaced by the resolution $(\eta_j)_{j\in\mathbb{Z}}$ of the identity on $(\mathbb{R}^n)^\bullet$. This implies that these authors can obtain an analogue of Mikhlin's theorem for homogeneous Besov spaces, whereas our result enables us to prove the analogue of Mikhlin's theorem given in Theorem 6.2 that is valid for nonhomogeneous Besov spaces. Note that the latter spaces are invariant under diffeomorphisms which is not true for homogeneous Besov spaces. Finally, it should be mentioned that, by using facts from Banach space geometry, Jung and Weis can reduce in their result the number of derivatives in estimate (4.24), provided E belongs to suitably restricted classes of Banach spaces.

Remark 4.6. Corollary 4.4 is close to being optimal in the sense that we cannot expect to obtain a similar result with $\varepsilon = 0$. Indeed, suppose that a belongs to $L_\infty(\mathbb{R}^n) \cap C^{n+1}\big((\mathbb{R}^n)^\bullet\big)$ and is a Fourier multiplier for $\mathfrak{F}(\mathbb{R}^n)$, where $\mathfrak{F}$ is any one of the symbols BUC, C_0, L_1, or L_∞. If a is positively homogeneous of degree 0 then a is constant. This has been shown by D. GUIDETTI [Gui93, Lemma 1.10] if $\mathfrak{F} = L_1$. The same proof applies to the other cases. □

5. Besov Spaces

Fix any radial $\psi \in \mathcal{D}(\mathbb{R}^n)$ satisfying (4.4) and denote by (ψ_k) the dyadic resolution of the identity on $\mathbb{R}^n$ induced by ψ. Given $s \in \mathbb{R}$ and $p, q \in [1,\infty]$, the **Besov space** $B^s_{p,q}(\mathbb{R}^n, E)$ of E-valued distributions on $\mathbb{R}^n$ is defined to be the vector subspace of $\mathcal{S}'(\mathbb{R}^n, E)$ consisting of all u satisfying

$$\|u\|_{B^s_{p,q}} := \big\|\big(2^{sk}\,\|\psi_k(D)u\|_{L_p(\mathbb{R}^n,E)}\big)_{k\in\mathbb{N}}\big\|_{\ell_q} < \infty \ . \tag{5.1}$$

It is a Banach space with respect to the norm defined by (5.1), and different choices of ψ lead to equivalent norms.

Throughout this section n and E are arbitrarily fixed. Thus, in order to simplify the writing, we usually omit $(\mathbb{R}^n, E)$ in the notation of the spaces of E-valued distributions on $\mathbb{R}^n$, that is, we simply write $\mathcal{S}$ or $B^s_{p,q}$ for $\mathcal{S}(\mathbb{R}^n, E)$ or $B^s_{p,q}(\mathbb{R}^n, E)$, respectively, etc.

If $-\infty < s_0 < s_1 < \infty$ and $p, q_0, q_1 \in [1,\infty]$ then

$$\mathcal{S} \hookrightarrow B^{s_1}_{p,q_1} \hookrightarrow B^{s_0}_{p,q_0} \hookrightarrow \mathcal{S}' \ . \tag{5.2}$$

Moreover,

$$B^s_{p,q_0} \hookrightarrow B^s_{p,q_1} \ , \qquad s \in \mathbb{R} \ , \quad 1 \le q_0 < q_1 \le \infty \ , \tag{5.3}$$

and

$$B^{s_1}_{p_1,q} \hookrightarrow B^{s_0}_{p_0,q} \ , \qquad -\infty < s_0 < s_1 < \infty \ , \quad p_0, p_1, q \in [1,\infty] \ , \tag{5.4}$$

provided $s_1 - n/p_1 = s_0 - n/p_0$.

Put $\Delta_t := \tau_t - 1$ for $t \in \mathbb{R}$. For $p \in [1,\infty)$, $q \in [1,\infty]$, and $\theta \in (0,1)$ set

$$[u]_{\theta,p,q} := \sum_{j=1}^n \big\|t^{-\theta}\,\|\Delta_{te_j}u\|_{L_p(\mathbb{R}^n,E)}\,\big\|_{L_q(\dot{\mathbb{R}}^+,dt/t)}$$

and

$$[u]_{1,p,q} := \sum_{j=1}^n \big\|t^{-1}\,\|\Delta^2_{te_j}u\|_{L_p(\mathbb{R}^n,E)}\,\big\|_{L_q(\dot{\mathbb{R}}^+,dt/t)} \ .$$

Then it can be shown that

$$u \mapsto \Big(\sum_{|\alpha| \le [s]} \|\partial^\alpha u\|_p^p + \sum_{|\alpha|=[s]} [\partial^\alpha u]_{s-[s],p,q}^p \Big)^{1/p} \tag{5.5}$$

is an equivalent norm on $B_{p,q}^s$, provided $s \in \mathbb{R}^+ \backslash \mathbb{N}$. If $s \in \dot{\mathbb{N}}$ then

$$u \mapsto \Big(\sum_{|\alpha| \le s-1} \|\partial^\alpha u\|_p^p + \sum_{|\alpha|=s-1} [\partial^\alpha u]_{1,p,q}^p \Big)^{1/p} \tag{5.6}$$

is an equivalent norm on $B_{p,q}^s$.

Besov spaces are stable under real interpolation. More precisely, if $0 < \theta < 1$ and $-\infty < s_0 < s_1 < \infty$ then

$$(B_{p,q_0}^{s_0}, B_{p,q_1}^{s_1})_{\theta,q} \doteq B_{p,q}^{(1-\theta)s_0+\theta s_1} , \qquad p,q,p_0,q_0 \in [1,\infty] , \tag{5.7}$$

where $(\cdot,\cdot)_{\theta,q}$ is the standard real interpolation functor of exponent θ. They are related to classical function spaces as follows: if $s \in \mathbb{R}^+ \backslash \mathbb{N}$ then

$$B_{p,p}^s \doteq \begin{cases} W_p^s , & 1 \le p < \infty , \\ BUC^s , & p = \infty , \end{cases} \tag{5.8}$$

where $\doteq$ means 'equivalent norms'. However, $B_{p,p}^k \ne W_p^k$ for $1 \le p < \infty$ and $k \in \mathbb{N}$ unless $p = 2$ and E is a Hilbert space, and $B_{\infty,\infty}^k \ne BUC^k$ for $k \in \mathbb{N}$. Moreover, if $0 < s < m$ and $m \in \mathbb{N}$ then

$$B_{p,q}^s \doteq (L_p, W_p^m)_{s/m,q} , \qquad 1 \le p < \infty , \quad 1 \le q \le \infty . \tag{5.9}$$

Lastly, the spaces $B_{p,\infty}^s$ coincide — except for equivalent norms — for $s > 0$ and $1 \le p < \infty$ with the Nikol'skii spaces (cf. [KJF77], for example, for definitions).

The above definition of $B_{p,q}^s$ and the stated properties of these spaces are literally the same as in the classical scalar case, for which we refer to [Tri83]. Vector-valued Besov spaces have been studied by several authors. In particular, Grisvard [Gri66] used (5.9) as defining relation to introduce vector-valued Besov spaces and deduced many of their properties by interpolation techniques. The most common way of introducing vector-valued Besov spaces (of positive order) is to define them via the norms (5.5) and (5.6) (e.g., [Mur74], [Prü93]).

In the scalar case the definition of Besov spaces through (5.1) goes back to Peetre [Pee67]. We refer to [Pee76], [Tri83], and [KJF77] for further historical remarks. It has been shown by Schmeisser [Sch86] that this approach can be carried over to the E-valued case if $1 \le p < \infty$. Given the latter restriction, Schmeisser established most of the properties stated above. For a coherent treatment (including the case $p = \infty$) and for further results we refer to [Ama97, Chapter VII].

We also set

$$\mathring{B}_{p,q}^s := \mathring{B}_{p,q}^s(\mathbb{R}^n, E) := \text{closure of } \mathcal{S} \text{ in } B_{p,q}^s$$

and we define the '**little Besov spaces**' by

$$b^s_{p,q} := b^s_{p,q}(\mathbb{R}^n, E) := \text{closure of } B^{s+1}_{p,q} \text{ in } B^s_{p,q}$$

for $s \in \mathbb{R}$ and $1 \le p,q \le \infty$. It will be shown below that

$$\mathring{B}^s_{p,q} = B^s_{p,q}\ , \qquad s \in \mathbb{R}\ , \quad 1 \le p,q < \infty\ , \tag{5.10}$$

and it follows from (1.3) and (5.8) that

$$\mathring{B}^s_{\infty,\infty} = C^s_0\ , \qquad s \in \mathbb{R}^+ \backslash \mathbb{N}\ . \tag{5.11}$$

Below we also show that

$$b^s_{p,q} = \begin{cases} B^s_{p,q}\ , & 1 \le p,q < \infty\ , \\ \mathring{B}^s_{p,\infty}\ , & 1 \le p < \infty\ , \qquad q = \infty\ , \end{cases} \tag{5.12}$$

for $s \in \mathbb{R}$. We refer to [Ama97, Chapter VII] for a proof of the fact that, given $0 < \theta < 1$ and $-\infty < s_0 < s_1 < \infty$,

$$(B^{s_0}_{p,q_0}, B^{s_1}_{p,q_1})^0_{\theta,\infty} \doteq (b^{s_0}_{p,q_0}, b^{s_1}_{p,q_1})^0_{\theta,\infty} \doteq b^{(1-\theta)s_0+\theta s_1}_{p,\infty} \tag{5.13}$$

for $p, q_0, q_1 \in [1,\infty]$, where $(\cdot,\cdot)^0_{\theta,\infty}$ denotes the 'continuous interpolation functor' of exponent θ introduced by Da Prato and Grisvard in [DG79] (cf. [Ama95, Section I.2] for a brief description of the basic facts from interpolation theory). From this we infer, in particular, that $b^s_{p,\infty}$ coincides for $s > 0$ and $1 < p < \infty$ with the 'little Nikol'skii space' introduced (in the scalar case) in [DG79, Section 6].

In [Ama97, Chapter VII] it is also shown that, given $-\infty < s_0 < s_1 < \infty$,

$$[\mathcal{B}^{s_0}_{p,q}, \mathcal{B}^{s_1}_{p,q}]_\theta \doteq \mathcal{B}^{(1-\theta)s_0+\theta s_1}_{p,q}\ , \qquad p,q \in [1,\infty]\ , \quad 0 < \theta < 1\ , \tag{5.14}$$

for $\mathcal{B} \in \{\mathring{B}, b\}$, where $[\cdot,\cdot]_\theta$ is the standard complex interpolation functor of exponent θ. Moreover,

$$B^k_{p,1} \overset{d}{\hookrightarrow} W^k_p \overset{d}{\hookrightarrow} \mathring{B}^k_{p,\infty}\ , \qquad 1 \le p < \infty\ , \tag{5.15}$$

and

$$B^k_{\infty,1} \overset{d}{\hookrightarrow} BUC^k \overset{d}{\hookrightarrow} b^k_{\infty,\infty} \tag{5.16}$$

as well as

$$\mathring{B}^k_{\infty,1} \overset{d}{\hookrightarrow} C^k_0 \overset{d}{\hookrightarrow} \mathring{B}^k_{\infty,\infty}\ , \tag{5.17}$$

provided $k \in \mathbb{N}$. In addition, if $\mathcal{B} \in \{B, \mathring{B}, b\}$ and $m \in \dot{\mathbb{N}}$ then

$$u \in \mathcal{B}^s_{p,q} \iff \partial^\alpha u \in \mathcal{B}^{s-m}_{p,q}\ ,\ |\alpha| \le m\ , \tag{5.18}$$

and

$$u \mapsto \sum_{|\alpha| \le m} \|\partial^\alpha u\|_{\mathcal{B}^{s-m}_{p,q}} \tag{5.19}$$

is an equivalent norm on $\mathcal{B}^s_{p,q}$ for $s \in \mathbb{R}$ and $1 \le p,q \le \infty$.

Furthermore, it is shown that $E_1 \hookrightarrow E_0$ implies

$$B^s_{p,q}(\mathbb{R}^n, E_1) \hookrightarrow B^s_{p,q}(\mathbb{R}^n, E_0) \tag{5.20}$$

and

$$\mathring{B}^s_{p,q}(\mathbb{R}^n, E_1) \hookrightarrow \mathring{B}^s_{p,q}(\mathbb{R}^n, E_0)\ , \quad b^s_{\infty,q}(\mathbb{R}^n, E_1) \hookrightarrow b^s_{\infty,q}(\mathbb{R}^n, E_0) \tag{5.21}$$

for $s \in \mathbb{R}$ and $p, q \in [1, \infty]$. If E_1 is also dense in E_0 then the injections (5.21) are dense as well.

Lastly, the following duality result is valid: if E is reflexive or E' is separable (more generally: if E' possesses the Radon-Nikodym property) then

$$\big(\mathring{B}^s_{p,q}(\mathbb{R}^n, E)\big)' \doteq B^{-s}_{p',q'}(\mathbb{R}^n, E')\ , \qquad s \in \mathbb{R}\ , \quad p, q \in [1, \infty]\ , \tag{5.22}$$

with respect to the $\mathcal{S}'$-$\mathcal{S}$-duality pairing, that is,

$$\langle u', u\rangle_{\mathring{B}^s_{p,q}(\mathbb{R}^n,E)} = \langle u', u\rangle_{\mathcal{S}(\mathbb{R}^n,E)}\ , \qquad u' \in B^{-s}_{p',q'}(\mathbb{R}^n, E)\ , \quad u \in \mathcal{S}(\mathbb{R}^n, E)\ .$$

In fact, if $p = \infty$ then (5.22) holds without any restriction on E.

It should be remarked that the duality Theorem (5.22) relies heavily on Corollary 2.5. Indeed, without that corollary, Theorem (5.22) cannot even be properly formulated.

In order to prove Fourier multiplier theorems for $\mathcal{B}^s_{p,q}$, where $\mathcal{B} \in \{B, \mathring{B}, b\}$, we need another representation of these spaces. For this we introduce the Banach spaces $\ell^s_q(E)$, $s \in \mathbb{R}$, $1 \le q \le \infty$, to be the subspace of $E^{\mathbb{N}}$ consisting of all $u := (u_k)$ satisfying

$$\|u\|_{\ell^s_q(E)} := \|(2^{sk}u_k)\|_{\ell_q} < \infty\ ,$$

endowed with the norm $\|\cdot\|_{\ell^s_q(E)}$. Furthermore, $c^s_0(E)$ denotes the closed linear subspace of $\ell^s_\infty(E)$ consisting of all $u = (u_k)$ for which $2^{sk}u_k \to 0$ as $k \to \infty$. It is not difficult to see that

$$\ell^s_p(E) \overset{d}{\hookrightarrow} \ell^t_q(E) \overset{d}{\hookrightarrow} \ell^t_r(E) \overset{d}{\hookrightarrow} c^t_0(E) \hookrightarrow \ell^t_\infty(E) \tag{5.23}$$

for $-\infty < t < s < \infty$, $1 \le p \le \infty$, and $1 \le q \le r < \infty$.

Now suppose that $\mathfrak{F} \in \{\, BUC, C_0, L_p\ ;\ 1 \le p < \infty \,\}$ and put

$$\ell^s_q\mathfrak{F} := \ell^s_q\mathfrak{F}(\mathbb{R}^n, E) := \Big\{\, u \in \mathcal{S}'(\mathbb{R}^n, E)\ ;\ \big(\psi_k(D)u\big) \in \ell^s_q\big(\mathfrak{F}(\mathbb{R}^n, E)\big) \,\Big\}\ ,$$

and

$$\|u\|_{\ell^s_q\mathfrak{F}} := \big\|\big(\psi_k(D)u\big)\big\|_{\ell^s_q(\mathfrak{F})}$$

for $s \in \mathbb{R}$ and $q \in [1, \infty]$. We define $c^s_0\mathfrak{F}$ analogously. Note that $\ell^s_q L_p = B^s_{p,q}$.

