On the Strong Solvability of the Navier-Stokes Equations

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Abstract. In this paper we study the strong solvability of the Navier-Stokes equations for rough initial data. We prove that there exists essentially only one maximal strong solution and that various concepts of generalized solutions coincide. We also apply our results to Leray-Hopf weak solutions to get improvements over some known uniqueness and smoothness theorems. We deal with rather general domains including, in particular, those having compact boundaries.

Mathematical Subject Classification (1991). 35Q30, 76D05, 35K55.

Keywords. Weak, very weak, and strong solutions; existence and uniqueness theorems with rough initial data.

0. Introduction

Throughout this paper $m \geq 2$ and either $\Omega = \mathbb{R}^m$ or Ω is a subdomain of $\mathbb{R}^m$ with a smooth boundary $\partial\Omega$. We consider the nonstationary Navier-Stokes equations

$$\begin{aligned} \nabla \cdot v &= 0 && \text{in } \Omega\ , \\ \partial_t v + (v \cdot \nabla)v - \nu\Delta v &= -\nabla p + f && \text{in } \Omega\ , \\ v &= 0 && \text{on } \partial\Omega\ , \\ v(\cdot, 0) &= v^0 && \text{in } \Omega\ , \end{aligned} \tag{0.1}$$

describing the motion of a viscous incompressible Newtonian fluid with a non-slip boundary condition (if $m = 2$ or $m = 3$, of course). Here $f : \Omega \times \mathbb{R}^+ \to \mathbb{R}^m$ is a given outer force field, $v^0 : \Omega \to \mathbb{R}^m$ is the prescribed initial velocity, and $v : \Omega \times \mathbb{R}^+ \to \mathbb{R}^m$ and $p : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ are the unknown velocity and pressure field, respectively. Moreover, ν is a given positive constant, the kinematic viscosity, and the (constant) density has been normalized to 1.

Assuming that Ω admits Helmholtz decompositions of $L_q := L_q(\Omega, \mathbb{R}^m)$ for each $q \in (1, \infty)$ we denote by P the projector onto the solonoidal vector fields. (We refer to the main body of this paper for precise assumptions and definitions of all the (more or less standard) concepts and function spaces of which we make free use in this introduction.) Then we eliminate the pressure field p by applying P to the second equation of (0.1) and arrive at the reduced Navier-Stokes system

$$\begin{aligned} \nabla\cdot v &= 0 && \text{in } \Omega\ , \\ \partial_t v + P(v\cdot\nabla)v - \nu P\Delta v &= Pf && \text{in } \Omega\ , \\ v &= 0 && \text{on } \partial\Omega\ , \\ v(\cdot,0) &= v^0 && \text{in } \Omega\ . \end{aligned} \tag{0.2}$$

By a solution (in any sense) of the Navier-Stokes equations, that is, of system (0.1), we mean a velocity field v satisfying (0.2). This is justified since any such solution v determines the pressure field p up to an inessential constant (if v is regular enough).

In this paper we are mainly interested in strong solutions (in a sense made precise below) for rough initial data, that is, if v^0 belongs to an appropriately wide class of functions.

In order to describe our main results and to compare them with the work of other authors we restrict ourselves in this introduction to the case

$$m \geq 3\ , \quad f = 0 \tag{0.3}$$

(more precisely, to the case where f is a conservative force field so that $Pf = 0$). The general situation is dealt with in the following sections.

First we consider the simplest case:

$$\Omega = \mathbb{R}^m\ . \tag{0.4}$$

Then (0.2) reduces to the system

$$\begin{aligned} \nabla\cdot v &= 0\ , \\ \partial_t v - \nu\Delta v &= -P(v\cdot\nabla)v\ , \\ v(\cdot,0) &= v^0 \end{aligned} \tag{0.5}$$

in $\mathbb{R}^m$ since P commutes with the Laplace operator. This case has been widely studied and there is an enormous amount of literature on this subject.

The first result on the solvability of (0.5), when v^0 belongs to L_q, is due to Fabes, Jones, and Rivière [18]. These authors show that the Navier-Stokes equations possess a unique local solution v in the class $L_r\big((0,T),L_s\big)$ with q, r, and s satisfying $s > m$ and

$$m/q < 2/r + m/s \leq 1\ , \tag{0.6}$$

provided $v^0 \in L_{q,\sigma} := PL_q$. By a solution in $L_r\big((0,T),L_s\big)$ they mean a **very weak solution** in $L_r\big((0,T),L_{s,\sigma}\big)$, that is, a function $v \in L_r\big((0,T),L_{s,\sigma}\big)$ satisfying

$$\int_0^T \big\{\big\langle(\partial_t+\nu\Delta)\varphi,v\big\rangle + \langle\nabla\varphi, v\otimes v\rangle\big\}\,dt = -\big\langle\varphi(0),v^0\big\rangle \tag{0.7}$$

for all $\varphi \in \mathcal{D}\big([0,T),\mathcal{D}_\sigma\big)$, where $\mathcal{D}_\sigma := \big\{\,\varphi\in\mathcal{D} := \mathcal{D}(\Omega,\mathbb{R}^m)\ ;\ \nabla\cdot\varphi = 0\,\big\}$ and $\langle\cdot,\cdot\rangle$ denotes the usual L_s-duality pairing (cf. Remark 7.1(a)). It is also shown in [18] that $T=\infty$, provided v^0 is sufficiently small in $L_q\cap L_{q'}$.

If $v^0 \in L_{q,\sigma}$ with $q > m$ then the existence of a unique very weak local solution v, being a weakly continuous function from $[0,T)$ into $L_{q,\sigma}$, has also been shown, by different techniques, by Beirão da Veiga [6]. Furthermore,

$$v \in C\big([0,T), L_{2,\sigma} \cap L_{q,\sigma}\big) \cap L_2\big((0,T), H_2^1 \cap L_{q,\sigma}\big) \tag{0.8}$$

if $v^0 \in L_{2,\sigma} \cap L_{q,\sigma}$. In addition, Beirão da Veiga gives an estimate for the maximal existence time (depending on $\|v^0\|_{L_q}$) and proves that the solution exists globally, that is, for all time, if the norm of v^0 in $L_2 \cap L_q$ is sufficiently small.

