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# Navier-Stokes Equations with Nonhomogeneous Dirichlet Data

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**Abstract**

We discuss the solvability of the time-dependent incompressible Navier-Stokes equations with nonhomogeneous Dirichlet data in spaces of low regularity.

## 1 Introduction

Throughout this paper $\Omega$ is a subdomain of $\mathbb{R}^3$ having a nonempty compact smooth boundary, $\Gamma$. We consider the nonhomogeneous nonstationary incompressible Navier-Stokes equations

$$\begin{aligned} \nabla \cdot v &= 0 \\ \partial_t v + (v\cdot\nabla)v - \Delta v &= -\nabla\pi + f \end{aligned} \quad \text{in } \Omega\times(0,\infty)\ , \qquad \begin{aligned} v &= g \quad \text{on } \Gamma\times(0,\infty)\ , \\ v(\cdot,0) &= v^0 \quad \text{on } \Omega\ . \end{aligned} \tag{1.1}$$

The exterior force $f$, the boundary velocity $g$, and the initial velocity $v^0$ are the given data, and the velocity $v$ and pressure $\pi$ are the unknowns.

It is well-known that these equations are a mathematical model for the motion of a viscous incompressible fluid, where we have normalized the (mathematically irrelevant) constant viscosity and density to 1.

In this paper we are interested in the case where the boundary data $g$ is different from zero and $v^0$, $f$, and $g$ possess little regularity. In particular, we do not assume that $g$ is a tangential vector field.

Our results are expected to be useful in control problems since we can guarantee the existence of a unique maximal solution (in a natural weak sense) depending continuously on the data, measured in a rather weak topology. The fact that we do not have to worry about compatibility conditions and can work in spaces of low regularity facilitates topological considerations which are necessary in the minimization of cost functionals, for example. To give an idea, we can guarantee the existence of a unique maximal (very weak) solution $(v, \nabla\pi)$ of (1.1) such that $v$ belongs to $L_r\big(J^+, L_q(\Omega)\big)$, where $J^+$ is the maximal interval of existence, provided

$$3 < q < r < \infty \ , \quad 1/r + 3/q \le 1 \ ,$$

and

$$\big(v^0, (f,g)\big) \in L_q(\Omega) \times L_{r,\mathrm{loc}}\big(\mathbb{R}^+, L_q(\Omega) \times L_q(\Gamma)\big) \ .$$

Navier-Stokes equations with nonhomogeneous boundary data in spaces of low regularity have been studied intensively by G. Grubb and V.A. Solonnikov [10], and, in particular, by G. Grubb (see [6]–[9]). These authors use pseudodifferential operator techniques and study the equations in anisotropic Bessel potential and Besov spaces. Thus they obtain fractional time derivatives only, in general. Moreover, they have to require that $v^0$ and $g$ belong to spaces of positive smoothness and that compatibility conditions have to be satisfied.

Our approach is different, based on semigroup and interpolation-extrapolation methods, developed by the author and already applied to the Navier-Stokes equations in [2]–[4]. Our solutions possess time derivatives with respect to some weak topology. Furthermore, we can separate space and time regularity to get very precise results.

We recall the main results of [4], restricting ourselves to the situation where $g = 0$ is not required, and draw some consequences. Then we deduce an improved version of the main existence and uniqueness theorem of [3]. Finally, we give sufficient conditions — a priori estimates in very weak topologies — for the solutions to exist globally. We also show that our results are optimal in the sense that our spaces of initial values cannot be enlarged within their classes.

## 2 Function spaces

We use standard notation and employ the following convention: If $\mathfrak{F}(\Omega, \mathbb{R}^3)$ is a vector space of $\mathbb{R}^3$-valued distributions on $\Omega$ then we simply denote it by $\mathfrak{F}$. If $X$ is a subset of $\mathbb{R}^3$ different from $\Omega$ then we put $\mathfrak{F}(X) := \mathfrak{F}(X, \mathbb{R}^3)$. For example, $\mathcal{D}$, resp. $\mathcal{D}(\overline{\Omega})$, is the space of smooth $\mathbb{R}^3$-valued functions having compact support in $\Omega$, resp. $\overline{\Omega}$, and $W_q^s(\Gamma)$ is the Sobolev-Slobodeckii space of $\mathbb{R}^3$-valued distributions on $\Gamma$.