We endow the sequence space $(\mathcal{S}')^{\mathbb{N}}$ with the topology of point-wise convergence. Then it is a consequence of (3.2) that the map

$$R^c : \mathcal{S}' \to (\mathcal{S}')^{\mathbb{N}} , \quad u \mapsto \bigl(\psi_k(D)u\bigr)$$

is well-defined, linear, and continuous. Note that the diagram

$$\begin{tikzcd} \mathcal{S}' \arrow[r, "R^c"] & (\mathcal{S}')^{\mathbb{N}} \\ \cup & \\ \ell_q^s\mathfrak{F} \arrow[r, "R^c"] & \ell_q^s(\mathfrak{F}) \arrow[uu, hook] \end{tikzcd}$$

is commutative, where $\ell_q^s\mathfrak{F} \subset \mathcal{S}'$ means that $\ell_q^s\mathfrak{F}$ is a vector subspace of $\mathcal{S}'$. Observe that the lower arrow represents an isometry. We also consider the map

$$R : \bigcup_{s,q} \ell_q^s(\mathfrak{F}) \to \mathcal{S}' , \quad (v_k) \mapsto \sum_k \chi_k(D) v_k \ .$$

The next lemma shows that R is a well-defined continuous retraction, that is, it possesses a continuous right inverse, namely R^c.

Lemma 5.1. *The map R is (that is, restricts to) a continuous retraction from $\ell_q^s(\mathfrak{F})$ onto $\ell_q^s\mathfrak{F}$ and from $c_0^s(\mathfrak{F})$ onto $c_0^s\mathfrak{F}$, respectively, and R^c is a coretraction for R.*

Proof. Clearly, R is linear. Put $\psi_k = 0$ for $k < 0$. Since $\psi_k\chi_\ell = 0$ for $|k-\ell| \ge 3$,

$$\psi_k(D)R\boldsymbol{v} = \sum_\ell \psi_k(D)\chi_\ell(D)v_\ell = \sum_{j=-2}^{2} \psi_k(D)\chi_{k+j}(D)v_{k+j}$$

for $\boldsymbol{v} = (v_k) \in \ell_q^s(\mathfrak{F})$. Hence we infer from Proposition 4.5 (putting $a := 1_E$) that

$$\|\psi_k(D)R\boldsymbol{v}\|_{\mathfrak{F}} \le c \sum_{j=-2}^{2} \|v_{k+j}\|_{\mathfrak{F}} \ , \qquad k \in \mathbb{N} \ , \quad \boldsymbol{v} \in \ell_q^s(\mathfrak{F}) \ .$$

This implies

$$R \in \mathcal{L}\bigl(\ell_q^s(\mathfrak{F}), \ell_q^s\mathfrak{F}\bigr) \cap \mathcal{L}\bigl(c_0^s(\mathfrak{F}), c_0^s\mathfrak{F}\bigr) \ .$$

Moreover, thanks to (4.17),

$$RR^c u = \sum_k \chi_k(D)\psi_k(D)u = \sum_k (\psi_k\chi_k)(D)u = \sum_k \psi_k(D)u = u$$

for $u \in \ell_q^s\mathfrak{F} \subset \mathcal{S}'$. Hence R is a retraction and R^c is a coretraction. □

Using this lemma we can show that the choices $\mathfrak{F} = L_\infty$ and $\mathfrak{F} = BUC$ lead to the same spaces.

Lemma 5.2. *$\ell_q^s L_\infty = \ell_q^s BUC$ and $c_0^s L_\infty = c_0^s BUC$.*

Proof. It is clear that $\ell_q^s BUC$ is a closed linear subspace of $\ell_q^s L_\infty$. Given $u \in \ell_q^s L_\infty$, it follows that $\psi_k(D)u \in L_\infty$ for $k \in \mathbb{N}$. Hence $\chi_k(D)u \in L_\infty$ for $k \in \mathbb{N}$ and we infer from $\psi_k \chi_k = \psi_k$ and Proposition 4.5 that

$$\psi_k(D)u = \psi_k(D)\chi_k(D)u \in BUC \ , \qquad k \in \mathbb{N} \ .$$

Consequently, $u \in \ell_q^s BUC$. Now the assertion follows since $c_0^s \mathfrak{F}$ is a closed linear subspace of $\ell_\infty^s \mathfrak{F}$. □

Next we characterize the closure of $\mathcal{S}$ in $B_{p,q}^s$.

Proposition 5.3. *The following identities are valid:*

$$\mathring{B}_{p,q}^s = \begin{cases} B_{p,q}^s \ , & 1 \le p, q < \infty \ , \\ c_0^s L_p \ , & 1 \le p < \infty \ , \quad q = \infty \ , \\ \ell_q^s C_0 \ , & p = \infty \ , \quad 1 \le q < \infty \ , \\ c_0^s C_0 \ , & p = q = \infty \ . \end{cases}$$

Proof. It is clear that the spaces on the right-hand side of the asserted equality are closed linear subspaces of $B_{p,q}^s = \ell_q^s L_p$ for the given ranges of p and q. Hence it suffices to show that $\mathcal{S}$ is dense in these spaces.

Suppose that $u \in \ell_q^s \mathfrak{F}$ if $1 \le q < \infty$, or $u \in c_0^s \mathfrak{F}$ if $q = \infty$, where $\mathfrak{F} := L_p$ if $1 \le p < \infty$, and $\mathfrak{F} := C_0$ if $p = \infty$. Since sequences with compact supports are dense in $\ell_q^s(E)$ for $1 \le q < \infty$, and in $c_0^s(E)$, we can approximate $R^c u$ arbitrarily closely in $\ell_q^s(\mathfrak{F})$ if $q < \infty$, and in $c_0^s(\mathfrak{F})$ if $q = \infty$, by sequences $\boldsymbol{v} \in \ell_q^s(\mathfrak{F})$ if $q < \infty$, and $\boldsymbol{v} \in c_0^s(\mathfrak{F})$ if $q = \infty$, having compact supports. Since $\mathcal{S}$ is dense in $\mathfrak{F}$, we can assume that $\boldsymbol{v} \in \mathcal{S}^{\mathbb{N}}$. Hence we infer from (3.2) that $w := R\boldsymbol{v} \in \mathcal{S}$ and $u - w = R(R^c u - \boldsymbol{v})$. From this and from Lemma 5.1 it follows that $\mathcal{S}$ is dense in $\ell_q^s \mathfrak{F}$ if $q < \infty$, and in $c_0^s \mathfrak{F}$ if $q = \infty$. □

The next proposition characterizes the little Besov space. In addition, it shows that in the definition of $b_{p,q}^s$ we could have replaced $B_{p,q}^{s+1}$ by $B_{p,q}^t$ for any $t > s$.

Proposition 5.4. *The following identities are valid:*

$$b_{p,q}^s = \begin{cases} B_{p,q}^s \ , & 1 \le p \le \infty \ , \quad 1 \le q < \infty \ , \\ \mathring{B}_{p,\infty}^s \ , & 1 \le p < \infty \ , \quad q = \infty \ , \\ c_0^s BUC \ , & p = q = \infty \ . \end{cases}$$

Furthermore,

$$b_{p,q}^s \text{ is the closure of } B_{p,q}^t \text{ in } B_{p,q}^s \tag{5.24}$$

for $-\infty < s < t < \infty$ *and* $p, q \in [1, \infty]$.

Proof. If $p \vee q < \infty$, it follows from $\mathcal{S} \overset{d}{\hookrightarrow} \mathring{B}_{p,q}^s = B_{p,q}^s$ and (5.4) that

$$B_{p,q}^t \overset{d}{\hookrightarrow} B_{p,q}^s \ , \qquad t > s \ . \tag{5.25}$$

This implies $b^s_{p,q} = B^s_{p,q} = \mathring{B}^s_{p,q}$ and (5.24) for $p \vee q < \infty$. Next we infer from (5.23) that

$$B^t_{\infty,q} = \ell^t_q L_\infty \overset{d}{\hookrightarrow} B^s_{\infty,q} \ , \qquad t > s \ , \quad 1 \le q < \infty \ .$$

This proves that $b^s_{\infty,q} = B^s_{\infty,q}$ as well as (5.24) if $1 \le q < \infty$. From (5.23) we deduce that

$$\ell^t_\infty(L_p) \overset{d}{\hookrightarrow} c^s_0(L_p) \ , \qquad t > s \ .$$

Hence we obtain $b^s_{p,\infty} = c^s_0 L_p$ and (5.24) for $1 \le p \le \infty$ and $q = \infty$ from the fact that $c^s_0 L_p$ is a closed linear subspace of $\ell^s_\infty L_p = B^s_{p,\infty}$. Now the assertion follows from Lemma 5.2 and Proposition 5.3. □

Remarks 5.5. (a) It may be worthwhile to point out the following dense injections:

$$\mathcal{S} \overset{d}{\hookrightarrow} B^{s_1}_{p,q_1} \overset{d}{\hookrightarrow} B^{s_0}_{p,q_0} \overset{d}{\hookrightarrow} \mathring{B}^{s_0}_{p,\infty} = b^{s_0}_{p,\infty} \overset{d}{\hookrightarrow} \mathcal{S}' \ , \qquad p < \infty \ , \tag{5.26}$$

if either $s_1 = s_0$ and $1 \le q_1 \le q_0 < \infty$, or $s_1 > s_0$ and $q_0 \vee q_1 < \infty$. Moreover,

$$\mathcal{S} \overset{d}{\hookrightarrow} \mathring{B}^s_{\infty,\infty} \hookrightarrow b^s_{\infty,\infty} \hookrightarrow B^s_{\infty,\infty} \overset{d}{\hookrightarrow} \mathcal{S}' \ , \qquad s \in \mathbb{R} \ . \tag{5.27}$$

Proof. The first injection in (5.26) follows from Proposition 5.3. The second one has already been observed in (5.25), provided $s_1 > s_0$. If $s_1 = s_0$ then it follows from (5.2) and the density of $\mathcal{S}$ in $B^{s_0}_{p,q_0}$. The last argument and Proposition 5.4 imply the third injection. The final embedding in (5.26) is entailed by $b^{s_0}_{p,\infty} \hookrightarrow B^{s_0}_{p,\infty} \hookrightarrow \mathcal{S}'$ and the density of $\mathcal{S}$ in $\mathcal{S}'$. In (5.27) all but the second injection are now clear. Since $\mathring{B}^s_{\infty,\infty} = c^s_0 C_0$ and $b^s_{\infty,\infty} = c^s_0 BUC$ by Propositions 5.3 and 5.4, respectively, the second injection in (5.27) is a consequence of (5.23). □

(b) $b^s_{\infty,\infty} \doteq buc^s$ for $s \in \mathbb{R}^+ \backslash \mathbb{N}$.

Proof. It is known that buc^s is the closure of BUC^k in BUC^s for $k \in \dot{\mathbb{N}}$ and $k - 1 < s < k$ (e.g., [Lun95, Section 0.2] or [Ama97, Chapter VII]). Hence the assertion follows from (5.16) and (5.24). □

From the above injection results we see, in particular, that

$$\mathring{B}^s_{p,q} \overset{d}{\hookrightarrow} \mathring{B}^t_{p,q} \ , \quad b^s_{p,q} \overset{d}{\hookrightarrow} b^t_{p,q} \ , \qquad -\infty < t < s < \infty \ , \quad p, q \in [1, \infty] \ .$$

These density results are the reason for introducing the spaces $\mathring{B}^s_{p,q}$ and $b^s_{p,q}$, since densely injected Banach scales are very useful in applications, for instance in connection with evolution problems.

The spaces $\mathring{B}^s_{p,q}$ are well-known in the scalar case (e.g., [Tri83]). In contrast to this it seems that, so far, little Besov spaces have not been introduced and studied.

6. Multiplier Theorems in Besov Spaces

Throughout this section we fix again $n \in \dot{\mathbb{N}}$ and the Banach space E and omit $(\mathbb{R}^n, E)$ in the notations for the spaces under consideration, if no confusion seems possible. We put

$$J^s := \Lambda^s(D) = (1-\Delta)^{s/2} \ , \qquad s \in \mathbb{R}^n \ .$$

Then we can generalize the 'lifting theorem' for scalar Besov spaces to the vector-valued case, and to the spaces $\mathring{B}^s_{p,q}$ and $b^s_{p,q}$ as well.

Theorem 6.1. *If $\mathcal{B} \in \{B, \mathring{B}, b\}$ then $J^s \in \mathcal{L}\mathrm{is}(\mathcal{B}^{s+t}_{p,q}, \mathcal{B}^t_{p,q})$ and $(J^s)^{-1} = J^{-s}$ for $s,t \in \mathbb{R}$ and $p,q \in [1,\infty]$.*

Proof. By means of Leibniz' rule, it is not difficult to verify that

$$|\partial^\alpha(\Lambda^s \psi_k)| \le c(\alpha,s) \max_{\beta\le\alpha} \bigl(\Lambda^{s-|\beta|}\, |\partial^{\alpha-\beta}\psi_k|\bigr) \ , \qquad \alpha \in \mathbb{N}^n \ .$$

Hence (4.16) implies

$$\Lambda^{|\alpha|}\, |\partial^\alpha(\Lambda^s\psi_k)| \le c(\alpha,s) 2^{ks} \ , \qquad \alpha \in \mathbb{N}^n \ , \quad k \in \mathbb{N} \ .$$

This shows that $\Lambda^s\psi_k \in S^0$ and

$$\|2^{-ks}\Lambda^s\psi_k\|_{S^0} \le c(s) \ , \qquad k \in \mathbb{N} \ .$$

Suppose that $\mathfrak{F} \in \{\, BUC, C_0, L_p \ ; \ 1 \le p < \infty \,\}$. Then Proposition 4.5 implies

$$\|2^{-ks}(\Lambda^s\psi_k\chi_k)(D)\|_{\mathcal{L}(\mathfrak{F})} \le c(s) \ , \qquad k \in \mathbb{N} \ .$$

Hence, using $\psi_k = \psi_k\chi_k^2$,

$$\begin{aligned}\|\psi_k(D)J^s u\|_{\mathfrak{F}} &= 2^{ks}\, \|2^{-ks}(\Lambda^s\psi_k\chi_k)(D)\chi_k(D)u\|_{\mathfrak{F}} \\ &\le c(s)2^{ks}\, \|\chi_k(D)u\|_{\mathfrak{F}} \le c(s) \sum_{j=-1}^{1} 2^{(k+j)s}\, \|\psi_{k+j}(D)u\|_{\mathfrak{F}}\end{aligned}$$

for $k \in \mathbb{N}$, provided $\psi_k(D)u \in \mathfrak{F}$ for $k \in \mathbb{N}$. From this we infer that

$$\|J^s u\|_{\ell^t_q\mathfrak{F}} \le c(s)\, \|u\|_{\ell^{s+t}_q\mathfrak{F}} \ , \qquad u \in \ell^{s+t}_q\mathfrak{F} \ ,$$

and

$$\|J^s u\|_{c^t_0\mathfrak{F}} \le c(s)\, \|u\|_{c^{s+t}_0\mathfrak{F}} \ , \qquad u \in c^{s+t}_0\mathfrak{F} \ .$$

Thus the assertion follows from Lemma 5.2 and Propositions 5.3 and 5.4. □

Now we are in a position to prove the following Fourier-multiplier theorem for vector-valued Besov spaces. Observe that there is no restriction whatsoever on the Banach spaces E_j.