More recently, C.P. Calderón (see [10], [11], [12]) obtained the existence of very weak local solutions of (0.1) for $q = m$ also.

The case $q = m$ is critical since then the nonlinear term $P(v \cdot \nabla)v$ has the 'same strength' as the Laplace operator, that is, $P(v \cdot \nabla)v$ is not subordinate to $-\Delta v$. Thus one cannot take advantage of the regularizing effect of the heat semigroup which is the basis of practically all known existence proofs. The criticality of L_m is also manifest in the scaling invariance $\|\lambda u(\lambda\cdot)\|_{L_m} = \|u\|_{L_m}$ for $\lambda > 0$ (see [13] for a detailed exposition of 'critical spaces' for Navier-Stokes equations).

The critical case $q = m$ has first been treated by Kato [45]. He showed, by using some ideas developed earlier, jointly with Fujita ([23], [46]), that, given $v^0 \in L_{m,\sigma}$, there exist $T > 0$ and a unique solution v of (0.1) in the class

$$C\big([0,T], L_{m,\sigma}\big) \cap C_{(1-m/q)/2}\big((0,T], L_{q,\sigma}\big) \ , \qquad m < q < \infty \ . \tag{0.9}$$

Here, given any Banach space E, any $\mu \in \mathbb{R}$, and any subinterval J of $\mathbb{R}^+$ containing 0 such that $\dot{J} := J \backslash \{0\} \neq \emptyset$, we denote by $C_\mu(\dot{J}, E)$ the Banach space consisting of all $u \in C(\dot{J}, E)$ such that $\big(t \mapsto t^\mu u(t)\big) \in BC(\dot{J}, E)$ and $t^\mu u(t) \to 0$ as $t \to 0$, equipped with the obvious norm.

A solution on $[0,T]$ is in this case a function v in (0.9) satisfying

$$v(t) = e^{-tS} v^0 + \int_0^t e^{-(t-\tau)S} b(v,v)(\tau)\, d\tau \ , \qquad 0 \le t \le T \ , \tag{0.10}$$

where S is the Stokes operator $-\nu P \Delta$ (hence $S = -\nu\Delta$ in the case under consideration) and

$$b(u,v) := -P(u \cdot \nabla)v = -P\nabla \cdot (u \otimes v)$$

on solonoidal vector fields. (In fact, this is a simplified version of Kato's result (see [13] and [85]), since in [45] class (0.9) is more restricted.) In [45] it is also shown that v is global if $\|v^0\|_{L_m}$ is sufficiently small.

The case $m = 3$ has recently been extensively studied by Cannone and Meyer [15] and Cannone [13]. Motivated by a wavelet approach of Federbush [21], in [15] there is proven an abstract local existence and uniqueness theorem for mild solutions of (0.1). By a **mild solution** (in E) of (0.1) on J we mean a function $v \in C(J, E)$ satisfying (0.10) on J, where E is a Banach space of distributions on which the Stokes semigroup $\{ e^{-tS} \ ; \ t \geq 0 \}$ is strongly continuous and the integral in (0.10) is well-defined. By means of Littlewood-Paley decompositions, that is, techniques from harmonic analysis, these authors introduce the concept of a Banach space 'well-suited for the Navier-Stokes equations'. Then they show that their local existence and uniqueness result holds whenever E is well-suited. In particular, in [15] (also see [13]) it is shown that L_q is well-suited if $q > m = 3$. In the same paper it is also shown that the Sobolev spaces $H_2^s := H_2^s(\Omega, \mathbb{R}^m)$ are well-suited for the Navier-Stokes equations if $s > 1/2$ (and, of course, $m = 3$ and $\Omega = \mathbb{R}^3$). Thus, if $m = 3$, there exists for each $s > 1/2$ and each

$$v^0 \in H_{2,0,\sigma}^s := \{ u \in H_2^s \ ; \ \nabla \cdot u = 0 \}$$

a unique mild solution

$$v \in C\big([0,T), H_{2,0,\sigma}^s\big)$$

of the Navier-Stokes equations. (The reason for the index 0 in $H_{2,0,\sigma}^s$ will become clear in (0.17).) Moreover, the existence time T depends on $\|v^0\|_{H_2^s}$ only. This extends an earlier result of Kato [44] who had to suppose that $s > 5/2$.

The more general case where v^0 belongs to a Bessel potential space

$$H_q^s := H_q^s(\Omega, \mathbb{R}^m)$$

has been investigated by Kato and Ponce [47] for $1 < q < \infty$ and $s > 1 + m/q$ if $m = 3$, and, by different techniques, by Ribaud [66]. The last author assumes that

$$1 < q < \infty \ , \quad -1 + m/q < s < (m/q) \wedge (1 + m/q)/2$$

and

$$v \in H_{q,0,\sigma}^s := \{ u \in H_q^s \ ; \ \nabla \cdot u = 0 \} \ .$$

Then he proves that (0.1) possesses a local mild solution

$$v \in C\big([0,T), H_{q,0,\sigma}^s\big) \ . \tag{0.11}$$

It is unique if

$$s \geq m(1/q - 1/2)_+ \ . \tag{0.12}$$

Otherwise, it is the unique solution in $L_r\big((0,T), L_{2q}\big)$, where

$$2/r + m/2q < 1 \ . \tag{0.13}$$

Moreover, u is smooth for $t > 0$. This result generalizes, in particular, the one of Kato and Ponce. (In [66] the case of certain parabolic equations is considered as well, as is being done in many other works. Since here we are interested in the Navier-Stokes equations we do not comment on those results.) Note that $H_{q,0,\sigma}^s$ contains non-regular tempered distributions if $s < 0$, which is possible if $q > m$.

The situation described so far is not very satisfactory. Indeed, uniqueness is always, except in Ribaud's result (0.11), (0.12), proven under additional restrictions (eg., (0.6) or (0.9)) which are artificial as far as the natural concepts of solutions are concerned. Uniqueness is only guaranteed if specific classes of functions are specified a priori, and there is no relation between the different uniqueness theorems. This amounts to the fact that 'each author has his own solutions' and, indeed, as many solutions as he can specify uniqueness classes. (We recall that in the introduction to Chapter VI of her book [55] Ladyženskaya already points out the fact that there are infinitely many 'generalized solutions' due to the possible choices of the underlying function spaces.) It is one of the purposes of this paper to rectify this unpleasant situation.