We always assume that $q, r \in (1,\infty)$. Then $H_q^s$ and $B_{q,\rho}^s$, $1 \le \rho \le \infty$, are the usual Bessel potential and Besov spaces, respectively, (of $\mathbb{R}^3$-valued distributions on $\Omega$) for $s \in \mathbb{R}$. (See [2] for more detailed explanations.) We set

$$\langle v, w\rangle := \int_\Omega v \cdot w \, dx \ , \qquad v, w \in \mathcal{D}(\overline{\Omega}) \ ,$$

and, denoting by $d\sigma$ the volume measure of $\Gamma$,

$$\langle v, w\rangle_\Gamma := \int_\Gamma v \cdot w \, d\sigma \ , \qquad v, w \in C(\Gamma) \ .$$

We also use $\langle\cdot,\cdot\rangle$ to denote the standard duality pairings between various spaces of (scalar- and vector-valued) distributions without fearing confusion. Similar conventions hold for $\langle\cdot,\cdot\rangle_\Gamma$. We write $\partial_\nu$ for the derivative on $\Gamma$ with respect to the outer unit normal $\nu$, denote by $\gamma$ the trace, and by $\gamma_\nu$ the normal trace operator, that is, $\gamma_\nu u = \nu\cdot\gamma u$.

We set

$$\boldsymbol{H}_q^s := \begin{cases} \{\, u \in H_q^s \ ;\ \gamma u = 0 \,\} \ , & 1/q < s \le 2 \ , \\ \{\, u \in H_q^{1/q}(\mathbb{R}^3) \ ;\ \mathrm{supp}(u) \subset \overline{\Omega} \,\} \ , & s = 1/q \ , \\ H_q^s \ , & 0 \le s < 1/q \ , \\ (\boldsymbol{H}_{q'}^{-s})' \ , & -2 \le s < 0 \ , \end{cases} \tag{2.1}$$

where the dual space is determined by means of the duality pairing $\langle\cdot,\cdot\rangle$. It follows (cf. [14, Theorems 4.7.1(a) and 4.8.1]) that

$$\boldsymbol{H}_q^s = H_q^s \ , \qquad -2 + 1/q < s < 1/q \ . \tag{2.2}$$

(In [14] the case of a bounded $\Omega$ is considered only. However, it is easy to verify that all results in that book cited here and below continue to hold if it is only assumed that $\Gamma$ is compact.) In [3, Remark 1.5] it is shown that

$$\mathcal{D}_0(\overline{\Omega}) := \{\, \varphi \in \mathcal{D}(\overline{\Omega}, \mathbb{R}) \ ;\ \gamma\varphi = 0 \,\}$$

is dense in $\boldsymbol{H}_q^s$ for $|s| \le 2$.

We denote by $\mathbb{H}_q$ the closure of $\mathcal{D}_\sigma := \{\, u \in \mathcal{D} \ ;\ \nabla\cdot u = 0 \,\}$ in $L_q$. Recall (e.g., [5], [11]–[13]) that

$$\mathbb{H}_q = \{\, u \in L_q \ ;\ \nabla\cdot u = 0,\ \gamma_\nu u = 0 \,\} \ .$$

We put

$$\mathbb{H}_q^s := \begin{cases} \boldsymbol{H}_q^s \cap \mathbb{H}_q \ , & 0 \le s \le 2 \ , \\ (\mathbb{H}_{q'}^{-s})' \ , & -2 \le s < 0 \ , \end{cases} \tag{2.3}$$

the dual spaces being determined by means of the duality pairing $\langle\cdot,\cdot\rangle_\sigma$, obtained by restricting $\langle\cdot,\cdot\rangle$ to $\mathbb{H}_{q'} \times \mathbb{H}_q$.

Similarly,

$$\boldsymbol{B}_{q,r}^s := \begin{cases} \{\, u \in B_{q,r}^s \ ;\ \gamma u = 0 \,\} \ , & 1/q < s \le 2 \ , \\ \{\, u \in B_{q,r}^{1/q}(\mathbb{R}^3) \ ;\ \mathrm{supp}(u) \subset \overline{\Omega} \,\} \ , & s = 1/q \ , \\ B_{q,r}^s \ , & 0 \le s < 1/q \ , \\ (\boldsymbol{B}_{q',r'}^{-s})' \ , & -2 \le s < 0 \ , \end{cases} \tag{2.4}$$

the dual space being determined by means of $\langle\cdot,\cdot\rangle$, and

$$\mathbb{B}_{q,r}^s := \begin{cases} \boldsymbol{B}_{q,r}^s \cap \mathbb{H}_q \ , & 0 < s \le 2 \ , \\ \text{the closure of } \mathcal{D}_\sigma \text{ in } B_{q,r}^0 \ , & s = 0 \ , \\ (\mathbb{B}_{q',r'}^{-s})' \ , & -2 \le s < 0 \ , \end{cases} \tag{2.5}$$