Theorem 6.2. *Suppose that $\mathcal{B} \in \{B, \mathring{B}, b\}$ and $m \in \mathbb{R}$. Then*

$$\bigl(a \mapsto a(D)\bigr) \in \mathcal{L}\Bigl(S^m(\mathbb{R}^n, E_1), \mathcal{L}\bigl(\mathcal{B}^{s+m}_{p,q}(\mathbb{R}^n, E_2), \mathcal{B}^s_{p,q}(\mathbb{R}^n, E_0)\bigr)\Bigr)$$

for $s \in \mathbb{R}$ and $p,q \in [1,\infty]$.

Proof. Note that $\Lambda^{-m}a \in S^0(\mathbb{R}^n, E_1)$ and that $a(D) = J^m(\Lambda^{-m}a)(D)$. Thus we infer from Theorem 6.1 that we can assume $m = 0$.

Suppose that $\mathfrak{F} \in \{\, BUC, C_0, L_p \ ; \ 1 \le p < \infty \,\}$. Then it follows from Proposition 4.5 and $\psi_k a = \psi_k a \chi_k$ that

$$\|\psi_k(D)a(D)u\|_{\mathfrak{F}} \le c\,\|a\|_{S^0}\,\|\chi_k(D)u\|_{\mathfrak{F}}\ , \qquad k \in \mathbb{N}\ ,$$

provided $\chi_k(D)u \in \mathfrak{F}(\mathbb{R}^n, E_2)$ for $k \in \mathbb{N}$. From this we infer that

$$\|a(D)u\|_{\ell_q^0\mathfrak{F}} \le c\,\|a\|_{S^0}\,\|u\|_{\ell_q^0\mathfrak{F}}\ , \qquad u \in \ell_q^0\mathfrak{F}(\mathbb{R}^n, E_2)\ , \tag{6.1}$$

and

$$\|a(D)u\|_{c_0^0\mathfrak{F}} \le c\,\|a\|_{S^0}\,\|u\|_{c_0^0\mathfrak{F}}\ , \qquad u \in c_0^0\mathfrak{F}(\mathbb{R}^n, E_2)\ . \tag{6.2}$$

Thus Lemma 5.2 and (6.1) imply

$$\|a(D)u\|_{B^0_{p,q}} \le c\,\|a\|_{S^0}\,\|u\|_{B^0_{p,q}}\ , \qquad u \in B^0_{p,q}(\mathbb{R}^n, E_2)\ .$$

Since $a(D)$ and J^s commute it follows from Theorem 6.1 that

$$\|a(D)u\|_{B^s_{p,q}} \le c\,\|a\|_{S^0}\,\|u\|_{B^s_{p,q}}\ , \qquad u \in B^s_{p,q}(\mathbb{R}^n, E_2)\ .$$

From Propositions 5.3 and 5.4 and from (6.1) and (6.2) we also obtain that

$$\|a(D)u\|_{\mathcal{B}^0_{p,q}} \le c\,\|a\|_{S^0}\,\|u\|_{\mathcal{B}^0_{p,q}}\ , \qquad u \in \mathcal{B}^0_{p,q}(\mathbb{R}^n, E_2)\ , \quad \mathcal{B} \in \{\overset{\circ}{B}, b\}\ .$$

From this and the fact that J^s is an isomorphism from $\mathcal{B}^s_{p,q}$ onto $\mathcal{B}^0_{p,q}$ we deduce

$$\|a(D)u\|_{\mathcal{B}^s_{p,q}} \le c\,\|a\|_{S^0}\,\|u\|_{\mathcal{B}^s_{p,q}}\ , \qquad u \in \mathcal{B}^s_{p,q}(\mathbb{R}^n, E)\ ,$$

hence the assertion. □

Occasionally, the following much simpler multiplier theorem is also useful.

Theorem 6.3. *Suppose that* $\mathcal{B} \in \{B, \overset{\circ}{B}, b\}$. *Then*

$$\big(a \mapsto a(D)\big) \in \mathcal{L}\Big(\mathcal{F}L_1(\mathbb{R}^n, E_1), \mathcal{L}\big(\mathcal{B}^s_{p,q}(\mathbb{R}^n, E_2), \mathcal{B}^s_{p,q}(\mathbb{R}^n, E_0)\big)\Big)\ ,$$

for $s \in \mathbb{R}$ *and* $p, q \in [1, \infty]$.

Proof. Suppose that $\mathfrak{F} \in \{\, BUC, C_0, L_p \ ; \ 1 \le p < \infty \,\}$. Since $a(D)$ and $\psi_k(D)$ commute it follows from Theorem 4.1 that

$$\|\psi_k(D)a(D)u\|_{\mathfrak{F}} = \|\mathcal{F}^{-1}(a) * \psi_k(D)u\|_{\mathfrak{F}} \le \|a\|_{\mathfrak{F}L_1}\,\|\psi_k(D)u\|_{\mathfrak{F}}$$

for $k \in \mathbb{N}$, provided $\psi_k(D)u \in \mathfrak{F}(\mathbb{R}^n, E_2)$. Now the arguments of the proof of Theorem 6.2 give the assertion. □

Corollary 6.4. *Suppose that* $\mathcal{B} \in \{B, \overset{\circ}{B}, b\}$ *and* $\Lambda^{-m}a \in \mathfrak{F}L_1(\mathbb{R}^n, E_1)$ *for some* $m \in \mathbb{R}$. *Then*

$$a(D) \in \mathcal{L}\big(\mathcal{B}^{s+m}_{p,q}(\mathbb{R}^n, E_2), \mathcal{B}^s_{p,q}(\mathbb{R}^n, E_0)\big)$$

and

$$\|a(D)\|_{\mathcal{L}(\mathcal{B}^{s+m}_{p,q},\mathcal{B}^{s}_{p,q})} \le c\,\|\Lambda^{-m}a\|_{\mathcal{F}L_1}\ ,$$

for $s\in\mathbb{R}$ *and* $p,q\in[1,\infty]$.

Proof. Theorem 6.3 guarantees that $a(D)J^{-m}$ is a bounded linear operator from $\mathcal{B}^s_{p,q}(\mathbb{R}^n,E_2)$ into $\mathcal{B}^s_{p,q}(\mathbb{R}^n,E_0)$, uniformly with respect to all parameters. Now, again, the assertion follows from Theorem 6.1. □

7. Resolvent Estimates and Semigroups

Throughout the remainder of this paper $\mathcal{B}\in\{B,\mathring{B},b\}$, $s\in\mathbb{R}$, and $p,q\in[1,\infty]$ are arbitrarily fixed, unless explicit restrictions are given.

Suppose that $E_i\hookrightarrow E_0$ and $m_i\in\mathbb{R}^+$ for $1\le i\le\ell$. Then $\boldsymbol{E}_1:=\bigcap_{i=1}^{\ell}E_i$ is a well-defined Banach space, and $\boldsymbol{E}_1\hookrightarrow E_i\hookrightarrow E_0$ for $1\le i\le\ell$. For abbreviation we set $m_0:=0$ and

$$\mathcal{B}^{s+m_j}_{p,q}:=\mathcal{B}^{s+m_j}_{p,q}(\mathbb{R}^n,E_j)\ ,\quad 0\le j\le\ell\ ,\qquad \boldsymbol{\mathcal{B}}^{s+\boldsymbol{m}}_{p,q}:=\bigcap_{i=1}^{\ell}\mathcal{B}^{s+m_i}_{p,q}\ .$$

Note that

$$\boldsymbol{\mathcal{B}}^{s+\boldsymbol{m}}_{p,q}\hookrightarrow\mathcal{B}^{s+m_i}_{p,q}\hookrightarrow\mathcal{B}^{s}_{p,q}\ ,\qquad 1\le i\le\ell\ .$$

For $\vartheta\in(0,\pi]$ we let S_ϑ be the closure of the sector $\{\,z\in\mathbb{C}\ ;\ |\arg z|<\vartheta\,\}$ in $\mathbb{C}$, and $S_0:=\{0\}$. We denote by $\rho(A)$ the resolvent set of the linear operator A. If $A\in\mathcal{L}(E_1,E_0)$ and we refer to $\rho(A)$, it is always understood that A is interpreted as a linear operator in E_0 (with domain E_1). Given $\kappa\ge1$ and $\vartheta\in[0,\pi]$, we write $A\in\mathcal{P}(E_1,E_0;\kappa,\vartheta)$ if $A\in\mathcal{L}(E_1,E_0)$ with $S_\vartheta\subset\rho(-A)$ and

$$(1+|\lambda|)^{1-j}\,\|(\lambda+A)^{-1}\|_{\mathcal{L}(E_0,E_j)}\le\kappa\ ,\qquad \lambda\in S_\vartheta\ ,\quad j=0,1\ . \tag{7.1}$$

We also set

$$\mathcal{P}(E_1,E_0;\vartheta):=\bigcup_{\kappa\ge1}\mathcal{P}(E_1,E_0;\kappa,\vartheta)\ ,\quad \mathcal{P}(E_1,E_0):=\bigcup_{0<\vartheta\le\pi}\mathcal{P}(E_1,E_0;\vartheta)\ .$$

Using these notations and conventions we can prove the following general resolvent estimate.

Theorem 7.1. *Suppose that*

$$b_i\in S^{m_i}\big(\mathbb{R}^n,\mathcal{L}(E_i,E_0)\big)\ ,\qquad 1\le i\le\ell\ , \tag{7.2}$$

and put $b:=b_1+\cdots+b_\ell$. *Let there exist* $\kappa\ge1$, $\omega\in\mathbb{R}$, *and* $\vartheta\in[0,\pi]$ *such that* $\omega+S_\vartheta$ *belongs to* $\rho\big(-b(\xi)\big)$ *and*

$$(1+|\xi|)^{m_i}\,\big|\big(\lambda+b(\xi)\big)^{-1}\big|_{\mathcal{L}(E_0,E_i)}+(1+|\lambda|)\,\big|\big(\lambda+b(\xi)\big)^{-1}\big|_{\mathcal{L}(E_0)}\le\kappa \tag{7.3}$$

for $\xi \in (\mathbb{R}^n)^\bullet$, $1 \le i \le \ell$, *and* $\lambda \in \omega + S_\vartheta$. *Then*

$$\omega + b(D) \in \mathcal{P}(\boldsymbol{\mathcal{B}}^{s+\boldsymbol{m}}_{p,q}, \mathcal{B}^s_{p,q}; c, \vartheta) ,$$

where $c := c(\kappa)$ *is independent of* b. *Furthermore,*

$$\big(\lambda + b(D)\big)^{-1} = (\lambda + b)^{-1}(D) , \qquad \lambda \in \omega + S_\vartheta .$$

Proof. Theorem 6.2 implies that $b(D) \in \mathcal{L}(\boldsymbol{\mathcal{B}}^{s+\boldsymbol{m}}_{p,q}, \mathcal{B}^s_{p,q})$. Since the inversion map $B \mapsto B^{-1}$ is analytic, we see that

$$(\lambda + b)^{-1} \in C^{n+1}\big((\mathbb{R}^n)^\bullet, \mathcal{L}(E_0, E_i)\big) , \qquad 1 \le i \le \ell , \quad \lambda \in \omega + S_\vartheta .$$

From $\partial_j(\lambda + b)^{-1} = -(\lambda + b)^{-1}(\partial_j b)(\lambda + b)^{-1}$ we infer by induction that $\partial^\alpha(\lambda + b)^{-1}$ is for each $\alpha \in \mathbb{N}^n$ with $|\alpha| \le n + 1$ a finite sum of terms of the form

$$\pm(\lambda + b)^{-1}(\partial^{\alpha_1} b)(\lambda + b)^{-1} \cdots (\partial^{\alpha_k} b)(\lambda + b)^{-1} \tag{7.4}$$

for $\alpha_j \in \mathbb{N}^n$ with $\alpha_1 + \cdots + \alpha_k = \alpha$. Note that (7.2) and (7.3) imply

$$\Lambda^{|\alpha|}(\xi) \left|\partial^\alpha b_i(\xi)\big(\lambda + b(\xi)\big)^{-1}\right|_{\mathcal{L}(E_0)} \le c(\kappa) , \qquad |\alpha| \le n + 1 , \quad \xi \in (\mathbb{R}^n)^\bullet ,$$

for $1 \le i \le \ell$. Hence it follows from (7.4) that

$$\Lambda^{|\alpha|}(\xi) \, |\partial^\alpha(\lambda + b)^{-1}(\xi)|_{\mathcal{L}(E_0, E_j)} \le c(\kappa) \, |(\lambda + b)^{-1}(\xi)|_{\mathcal{L}(E_0, E_j)} , \qquad |\alpha| \le n + 1 ,$$

for $\xi \in (\mathbb{R}^n)^\bullet$ and $0 \le j \le \ell$. Thus (7.3) implies

$$(\lambda + b)^{-1} \in S^{-m_j}\big(\mathbb{R}^n, \mathcal{L}(E_0, E_j)\big) , \qquad 0 \le j \le \ell ,$$

and

$$\|(\lambda + b)^{-1}\|_{S^{-m_i}(\mathbb{R}^n, \mathcal{L}(E_0, E_i))} + (1 + |\lambda|) \, \|(\lambda + b)^{-1}\|_{S^0(\mathbb{R}^n, \mathcal{L}(E_0))} \le c(\kappa)$$

for $1 \le i \le \ell$ and $\lambda \in \omega + S_\vartheta$. Now Theorem 6.2 guarantees that

$$(\lambda + b)^{-1}(D) \in \mathcal{L}(\mathcal{B}^s_{p,q}, \boldsymbol{\mathcal{B}}^{s+\boldsymbol{m}}_{p,q})$$

and

$$\|(\lambda + b)^{-1}(D)\|_{\mathcal{L}(\mathcal{B}^s_{p,q}, \boldsymbol{\mathcal{B}}^{s+\boldsymbol{m}}_{p,q})} + (1 + |\lambda|) \, \|(\lambda + b)^{-1}(D)\|_{\mathcal{L}(\mathcal{B}^s_{p,q})} \le c(\kappa)$$

for $\lambda \in \omega + S_\vartheta$.

Lastly,

$$\begin{aligned}\big(\lambda + b(D)\big)(\lambda + b)^{-1}(D) &= \lambda(\lambda + b)^{-1}(D) + \sum_{i=1}^{\ell} b_i(D)(\lambda + b)^{-1}(D) \\ &= \mathcal{F}^{-1}\lambda(\lambda + b)^{-1}\mathcal{F} + \sum_{i=1}^{\ell} \mathcal{F}^{-1} b_i(\lambda + b)^{-1}\mathcal{F} \\ &= \mathcal{F}^{-1}(\lambda + b)(\lambda + b)^{-1}\mathcal{F} = 1_{\mathcal{B}^s_{p,q}}\end{aligned}$$

and, similarly,

$$(\lambda + b)^{-1}(D)\big(\lambda + b(D)\big) = 1_{\mathcal{B}^{s+\boldsymbol{m}}_{p,q}} , \qquad \lambda \in \omega + S_\vartheta .$$

This shows that $\omega + S_\vartheta$ belongs to $\rho\big(-b(D)\big)$ and that $\big(\lambda + b(D)\big)^{-1} = (\lambda + b)^{-1}(D)$ for $\lambda \in \omega + S_\vartheta$, which proves the theorem. □

As a consequence of Theorem 7.1 we obtain solvability results for the equation $\big[\omega + a(D)\big]u = f$ in more conventional function spaces.