In order to formulate our main results we introduce suitable subspaces of particular Besov spaces as follows: if $0 < |s| < 2$ we set

$$n_q^s := \text{closure of } H_q^s \text{ in } B_{q,\infty}^s \ ,$$

denoting by $B_{q,r}^s := B_{q,r}^s(\Omega, \mathbb{R}^m)$ Besov spaces. Since

$$B_{q,1}^s \hookrightarrow H_q^s \hookrightarrow B_{q,\infty}^s \ , \qquad 1 < q < \infty \ , \quad s \in \mathbb{R} \ ,$$

the 'little Nikol'skii spaces' n_q^s are well-defined. We also set

$$n_{q,0,\sigma}^s := \{\, u \in n_q^s \ ; \ \nabla \cdot u = 0 \,\} \ , \qquad 1 < q < \infty \ , \quad 0 < |s| < 2 \ ,$$

(if $\Omega = \mathbb{R}^m$), and $n_{q,0,\sigma}^0 := L_{q,\sigma}$.

Now we can formulate a preliminary version of our main existence and uniqueness result. It is implied by Theorem 6.1 and Proposition 6.5.

Proposition 0.1. *Suppose that $m < q \leq r < \infty$ and*

$$v^0 \in n_{q,0,\sigma}^{-1+m/q} \ .$$

Then there exists a unique maximal solution $v := v(\cdot, v^0)$ of the Navier-Stokes equations such that

$$v \in C\big((0,t^+), H_{r,0,\sigma}^2\big) \cap C^1\big((0,t^+), L_{r,\sigma}\big) \tag{0.14}$$

and

$$\lim_{t \to 0} v(t) = v^0 \quad \text{in } n_{q,0,\sigma}^{-1+m/q}$$

as well as

$$\lim_{t \to 0} t^{(1-m/q)/2} v(t) = 0 \quad \text{in } L_q \ .$$

Observe that, a priori, Proposition 0.1 guarantees for each $r \geq q$ a unique maximal solution v_r on the maximal interval of existence $[0, t_r^+)$. Since the spaces (0.14) are not comparable for different values of r it is conceivable that $v_r \neq v_s$ if $r \neq s$. In Proposition 6.5 it is shown that $v_s \supset v_r$ if $s > r$. This means, in particular, that $t_r^+ \leq t_s^+$ for $r < s$. Thus, although the solution v_r ceases to exist in class (0.14) at t_r^+, if $t_r^+ < \infty$, it can be continued to the possibly larger interval $[0, t_s^+)$ in the class which is obtained by replacing $H_{r,\sigma}^2$ and $L_{r,\sigma}$ in (0.14) by $H_{s,\sigma}^2$ and $L_{s,\sigma}$, respectively. Thus we should obtain a unique maximal solution v, independently of $r > q$, by letting $r \to \infty$. To give a precise formulation we need some preparation.

Suppose that $\{ E_\alpha \ ; \ \alpha_0 < \alpha < \infty \}$ is a family of Banach spaces such that $E_\alpha \hookrightarrow E_\beta$ if $\alpha < \beta$. Then $\bigcup E_\alpha := \bigcup_{\alpha > \alpha_0} E_\alpha$ is a vector space with the obvious definition of its linear structure. There exists a finest locally convex topology on $\bigcup E_\alpha$ such that each one of the natural inclusions

$$E_\beta \to \bigcup E_\alpha \ , \quad x \mapsto x$$

is continuous. The space $\bigcup E_\alpha$, endowed with this topology, is said to be the direct limit of the family $\{ E_\alpha \ ; \ \alpha > \alpha_0 \}$ and denoted by

$$\varinjlim E_\alpha \quad \text{or} \quad \varinjlim_\alpha E_\alpha \ .$$

Clearly, $E_\beta \hookrightarrow \varinjlim E_\alpha$ for $\beta > \alpha_0$ (cf. [17, Appendix Two], [42, Section 2, § 12]).

In Section 3 it is shown that

$$n_{r,0,\sigma}^{-1+m/r} \hookrightarrow n_{s,0,\sigma}^{-1+m/s} \ , \qquad m/3 < r < s < \infty \ .$$

Thus the direct limit

$$n_{\infty,0,\sigma}^{-1} := \varinjlim_r n_{r,0,\sigma}^{-1+m/r}$$

is well-defined, and

$$n_{r,0,\sigma}^{-1+m/r} \hookrightarrow n_{\infty,0,\sigma}^{-1} \ , \qquad m/3 < r < \infty \ .$$

Now suppose that $u \in C\big([0, t^+), \varinjlim E_\alpha\big)$ for some $t^+ \in (0, \infty]$. Then we say that u is well-adapted to $\varinjlim E_\alpha$ if there exist $\alpha_1 > \alpha_0$ and, for each $\alpha \geq \alpha_1$, a number $t_\alpha^+ \in (0, t^+]$ such that

$$t_\alpha^+ = \sup\big\{ t \in [0, t^+) \ ; \ u(\tau) \in E_\alpha, \ 0 \leq \tau \leq t \big\}$$

and

$$u \,|\, [0, t_\alpha^+) \in C\big([0, t_\alpha^+), E_\alpha\big) \ .$$

Note that $t_\alpha^+ \leq t_\beta^+$ if $\alpha < \beta$. We call t_α^+ time of maximal existence of u in E_α.

We say that v is a **maximal strong solution** of the Navier-Stokes equations if there exists a maximal $t^+ \in (0,\infty]$, the maximal existence time, such that

$$v \in C\big([0,t^+), n^{-1}_{\infty,0,\sigma}\big) \ ,$$

v is well-adapted to $n^{-1}_{\infty,0,\sigma}$,

$$v \in C\big((0,t^+_r), H^2_{r,0,\sigma}\big) \cap C^1\big((0,t^+_r), L_{r,\sigma}\big)$$

for each sufficiently large $r > m$ with t^+_r being the maximal existence time of v in $n^{-1+m/r}_{r,0,\sigma}$, and v satisfies (0.2).