where now the dual spaces are determined by the pairing $\langle\cdot,\cdot\rangle_\sigma$. Similarly as for the Bessel potential spaces,

$$\boldsymbol{B}^s_{q,r} = B^s_{q,r}\ , \qquad -2+1/q < s < 1/q\ . \tag{2.6}$$

In general,

$$E^s_q \overset{d}{\hookrightarrow} E^t_q\ , \qquad E_q \in \{\, \boldsymbol{H}_q, \mathbb{H}_q, \boldsymbol{B}_{q,r}, \mathbb{B}_{q,r}\ ;\ 1<r<\infty \,\}\ , \quad s>t\ , \tag{2.7}$$

where the superscript $d$ means 'dense embedding'. Thus it follows from (2.2)–(2.4) and (2.6) that $\mathbb{H}^s_q$, resp. $\mathbb{B}^s_{q,r}$, is the closure of $\mathbb{H}_q$ in $H^s_q$, resp. $B^s_{q,r}$, for $-2+1/q<s\le 0$.

Lastly,

$$\mathbb{G}_q := \{\, v\in L_q\ ;\ v=\nabla\pi,\ \pi\in L_{q,\mathrm{loc}}(\overline{\Omega},\mathbb{R}) \,\}$$

and

$$\mathbb{G}^s_q := \boldsymbol{H}^s_q \cap \mathbb{G}_q\ , \qquad 0\le s\le 2\ ,$$

whereas

$$\mathbb{G}^s_q \text{ is the closure of } \mathbb{G}_q \text{ in } \boldsymbol{H}^s_q \text{ for } -2\le s<0\ .$$

Observe that we can also define spaces $\boldsymbol{B}^s_{q,\rho}$ and $\mathbb{B}^s_{q,\rho}$ for $\rho\in\{1,\infty\}$ and $0\le s\le 2$ by replacing $r$ in (2.4) and (2.5), respectively, by $\rho$. Thus we obtain well-defined scales $\boldsymbol{B}^s_{q,\infty}$ and $\mathbb{B}^s_{q,\infty}$ for $|s|\le 2$ by setting

$$\boldsymbol{B}^s_{q,\infty} := (B^{-s}_{q',1})'\ , \quad \mathbb{B}^s_{q,\infty} := (\mathbb{B}^{-s}_{q',1})'\ , \qquad -2\le s<0\ ,$$

with respect to the duality pairings $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_\sigma$, respectively. Finally, we put

$$\mathring{\boldsymbol{B}}^s_{q,\infty} = \text{closure of } \boldsymbol{H}^s_q \text{ in } \boldsymbol{B}^s_{q,\infty}\ , \quad \mathring{\mathbb{B}}^s_{q,\infty} := \text{closure of } \mathbb{H}^s_q \text{ in } \mathbb{B}^s_{q,\infty}$$

for $|s|\le 2$. For $|s|<2$ with $s\ne 0$ the spaces $\mathring{\boldsymbol{B}}^s_{q,\infty}$ and $\mathring{\mathbb{B}}^s_{q,\infty}$ are denoted in [2] by $n^s_{q,0}$ and $n^s_{q,0,\sigma}$, respectively, and called little Nikol'skii spaces.

# 3 Integrable data

Throughout this section we suppose that

- $3<q<r<\infty,\ 1/r+3/q\le 1$ ; (3.1a)
- $(f,g)\in L_{r,\mathrm{loc}}\big(\mathbb{R}^+, \boldsymbol{H}^{-2+1/r}_q \times W^{-1/q+1/r}_q(\Gamma)\big)$ ; (3.1b)
- $v^0\in \mathbb{B}^{-1/r}_{q,r}$ . (3.1c)

Let $J$ be a subinterval of $\mathbb{R}^+$ containing 0 such that $\dot{J} := J\backslash\{0\}\ne\emptyset$. Set $J^* := J\backslash\{\sup J\}$. The pair $(v,w)$ is said to be a (*very weak*) $L_r(H^{1/r}_q)$-*solution* of the Navier-Stokes equations (1.1) on $J$ if

$$(v,w)\in L_{r,\mathrm{loc}}(J^*, \mathbb{H}^{1/r}_q \times \mathbb{G}^{-2+1/r}_q) \tag{3.2}$$

and

$$\begin{aligned}&\int_J \big\{ \langle \partial_t\varphi + \Delta\varphi, v\rangle + \langle \nabla\varphi, v\otimes v\rangle \big\}\, dt \\ &= \int_J \big\{ \langle w,\varphi\rangle - \langle f,\varphi\rangle + \langle g, \partial_\nu\varphi\rangle_\Gamma \big\}\, dt - \big\langle v^0, \varphi(0)\big\rangle\end{aligned} \tag{3.3}$$

for all $\varphi \in \mathcal{D}\big(J^*, \mathcal{D}_0(\overline{\Omega})\big)$.