Corollary 7.2. *Let the hypotheses of Theorem* 7.1 *be satisfied. Then*

$$\omega + b(D) \in \mathcal{P}\Big(\bigcap_{i=1}^{\ell} W_p^{s+m_i}(\mathbb{R}^n, E_i), W_p^s(\mathbb{R}^n, E_0); \vartheta\Big) , \qquad 1 \le p < \infty ,$$

and

$$\omega + b(D) \in \mathcal{P}\Big(\bigcap_{i=1}^{\ell} buc^{s+m_i}(\mathbb{R}^n, E_i), buc^s(\mathbb{R}^n, E_0); \vartheta\Big) ,$$

as well as

$$\omega + b(D) \in \mathcal{P}\Big(\bigcap_{i=1}^{\ell} C_0^{s+m_i}(\mathbb{R}^n, E_i), C_0^s(\mathbb{R}^n, E_0); \vartheta\Big)$$

for $s \in \mathbb{R}^+ \backslash \mathbb{N}$ *with* $s + m_i \notin \mathbb{N}$ *for* $1 \le i \le \ell$. *Moreover, given* $k \in \mathbb{N}$,

$$\big(\omega + b(D)\big)^{-1} \in \mathcal{L}\Big(W_p^k(\mathbb{R}^n, E_0), \bigcap_{m_i > 0} W_p^{k+\sigma_i}(\mathbb{R}^n, E_i)\Big) , \qquad 1 \le p < \infty ,$$

and

$$\big(\omega + b(D)\big)^{-1} \in \mathcal{L}\Big(BUC^k(\mathbb{R}^n, E_0), \bigcap_{m_i > 0} buc^{k+\sigma_i}(\mathbb{R}^n, E_i)\Big) ,$$

as well as

$$\big(\omega + b(D)\big)^{-1} \in \mathcal{L}\Big(C_0^k(\mathbb{R}^n, E_0), \bigcap_{m_i > 0} C_0^{k+\sigma_i}(\mathbb{R}^n, E_i)\Big)$$

for $k \in \mathbb{N}$ *and* $0 \le \sigma_i < m_i$, *provided* $\max m_i > 0$.

Proof. The first three assertions are special cases of Theorem 7.1, thanks to (5.8) and Remark 5.5(b). Since, thanks to (5.15) and Theorem 7.1

$$W_p^k(\mathbb{R}^n, E_0) \hookrightarrow \mathring{B}^k_{p,\infty}(\mathbb{R}^n, E_0) \xrightarrow{(\omega + b(D))^{-1}} \mathring{B}^{k+m_i}_{p,\infty}(\mathbb{R}^n, E_i) , \qquad 1 \le i \le \ell ,$$

and since

$$\mathring{B}^{k+m_i}_{p,\infty}(\mathbb{R}^n, E_i) \hookrightarrow \mathring{B}^{k+\sigma_i}_{p,1}(\mathbb{R}^n, E_i) \hookrightarrow W_p^{k+\sigma}(\mathbb{R}^n, E_i) , \qquad 0 \le \sigma_i < m_i ,$$

by (5.2), (5.3), (5.8), and (5.15), we see that the fourth assertion is true. By replacing in this argument (5.15) by (5.17) we obtain the last assertion. Finally, we infer the validity of the fifth assertion by replacing in the above deduction (5.15) by (5.16), and $\mathring{B}$ by b, respectively. $\square$

Suppose that $E_1 \overset{d}{\hookrightarrow} E_0$. Then we write $A \in \mathcal{H}(E_1, E_0)$ iff $A \in \mathcal{L}(E_1, E_0)$ and $-A$, considered as a linear operator in E_0, is the infinitesimal generator of a strongly continuous analytic semigroup $\{\, e^{-tA} \ ; \ t \geq 0 \,\}$ on E_0, that is, in $\mathcal{L}(E_0)$. It is known that this is the case iff $\omega + A \in \mathcal{P}(E_1, E_0; \pi/2)$ for some $\omega \in \mathbb{R}$. Then there exists $\vartheta > \pi/2$ with $\omega + A \in \mathcal{P}(E_1, E_0; \vartheta)$. Moreover, there are constants $\omega_0 < \omega$ and $M \geq 1$ such that

$$\|e^{-tA}\|_{\mathcal{L}(E_0)} \leq M e^{\omega_0 t} \ , \qquad t \geq 0 \ , \tag{7.5}$$

(cf. [Ama95, Section I.1] for proofs and more details). Using these facts we obtain from Theorem 7.1 the following generation result for analytic semigroups:

Theorem 7.3. *Let the hypotheses of Theorem* 7.1 *be satisfied with $\vartheta \geq \pi/2$, and suppose that $\boldsymbol{E}_1$ is dense in E_0 and $\mathcal{B} \in \{\mathring{B}, b\}$. Then $b(D) \in \mathcal{H}(\boldsymbol{\mathcal{B}}_{p,q}^{s+\boldsymbol{m}}, \mathcal{B}_{p,q}^{s})$.*

Proof. The assertion is an immediate consequence of Theorem 7.1 and the above remarks, provided we show that $\boldsymbol{\mathcal{B}}_{p,q}^{s+\boldsymbol{m}}$ is dense in $\mathcal{B}_{p,q}^{s}$.
Since $\boldsymbol{E}_1 \hookrightarrow E_i$ for $1 \leq i \leq \ell$, we infer from (5.2), (5.20), and (5.21) that

$$\mathcal{B}_{p,q}^{t}(\mathbb{R}^n, \boldsymbol{E}_1) \hookrightarrow \boldsymbol{\mathcal{B}}_{p,q}^{s+\boldsymbol{m}} \ , \quad \mathcal{B}_{p,q}^{t}(\mathbb{R}^n, \boldsymbol{E}_1) \overset{d}{\hookrightarrow} \mathcal{B}_{p,q}^{s} \ , \qquad t \geq s + \max m_i \ ,$$

thanks to the density of $\boldsymbol{E}_1$ in E_0. This implies that $\boldsymbol{\mathcal{B}}_{p,q}^{s+\boldsymbol{m}}$ is dense in $\mathcal{B}_{p,q}^{s}$. $\square$

Corollary 7.4. *Let the hypotheses of Theorem* 7.1 *be satisfied with $\vartheta \geq \pi/2$, and suppose that $\boldsymbol{E}_1$ is dense in E_0. Then*

$$b(D) \in \mathcal{H}\Bigl(\bigcap_{i=1}^{\ell} W_p^{s+m_i}(\mathbb{R}^n, E_i), W_p^{s}(\mathbb{R}^n, E_0)\Bigr) \ , \qquad 1 \leq p < \infty \ ,$$

and

$$b(D) \in \mathcal{H}\Bigl(\bigcap_{i=1}^{\ell} buc^{s+m_i}(\mathbb{R}^n, E_i), buc^{s}(\mathbb{R}^n, E_0)\Bigr) \ ,$$

as well as

$$b(D) \in \mathcal{H}\Bigl(\bigcap_{i=1}^{\ell} C_0^{s+m_i}(\mathbb{R}^n, E_i), C_0^{s}(\mathbb{R}^n, E_0)\Bigr)$$

for $s \in \mathbb{R}^+ \backslash \mathbb{N}$ with $s + m_i \notin \mathbb{N}$ for $1 \leq i \leq \ell$.

Remark 7.5. Suppose that $\omega \leq 0$ (or, equivalently, replace b by $\omega + b$). Then we deduce from (7.3) that $-b(\xi)$ generates an exponentially decaying strongly continuous analytic semigroup $\{\, e^{-tb(\xi)} \ ; \ t \geq 0 \,\}$ on E_0 for $\xi \in (\mathbb{R}^n)^{\bullet}$. Thus

$$e^{-tb} \in L_\infty\bigl(\mathbb{R}^n, \mathcal{L}(E_0)\bigr) \hookrightarrow \mathcal{S}'\bigl(\mathbb{R}^n, \mathcal{L}(E_0)\bigr)$$

and, consequently, $\mathcal{F}^{-1}(e^{-tb})$ is well-defined in $\mathcal{S}'\big(\mathbb{R}^n, \mathcal{L}(E_0)\big)$. In [Ama97, Chapter VII] the following *representation theorem* is proven:

$$e^{-tb(D)} = \mathcal{F}^{-1} e^{-tb} \mathcal{F} = \mathcal{F}^{-1}(e^{-tb}) * \ , \qquad t \geq 0 \ ,$$

if the hypotheses of Theorem 7.1 are satisfied with $\omega \leq 0$ and $\vartheta \geq \pi/2$, and if $\boldsymbol{E}_1$ is dense in E_0. Thus $\mathcal{F}^{-1}(e^{-tb}) \in \mathcal{S}'\big(\mathcal{L}(E_0)\big)$ is the 'kernel' of the semigroup $\{\, e^{-tb(D)} \ ; \ t \geq 0 \,\}$. □

Suppose that A is a closed linear operator in E such that $\omega + S_\vartheta \subset \rho(-A)$ for some $\omega \in \mathbb{R}$ and $\vartheta \in (\pi/2, \pi)$. Also suppose that there exist $\alpha \in (0,1]$ and $c > 0$ such that

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(E)} \leq c(1 + |\lambda|)^{-\alpha} \ , \qquad \lambda \in \omega + S_\vartheta \ . \tag{7.6}$$

Denote by Γ the negatively oriented boundary of S_ϑ. Then

$$e^{-tA} := \frac{1}{2\pi i} \int_{\omega + \Gamma} e^{\lambda t} (\lambda + A)^{-1} \, d\lambda \in \mathcal{L}(E) \ , \qquad t > 0 \ , \tag{7.7}$$

and

$$(t \mapsto e^{-tA}) \in C^\infty\big((\mathbb{R}^+)^\bullet, \mathcal{L}(E)\big) \ .$$

Moreover, $\operatorname{im}(e^{-tA}) \subset \operatorname{dom}(A^k)$ and $\partial^k e^{-tA} = (-1)^k A^k e^{-tA}$ with

$$\|\partial^k e^{-tA}\|_{\mathcal{L}(E)} \leq c(k, \alpha) t^{\alpha - k - 1} e^{\omega t}$$

for $t > 0$ and $k \in \mathbb{N}$. Also, putting $e^{-0A} := 1_E$, it follows that $\{\, e^{-tA} \ ; \ t \geq 0 \,\}$ is a semigroup on E satisfying

$$e^{-tA} x \to x \text{ as } t \to 0 \ , \qquad x \in \operatorname{dom}(A) \ ,$$

(cf. [Kre72, § I.3] and [Ama95, Lemma II.4.1.1]). From (7.7) we infer that $(\lambda + \omega + A)^{-1}$ is the Laplace transform of $\{\, e^{-t(\omega + A)} = e^{-\omega t} e^{-tA} \ ; \ t \geq 0 \,\}$ for $\lambda \in S_\vartheta$. This implies that the semigroup $\{\, e^{-tA} \ ; \ t \geq 0 \,\}$ uniquely determines the infinitesimal generator $-A$. Note, however, that this semigroup is not strongly continuous at $t = 0$ if $\alpha < 1$. If $\alpha = 1$ then (7.6) defines an analytic semigroup in the sense that the function $t \mapsto e^{-tA}$ has a holomorphic extension over the sector $\mathring{S}_{\vartheta - \pi/2}$ (e.g., [Lun95, Proposition 2.1.1]). In this case $\{\, e^{-tA} \ ; \ t \geq 0 \,\}$ is a strongly continuous semigroup iff A is densely defined.

For easy reference we say that $-A$ generates a C^∞-semigroup $\{e^{-tA} \ ; \ t \geq 0 \,\}$ on E of singular type α if A is a closed linear operator in E satisfying (7.6) for some $\omega \in \mathbb{R}$. In this case it is always understood that the semigroup is defined by (7.7).

Using the above definition and the corresponding convention we can now easily prove a generation theorem for some more conventional function spaces.

Theorem 7.6. *Let the hypotheses of Theorem* 7.1 *be satisfied with* $\vartheta \geq \pi/2$ *and* $\max m_i > 0$, *and let* $k \in \mathbb{N}$ *and* $\alpha \in (0,1)$. *Then* $-b(D)$ *generates a* C^∞*-semigroup of singular type* α *on each one of the spaces* $W_p^k(\mathbb{R}^n, E_0)$, $1 \leq p < \infty$, $BUC^k(\mathbb{R}^n, E_0)$, *and* $C_0^k(\mathbb{R}^n, E_0)$.

Proof. Set $\mathcal{B} := \mathring{B}$ if $\mathfrak{F}^k \in \{ C_0^k, W_p^k \ ; \ 1 \le p < \infty \}$, and $\mathcal{B} := b$ if $\mathfrak{F}^k = BUC^k$ for $k \in \mathbb{N}$. Also put $r := p$ if $\mathfrak{F}^k = W_p^k$, and $r := \infty$ otherwise. From (5.15)–(5.17) we know that

$$\mathcal{B}^k_{r,1}(\mathbb{R}^n, E_0) \overset{d}{\hookrightarrow} \mathfrak{F}^k(\mathbb{R}^n, E_0) \overset{d}{\hookrightarrow} \mathcal{B}^k_{r,\infty}(\mathbb{R}^n, E_0) \ . \tag{7.8}$$

Theorem 7.1 implies that $b(D)$ is a closed linear operator in $\mathcal{B}^k_{r,\infty}(\mathbb{R}^n, E_0)$ such that $\omega + S_\vartheta \subset \rho\big(-b(D)\big)$ and

$$\big\| \big(\lambda + b(D)\big)^{-1} \big\|_{\mathcal{L}(\mathcal{B}^k_{r,\infty}(\mathbb{R}^n,E_0),\mathcal{B}^{k+m_i}_{r,\infty}(\mathbb{R}^n,E_i))} \le c \tag{7.9}$$

and

$$(1 + |\lambda|)\, \big\| \big(\lambda + b(D)\big)^{-1} \big\|_{\mathcal{L}(\mathcal{B}^k_{r,\infty}(\mathbb{R}^n,E_0))} \le c \tag{7.10}$$

for $\lambda \in \omega + S_\vartheta$ and $1 \le i \le \ell$. From (5.2), (5.20), (5.21), and (7.8) we infer that

$$\mathcal{B}^{k+m_i}_{r,\infty}(\mathbb{R}^n, E_i) \hookrightarrow \mathcal{B}^{k+\sigma_i}_{r,1}(\mathbb{R}^n, E_0) \hookrightarrow \mathfrak{F}^k(\mathbb{R}^n, E_0) \ ,$$

provided $0 \le \sigma_i < m_i$. Hence we see from (7.8)–(7.10) that the $\mathfrak{F}^k(\mathbb{R}^n, E_0)$-realization of $b(D)$, which we again denote by $b(D)$, is a closed linear operator in $\mathfrak{F}^k(\mathbb{R}^n, E_0)$ with $\omega + S_\vartheta \subset \rho\big(-b(D)\big)$, satisfying the estimate

$$\big\| \big(\lambda + b(D)\big)^{-1} \big\|_{\mathcal{L}(\mathfrak{F}^k,\mathcal{B}^{k+\sigma_i}_{r,1})} + (1 + |\lambda|)\, \big\| \big(\lambda + b(D)\big)^{-1} \big\|_{\mathcal{L}(\mathfrak{F}^k,\mathcal{B}^k_{r,\infty})} \le c$$

for $\lambda \in \omega + S_\vartheta$, provided $0 \le \sigma_i < m_i$, where we have set $\mathfrak{F}^k := \mathfrak{F}^k(\mathbb{R}^n, E_0)$ and, similarly, $\mathcal{B}^s_{r,q} := \mathcal{B}^s_{r,q}(\mathbb{R}^n, E_0)$. Fix $0 < \varepsilon < \delta < m_1$, assuming without loss of generality that $m_1 > 0$. Then, using (5.2) once more, we infer from the last estimate that

$$\big\| \big(\lambda + b(D)\big)^{-1} \big\|_{\mathcal{L}(\mathfrak{F}^k,\mathcal{B}^{k+\delta}_{r,1})} + (1 + |\lambda|)\, \big\| \big(\lambda + b(D)\big)^{-1} \big\|_{\mathcal{L}(\mathfrak{F}^k,\mathcal{B}^{k-\varepsilon}_{r,1})} \le c \tag{7.11}$$

for $\lambda \in \omega + S_\vartheta$. Thanks to (5.7) and (5.14) we see that

$$\|x\|_{\mathcal{B}^{k+\varepsilon}_{r,1}} \le c(\varepsilon, \delta)\, \|\|x\|^{1-\theta}_{\mathcal{B}^{k-\varepsilon}_{r,1}}\, \|x\|^{\theta}_{\mathcal{B}^{k+\delta}_{r,1}} \ , \qquad x \in \mathcal{B}^{k+\delta}_{r,1} \ ,$$

where $\theta := 2\varepsilon(\varepsilon + \delta)^{-1}$. Consequently, by (7.11),

$$\big\| \big(\lambda + b(D)\big)^{-1} \big\|_{\mathcal{L}(\mathfrak{F}^k)} \le c(\varepsilon)\, \big\| \big(\lambda + b(D)\big)^{-1} \big\|_{\mathcal{L}(\mathfrak{F}^k,\mathcal{B}^{k+\varepsilon}_{r,1})} \le c(\varepsilon,\delta)(1 + |\lambda|)^{(\varepsilon-\delta)/(\varepsilon+\delta)}$$

for $\lambda \in \omega + S_\vartheta$. Thus, given $\alpha \in (0,1)$ and $\delta \in (0, m_1)$, we obtain the estimate

$$\big\| \big(\lambda + b(D)\big)^{-1} \big\|_{\mathcal{L}(\mathfrak{F}^k)} \le c(1 + |\lambda|)^{-\alpha} \ , \qquad \lambda \in \omega + S_\vartheta \ ,$$

by putting $\varepsilon := (1 - \alpha)\delta/(1 + \alpha)$. Since we can assume that $\vartheta > \pi/2$, the assertion follows. □