If $q > m/3$ then v is said to be a **strong q-solution** on J if

$$v \in C(J, n^{-1+m/q}_{q,0,\sigma}) \cap C(\dot{J}, H^2_{q,0,\sigma}) \cap C^1(\dot{J}, L_{q,\sigma})$$

and v satisfies (0.2). If $v^0 \in n^{-1+m/q}_{q,0,\sigma}$ and v is a strong q-solution on J then it is a strong r-solution on J for each $r > q$. In particular, if $v^0 \in n^{-1+m/q}_{q,0,\sigma}$ and v is a maximal strong solution then v is a strong r-solution on $[0, t^+_r)$ for each $r \geq q$.

After these preparations we can formulate a simplified version of our main existence and uniqueness theorems for strong solutions of (0.1).

Theorem 0.2. *Suppose that $m/3 < q < \infty$ and $v^0 \in H^{-1+m/q}_{q,0,\sigma}$.*

(i) *There exists a unique maximal strong solution $v := v(\cdot, v^0)$ of the Navier-Stokes equations satisfying*

$$\lim_{t\to 0} v(t) = v^0 \quad \text{in } H^{-1+m/q}_q$$

and, if $q > m$,

$$\lim_{t\to 0} t^{(1-m/q)/2} v(t) = 0 \quad \text{in } L_q \ .$$

It is smooth for $t > 0$, that is,

$$v \in C^\infty\big(\overline{\Omega} \times (0,t^+), \mathbb{R}^m\big) \ ,$$

where $t^+ := t^+(v^0)$ is the maximal existence time of v.

(ii) *If*

$$v^0 \in F^s_{q,0,\sigma} \in \{\, H^s_{q,0,\sigma}, B^s_{q,r,0,\sigma}, n^s_{q,0,\sigma} \ ; \ 1 \leq r < \infty \,\}$$

for some $s \in \big((-1+m/q)_+, 2\big]$ then

$$\lim_{t\to 0} v(t) = v^0 \quad \text{in } F^s_{q,0,\sigma} \ .$$

(iii) *If $q \geq m$ then*

$$v \in L_r\big((0,T), L_s\big) \ , \qquad 0 < T < t^+_q \ ,$$

for all $r \in [2,\infty]$ and $s \in [m,\infty)$ satisfying $2/r + m/s = m/q$.

(iii) *Given $T > 0$, there exists $R > 0$ such that*

$$t^+(v^0) > T \qquad \textit{for} \quad \|v^0\|_{n^{-1+m/q}_{q,0,\sigma}} \leq R \ .$$

Proof. This is a consequence of Theorem 11.1, Corollary 11.2, and Remarks 9.5(a) and 11.3(e). □

Of course,

$$B^s_{q,r,0,\sigma} = \{\, u \in B^s_{q,r} \ ; \ \nabla \cdot u = 0 \,\}$$

(if $\Omega = \mathbb{R}^m$). Also note that

$$F^s_{q,0,\sigma} \hookrightarrow H^{-1+m/q}_{q,0,\sigma} \ , \qquad s > -1 + m/q \ . \tag{0.15}$$

Next we show that all solutions described above coincide on their intervals of existence with the strong solution $v(\cdot, v^0)$. This will be entailed by Theorem 0.2 and the following result.

Theorem 0.3. *Suppose that $q \geq m$ and $v^0 \in L_{q,\sigma}$.*
(a) *The following are equivalent:*
(i) *u is a very weak solution in $C\big([0,T), L_{q,\sigma}\big)$.*
(ii) *u is a mild solution in $L_{q,\sigma}$ on $[0,T)$.*
(iii) *u is a strong q-solution in $C\big([0,T), L_{q,\sigma}\big)$.*
(b) *$v(\cdot, v^0)$ is a mild solution in $L_{q,\sigma}$ on $[0, t_q^+)$.*

Proof. (a) follows from Theorems 6.1 and 7.2 (also see Remark 5.7(a)), and (b) is implied by (a) and Theorem 7.2. □

A result related to parts (i) and (ii) of this theorem has also been shown by Fabes, Jones, and Rivière [18] using crucially the fact that the Stokes semigroup as well as the Helmholtz projector possess rather explicit representations on $\mathbb{R}^m$.

Now it is easy to derive the desired uniqueness result guaranteeing that all solutions described up to now coincide on their respective intervals of existence if their initial values coincide.

Theorem 0.4. *Suppose that $q > m/3$. Then $v(\cdot, v^0) \supset v$ whenever v is one of the solutions described above and $v^0 = v(0)$.*

Proof. (i) Suppose that $q > m$ and $v^0 \in L_{q,\sigma}$. It follows from Theorem 0.2(iii), Hölder's inequality (see the proof of Remark 9.5(b)), and Theorem 0.3 that $v(\cdot, v^0)$ is a very weak solution in $L_r\big((0,T), L_s\big)$ for $0 < T < t_q^+$, where r and s satisfy (0.6). Since the Fabes, Jones, and Rivière solution v is the only one in this class it follows that $v(\cdot, v^0) \supset v$. The same argument applies to Calderón's solution if $q = m$.

(ii) If $q > m$ then $v(\cdot, v^0)$ is continuous from $[0, t_q^+)$ into $L_{q,\sigma}$. Hence it is continuous from $[0, t_q^+)$ into the weak topology of $L_{q,\sigma}$. Theorem 0.3 guarantees that $v(\cdot, v^0)$ is a very weak solution on $[0, t_q^+)$. Since the solution v constructed by Beirão da Veiga is the only very weak solution in this class we infer that $v(\cdot, v^0) \supset v$.

(iii) Let $v^0 \in H^s_{q,0,\sigma}$ with $s \geq (-1 + m/q)_+$. Then $v^0 \in L_{q,\sigma}$ if $q \geq m$, and Theorem 3.10 guarantees that $v^0 \in L_{m,\sigma}$ if $q < m$. Thus $v(\cdot, v^0) \in C\big([0, t_q^+), L_{q \vee m}\big)$ and is the only mild solution in this class by Theorem 0.2(i) and Theorem 0.3. Hence, if v is any one of the solutions obtained by Kato [44], [45], Kato and Ponce [47], Ribaud [66], Cannone and Meyer [15], and Cannone [13], it follows from

$$H^s_{q,0,\sigma} \hookrightarrow H^{(-1+m/q)_+}_{q,0,\sigma} \hookrightarrow L_{q \vee m, \sigma}$$

and the continuity of v as a map into $H^s_{q,0,\sigma}$ that $v(\cdot, v^0) \supset v$. $\square$

The following remark implies that, in a suitable sense, $v(\cdot, v^0)$ is also independent of q.