A solution is maximal if there does not exist another solution (of the same type) being a proper extension of it.

Clearly, (3.3) is *formally* obtained from the second differential equation in (1.1), the momentum equation, by multiplying it by $\varphi$, integrating by parts, using Green's formula, the boundary and initial data, and setting $w := \nabla\pi$.

By admitting in (3.3) standard test functions $\varphi \in \mathcal{D}(J^*, \mathcal{D}) = \mathcal{D}(\Omega\times J^*)$ only, it follows that a very weak solution is a distributional solution of the momentum equation.

In the remainder of this paper we write $\nabla\pi$ for $w$ without fearing confusion.

The following theorem guarantees the existence of a unique maximal $L_r(H_q^{1/r})$-solution of (1.1) and gives further regularity properties.

**Theorem 1.** *Let assumptions* (3.1) *be satisfied. Then the Navier-Stokes equations* (1.1) *possess a unique maximal* $L_r(H_q^{1/r})$*-solution,* $(v,\nabla\pi)$*. The maximal interval of existence,* $J^+$*, that is,* $\mathrm{dom}(v,\nabla\pi)$*, is open in* $\mathbb{R}^+$*. Moreover,*

$$v \in C(J^+, \mathbb{B}_{q,r}^{-1/r})\ , \qquad (\dot v, \nabla\pi) \in L_{r,\mathrm{loc}}(J^+, \mathbb{H}_q^{-2+1/r} \times \mathbb{G}_q^{-2+1/r})\ . \tag{3.4}$$

*If* $J^+ \neq \mathbb{R}^+$ *then* $v$ *is not uniformly continuous on* $J^+$*.*

**Proof.** This is a particular case of [4, Theorem]. ■

**Remarks 1. (a)** In [3] it is shown that

$$\boldsymbol{H}_q^{-2+1/r} \cong H_q^{-2+1/r} \times W_q^{-1/q+1/r}(\Gamma)\ . \tag{3.5}$$

Thus $\boldsymbol{H}_q^{-2+1/r}$ is not a space of distributions on $\Omega$ but contains also distributions being supported on $\Gamma$. This explains why there is no compatibility condition for $g$ guaranteeing that $g$ is a tangential vector field. Indeed, a possible nontrivial normal component of $g$ is compensated by the 'boundary part' of the (generalized) pressure gradient $w$.

**(b)** Due to (3.5) our hypotheses on $(f,g)$ are ambiguous in the sense that $f$ may 'contain a part on $\Gamma$' which should be covered by $g$. It is tempting to assume that

$$f \in L_{r,\mathrm{loc}}(\mathbb{R}^+, H_q^{-2+1/r})\ . \tag{3.6}$$

However, then

$$\int_J \langle f,\varphi\rangle\, dt \tag{3.7}$$

is not necessarily well-defined for all test functions $\varphi \in \mathcal{D}\big(J^*, \mathcal{D}_0(\overline{\Omega})\big)$. To guarantee that (3.7) is meaningful for a given $f$ with (3.6), it suffices, for example, to assume that

$f = f_0 + f_1$ with $\operatorname{dist}\bigl(\operatorname{supp}(f_0), \Gamma \times \mathbb{R}^+\bigr) > 0$ and $f_1 \in L_{r,\mathrm{loc}}(\mathbb{R}^+, H_q^\sigma)$ with $\sigma > -2 + 1/q$ (see [4]).

**(c)** Given $T > 0$, the velocity part of the maximal solution satisfies

$$v \in BUC\bigl(J^+ \cap [0,T], \mathbb{B}_{q,r}^{-1/r}\bigr) ,$$

where $BUC$ is the space of bounded and uniformly continuous functions, iff

$$v \in L_r\bigl(J^+ \cap [0,T], \mathbb{H}_q^{1/r}\bigr) . \tag{3.8}$$

Thus, if it can be shown that

- either $v \in BUC\bigl(J^+ \cap [0,T], B_{q,r}^{-1/r}\bigr)$ ,
- or $v \in L_r\bigl(J^+ \cap [0,T], H_q^{1/r}\bigr)$

for every $T > 0$, then $(v, \nabla\pi)$ is a global solution.