Remarks 7.7. (a) From Theorem 7.1 we infer that $-b(D)$ generates an analytic semigroup on $\mathcal{B}^k_{r,\infty}(\mathbb{R}^n, E_0)$ if $\vartheta \ge \pi/2$, which is strongly continuous iff $\boldsymbol{E}_1$ is dense in E_0. It is obvious that the C^∞-semigroup on $\mathfrak{F}^k(\mathbb{R}^n, E_0)$ is the restriction of this analytic semigroup if $\max m_i > 0$.

(b) Suppose that $\mathfrak{F} \in \{BUC, C_0, L_p \ ;\ 1 \le p < \infty\}$, and that

$$(u^0, f) \in \mathfrak{F}(\mathbb{R}^n, E_0) \times C\bigl(\mathbb{R}^+, \mathfrak{F}(\mathbb{R}^n, E_0)\bigr) ,$$

for example. Consider the evolution equation

$$\dot{u} + b(D)u = f(t) \ , \quad t > 0 \ , \qquad u(0) = u^0 \ , \tag{7.12}$$

where the hypotheses of Theorem 7.1 are satisfied with $\vartheta \ge \pi/2$ and $\max m_i > 0$. Then one can, in principle, consider (7.12) as an equation in $\mathfrak{F}(\mathbb{R}^n, E_0)$ and use Theorem 7.6 to derive solvability results for (7.12) (cf. [Yag89]).

On the other hand we can use the fact that

$$\mathfrak{F}(\mathbb{R}^n, E_0) \hookrightarrow \mathcal{B}^0_{r,\infty}(\mathbb{R}^n, E_0) ,$$

where $r := \infty$ and $\mathcal{B} := b$ if $\mathfrak{F} = BUC$, and $r := \infty$ and $\mathcal{B} := \mathring{B}$ if $\mathfrak{F} = C_0$, and where $r := p$ and $\mathcal{B} := B$ if $\mathfrak{F} = L_p$, and consider equation (7.12) as an evolution equation in $\mathcal{B}^0_{r,\infty}(\mathbb{R}^n, E_0)$. Assuming, for simplicity, that $\boldsymbol{E}_1$ is dense in E_0, we can appeal to Corollary 7.4 which tells us that $-b(D)$ generates a strongly continuous analytic semigroup on $\mathcal{B}^0_{r,\infty}(\mathbb{R}^n, E_0)$. Thus we obtain existence and regularity for equation (7.12) by invoking the well-established theory of linear parabolic evolution equations (e.g., [Paz83], [Tan79], [Ama95]). That approach has two advantages. First: it relies completely on the 'classical' theory of parabolic evolution equations that is well-understood. Second: the domain of the generator is explicitly known, thanks to Corollary 7.4. The latter fact is particularly important in connection with quasilinear problems for which equation (7.12) occurs by linearization. Of course, given suitable regularity assumptions for u^0 and f, the final solvabiltity result can be formulated without any reference to $\mathcal{B}^0_{r,\infty}(\mathbb{R}^n, E_0)$, that is, completely within the framework of 'conventional' function spaces.

Suppose that $E_1 \hookrightarrow E \hookrightarrow E_0$ and $0 \le \theta \le 1$. Then we write $E \in J_\theta(E_0, E_1)$ if there exists $\kappa \in \mathbb{R}^+$ such that

$$\|x\|_E \le \kappa \, \|x\|_{E_0}^{1-\theta} \, \|x\|_{E_1}^{\theta} \ , \qquad x \in E_1 \ . \tag{7.13}$$

It is known that this is the case with $\theta \in (0,1)$ iff $(E_0, E_1)_{\theta,1} \hookrightarrow E$ (e.g., [BL76, Section 3.5]). Moreover,

$$B \in \mathcal{P}(E_1, E_0) \Rightarrow E_\theta(B) := \bigl(\mathrm{dom}(B), \|B^\theta \cdot\|_{E_0}\bigr) \in J_\theta(E_0, E_1) \tag{7.14}$$

for $0 < \theta < 1$ (cf. [Ama95, Theorem V.1.2.4]).

Using this concept we can now prove a simplified version of Theorem 7.1.

Theorem 7.8. *Suppose that $E_1 \hookrightarrow E_i \hookrightarrow E_0$ and there are numbers $\theta_i \in [0,1)$ with $E_i \in J_{\theta_i}(E_0, E_1)$ for $2 \le i \le \ell$. Also suppose that*

$$b_i \in S^{m_i}\bigl(\mathbb{R}^n, \mathcal{L}(E_i, E_0)\bigr) \ , \qquad 1 \le i \le \ell \ , \tag{7.15}$$

where $0 \le m_i \le \theta_i m_1$ *for* $2 \le i \le \ell$. *Finally, let there be* $\kappa_1 \ge 1$ *and* $\vartheta \in (0,\pi]$ *such that*

$$(1+|\xi|)^{m_1}\,|(\lambda+b_1)^{-1}(\xi)|_{\mathcal{L}(E_0,E_1)} + (1+|\lambda|)\,|(\lambda+b_1)^{-1}(\xi)|_{\mathcal{L}(E_0)} \le \kappa_1 \tag{7.16}$$

for $\xi \in (\mathbb{R}^n)^\bullet$ *and* $\lambda \in S_\vartheta$. *Then there exists* $\omega \ge 0$ *such that*

$$\omega + b(D) \in \mathcal{P}\big(\mathcal{B}^{s+m_1}_{p,q}(\mathbb{R}^n,E_1), \mathcal{B}^s_{p,q}(\mathbb{R}^n,E_0);\vartheta\big) \ .$$

Proof. Let μ_i be a bound for the supremum norm of $(1+|\xi|)^{-m_i}\,|b_i|_{\mathcal{L}(E_i,E_0)}$ for $2 \le i \le \ell$. Then, denoting by κ_i the constant in (7.13) belonging to E_i, it follows from (7.13), (7.15), and (7.16) that

$$|b_i(\lambda+b_1)^{-1}(\xi)|_{\mathcal{L}(E_0)} \le \kappa_1\mu_i\kappa_i(1+|\lambda|)^{\theta_i-1} \ , \qquad 2 \le i \le \ell \ , \quad \lambda \in S_\vartheta \ .$$

Hence

$$|(b_2+\cdots+b_\ell)(\lambda+b_1)^{-1}|_{\mathcal{L}(E_0)} \le \kappa_1(\mu_2\kappa_2+\cdots+\mu_\ell\kappa_\ell)(1+|\lambda|)^{\theta-1}$$

for $\lambda \in S_\vartheta$ and $\xi \in (\mathbb{R}^n)^\bullet$, where $\theta := \max\theta_i$. Fix $\sigma \in (0,1)$ and put

$$\omega(\sigma) := \Big(\big[\kappa_1(\mu_2\kappa_2+\cdots+\mu_\ell\kappa_\ell)/\sigma\big]^{1/(1-\theta)} - 1\Big)_+ \ . \tag{7.17}$$

Then

$$(\lambda+b) = \big(1+(b_2+\cdots+b_\ell)(\lambda+b_1)^{-1}\big)(\lambda+b_1)$$

implies that $\omega(\sigma)+S_\vartheta \subset \rho\big(-b(\xi)\big)$ and

$$|(\lambda+b)^{-1}(\xi)|_{\mathcal{L}(E_0,E_i)} \le (1-\sigma)^{-1}\,|(\lambda+b_1)^{-1}(\xi)|_{\mathcal{L}(E_0,E_i)} \ , \qquad 1 \le i \le \ell \ ,$$

for $\xi \in (\mathbb{R}^n)^\bullet$ and $\lambda \in \omega(\sigma)+S_\vartheta$. We infer from (7.16) and $E_1 \hookrightarrow E_i$ that

$$|(\lambda+b_1)^{-1}(\xi)|_{\mathcal{L}(E_0,E_i)} \le c(1+|\xi|)^{-m_1} \ , \qquad \xi \in (\mathbb{R}^n)^\bullet \ , \quad \lambda \in \omega+S_\vartheta \ ,$$

so that (7.3) is satisfied. Now the assertion follows from Theorem 7.1 thanks to the fact that (5.2), (5.20), and (5.21) imply $\boldsymbol{\mathcal{B}}^{s+\boldsymbol{m}}_{p,q} = \mathcal{B}^{s+m_1}_{p,q}(\mathbb{R}^n,E_1)$. □

Clearly, $\omega(\sigma)$ is an upper bound for ω. Thus (7.17) can sometimes be used to show that ω can be taken to be zero.

8. Applications

In this section we illustrate the preceding general results by some model problems. In concrete examples we do not strive for the most general formulations but rather choose simple settings. The reader will have no problem seeing the underlying general structure which will enable him to obtain further applications in the same spirit.

A. Degenerate Boundary Value Problems

Let X be a bounded open subset of $\mathbb{R}^d$, which is smooth, that is, $\overline{X}$ is a d-dimensional C^∞-submanifold of $\mathbb{R}^n$ with boundary ∂X, and let ν be the outer unit-normal vector field on ∂X. Suppose that $a_{\alpha,\beta} \in C^\infty(\overline{X})$ for $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq 2$, and $\beta \in \mathbb{N}^n$ with $|\beta| \leq 1$, and set

$$\mathcal{A} := \Delta_x^2 - \Delta_x^2 \Delta_y + \sum_{|\alpha|\leq 3} \sum_{|\beta|\leq 1} a_{\alpha,\beta}(x) \partial_x^\alpha \partial_y^\beta \ , \tag{8.1}$$

where $x \in X$ and $y \in \mathbb{R}^n$. Consider the boundary value problem (BVP) on the cylinder $X \times \mathbb{R}^n$:

$$\mathcal{A}u = f \text{ in } X \times \mathbb{R}^n \ , \qquad u = \partial_\nu u = 0 \text{ on } \partial X \times \mathbb{R}^n \ . \tag{8.2}$$

Fix $r \in (1,\infty)$, denote by γ_∂ the trace operator on ∂X, and put

$$W_{r,*}^t(X) := \{\, v \in W_r^t(X) \ ; \ \gamma_\partial v = \partial_\nu v = 0 \,\} \ , \qquad t \geq 2 \ .$$

Proposition 8.1. *Put* $\mathbb{E}_0 := \mathcal{B}_{p,q}^s\big(\mathbb{R}^n, L_r(X)\big)$ *and* $\mathbb{E}_1 := \mathcal{B}_{p,q}^{s+2}\big(\mathbb{R}^n, W_{r,*}^4(X)\big)$. *Then* $\mathcal{A} \in \mathcal{P}(\mathbb{E}_1, \mathbb{E}_0; \vartheta)$ *for* $0 < \vartheta < \pi$. *In particular,* $\mathcal{A} \in \mathcal{H}(\mathbb{E}_1, \mathbb{E}_0)$ *if* $\mathcal{B} \in \{\overset{\circ}{B}, b\}$.

Proof. Set

$$E_0 := L_r(X) \ , \quad E_1 := W_{r,*}^4(X) \ , \quad E_2 := W_{r,*}^3(X) \ .$$

Then $E_1 \overset{d}{\hookrightarrow} E_2 \overset{d}{\hookrightarrow} E_0$. By a result of Seeley [See72] it is known that $E_2 \doteq [E_0, E_1]_{3/4}$. Consequently, $E_2 \in J_{3/4}(E_0, E_1)$.

Denote by B the E_0-realization of the elliptic BVP $\{\Delta^2; \gamma_\partial, \partial_\nu\}$ on X, that is,

$$Bv := \Delta^2 v \ , \qquad v \in W_{r,*}^4(X) \ .$$

Then it is well-known that

$$B \in \mathcal{P}(E_1, E_0; \vartheta) \ , \qquad 0 < \vartheta < \pi \ . \tag{8.3}$$

Set $b_1(\eta) := \Lambda^2(\eta)B$ for $\eta \in \mathbb{R}^n$ and observe that

$$b_1 \in S^2\big(\mathbb{R}^n, \mathcal{L}(E_1, E_0)\big) \ .$$

We infer from (8.3) that

$$\begin{aligned} |(\lambda + b_1)^{-1}(\eta)|_{\mathcal{L}(E_0,E_j)} &= \Lambda^{-2}(\eta) \, \big|\big(\lambda\Lambda^{-2}(\eta) + B\big)^{-1}\big|_{\mathcal{L}(E_0,E_j)} \\ &\leq c\Lambda^{-2}(\eta)\big(1 + |\lambda|\,\Lambda^{-2}(\eta)\big)^{j-1} \end{aligned}$$

for $\eta \in \mathbb{R}^n$, $\lambda \in S_\vartheta$, and $j = 0, 1$. Hence b_1 satisfies (7.16) with $m_1 := 2$.