Remark 0.5. Let the hypotheses of Theorem 0.2 be satisfied and fix any $p > q$. Then

$$H^{-1+m/q}_{q,0,\sigma} \hookrightarrow H^{-1+m/p}_{p,0,\sigma}$$

by Theorem 3.10. Hence we obtain unique maximal strong solutions $v_q(\cdot, v^0)$ and $v_p(\cdot, v^0)$ if we apply Theorem 0.2 to $v^0 \in H^{-1+m/q}_{q,0,\sigma}$ or to $v^0 \in H^{-1+m/p}_{p,0,\sigma}$, respectively. However, $v_q(\cdot, v^0) \subset v_p(\cdot, v^0)$.

Proof. This follows from Proposition 6.5. $\square$

The following theorem, combined with (0.15), shows that $v(\cdot, v^0)$ blows up near t^+ in each norm which is stronger than the $H^{-1+m/q}_q$-norm if $v(\cdot, v^0)$ does not exist globally. In addition, it contains an estimate for the blow-up rate.

Theorem 0.6. *Suppose that $m/3 < q < \infty$ and $v^0 \in H^{-1+m/q}_{q,0,\sigma}$. Put $v := v(\cdot, v^0)$.*

(i) *If $t^+ < \infty$ then*

$$\lim_{t \to t^+} \|v(t)\|_{H^s_{r,0,\sigma}} = \infty$$

for every $r > m$ with $r \geq q$ and every $s > -1 + m/r$.

(ii) *Suppose that $r > m$ with $r \geq q$ and $-1 + m/r < s \leq 0$. Then*

$$\|v(t)\|_{H^s_{r,0,\sigma}} \geq c/(t^+ - t)^{(s+1-m/r)/2} , \qquad 0 < t^+ - t \leq 1 ,$$

where $c > 0$ is independent of v^0.

Proof. This follows from Remarks 11.3(a) and (b). $\square$

Now we turn to the much more complicated case $\Omega \neq \mathbb{R}^m$. For simplicity, we assume, in addition to (0.3), that

$$\begin{aligned}&\text{either } \Omega \text{ has a compact boundary}\\ &\text{or } \Omega \text{ is a half-space in } \mathbb{R}^m\ .\end{aligned} \tag{0.16}$$

Many of the results described below hold for more general domains. For this we refer to the main body of this paper.

First we have to give a meaning to $H^s_{q,0,\sigma} := H^s_{q,0,\sigma}(\Omega, \mathbb{R}^m)$ in this case. We set

$$H^s_{q,0,\sigma} := \begin{cases} \{\, u \in H^s_q\ ;\ \nabla \cdot u = 0,\ u\,|\,\partial\Omega = 0 \,\}\ , & 1/q < s \leq 2\ , \\ \{\, u \in H^s_q\ ;\ \nabla \cdot u = 0,\ u \cdot \vec{n} = 0 \,\}\ , & 0 \leq s < 1/q\ , \end{cases} \tag{0.17}$$

where $\vec{n}$ is the outer unit-normal field on $\partial\Omega$. Of course, $u \cdot \vec{n}$ and $u\,|\,\partial\Omega$ are to be understood in the sense of traces (see Sections 2 and 3). It follows that

$$H^0_{q,0,\sigma} = L_{q,\sigma}\ .$$

We also put

$$H^{1/q}_{q,0,\sigma} := \big\{\, u \in H^{1/q}_q(\mathbb{R}^m, \mathbb{R}^m)\ ;\ \mathrm{supp}(u) \subset \overline{\Omega},\ \nabla \cdot u = 0,\ u \cdot \vec{n} = 0 \,\big\}\ . \tag{0.18}$$

Of particular interest is the case $q = 2$ where Remark 2.5(a) implies

$$H^{1/2}_{2,0,\sigma} := \big\{\, u \in H^{1/2}_2;\ \nabla \cdot u = 0,\ u \cdot \vec{n} = 0,\ d^{-1/2}u \in L_2 \,\big\}$$

with $d(x) := 1 \wedge \mathrm{dist}(x, \partial\Omega)$ for $x \in \Omega$, and where $H^{1/2}_{2,0,\sigma}$ is given the norm

$$u \mapsto \big(\|u\|^2_{H^{1/2}_2} + \|d^{-1/2}u\|^2_{L_2}\big)^{1/2}\ .$$

Similarly, we define $B^s_{q,r,0,\sigma}$, $1 \leq r < \infty$, and $n^s_{q,0,\sigma}$ for $0 \leq s \leq 2$ by replacing H^s_q in (0.17) and (0.18) by $B^s_{q,r}$ and n^s_q, respectively.

Next we set

$$H^{-s}_{q,0,\sigma} := (H^s_{q',0,\sigma})'\ , \qquad 0 < s \leq 2\ ,$$

by means of the $L_{q,\sigma}$-duality pairing, where $q' := q/(q-1)$ is the dual exponent of $q \in (1,\infty)$. Due to the presence of a nonempty boundary these 'negative' spaces do not allow an easy characterization similar to the one for the case $\Omega = \mathbb{R}^m$. In fact, it follows from Proposition 2.4 and Theorem 3.5 that in the presently most interesting case where $s \leq 1$, the elements of $H^{-s}_{q,0,\sigma}$ can be identified with equivalence classes of distributions in H^{-s}_q, where two distributions in H^{-s}_q are equivalent if they differ by the gradient of an appropriately smooth function only. For a useful characterization of the distributions belonging to H^{-s}_q for $0 < s \leq 1$ we refer to Theorem 2.1.

We also put

$$B^{-s}_{q,r,0,\sigma} := (B^{s}_{q',r',0,\sigma})' \ , \qquad 1 < r \le \infty \ , \quad 0 < s \le 2 \ ,$$

by means of the $L_{q,\sigma}$-duality pairing, and

$$n^{-s}_{q,0,\sigma} := \text{closure of } L_{q,\sigma} \text{ in } B^{-s}_{q,\infty,0,\sigma} \ , \qquad 0 < s \le 2 \ ,$$

(see Sections 2 and 3 for more details).