**Proof.** Since $v$ satisfies the differential equation in [4, (5.8)], it is easily verified that (3.8) implies that

$$\dot{v} \in L_r\bigl(J^+ \cap [0,T], \mathbb{H}_q^{-2+1/r}\bigr).$$

Now the assertion is a consequence of [4, Theorem 2.2(i)], Theorem 1, and of (2.2) and (2.6). ■

**(d)** The maximal solution $(v, \nabla\pi)$ depends in the topologies described by (3.2) and (3.4) continuously on $\bigl(v^0, (f,g)\bigr)$, with respect to the topologies specified in (3.1).

**Proof.** See [4, Remark 2.3(c)]. ■

**(e)** In [4, Proposition 3.4] it is shown that

$$\mathbb{B}_{q_1,r_1}^{s_1} \overset{d}{\hookrightarrow} \mathbb{B}_{q_0,r_0}^{s_0} , \tag{3.9}$$

provided

$$s_1 - 3/q_1 \ge s_0 - 3/q_0 , \quad 1 > 1/q_1 \ge 1/q_0 > 0 , \quad 1 > 1/r_1 \ge 1/r_0 > 0 . \tag{3.10}$$

It is known that the corresponding embeddings for the standard Besov spaces are optimal. This implies that conditions (3.9), (3.10) are sharp as well. From this we deduce that the two spaces $\mathbb{B}_{q_1,r_1}^{-1/r_1}$ and $\mathbb{B}_{q_0,r_0}^{-1/r_0}$ are incomparable if $(q_1, r_1) \ne (q_0, r_0)$. In this sense each one of the spaces $\mathbb{B}_{q,r}^{-1/r}$, where $(q,r)$ satisfies (3.1a), is an optimal space of initial values. ■

## 4 Time-continuous data

In this section we suppose that

- $3 < q < \tau < r < \infty,\ -1 + 3/q \le s < 1/r$ ; (4.1a)
- $(f, g) \in C\big(\mathbb{R}^+, \boldsymbol{H}_q^{-2+1/\tau} \times W_q^{-1/q+1/\tau}(\Gamma)\big)$ ; (4.1b)
- $v^0 \in \mathring{\mathbb{B}}^s_{q,\infty}$ . (4.1c)

Thus, in comparison with (3.1b), we now require slightly more smoothness for $f$ and $g$.

The pair $(v, \nabla\pi)$ is said to be a (very weak) $C(H_q^{1/r})$*-solution* of the Navier-Stokes equations (1.1) on $J$ if (3.3) holds (with $w = \nabla\pi$ and) with (3.2) being replaced by

$$(v, \nabla\pi) \in C(\dot{J}, \mathbb{H}_q^{1/r} \times \mathbb{G}_q^{-2+1/r}) \ . \tag{4.2}$$

**Theorem 2.** *Let* (4.1) *be satisfied. Then the Navier-Stokes equations* (1.1) *possess a unique maximal* $C(H_q^{1/r})$*-solution,* $(v, \nabla\pi)$*, satisfying*

$$\lim_{t\to 0} t^{(1/r-s)/2} \, \|v(t)\|_{H_q^{1/r}} = 0 \ . \tag{4.3}$$

*The maximal interval of existence,* $J^+$*, is open in* $\mathbb{R}^+$*,*

$$v \in C^1(\dot{J}, \mathbb{H}_q^{-2+1/r}) \cap C(J, \mathring{\mathbb{B}}^s_{q,\infty}) \ ,$$

*and*

$$\lim_{t\to 0} t^{1/r-s} \, \|\nabla\pi(t)\|_{\boldsymbol{H}_q^{-2+1/r}} = 0 \ . \tag{4.4}$$

**Proof.** Given Banach spaces $E$ and $F$, we write $\mathcal{L}(E, F)$ for the Banach space of all bounded linear operators from $E$ into $F$, and $\mathcal{L}^2(E, F)$ is the Banach space of all continuous bilinear maps from $E$ into $F$.