Next we put

$$b_2(\eta) := \sum_{|\alpha|\le 3} \sum_{|\beta|\le 1} \eta^\beta a_{\alpha,\beta} \partial_x^\alpha \ .$$

It is obvious that

$$b_2 \in S^1\big(\mathbb{R}^n, \mathcal{L}(E_2, E_0)\big) \ .$$

Hence Theorems 7.3 and 7.8 imply that $b(D) \in \mathcal{H}(\mathbb{E}_1, \mathbb{E}_0)$, which, thanks to $b(D) = \mathcal{A}$, proves the theorem. □

It follows from Proposition 8.1, for example, that the initial-boundary value problem (IBVP)

$$\begin{aligned} \partial_t u + \mathcal{A}u &= f(t) && \text{in } X \times \mathbb{R}^n \ , \\ u = \partial_\nu u &= 0 && \text{on } \partial X \times \mathbb{R}^n \ , \\ u(\cdot, 0) &= u^0 \end{aligned} \quad t > 0 \ , \tag{8.4}$$

has for each $f \in C^\theta(\mathbb{R}^+, \mathbb{E}_0)$, where $\theta \in (0,1)$, and for each $u^0 \in \mathbb{E}_0$ a unique solution

$$u \in C(\mathbb{R}^+, \mathbb{E}_0) \cap C(\dot{\mathbb{R}}^+, \mathbb{E}_1) \cap C^1(\dot{\mathbb{R}}^+, \mathbb{E}_0)$$

(cf. [Ama95, Theorem II.1.2.1]). Also see Subsection D for maximal regularity results.

Instead of using Sobolev spaces we could also use little Hölder spaces, that is, we could set

$$E_0 := buc_*^\theta(\overline{X}) \ , \quad E_1 := buc_*^{4+\theta}(\overline{X}) \ , \quad E_2 := buc_*^{3+\theta}(\overline{X})$$

for some $\theta \in (0,1)$, where the subscript $*$ indicates that appropriate boundary conditions are satisfied. Then, using results from [Ama97], the above proof remains valid in this case as well. Thus choosing $\mathcal{B} \in \{\mathring{B}_{\infty,\infty}, b_{\infty,\infty}\}$, we obtain the solvability of BVP (8.2) and IBVP (8.4) in suitable Hölder classes. In addition, we can invoke Theorem 7.6 to derive solvability results from (8.4) in more conventional function spaces. It should also be noted that the use of perturbation theorems for generators of analytic semigroups (cf. [Ama95, Theorem [I.1.3.1]) allows for more general lower order terms than $b_2(D_y)$. In particular, the coefficients $a_{\alpha,\beta}$ can depend on $y \in \mathbb{R}^n$ as well. We leave details to the interested reader (also recall Remark 7.7(b)).

Note that the principal symbol of the differential operator $\mathcal{A}$ equals $|\xi|^4\,|\eta|^2$ for $(\xi,\eta) \in \mathbb{R}^d \times \mathbb{R}^n$. Hence $\mathcal{A}$ is degenerate elliptic.

B. Operator Convolution Equations on the Line

Suppose that $E_1 \hookrightarrow E_0$ and consider the convolution equation

$$A * u = f \qquad \text{in } \mathcal{S}'(\mathbb{R}, E_0) \ , \tag{8.5}$$

where $(A, f) \in \mathcal{S}'\big(\mathbb{R}, \mathcal{L}(E_1, E_0) \times E_0\big)$. More precisely, assume that

$$A = \partial A_0 + A_1 \ , \qquad A_j \in \mathcal{S}'\big(\mathbb{R}, \mathcal{L}(E_j, E_0)\big) \ , \quad j = 0, 1 \ . \tag{8.6}$$

Then (8.5) takes the form

$$A_0 * \dot{u} + A_1 * u = f \ , \tag{8.7}$$

where $\dot{u} := \partial u$.

Theorem 8.2. *Suppose that $\widehat{A}_j$ are regular distributions such that*

$$\widehat{A}_j \in S^0\big(\mathbb{R}, \mathcal{L}(E_j, E_0)\big) \ , \qquad j = 0, 1 \ . \tag{8.8}$$

Put $a(\xi) := i\xi\widehat{A}_0(\xi) + \widehat{A}_1(\xi)$ and suppose that $a(\xi) \in \mathcal{L}\mathrm{is}(E_1, E_0)$ for $\xi \in \dot{\mathbb{R}}$ and

$$\sup_{\xi \in \dot{\mathbb{R}}} (1 + |\xi|)^{1-j} \, |a^{-1}(\xi)|_{\mathcal{L}(E_0, E_j)} < \infty \ , \qquad j = 0, 1 \ . \tag{8.9}$$

Then

$$(u \mapsto A * u) \in \mathcal{L}\mathrm{is}\big(\mathcal{B}^{s+1}_{p,q}(\mathbb{R}, E_0) \cap \mathcal{B}^s_{p,q}(\mathbb{R}, E_1), \mathcal{B}^s_{p,q}(\mathbb{R}, E_0)\big) \tag{8.10}$$

and the inverse of the map (8.10) *is given by $v \mapsto G * v$, where $G := \mathcal{F}^{-1}(a^{-1})$.*

Proof. Set $b_1(\xi) := i\xi\widehat{A}_0(\xi)$ and $b_2(\xi) := \widehat{A}_1(\xi)$ for $\xi \in \dot{\mathbb{R}}$. Also set $\widetilde{E}_0 := E_0$ and $\widetilde{E}_i := E_{i-1}$, $1 \le i \le 2 := \ell$. Then (8.8) implies

$$b_i \in S^{m_i}\big(\mathbb{R}, \mathcal{L}(\widetilde{E}_i, \widetilde{E}_0)\big) \ , \qquad i = 1, 2 \ ,$$

where $m_1 := 1$ and $m_2 := 0$. From (8.9) we see that (7.3) is satisfied with $\vartheta := 0$ (and E_i replaced by $\widetilde{E}_i$). Hence Theorem 7.1 implies the assertion. □

Corollary 8.3. *Let the hypotheses of Theorem* 8.2 *be satisfied. Then the convolution equation*

$$A_0 * \dot{u} + A_1 * u = f$$

has for each $f \in L_p(\mathbb{R}, E_0)$, $1 \le p < \infty$, a unique solution

$$u = G * f \in W^s_p(\mathbb{R}, E_0) \ , \qquad 0 \le s < 1 \ .$$

If $f \in BUC(\mathbb{R}, E_0)$ then $G * f \in buc^s(\mathbb{R}, E_0)$ for $0 \le s < 1$, and if $f \in C_0(\mathbb{R}, E_0)$ then $G * f \in C^s_0(\mathbb{R}, E_0)$ for $0 \le s < 1$. In each case the map $f \mapsto G * f$ is continuous between the respective spaces.

Proof. This is an easy consequence of (8.10), the embedding results (5.2), (5.3), and (5.15)–(5.17), and the characterizations of Besov spaces given in (5.8), (5.11), and Remark 5.5(b). □

Note that we have the more precise information $u \in B^1_{p,\infty}(\mathbb{R}, E_0) \cap B^0_{p,\infty}(\mathbb{R}, E_0)$ if $f \in L_p(\mathbb{R}, E_0)$. Similar assertions are true in the two other cases.

Suppose that

$$A_j = \delta \otimes A_{j,0} + A_{j,1}\ , \qquad A_{j,0} \in \mathcal{L}(E_j, E_0)\ , \quad A_{j,1} \in \mathcal{S}'\big(\mathbb{R}, \mathcal{L}(E_j, E_0)\big)$$

such that $\widehat{A}_{j,1}$ are regular distributions for $j = 0, 1$. Then (8.7) reduces to

$$A_{0,0}\dot{u} + A_{0,1} * \dot{u} + A_{1,0}u + A_{1,1} * u = f\ . \tag{8.11}$$

Since $(\delta \otimes B)\widehat{\ } = \mathbf{1} \otimes \widehat{B}$, we see that (8.8) is equivalent to

$$\widehat{A}_{j,1} \in S^0\big(\mathbb{R}, \mathcal{L}(E_j, E_0)\big)\ , \qquad j = 0, 1\ , \tag{8.12}$$

and (8.9) takes the form

$$(1 + |\xi|)^{1-j}\, \big|\big(i\xi\big[A_{0,0} + \widehat{A}_{0,1}(\xi)\big] + A_{1,0} + \widehat{A}_{1,1}(\xi)\big)^{-1}\big|_{\mathcal{L}(E_0, E_j)} \le c < \infty \tag{8.13}$$

for $j = 0, 1$ and $\xi \in \dot{\mathbb{R}}$. The situation simplifies considerably if we restrict ourselves to 'equations of scalar type', that is, if we assume that $A_{0,j} = \alpha_j 1_{E_0}$ and $A_{1,j} = \beta_j A$, where $\alpha_j \in \mathbb{C}$ and $\beta_j \in \mathcal{S}'(\mathbb{R})$, for $j = 0, 1$, and $A \in \mathcal{L}(E_1, E_0)$. In this case Theorem 8.2 implies the following 'maximal regularity theorem' for 'parabolic integro-differential equations of scalar type'.

Proposition 8.4. *Suppose that $A \in \mathcal{P}(E_1, E_0; \vartheta)$ for some $\vartheta \in (0, \pi)$. Also suppose that $\alpha_0, \beta_0 \in \mathbb{C}$ and $\alpha_1, \beta_1 \in \mathcal{S}'(\mathbb{R})$ such that $\widehat{\alpha}_1, \widehat{\beta}_1 \in L_{1,\mathrm{loc}}(\mathbb{R}) \cap C^2(\dot{\mathbb{R}})$ with*

$$|\partial^j \widehat{\alpha}_1(\xi)| + |\partial^j \widehat{\beta}_1(\xi)| \le c(1 + |\xi|)^{-j}\ , \qquad \xi \in \dot{\mathbb{R}}\ , \quad j = 0, 1, 2\ . \tag{8.14}$$

Assume that

$$\liminf_{|\xi| \to \infty} |\alpha_0 + \widehat{\alpha}_1(\xi)| > 0 \tag{8.15}$$

and

$$\inf_{\xi \in \dot{\mathbb{R}}} |\beta_0 + \widehat{\beta}_1(\xi)| > 0\ , \qquad \xi \in \dot{\mathbb{R}}\ . \tag{8.16}$$

Finally, assume that

$$\frac{i\xi\big(\alpha_0 + \widehat{\alpha}_1(\xi)\big)}{\beta_0 + \widehat{\beta}_1(\xi)} \in S_\vartheta\ , \qquad \xi \in \dot{\mathbb{R}}\ . \tag{8.17}$$

Then the convolution equation

$$\alpha_0\dot{u} + \alpha_1 * \dot{u} + \beta_0 A u + \beta_1 * A u = f \tag{8.18}$$

has for each $f \in \mathcal{B}^s_{p,q}(\mathbb{R}, E_0)$ a unique solution $u \in \mathcal{B}^{s+1}_{p,q}(\mathbb{R}, E_0) \cap \mathcal{B}^s_{p,q}(\mathbb{R}, E_1)$, and the estimate

$$\|\dot{u}\|_{\mathcal{B}^s_{p,q}(\mathbb{R}, E_0)} + \|u\|_{\mathcal{B}^s_{p,q}(\mathbb{R}, E_1)} \le c\, \|f\|_{\mathcal{B}^s_{p,q}(\mathbb{R}, E_0)}$$

is valid, where c is independent of f.

Proof. Since $A \in \mathcal{P}(E_1, E_0; \vartheta)$, it follows from (7.1) and (8.17) that

$$a(\xi) := i\xi\big(\alpha_0 + \widehat{\alpha}_1(\xi)\big) + \big(\beta_0 + \widehat{\beta}_1(\xi)\big)A \in \mathcal{L}\mathrm{is}(E_1, E_0)$$

and

$$|a^{-1}(\xi)|_{\mathcal{L}(E_0,E_j)} \leq \begin{cases} \kappa\big(|\beta_0 + \widehat{\beta}_1(\xi)| + |\xi|\,|\alpha_0 + \widehat{\alpha}_1(\xi)|\big)^{-1}, & j = 0, \\ \kappa\ |\beta_0 + \widehat{\beta}_1(\xi)|^{-1}, & j = 1, \end{cases}$$

for $\xi \in \dot{\mathbb{R}}$. Now we deduce from (8.15) and (8.16) that (8.13) is satisfied. Hence the assertion follows from Theorem 8.2, thanks to the fact that (8.14) implies (8.12). □

We leave it to the interested reader to formulate the analogue to Corollary 8.3 for equation (8.18).

It should be noted that conditions (8.15)–(8.17) are trivially true if $\alpha_0, \beta_0 > 0$ and $\widehat{\alpha}_1, \widehat{\beta}_1 \geq 0$. Recall that, thanks to Bochner's theorem, $\widehat{\alpha}_1, \widehat{\beta}_1 \geq 0$ iff α_1 and β_1 are 'positive definite'.

Of course, in concrete situations A can be the $L_r(X)$-realization of an elliptic BVP on a smooth bounded domain $X \subset \mathbb{R}^d$, for example. In particular, if A corresponds to a second order elliptic BVP equation then (8.18) can be interpreted as a heat-conduction problem with memory (cf. [Prü93, Section 5] for more details and bibliographic references).

Theorem 8.2 and Proposition 8.4 are related to a maximal regularity result of Prüss (namely [Prü93, Theorem 12.5]), which gives sufficient conditions guaranteeing that (8.11) has for $f \in B^s_{p,q}(\mathbb{R}, E_0)$ a unique solution $u \in B^s_{p,q}(\mathbb{R}, E_0)$ with $\dot{u} \in B^s_{p,q}(\mathbb{R}, E_0)$, where $s \in (0,1)$ and $1 \leq p, q \leq \infty$. More recently, equations of the form (8.11) have also been studied by Guidetti [Gui96]. This author establishes maximal regularity results in Slobodeckii spaces $W^s_p(\mathbb{R}, E_0)$, for $1 \leq p < \infty$ and $0 < s < 1$, under conditions that are closely related to (8.12) and (8.13). For earlier maximal regularity results for linear integrodifferential equations in Hölder spaces we refer to [DL88]. Note that these authors cannot derive solvability conditions from their results corresponding to Corollary 8.3 since their theorems are restricted to Besov spaces of positive order.

C. Operator Convolution Equations on the Half-Line

In the following we put $\mathcal{S}'_+(E) := \mathcal{S}'(\mathbb{R}, E) \cap \mathcal{D}_+(E)$, and we define the closed linear subspace $\mathcal{B}^s_{p,q,+}(E)$ of $\mathcal{B}^s_{p,q}(\mathbb{R}, E)$ by

$$\mathcal{B}^s_{p,q,+}(E) := \mathcal{B}^s_{p,q}(\mathbb{R}, E) \cap \mathcal{S}'_+(E) .$$

Suppose that $E_1 \hookrightarrow E_0$ and $(A, f) \in \mathcal{S}'_+\big(\mathcal{L}(E_1, E_0) \times E_0\big)$. Then we consider the convolution equation

$$A * u = f \qquad \text{in } \mathcal{S}'_+(E_0) , \tag{8.19}$$

that is, on the half-line $\mathbb{R}^+$. Thus by a solution of (8.19) we mean a distribution $u \in \mathcal{S}'_+(E_1)$ satisfying (8.19), provided $f \in \mathcal{S}'_+(E_0)$ is given. Note that this implies that

u satisfies the 'initial condition' $u(0)$ in some generalized sense. It follows from (5.2)–(5.4) and (5.16) that $\mathcal{B}^s_{p,q} \hookrightarrow BUC$, provided either $s > n/p$ and $q \in [1,\infty]$, or $s = n/p$ and $q = 1$. This implies that a solution u of (8.19) satisfies the initial condition $u(0) = 0$ in the classical sense if $u \in \mathcal{B}^s_{p,q,+}(E_1)$ and s, p, and q satisfy the above conditions.

Again we assume that $A = \partial A_0 + A_1$, where now

$$A_j \in \mathcal{S}'_+\big(\mathcal{L}(E_j, E_0)\big) \ , \qquad j = 0, 1 \ , \tag{8.20}$$

so that (8.19) is the convolution evolution equation

$$A_0 * \dot{u} + A_1 * u = f \qquad \text{in } \mathcal{S}'_+(E_0) \ . \tag{8.21}$$

It follows from (8.20) that the Laplace transform $\widetilde{A}_j$ of A_j is well-defined on $\mathbb{R}^+ + i\,\mathbb{R}$ and holomorphic on $\dot{\mathbb{R}}^+ + i\,\mathbb{R}$.