Lastly, we have to redefine the concept of a very weak solution by taking into account the presence of the boundary. For simplicity, we restrict ourselves here to the case $q \ge m$. Then a **very weak q-solution** on J of (0.1) (with $f = 0$) is a function $v \in C(J, L_{q,\sigma})$ satisfying (0.7) for all

$$\varphi \in L_1(J, H^2_{q',0,\sigma}) \cap W^1_1(J, L_{q',\sigma})$$

having compact support in $J^* := J \backslash \sup(J)$. (In Remark 7.1(c) it is shown that this definition coincides with the earlier one if $\Omega = \mathbb{R}^m$.)

With these definitions Proposition 0.1 remains true if assumption (0.4) is replaced by (0.16). Hence the concept of a maximal strong solution of the Navier-Stokes equations is well-defined in this case also.

Theorem 0.7. *Let condition* (0.16) *be satisfied. Then Theorems* 0.2, 0.3, 0.6, *and Remark* 0.5 *are true.*

Proof. This follows from the fact that all theorems from the main body of this paper, referred to so far, are also valid if (0.16) is satisfied. □

In the remainder, Ω is said to be a **standard domain** if either $\Omega = \mathbb{R}^m$ or Ω satisfies (0.16).

The most important concept of a solution in the theory of the Navier-Stokes equations is probably the one of a weak solution. If $v^0 \in L_{2,\sigma}$ then

$$u \in L_\infty(J, L_{2,\sigma}) \cap L_2(J, H^1_2)$$

is a **weak solution** on J of (0.1) (with $f = 0$), provided

$$\int_J \bigl\{ -\langle \dot\varphi, u \rangle + \nu \langle \nabla\varphi, \nabla u \rangle + \bigl\langle \varphi, (u \cdot \nabla) u \bigr\rangle \bigr\} \, dt = \bigl\langle \varphi(0), u^0 \bigr\rangle$$

for all $\varphi \in \mathcal{D}(J^*, \mathcal{D}_\sigma)$. It is a **global** weak solution if it is a weak solution on $[0, T]$ for every $T > 0$. As is well-known, thanks to Leray [57] and Hopf [41], there exists for each $v^0 \in L_{2,\sigma}$ at least one global weak solution v satisfying the energy inequality

$$\|v(t)\|^2_{L_2} + 2\nu \int_0^t \|\nabla v(\tau)\|_{L_2} \, d\tau \le \|v^0\|^2_{L_2} \ , \qquad t > 0 \ , \tag{0.19}$$

a **Leray-Hopf weak solution**. Uniqueness and smoothness are open problems. (We refer to the surveys by Galdi [27] and Wiegner [85] for more details, as well as to [27], Lions [58], and Temam [75] for (modernized versions of) the existence proofs.)

The following theorem guarantees uniqueness and smoothness on the maximal existence interval of the strong solution $v(\cdot, v^0)$.

Theorem 0.8. *Suppose that Ω is a standard domain and $v^0 \in L_{2,\sigma} \cap L_{q,\sigma}$ for some $q \geq m$. Then*

(i) *$v := v(\cdot, v^0)$ is a weak solution on $[0, T]$ for every $T \in (0, t^+)$. It belongs to $C\big([0, t^+), L_2\big)$ and satisfies the* strong energy equality

$$\|v(t)\|_{L_2}^2 + 2\nu \int_s^t \|\nabla v(\tau)\|_{L_2}^2 \, d\tau = \|v(s)\|_{L_2}^2 \,, \qquad 0 \leq s < t < t^+ \,.$$

(ii) *If u is any Leray-Hopf weak solution then $u \supset v(\cdot, v^0)$. In particular, u is smooth on $(0, t^+)$.*

Proof. (i) follows from Remarks 10.2(a) and 11.3(d), and (ii) is a special case of Theorem 11.4. □

Given the postulated hypotheses, Theorem 0.8 guarantees local uniqueness and smoothness for Leray-Hopf weak solutions without further restrictions. In particular, if $v(\cdot, v^0)$ exists globally then there is a unique Leray-Hopf weak solution and it is smooth for $t > 0$. This is in contrast to the known uniqueness theorems of Foias [22], Prodi [64], Serrin [68], Fabes, Jones, and Rivière [18], Masuda [60], Sohr and von Wahl [73], von Wahl [80], Kozono and Sohr [51] which are conditional in the sense that they require the solution to belong to more restricted classes.

It is well-known that Leray-Hopf weak solutions with appropriately regular initial values are smooth as long as there exist strong solutions with the same initial values (e.g., [31], [45] and others; see the surveys by Galdi [27] and Wiegner [85]). The new fact is the (unconditional) uniqueness assertion.

The first local regularity results for weak solutions of the Navier-Stokes equations are due to Kiselev and Ladyženskaya [48] (also see the exposition in [55] and related work of Sobolevskii [70], [71]). This research has been considerably improved by Solonnikov [74] who, by means of potential theoretic estimates, established the local existence of strong q-solutions in Sobolev and Hölder spaces under the assumption that $m = 3$ and Ω has a compact boundary, provided v^0 is sufficiently regular. Using those results which, by the way, are optimal as far as regularity in the classes under consideration goes, Solonnikov could also prove existence, but not uniqueness, of a local solution for $v^0 \in L_{3,\sigma}$.

Temporarily, we now suppose that

$$\Omega \text{ is bounded }.$$

Given this assumption, Sobolevskii and, independently, Kato and Fujita [23], [46] were the first to employ semigroup theory and, in particular, the technique of fractional powers in the study of the Navier-Stokes equations. In [23], improving the results of [46], it is shown that, if $m = 3$ and $v^0 \in H^{1/2}_{2,0,\sigma}$, there exist $T > 0$ and a unique strong 2-solution v in $C\big([0,T], L_{2,\sigma}\big)$ satisfying

$$\lim_{t\to 0} t^{1/4}\, \|v(t)\|_{H^1_2} = 0 \ .$$

Extending the Kato-Fujita approach from the Hilbert to a Banach space setting, Miyakawa [61] assumed that $q > m$ and $v^0 \in L_{q,\sigma}$ and proved the existence of $T > 0$ and of a unique mild solution v in

$$C\big([0,T], L_{q,\sigma}\big) \cap C_{s/2}\big((0,T], H^s_{q,0,\sigma}\big)$$

for some $s \in (1, 3/2)$. Miyakawa also showed that v is a weak solution on $[0,T]$ satisfying the energy inequality.