We denote by $\mathfrak{R} \in \mathcal{L}\big(W_q^{-1/q+1/\tau}(\Gamma), \boldsymbol{H}_q^{-2+1/\tau}\big)$ the dual of the interior normal derivative operator

$$-\partial_\nu \in \mathcal{L}\big(\boldsymbol{H}_{q'}^{2-1/\tau}, W_{q'}^{1/q-1/\tau}(\Gamma)\big) \ . \tag{4.5}$$

The validity of (4.5) is a consequence of the trace theorem. Then

$$f + \mathfrak{R}g \in C(\mathbb{R}^+, \boldsymbol{H}_q^{-2+1/\tau}) \ . \tag{4.6}$$

We also set $B(v, w) := \nabla \cdot (v \otimes w)$ and recall from [4, Lemma 4.1] that

$$B \in \mathcal{L}^2(\mathbb{H}_q^{1/r}, \boldsymbol{H}_q^{2/r-1-3/q}) \ , \tag{4.7}$$

By $\boldsymbol{A}$ we mean the unique extension in $\mathcal{L}(\boldsymbol{H}_q^{1/r}, \boldsymbol{H}_q^{-2+1/r})$ of $-\Delta | \boldsymbol{H}_q^2 \in \mathcal{L}(\boldsymbol{H}_q^2, L_q)$, the negative Laplace operator in $L_q$ with Dirichlet boundary conditions. Then $\boldsymbol{A}$ is well-defined and, considered as an unbounded linear operator in $\boldsymbol{H}_q^{-2+1/r}$, it generates a strongly continuous analytic semigroup on $\boldsymbol{H}_q^{-2+1/r}$ (see [2, Section 2 ]).

Now we consider the differential equation

$$\dot{v} + \boldsymbol{A}v = -\nabla\pi + B(v,v) + f + \mathfrak{R}g \quad \text{in } \dot{J}\ , \qquad v(0) = v^0 \tag{4.8}$$

in $\boldsymbol{H}_q^{-2+1/r}$. From [3, Theorem 2.2] we know that

$$\boldsymbol{H}_q^{-2+1/r} = \mathbb{H}_q^{-2+1/r} \oplus \mathbb{G}_q^{-2+1/r}$$

and that the corresponding projection $\boldsymbol{P}$ from $\boldsymbol{H}_q^{-2+1/r}$ onto $\mathbb{H}_q^{-2+1/r}$ is the unique continuous extension of the Helmholtz projector $P:\ L_q \to \mathbb{H}_q$. Moreover, $\mathbb{A} := \boldsymbol{P}\boldsymbol{A}\,|\,\mathbb{H}_q^{1/r}$ is the unique extension in $\mathcal{L}(\mathbb{H}_q^{1/r}, \mathbb{H}_q^{-2+1/r})$ of the Stokes operator $-P\Delta\,|\,\mathbb{H}_q^2 \in \mathcal{L}(\mathbb{H}_q^2, \mathbb{H}_q)$, and $-\mathbb{A}$ is the infinitesimal generator of an analytic semigroup $\{\, \mathbb{U}(t)\ ;\ t \geq 0 \,\}$ on $\mathbb{H}_q^{-2+1/r}$, the unique continuous extension over $\mathbb{H}_q^{-2+1/r}$ of the Stokes semigroup on $\mathbb{H}_q$.

Now we set $b := \boldsymbol{P}B$ and $h := \boldsymbol{P}(g + \mathfrak{R}g)$ and consider the evolution equation

$$\dot{v} + \mathbb{A}v = b(v,v) + h \quad \text{in } \dot{J}\ , \qquad v(0) = v^0 \tag{4.9}$$

in $\mathbb{H}_q^{-2+1/r}$, which is obtained by projecting (4.8) into $\mathbb{H}_q^{-2+1/r}$. From (4.7) we deduce that

$$b \in \mathcal{L}^2(\mathbb{H}_q^{1/r}, \mathbb{H}_q^{2/r-1-3/q})\ , \tag{4.10}$$

and (4.6) implies

$$h \in C(\mathbb{R}^+, \mathbb{H}_q^{-2+1/\tau})\ . \tag{4.11}$$

Put $2\gamma := (1/r + 1 - 3/q) \wedge (1/\tau - 1/r)$. Then (2.7), (4.10), and (4.11) imply

$$b \in \mathcal{L}^2(\mathbb{H}_q^{1/r}, \mathbb{H}_q^{-2+1/r+2\gamma})\ , \quad h \in C(\mathbb{R}^+, \mathbb{H}_q^{-2+1/r+2\gamma})\ . \tag{4.12}$$

Note that $0 < 2\gamma < 1$.