The following theorem is the analogue to Theorem 8.2, where we now consider convolution equations on the half-line.

Theorem 8.5. *Suppose that*

$$\widetilde{A}_j(i\,\cdot) \in S^0\big(\mathbb{R}, \mathcal{L}(E_j, E_0)\big) \ , \qquad j = 0, 1 \ .$$

Put

$$\widetilde{a}(z) := z\widetilde{A}_0(z) + \widetilde{A}_1(z) \ , \qquad z \in (\mathbb{R}^+ + i\,\mathbb{R})^\bullet \ ,$$

and assume that $\widetilde{a}\big((\mathbb{R}^+ + i\,\mathbb{R})^\bullet\big) \subset \mathcal{L}\mathrm{is}(E_1, E_0)$, *that*

$$\sup_{\eta \in \dot{\mathbb{R}}} (1 + |\eta|)^{1-j} \, |\widetilde{a}^{-1}(i\eta)|_{\mathcal{L}(E_0, E_j)} < \infty \ , \qquad j = 0, 1 \ ,$$

and that $z \mapsto |\widetilde{a}^{-1}(z)|_{\mathcal{L}(E_0)}$ *is polynomially bounded. Then*

$$(u \mapsto A * u) \in \mathcal{L}\mathrm{is}\big(\mathcal{B}^{s+1}_{p,q,+}(E_0) \cap \mathcal{B}^s_{p,q,+}(E_1), \mathcal{B}^s_{p,q,+}(E_0)\big) \ .$$

The inverse of this map is given by $v \mapsto G * v$, *where*

$$G := \mathcal{F}^{-1}\big(\widetilde{a}^{-1}(i\,\cdot)\big) \in \mathcal{S}_+\big(\mathcal{L}(E_0)\big) \ .$$

Proof. Thanks to $\widetilde{u}(i\eta) = \widehat{u}(\eta)$ for $\eta \in \mathbb{R}$ and to Theorem 8.2 the assertion follows from Theorem 3.4, provided we show that $\mathrm{supp}(G) \subset \mathbb{R}^+$. Since the inversion map $B \mapsto B^{-1}$ is analytic and since $\widetilde{a}$ is analytic on $\dot{\mathbb{R}}^+ + i\,\mathbb{R}$, it follows that $\widetilde{a}^{-1}$ is analytic on $\dot{\mathbb{R}}^+ + i\,\mathbb{R}$. Thus the Paley-Wiener theorem theorem guarantees the desired support restriction. □

It is easy to prove the analogue to Proposition 8.4 for convolution equations of scalar type on the half-line.

Proposition 8.6. *Suppose that* $A \in \mathcal{P}(E_1, E_0; \vartheta)$ *for some* $\vartheta \in (0, \pi)$. *Also suppose that* $\alpha_0, \beta_0 \in \mathbb{C}$ *and* $\alpha_1, \beta_1 \in \mathcal{S}'_+$ *such that* $\widetilde{\alpha}_1(i\,\cdot), \widetilde{\beta}_1(i\,\cdot) \in C^2(\dot{\mathbb{R}})$ *with*

$$|\partial^j \widetilde{\alpha}_1(i\eta)| + |\partial^j \widetilde{\beta}_1(i\eta)| \leq c(1+|\eta|)^{-j} , \qquad \eta \in \dot{\mathbb{R}} , \quad j = 0,1,2 .$$

Assume that

$$\liminf_{|\eta|\to\infty} |\alpha_0 + \widetilde{\alpha}_1(i\eta)| > 0$$

and

$$\inf_{\operatorname{Re} z > 0} |\beta_0 + \widetilde{\beta}_1(z)| > 0 , \tag{8.22}$$

as well as

$$\frac{z\big(\alpha_0 + \widetilde{\alpha}_1(z)\big)}{\beta_0 + \widetilde{\beta}_1(z)} \in S_\vartheta , \qquad \operatorname{Re} z \geq 0 , \quad z \neq 0 . \tag{8.23}$$

Then the convolution equation

$$\alpha_0 \dot{u} + \alpha_1 * \dot{u} + \beta_0 A u + \beta_1 * A u = f \tag{8.24}$$

has for each $f \in \mathcal{B}^s_{p,q,+}(E_0)$ *a unique solution* $u \in \mathcal{B}^{s+1}_{p,q,+}(E_0) \cap \mathcal{B}^s_{p,q,+}(E_1)$, *and the estimate*

$$\|\dot{u}\|_{\mathcal{B}^s_{p,q}(\mathbb{R},E_0)} + \|u\|_{\mathcal{B}^s_{p,q}(\mathbb{R},E_1)} \leq c\,\|f\|_{\mathcal{B}^s_{p,q}(\mathbb{R},E_0)}$$

is valid, where c *is independent of* f.

Proof. From (7.1), (8.22), and (8.23) we infer that

$$\widetilde{a}(z) := z\big(\alpha_0 + \widetilde{\alpha}_1(z)\big) + \big(\beta_0 + \widetilde{\beta}_1(z)\big)A \in \mathcal{L}\mathrm{is}(E_1, E_0) , \qquad \operatorname{Re} z \geq 0 , \quad z \neq 0 ,$$

and that $|\widetilde{a}^{-1}(\cdot)|_{\mathcal{L}(E_0)}$ is bounded on $(\mathbb{R}^+ + i\,\mathbb{R})^\bullet$. Since the other assumptions imply the validity of the remaining hypotheses of Theorem 8.5, that theorem implies the assertion. $\square$

Corollary 8.7. *Suppose that either the hypotheses of Theorem* 8.5 *or of Proposition* 8.6 *are satisfied. Then the convolution equation* (8.21) *or* (8.24), *respectively, has for each* $f \in L_p(\mathbb{R}^+, E_0)$, $1 \leq p < \infty$, *a unique solution*

$$u \in W^s_p(\mathbb{R}^+, E_0) , \qquad 0 \leq s < 1 ,$$

and the estimate

$$\|u\|_{W^s_p(\mathbb{R},E_0)} \leq c\,\|f\|_{L_p(\mathbb{R}^+,E_0)}$$

is valid, where c *is independent of* f.

Proof. This follows from Theorem 8.5 and Proposition 8.6, respectively, by arguments similar to the one in the proof of Corollary 8.3. $\square$

We leave it to the reader to prove and formulate the corresponding results if f belongs to $BUC(\mathbb{R}, E_0) \cap \mathcal{D}'_+(E_0)$ or if $f \in C_0(\mathbb{R}, E_0) \cap \mathcal{D}'_+(E_0)$, respectively.

There is an extensive literature on operator convolution equations on the half-line, and much of it is collected, unified, and generalized in the comprehensive treatise of PRÜSS [Prü93], to which we refer for details. Theorem 8.5 and Proposition 8.6 are related to — but different from — some maximal regularity theorems of PRÜSS [Prü93, Theorems 7.4 and 7.5] which, in turn, generalize earlier results of DA PRATO-IANNELLI [DI85], LUNARDI [Lun85], and others. The latter authors consider Volterra integral equations

$$\dot{u}(t) + Bu(t) + \int_0^t C(t-s)u(s)\,ds = f(t)\ , \qquad t \geq 0\ , \tag{8.25}$$

with $B \in \mathcal{P}(E_1, E_0; \pi/2)$ and $C : \mathbb{R}^+ \to \mathcal{L}(E_1, E_0)$ satisfying suitable assumptions. Note that (8.25) is a convolution equation of the form (8.21) if we put $A_0 := \delta \otimes 1_{E_0}$ and $A_1 := B \otimes \delta_0 + C$. Again it should be remarked that PRÜSS' theorems are restricted to Besov spaces of positive order so that he cannot deduce from them assertions comparable to the ones of Corollary 8.7.

CLÉMENT and DA PRATO [CD90] study a heat conduction problem for materials of fading memory type of the form

$$\partial_t\Big(b_0 u(t,\cdot) + \int_0^t \beta(t-s)u(s,\cdot)\,ds\Big) = c_0 \Delta u(t,\cdot)\ , \qquad t > 0\ , \tag{8.26}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ under zero Dirichlet boundary conditions. By means of Laplace transformation techniques they investigate existence and regularity questions where they choose $C(\overline{\Omega})$ as basic space E_0. The obtained results are then used to handle semilinear perturbations. Similar problems are treated in [Lun90]. The author of that paper also includes a term of the form

$$\int_0^t \gamma(t-s)\Delta u(s)\,ds\ , \qquad t \geq 0\ . \tag{8.27}$$

We refer to these papers for further references to earlier and related work.

Note that (8.26) has the form (8.21) with $A_0 := (b_0 \otimes \delta + \beta) \otimes 1_{E_0}$ and $A_1 := \delta \otimes A$, where A is the realization in E_0 of $-c_0\Delta$ under Dirichlet boundary conditions. Of course, a term of the form (8.27) can be included by putting $A_1 := (\delta + \gamma) \otimes A$.

Clearly, Corollary 8.7 is an assertion about generalized solutions if we restrict ourselves to L_p-spaces only, that is, do not consider Besov spaces at all. However, the use of Besov spaces allows to obtain rather precise inclusions for the range of the solution operator $f \mapsto u$ of the convolution equation under consideration.

D. Maximal Regularity for Parabolic Cauchy Problems

As an almost trivial corollary of Proposition 8.6 we obtain the following maximal regularity theorem for the parabolic Cauchy problem

$$\dot{u} + Au = f(t)\ , \quad t > 0\ , \qquad u(0) = 0\ , \tag{8.28}$$

where $A \in \mathcal{H}(E_1, E_0)$.

Theorem 8.8. *Suppose that $E_1 \hookrightarrow E_0$ and $A \in \mathcal{P}(E_1, E_0; \pi/2)$. Then*

$$(\partial + A) \in \mathcal{L}\mathrm{is}\big(\mathcal{B}^{s+1}_{p,q,+}(E_0) \cap \mathcal{B}^{s}_{p,q,+}(E_1), \mathcal{B}^{s}_{p,q,+}(E_0)\big) \ .$$

If $f \in \mathcal{B}^{s}_{p,q,+}(E_0) \cap L_{1,\mathrm{loc}}(\mathbb{R}, E_0)$ then

$$(\partial + A)^{-1} f(t) = \int_0^t e^{-(t-\tau)A} f(\tau)\, d\tau \ , \qquad t \geq 0 \ , \tag{8.29}$$

and it is the unique solution in $\mathcal{B}^{s+1}_{p,q,+}(E_0) \cap \mathcal{B}^{s}_{p,q,+}(E_1)$ of problem (8.28).

Proof. The first assertion follows from Proposition 8.6 by putting $\alpha_0 = \beta_0 = 1$ and $\alpha_1 = \beta_1 = 0$. Set $e_A(t) := e^{-tA}$ for $t \geq 0$, and $e_A(t) := 0$ for $t < 0$. Then

$$e_A \in \mathcal{S}'_+\big(\mathcal{L}(E_0)\big) \qquad \text{and} \qquad \widetilde{e_A}(z) = (z + A)^{-1} \ , \quad \mathrm{Re}\, z \geq 0 \ .$$

Hence, letting $a(z) := z + A$ for $\mathrm{Re}\, z \geq 0$, it follows that $a(z) \in \mathcal{L}\mathrm{is}(E_1, E_0)$ and

$$a^{-1}(i\eta) = (i\eta + A)^{-1} = \widetilde{e_A}(i\eta) = \widehat{e_A}(\eta) \ , \qquad \eta \in \mathbb{R} \ ,$$

thanks to (1.5). Thus

$$(\partial + A)^{-1} f = \mathcal{F}^{-1}\big(a^{-1}(i\,\cdot)\big) * f = e_A * f$$

and the second assertion is implied by Example 3.3. □

Let $\mathbb{E}$ be a Banach space of E-valued temperate distributions with suports in $\mathbb{R}^+$. Then $\mathbb{E}$ is said to be a space of maximal regularity for A if $f \in \mathbb{E}$ implies that (8.28) has a unique solution u such that $\dot{u}$ and Au belong to $\mathbb{E}$ as well, and such that

$$\|\dot{u}\|_{\mathbb{E}} + \|Au\|_{\mathbb{E}} \leq c\, \|f\|_{\mathbb{E}} \ ,$$

where c is independent of f.

Corollary 8.9. *Suppose that either $s > 0$, or $s = 0$ and $q = 1$. Then $\mathcal{B}^{s}_{p,q,+}(E_0)$ is a space if maximal regularity for A.*

Proof. From (5.2) and (5.15)–(5.17) we infer that $f \in L_{1,\mathrm{loc}}(\mathbb{R}, E_0)$ whenever f belongs to $\mathcal{B}^{s}_{p,q}(\mathbb{R}, E_0)$ and either $s > 0$, or $s = 0$ and $q = 1$. Hence the assertion follows from Theorem 8.8. □

Theorem 8.8 generalizes — and simplifies considerably the proofs of — earlier results of Da Prato and Grisvard [DG75], Di Blasio [Bla84], and Muramatu [Mur90]. Indeed, Da Prato and Grisvard established the facts that

$$\widehat{W}{}^s_p(\mathbb{R}^+, E_0) := W^s_p(\dot{\mathbb{R}}^+, E_0) \ , \qquad 0 < s < 1/p \ , \quad 1 \leq p < \infty \ ,$$

and

$$BUC^s_0(\mathbb{R}^+, E_0) := \big\{\, u \in BUC^s(\dot{\mathbb{R}}^+, E_0) \ ; \ u(0) = 0 \,\big\} \ , \qquad 0 < s < 1 \ ,$$

are spaces of maximal regularity for A. In [Bla84] it is shown that $\widehat{W}_p^s(\mathbb{R}^+, E_0)$ is a space of maximal regularity for A if $s \in [1/p, 1)$, $1 \le p < \infty$, where

$$\widehat{W}_p^{1/p}(\mathbb{R}^+, E) := \{\, u \in W_p^{1/p}(\dot{\mathbb{R}}^+, E) \ ;\ u \in L_p(\dot{\mathbb{R}}^+, E; dt/t) \,\}$$

and

$$\widehat{W}_p^s(\mathbb{R}^+, E) := \{\, u \in W_p^s(\dot{\mathbb{R}}^+, E) \ ;\ u(0) = 0 \,\} \ , \qquad 1/p < s < 1 \ .$$

It can be proven that

$$\widehat{W}_p^s(\mathbb{R}^+, E) \doteq B_{p,p,+}^s(E) \quad , \qquad 0 < s < 1 \ , \quad 1 \le p < \infty \ ,$$

and that

$$BUC_0^s(\mathbb{R}^+, E) \doteq B_{\infty,\infty,+}^s(E) \ , \qquad 0 < s < 1 \ ,$$

(cf. [Ama97]).

Finally, Muramatu proved that $e_A * f$ belongs to $B_{p,q,+}^{s+1}(E_0)$ whenever f lies in $B_{p,q,+}^s(E_0) \cap L_{1,\mathrm{loc}}(\mathbb{R}, E_0)$, and that $e_A * f$ solves (8.28), provided either $s > 1/p$, or $s = 1/p$ and $q = 1$.