Semigroup theory and fractional powers have also been used by v. Wahl [80] to get local strong solutions. He assumes that $v^0 \in L_{q,\sigma}$ with $q \geq m$ and establishes the existence of $r \in (0,1)$ and of a unique maximal mild solution v of (0.1) in

$$C\big([0,t_q^+), L_{q,\sigma}\big) \cap C_{r/2}\big((0,t_q^+), H^r_{q,0,\sigma}\big) \ .$$

In addition, v. Wahl proves higher regularity results and studies also the case $m/3 < q < m$ with $v^0 \in H^2_{q,0,\sigma}$.

Those results have been considerably improved by Giga and Miyakawa [32]: suppose that

$$q > m/3 \ , \quad -1 + m/q \leq s < 2 \ , \quad v^0 \in H^s_{q,0,\sigma} \ .$$

Then, given any $r \in \big(s, 2 \wedge (2+s)\big)$, in [32] it is shown that there exists a mild solution

$$v \in C\big([0,T], H^s_{q,0,\sigma}\big) \cap C_{(r-s)/2}\big([0,T], H^r_{q,0,\sigma}\big) \ . \tag{0.20}$$

Moreover, any mild solution satisfying (0.20) for some $r > |s|$ is unique. In [32] it is also shown that v is smooth for $t > 0$.

The results of Giga and Miyakawa also extend earlier results of Weissler [82]. Similar results involving other restrictions on r are contained in Grubb [37] who, however, treats other boundary conditions also.

Remarks 0.9. (a) In none of the papers of Kato and Fujita [23], [46], Miyakawa [61], v. Wahl [80], and Giga and Miyakawa [32] do the spaces $H^s_{q,0,\sigma}$ occur explicitly. Indeed, all results in those works are formulated in terms of fractional powers of the Stokes operator S_q on $L_{q,\sigma}$ (as is the case in many other papers). However, it follows from a result of Giga [30] on the boundedness of the imaginary powers of S_q that

$$D(S_q^\theta) \doteq [L_{q,\sigma}, H^2_{q,0,\sigma}]_\theta \ , \qquad 0 < \theta < 1 \ ,$$

where $[\cdot,\cdot]_\theta$, $0 < \theta < 1$, are the complex interpolation functors. Using this,

$$[L_{q,\sigma}, H^2_{q,0,\sigma}]_\theta \doteq H^{2\theta}_{q,0,\sigma} \ , \qquad 0 < \theta < 1 \ ,$$

(see Theorem 2.2) and Theorem 3.4, we obtain the statements given above.

(b) In all of the above work it is also shown that the respective solutions are global if v^0 is sufficiently small in the respective norm. Furthermore, a non-vanishing force f is admitted too. □

Kobayashi and Muramatu [49] have presented a variant of the work of Giga and Miyakawa by introducing a class of abstract Besov spaces constructed by means of fractional powers of the Stokes operator. Their results amount essentially to replacing $H^s_{q,0,\sigma}$ by $n^s_{q,0,\sigma}$, although this is shown nowhere. In particular, in [49] there is given no concrete characterization of these abstract Besov spaces.

Giga [31] and, more recently, Wiegner [85] present existence proofs using Kato's ideas [45]. (Also see the work of Weissler [81], [82], [83].) This approach is not based on fractional powers but on L_p-L_q-estimates for the Stokes semigroup (cf. [85], for example, for definitions). The required L_p-L_q-estimates are known to hold if Ω is a standard domain (e.g., see [45] if $\Omega = \mathbb{R}^m$, [7] and [78] if Ω is a half-space, [31] if Ω is bounded, and [43] if Ω is an exterior domain). Thus we assume that

$$\Omega \text{ is a standard domain .}$$

Then Giga's [31] results imply that, given $v^0 \in L_{q,\sigma}$ with $q \geq m$, there exist $T > 0$ and a unique mild solution v in

$$C\big([0,T], L_{q,\sigma}\big) \cap C_{1/r}\big((0,T], L_{s,\sigma}\big) \cap L_r\big((0,T), L_{s,\sigma}\big)$$

where $r, s \in (q,\infty)$ satisfy $2/r + m/s = m/q$. If v^0 is sufficiently small in L_m then $T = \infty$. If $u \in C\big([0,t^*), L_{r,\sigma}\big)$ is a maximal mild solution for some $r > m$ then

$$\|u(t)\|_{L_r} \geq c\big/(t^*_r - t)^{(1-m/r)/2} \ . \tag{0.21}$$

Furthermore, if $v^0 \in L_{2,\sigma} \cap L_{q,\sigma}$ then v is a weak solution satisfying the energy inequality. (To be more precise: assumption (A) in [31] is covered by the L_p-L_q-estimates. For condition (NI) one has to invoke Theorem 4.2 of this paper.)

Suppose that $v^0 \in L_{m,\sigma}$. Then Wiegner's [85] theorem implies the existence of a unique maximal mild solution $v \in C\big([0,t^*_m), L_{m,\sigma}\big)$ satisfying

$$\sup_{0<t<T} t^{1/2} \, \|\nabla v(t)\|_{L_m} + \sup_{0<t<T} t^{(1-m/s)/2} \, \|v(t)\|_{L_s} < \infty$$

for $0 < T < t_m^+$ and $s > m$. Moreover,

$$v \in L_r\big((0,T), L_s\big) \ , \qquad 0 < T < t_m^+ \ ,$$

provided $s > m$ and $2/r + s/m = 1$. Finally, $t^+ = \infty$ if v^0 is sufficiently small in L_m.

Kozono and Nakao [50] assume that $m \geq 4$ and either $\Omega = \mathbb{R}^m$, a half-space, or an exterior domain, and that $v^0 \in L_{m,\sigma} \cap L_{r,\sigma}$ with $r > m$. Then they prove the existence of a unique local strong m-solution v. They also show that $v \in C\big([0,T], L_r\big)$ and establish estimate (0.21).