From [2, Theorem 3.4] we infer that Status: O

$$\mathbb{H}_q^{-2+1/r+2\gamma} \doteq [\mathbb{H}_q^{-2+1/r}, \mathbb{H}_q^{1/r}]_\gamma\ , \tag{4.13}$$

where $[\cdot,\cdot]_\gamma$ denotes the complex interpolation functor of exponent $\gamma$. Furthermore, setting $2\alpha := 1/r - s$, we also infer from [2, Theorem 3.4] that

$$\mathring{\mathbb{B}}^s_{q,\infty} \doteq (\mathbb{H}_q^{-2+1/r}, \mathbb{H}_q^{1/r})^0_{1-\alpha,\infty} \tag{4.14}$$

with $(\cdot,\cdot)^0_{1-\alpha,\infty}$ being the continuous interpolation functor of exponent $1-\alpha$, and that $\mathbb{U}\,|\,\mathring{\mathbb{H}}^s_q$ is strongly continuous. Hence, setting

$$E_0 := \mathbb{H}_q^{-2+1/r}\ , \quad E_1 := \mathbb{H}_q^{1/r}\ , \quad F_{1-\alpha} := \mathring{\mathbb{B}}^s_{q,\infty}\ ,$$

it follows from (4.12)–(4.14) and [2, Theorem 5.6] that (4.9) possesses a unique maximal solution

$$v \in C(J^+, \mathring{\mathbb{B}}^s_{q,\infty}) \cap C(\dot{J}^+, \mathbb{H}_q^{1/r}) \cap C^1(\dot{J}^+, \mathbb{H}_q^{-2+1/r}) \tag{4.15}$$

satisfying (4.3), and that $J^+$ is open in $\mathbb{R}^+$.

Now we put

$$\nabla\pi := (1-\boldsymbol{P})\big(-\boldsymbol{A}v + B(v,v) + f + \mathfrak{R}g\big) \ .$$

Then it follows from (4.6), (4.7), the continuity of $\boldsymbol{A}$ as a map from $\mathbb{H}_q^{1/r}$ into ${\boldsymbol{H}_q}^{-2+1/r}$, the definition of $\boldsymbol{P}$, and (4.15) that

$$\nabla\pi \in C(\dot{J}^+, {\mathbb{G}_q}^{-2+1/r})$$

and that (4.4) is true. It is obvious that $(v,\nabla\pi)$ satisfies (4.8) with $J = J^+$.

Set $F := -\nabla\pi + B(v,v) + f + \mathfrak{R}g$. Then (4.2)–(4.7) imply

$$F \in C(\dot{J}^+, {\boldsymbol{H}_q}^{-2+1/r}) \ , \qquad \lim_{t\to 0} t^{2\alpha}\, \|F(t)\|_{{\boldsymbol{H}_q}^{-2+1/r}} = 0 \ . \tag{4.16}$$

Fix $p \in (1, 1/2\alpha)$. We deduce from (2.2), (4.3), and (4.16) that

$$v \in L_{p,\mathrm{loc}}(J^+, H_q^{1/r}) \ , \quad \boldsymbol{A}v, F \in L_{p,\mathrm{loc}}(J^+, {\boldsymbol{H}_q}^{-2+1/r}) \ .$$

Hence $\dot{v} = -\boldsymbol{A}v + F \in L_{p,\mathrm{loc}}(J^+, {\boldsymbol{H}_q}^{-2+1/r})$, so that $v$ is a $W_{p,\mathrm{loc}}^1$-solution of

$$\dot{v} + \boldsymbol{A}v = F \quad \text{in } \dot{J}^+ \ , \qquad v(0) = v^0 \tag{4.17}$$

in ${\boldsymbol{H}_q}^{-2+1/r}$ in the sense of [1, Section III.1.3]. Thus Theorem V.2.8.3 of that book (with $\alpha := 0$ and $E_0 := {\boldsymbol{H}_q}^{-2+1/r}$) guarantees that $v$ is the unique maximal $L_p(H_q^{1/r})$-solution of (1.1). This implies that $(v,\nabla\pi)$ is the unique maximal $C(H_q^{1/r})$-solution of (1.1) satisfying (4.3). ■

**Remarks 2.** **(a)** Suppose that for each $T > 0$ one of the following conditions is satisfied:

- $v \in BUC\big(J^+ \cap [0,T], B_{q,\infty}^{-1+3/q}\big)$ ;
- $v\big(J^+ \cap [0,T]\big)$ is relatively compact in $B_{q,\infty}^{-1+3/q}$ ;
- there exist $t_0 \in J^+$ and $\sigma > -1 + 3/q$ such that
$$\sup_{t\in J^+\cap[t_0,T]} \|v(t)\|_{W_q^\sigma} < \infty \ .$$

Then $J^+ = \mathbb{R}^+$.