E. Bounded Solutions of Parabolic Equations

Given $\alpha \in \mathbb{R}$, we define the weighted Besov space

$$e^\alpha \mathcal{B}_{p,q}^s := e^\alpha \mathcal{B}_{p,q}^s(\mathbb{R}, E) := \{\, u \in \mathcal{S}'(\mathbb{R}, E) \ ;\ e^{-\langle \alpha, \cdot \rangle} u \in \mathcal{B}_{p,q}^s(\mathbb{R}, E) \,\} \ ,$$

equipped with the norm

$$\|u\|_{e^\alpha \mathcal{B}_{p,q}^s} := \|e^{-\langle \alpha, \cdot \rangle} u\|_{\mathcal{B}_{p,q}^s} \ .$$

It is clear that $e^\alpha \mathcal{B}_{p,q}^s$ is a Banach space.

Suppose that $E_1 \hookrightarrow E_0$ and $A \in \mathcal{L}(E_1, E_0)$. We consider the linear evolution equation

$$\dot{u} + Au = f(t) \ , \qquad t \in \mathbb{R} \ . \tag{8.30}$$

The following proposition gives a sufficient condition for (8.30) to possess a unique solution in the weighted Besov spaces $e^\alpha \mathcal{B}_{p,q}^s$.

Proposition 8.10. *Suppose that $\alpha, \beta \in \mathbb{R}$ and $\theta \in [\pi/2, \pi)$. Also suppose that $\beta + A$ belongs to $\mathcal{P}(E_1, E_0; \theta)$, and that $\sigma(-A) \cap (\alpha + i\,\mathbb{R}) = \emptyset$. Then*

$$\partial + A \in \mathcal{L}\mathrm{is}\big(e^\alpha \mathcal{B}_{p,q}^{s+1}(\mathbb{R}, E_0) \cap e^\alpha \mathcal{B}_{p,q}^s(\mathbb{R}, E_1), e^\alpha \mathcal{B}_{p,q}^s(\mathbb{R}, E_0)\big) \ . \tag{8.31}$$

Proof. Clearly, u solves (8.30) iff $v := e^{-\langle \alpha, \cdot \rangle} u$ solves $\dot{v} + (\alpha + A)v = e^{-\langle \alpha, \cdot \rangle} f$. Moreover, $\sigma(-A) \cap (\alpha + i\,\mathbb{R}) = \emptyset$ iff $\sigma\big(-(\alpha + A)\big) \cap i\,\mathbb{R} = \emptyset$. Hence, by replacing A by $\alpha + A$, we can assume that $\alpha = 0$. Since $\beta + A \in \mathcal{P}(E_1, E_0; \theta)$, and since we can assume that

$\theta > \pi/2$, there exists $r > 0$ such that $i\xi \in \beta + S_\theta$ for $|\xi| \geq r$. Using this fact, the compactness of $i[-r,r] \subset \mathbb{C}$, and $\sigma(-A) \cap i\mathbb{R} = \emptyset$, it follows that

$$(1+|\xi|)^{1-j} \, \|(i\xi + A)^{-1}\|_{\mathcal{L}(E_0,E_j)} \leq c \;, \qquad \xi \in \mathbb{R} \;.$$

Hence, letting $b_1(\xi) := i\xi \, \mathrm{id}_{E_0}$ and $b_2(\xi) := A$ for $\xi \in \mathbb{R}$ as well as $\widetilde{E}_1 := E_0$ and $\widetilde{E}_2 := E_1$, we see that the assumptions of Theorem 7.1 are satisfied with $\ell = 2$, $m_1 = 1$, $m_2 = 0$, and $\omega = \vartheta = 0$. Thus that theorem gives the assertion. □

Proposition 8.10 generalizes considerably a result of Lunardi [Lun95, Theorem 4.4.7 and Proposition 4.4.13], who showed — by a completely different method — that the above hypotheses imply

$$\partial + A \in \mathcal{L}\mathrm{is}\big(e^\alpha BUC^{1+s}(\mathbb{R},E_0) \cap e^\alpha BUC^s(\mathbb{R},E_1), e^\alpha BUC^s(\mathbb{R},E_0)\big)$$

for $0 < s < 1$.

F. Singular Boundary Value Problems

Suppose that $n \geq 2$ and let Ω be an open subset of S^{n-1} such that $\overline{\Omega}$ is a smooth $(n-1)$-dimensional submanifold of S^{n-1} with boundary $\partial\Omega$. We assume that $\partial\Omega$ is the disjoint union of Γ_0 and Γ_1, where Γ_0 and Γ_1 are open and closed in $\partial\Omega$. Of course, if $\Gamma_j = \emptyset$ (in particular: if $\Omega = S^{n-1}$), then any reference to Γ_j has to be neglected in what follows.

We denote by C_Ω the open cone in $\mathbb{R}^n$ with base Ω and vertex at zero, that is,

$$C_\Omega := \big\{\, y \in \mathbb{R}^n \;;\; r := |y| > 0,\ \omega := y/|y| \in \Omega \,\big\} \;.$$

Then we consider the boundary value problem (BVP) for the Laplace operator on $\dot{C}_\Omega$:

$$-\Delta u = f \quad \text{in } \dot{C}_\Omega \;, \qquad \delta\partial_\nu u + (1-\delta)u = 0 \quad \text{on } \partial\dot{C}_\Omega \;, \tag{8.32}$$

where ν is the outer unit-normal on $\partial\dot{C}_\Omega$ and

$$\delta(r,\omega) := \begin{cases} 0 \;, & \omega \in \Gamma_0 \;, \\ 1 \;, & \omega \in \Gamma_1 \;, \end{cases}$$

for $r > 0$. Inserting polar coordinates this BVP transforms into

$$\begin{aligned} r^{-2}\big[(rD_r)^2 - i(n-2)rD_r - \Delta_\Omega\big]u &= f \qquad && \text{in } (\mathbb{R}^+)^\bullet \times \Omega \;, \\ u &= 0 && \text{on } (\mathbb{R}^+)^\bullet \times \Gamma_0 \;, \\ \partial_\nu u &= 0 && \text{on } (\mathbb{R}^+)^\bullet \times \Gamma_1 \;, \end{aligned}$$

where now ν is the outer normal on $(\mathbb{R}^+)^\bullet \times \Gamma_1$, and Δ_Ω denotes the Laplace-Beltrami operator on Ω. By the further substitution $x := \log r$ we arrive at the BVP on the cylinder $\mathbb{R} \times \Omega$:

$$(8.33)\qquad \begin{aligned} \big(D_x^2 - i\,(n-2)D_x - \Delta_\Omega\big)v = g &\qquad \text{in } \mathbb{R} \times \Omega\ , \\ v = 0 &\qquad \text{on } \mathbb{R} \times \Gamma_0\ , \\ \partial_\nu v = 0 &\qquad \text{on } \mathbb{R} \times \Gamma_1\ , \end{aligned}$$

where

$$v(x,\omega) := u(e^x,\omega)\ , \quad g(x,\omega) := e^{2x} f(e^x,\omega)\ .$$

We fix $\rho \in (1,\infty)$ and put

$$W^j_{\rho,\mathcal{B}}(\Omega) := \begin{cases} L_\rho(\Omega)\ , & j=0\ , \\ \{\, u \in W^1_\rho(\Omega)\ ;\ u\,|\,\Gamma_0 = 0 \,\}\ , & j=1\ , \\ \{\, u \in W^2_\rho(\Omega)\ ;\ u\,|\,\Gamma_0 = 0,\ \partial_\nu u\,|\,\Gamma_1 = 0 \,\}\ , & j=2\ , \end{cases}$$

where now ν is the outer unit-normal on $\partial\Omega$ and, of course, integration is performed with respect to the volume measure of Ω. Similarly, we fix $\theta \in (0,1)$ and set

$$h^{j+\theta}_{\mathcal{B}}(\overline{\Omega}) := \begin{cases} \{\, u \in buc^{j+\theta}(\Omega)\ ;\ u\,|\,\Gamma_0 = 0 \,\}\ , & j = 0,1\ , \\ \{\, u \in buc^{2+\theta}(\Omega)\ ;\ u\,|\,\Gamma_0 = 0,\ \partial_\nu u\,|\,\Gamma_1 = 0 \,\}\ , & j=2\ . \end{cases}$$

Then we put either

$$F_j := W^j_{\rho,\mathcal{B}}(\Omega) \quad \text{or} \quad F_j := h^{j+\theta}_{\mathcal{B}}(\overline{\Omega})\ , \qquad j = 0,1,2\ .$$

We also set $a_0 := -\Delta_\Omega\,|\,F_2$ and $a(\xi) := \xi^2 - i\,(n-2)\xi + a_0$ for $\xi \in \mathbb{R}$. Then we reformulate (8.33) as the operator equation

$$(8.34)\qquad a(D)v = g \qquad \text{on } \mathbb{R}\ .$$

Concerning its solvability we prove the following isomorphism theorem:

Proposition 8.11. *Suppose that $\mu := 0$ if $\Gamma_0 \neq \emptyset$, and $\mu > 0$ otherwise. Then*

$$\mu + a(D) \in \mathcal{P}\Big(\bigcap_{j=0}^{2} \mathcal{B}^{s+j}_{p,q}(\mathbb{R}, F_{2-j}), \mathcal{B}^s_{p,q}(\mathbb{R}, F_0); \pi/2\Big)\ .$$

Proof. It is a well-known fact that $\mu + a_0 \in \mathcal{P}(F_2, F_0; \pi/2)$ and $F_1 \in J_{1/2}(F_2, F_0)$ (e.g., [Ama97]). From this we infer that

$$(8.35)\qquad (1+|\lambda|)^{(2-j)/2}\,\|(\lambda + a_0)^{-1}\|_{\mathcal{L}(F_0,F_j)} \le c\ , \qquad j=0,1,2\ , \quad \operatorname{Re}\lambda \ge \omega\ .$$

Put $E_0 := F_0$ and $E_j := F_{3-j}$ for $j = 1,2,3$, and

$$b_1(\xi) := a_0\ , \quad b_2(\xi) := -i\,(n-2)\xi\ , \quad b_3(\xi) := \xi^2\ .$$

Then

$$b_j \in S^{j-1}\big(\mathbb{R}, \mathcal{L}(E_j, E_0)\big)\ , \qquad j=1,2,3\ ,$$

and $b := b_1 + b_2 + b_3 = a$. From (8.35) we deduce that $\lambda \in \rho\bigl(-b(\xi)\bigr)$ for $\xi \in \mathbb{R}$ and $\operatorname{Re}\lambda \geq \omega$, and that

$$(1+|\xi|)^{j-1}\,\bigl|\bigl(\lambda+b(\xi)\bigr)^{-1}\bigr|_{\mathcal{L}(E_0,E_j)} + (1+|\lambda|)^{-1}\,\bigl|\bigl(\lambda+b(\xi)\bigr)^{-1}\bigr|_{\mathcal{L}(E_0)} \leq c$$

for $\xi \in \mathbb{R}$, $\operatorname{Re}\lambda \geq \omega$, and $j = 1, 2, 3$. Hence the assertion follows from Theorem 7.1. □

Corollary 8.12. *Suppose that $\mathcal{B} \in \{\mathring{B}, b\}$. Then*

$$a(D) \in \mathcal{H}\Bigl(\bigcap_{j=0}^{2} \mathcal{B}^{s+j}_{p,q}(\mathbb{R}, F_{2-j}), \mathcal{B}^{s}_{p,q}(\mathbb{R}, F_0)\Bigr) \ .$$

As an application we obtain a solvability result for the singular parabolic initial-boundary value problem in $\dot{C}_\Omega$:

$$\text{(8.36)} \qquad \begin{aligned} \partial_t u - r^2 \Delta u &= f && \text{in } \dot{C}_\Omega \ , \\ \delta \partial_\nu u + (1-\delta) u &= 0 && \text{on } \partial \dot{C}_\Omega \ , \\ u(\cdot, 0) &= u^0 && \text{on } \dot{C}_\Omega \ . \end{aligned} \qquad t > 0 \ ,$$

By means of the above transformations we arrive at the Cauchy problem

$$\text{(8.37)} \qquad \partial_t v + a(D) v = g \ , \quad t > 0 \ , \qquad v(\cdot, 0) = v^0 \ .$$

Thanks to Corollary 8.12 and well-known results concerning parabolic evolution equations (e.g., [Ama95, Theorem II.1.2.1]) it follows that problem (8.37) has for each v^0 belonging to $\mathbb{F}_0 := \mathcal{B}^s_{p,q}(F_0)$ a unique solution

$$v \in C(\mathbb{R}^+, \mathbb{F}_0) \cap C\bigl((\mathbb{R}^+)^\bullet, \mathbb{F}_1\bigr) \cap C^1\bigl((\mathbb{R}^+)^\bullet, \mathbb{F}_0\bigr) \ ,$$

where $\mathbb{F}_1 := \bigcap_{j=0}^2 \mathcal{B}^{s+j}_{p,q}(\mathbb{R}, F_{2-j})$, provided $g \in C(\mathbb{R}^+, \mathbb{F}_0)$, for example. By invoking Theorem 8.8 we also obtain maximal regularity results for (8.37). In addition, by using Corollary 7.2 or Remark 7.7(b), respectively, we can formulate solvability results for equations (8.34) and (8.36) in more conventional function spaces. Lastly, by resubstituting the original variables we arrive at maximal regularity and solvability results in suitable weighted spaces on $\dot{C}_\Omega$. We leave all this to the interested reader.

Elliptic boundary value problems on manifolds with singularities already attracted much attention, and much of the work is collected in the monographs [Gri85], [Dau88], [Sch91], [Sch94a], and [NP94]. Problem (8.32) is one of the simplest model problems and is well understood (see [Gri85] and [NP94]). In particular, the transformation to an elliptic problem on the cylinder $\mathbb{R} \times \Omega$ goes back to the pioneering work of Kondratiev [Kon67]. It is the purpose of this subsection to show that our operator-theoretic approach works in this case as well. Admittedly, it has the drawback that it does not give maximal regularity results in familiar Sobolev spaces based on L_p as the classical approach does (e.g., [NP94]). However, it has the advantage that it opens a way to treat pseudodifferential problems with operator-valued symbols in a non-hilbertian setting. Pseudodifferential operators with operator-valued symbols have been very successfully employed by SCHULZE and coworkers (e.g., [Sch91], [Sch94a], [Sch94b], and [SS94]). However, this approach is restricted to a Hilbert space setting due to the lack of an operator-valued analogue of Mikhlin's theorem for vector-valued L_p-spaces. For the investigation of nonlinear problems on manifolds with singularities such a Hilbert space setting is too narrow.

An operator-valued approach to the Dirichlet problem for the Laplace equation in a cone has been proposed by GRISVARD in [Gri95]. He obtains maximal regularity results in L_p-Sobolev spaces for $1 < p < \infty$ by employing the Dore-Venni theorem [DV87] on the closedness of the sum of two closed operators (also see [DV90], [HP95] for a similar approach in two space dimensions). However, this method has several shortcomings. First: on has to know that the operators under consideration have bounded imaginary powers. The verification of this fact is rather difficult, in general. Indeed, for most operators induced by elliptic boundary value problems this is not known, at least if the coefficients are not infinitely smooth. Second: the underlying Banach spaces have to be UMD spaces, thus reflexive. This rules out the use of Hölder spaces, for example. Third: the operators have to commute. (We refer to [Ama95, Section III.4] for an exposition of the Dore-Venni theorem as well as for the basic results on UMD spaces.) Such a commutativity property cannot be expected, in general, if we consider differential operators whose coefficients depend on $\omega \in \Omega$, for example. Operators of that type occur naturally by freezing the radial variable in variable coefficients problems.

Our approach does not require any of these assumptions. But it uses Besov spaces which are less familiar and more difficult to handle. In this class of spaces it gives optimal results.

Finally, we mention that the singular parabolic problem (8.36) under Dirichlet boundary conditions (that is, the case $\delta = 0$) in a plane domain has been investigated in [Ven94] by means of the Dore-Venni theorem.

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