**Proof.** For $-2+1/q < t < 1/q$ one finds, similarly as (2.6), that $\boldsymbol{B}_{q,\infty}^t = B_{q,\infty}^t$. By interpolating (with $(\cdot,\cdot)_{q,\infty}$) one deduces from [3, Theorem 2.2] that $\mathbb{B}_{q,\infty}^t$ is a closed linear subspace of $\boldsymbol{B}_{q,\infty}^t = B_{q,\infty}^t$. From this and $v \in C(J^+, \mathring{\mathbb{B}}_{q,\infty}^{-1+3/q})$ it follows that $B$ can be replaced by $\mathring{\mathbb{B}}$ in the first two conditions above. Similarly, one can replace $W_q^\sigma = B_{q,q}^\sigma$ by $\mathbb{B}_{q,q}^\sigma$ in the third hypothesis. Now the assertion follows from the proof of Theorem 2 and [2, Remarks 5.9(b) and (d)], thanks to $\mathbb{B}_{q,q}^\sigma \hookrightarrow \mathbb{B}_{q,\infty}^\sigma$. ■

**(b)** Everything said above remains valid if we replace the continuity hypothesis for $(f,g)$ by

$$(f,g) \in C\big((0,\infty), \boldsymbol{H}_q^{-2+1/\tau} \times W_q^{-1/q+1/\tau}(\Gamma)\big)$$

and $\lim_{t\to 0} t^{\alpha+\gamma}\big(f(t), g(t)\big) = 0$ in $\boldsymbol{H}_q^{-2+1/\tau} \times W_q^{-1/q+1/\tau}(\Gamma)$.

**Proof.** This is a consequence of [2, Theorem 5.6] and the proof of Theorem 2. ■

**(c)** It can be shown that $\mathbb{B}^s_{q,r} \overset{d}{\hookrightarrow} \mathring{\mathbb{B}}^s_{q,\infty}$. Suppose that $3 < q < r < \infty$ and $1/r + 3/q < 1$, and that $(f, g)$ satisfies (4.1b). Then $(f, g)$ satisfies (3.1b) as well. Hence, given $v^0 \in \mathbb{B}^{-1/r}_{q,r}$, Theorem 1 guarantees the existence of a unique maximal $L_r(H^{1/r}_q)$-solution $(v_r, \nabla\pi_r)$ on $J^+_r$. Since $s := -1/r > -1 + 3/q$, assumptions (4.1) are satisfied as well. Hence Theorem 2 implies that there exists a unique maximal $C(H^{1/r}_q)$-solution $(v_\infty, \nabla\pi_\infty)$ on $J^+_\infty$ satisfying

$$\lim_{t\to 0} t^{1/r} \|v_\infty(t)\|_{H^{1/r}_q} = 0 \ .$$

Note that this does not imply that $v_\infty \in L_{r,\mathrm{loc}}(J^+_\infty, H^{1/r}_q)$ and, conversely, we cannot guarantee that $(v_r, \nabla\pi_r)$ is a $C(H^{1/r}_q)$-solution on $J^+_r$. This shows that Theorems 1 and 2 are independent of each other.

**(d)** If (3.1) is satisfied and $v$ is the unique maximal $L_r(H^{1/r}_q)$-solution of (1.1) then it follows that there are $q_0 \geq q$ and $r_0 \geq r$ satisfying $2/r_0 + 3/q_0 \leq 1$ such that

$$v \in L_{r_0,\mathrm{loc}}(J^+, L_{q_0}) \ . \tag{4.18}$$

Such a 'Serrin' condition is also valid if $v$ is a $C(H^{1/r}_q)$-solution, provided $(f, g)$ is appropriately smooth (cf. [2, Remark 9.5(b)]). This fact can be employed to obtain regularity results for these solutions.

**Proof.** Condition (4.18) is a consequence of [4, Theorm 3.3]. ■

**Corollary 1.** *Suppose that $v^0$ and $f$ are smooth and $g = 0$. If the hypothesis of Remark 2(a) is satisfied then the Navier-Stokes equations possess a unique global smooth solution.*

**Proof.** This is an immediate consequence of Remarks 2(a) and (d). ■

Of course, the assumption that $g = 0$ can be replaced by assuming that $g$ is a smooth tangential vector field. Furthermore,

$$B^{-1+3/q_0}_{q_0,\infty} \hookrightarrow B^{-1+3/q_1}_{q_1,\infty} \ , \qquad 3 < q_0 < q_1 < \infty \ ,$$

where these embeddings are proper. Increasing $g$ should thus make the task of verifying one of the conditions in Remark 2(a) easier.

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