Similarly as in the case where $\Omega = \mathbb{R}^m$, we can show that all solutions described above are restrictions of the maximal strong solution $v(\cdot, v^0)$ of Theorem 0.2.

Theorem 0.10. *Let Ω be a standard domain. If v is any one of the solutions described above then $v(\cdot, v^0) \supset v$, provided $v^0 = v(0)$.*

Proof. (a) Assume that $v^0 = v(0) \in H^s_{q,0,\sigma}$ with $s \geq (-1 + m/q)_+$. Then Theorem 0.2 guarantees that $v(\cdot, v^0)$ is the unique maximal strong solution u satisfying $u(t) \to v^0$ in $H^s_{q,0,\sigma}$ as $t \to 0$. In particular, $v(\cdot, v^0)$ is the only strong q-solution in $C\big([0, t_q^+), H^s_{q,0,\sigma}\big)$. By Theorem 0.3 and the cited results, v belongs to (a proper subclass of) $C\big([0, t_q^+), H^s_{q,0,\sigma}\big)$. Hence $v(t) = v(t, v^0)$ for $0 \leq t < t_q^+$.

(b) It remains to consider the case $v^0 \in H^s_{q,0,\sigma}$ with $-1 + m/q \leq s < 0$ which is admissible — under some restrictions — in the results of Giga and Miyakawa [32] and Grubb [37]. But in this case the assertion is a consequence of Theorem 6.1, Proposition 6.4, and the construction of $v(\cdot, v^0)$ in the proof of Theorem 11.1. $\square$

We point out that Kozono and and Yamazaki [52] study the well-posedness of the Navier-Stokes equations on $\mathbb{R}^n$ by assuming that v^0 belongs to certain spaces of Besov type which are constructed by means of Morrey instead of Lebesgue spaces. The same authors consider in [53] initial data in $L_{m,\infty} + L_q$ for some $q > m$, where $L_{m,\infty}$ is a Lorentz space and Ω is an exterior domain. Those results do not seem to be comparable to the ones of this paper.

Now we can comment on some of the improvements of our results over the existing ones.

• Our theorems generalize and improve almost all of the previous existence and uniqueness results (known to us).

• We give rather precise descriptions of the spaces of initial values, in contrast to abstract statements to the effect that v^0 belongs to some fractional power space.

• If $v^0 \in H^s_{q,0,\sigma}$ with $s \geq (-1 + m/q)_+$ then Theorem 0.2 guarantees existence and uniqueness under the sole (natural) assumption that the solution be continuous at $t = 0$ in $H^s_{q,0,\sigma}$. Almost all of the existence results known so far require additional restrictions.

• We can admit initial values in the negative space $H^{-1+m/q}_{q,0,\sigma}$, or even in the space $n^{-1+m/q}_{q,0,\sigma}$, with arbitrarily large $q > m$. This is the first result of this type if Ω is an exterior domain.

• The blow-up results given in Theorem 0.6 generalize (0.21) by showing that already weaker norms than the L_r-norm blow up if t^+ is finite.

• The equivalences given in Theorem 0.3(a) are of independent interest.

It follows from the results of Kato [45], Giga [31], and Wiegner [85], for example, and from Theorems 0.4 and 0.10 that $v(\cdot, v^0)$ exists globally if $v^0 \in L_{m,\sigma}$ and $\|v^0\|_{L_m}$ is sufficiently small. Recently, this has been considerably improved by Cannone [13], [14] if $\Omega = \mathbb{R}^3$. Under this assumption he shows that, given $q \in (3, 6]$, the Navier-Stokes equations (with $f = 0$) possess a unique global mild solution

$$v \in BC(\mathbb{R}^+, L_{3,\sigma}) \cap C\big((0,\infty), L_q\big)$$

satisfying

$$\|v(t)\|_{L_q} \leq ct^{-1+3/q} \ , \qquad t > 0 \ ,$$

whenever $v^0 \in L_{3,\sigma}$ and the norm of v^0 is sufficiently small in the homogeneous Besov space $\dot{B}^{-1+3/q}_{q,\infty}$. To be small in $\dot{B}^{-1+3/q}_{q,\infty}$ is a much weaker condition than to be small in L_3. Cannone's proof rests heavily on the fact that the boundary of Ω is empty since he uses that the Stokes semigroup reduces to the heat semigroup and the Helmholtz projector has a rather explicit representation by means of Riesz operators. In addition, essential use is made of the representations of Besov spaces by means of Littlewood-Paley decompositions on the Fourier side.

Using the particularly simple geometry of a half-space in $\mathbb{R}^3$, allowing a reflection argument, together with Ukai's representation formula for the Stokes operator [78] Cannone, Planchon, and Schonbek [16] extended this global existence theorem to the case where Ω is a half-space in $\mathbb{R}^3$.

The starting point for our paper was the question whether Cannone's result could be proven for other domains as well; for example, if Ω is bounded. In this case homogeneous Besov spaces are not meaningful anymore since they are not invariant under local diffeomorphisms and since they involve conditions on the behavior of their members at infinity. Natural substitutes are the (nonhomogeneous) spaces $B^{-1+m/q}_{q,\infty}$. However, due to the presence of a nonempty boundary, the situation is more complicated. Moreover, Fourier analysis is no longer useful directly. Thus we use semigroup theory to prove the following result.

Theorem 0.11. *Suppose that Ω is bounded, $q > m/3$, and $0 \leq \omega < \lambda_0$, where λ_0 is the smallest eigenvalue of the Stokes operator. Then, given any $r > m$ with $r \geq q$ and $v^0 \in H^{-1+m/q}_{q,0,\sigma}$, the solution $v(\cdot, v^0)$ exists globally and satisfies*

$$\|v(t, v^0)\|_{L_r} \leq ct^{-1+m/r} e^{-\omega t} \ , \qquad t > 0 \ , \tag{0.22}$$

provided the norm of v^0 is sufficiently small in $n_{r,0,\sigma}^{-1+m/r}$.

Proof. This is a special case of Theorem 11.6. $\square$

Remarks 0.12. (a) Suppose that $v^0 \in H_{q,0,\sigma}^{(-1+m/q)+}$. Then

$$v^0 \in H_{q,0,\sigma}^{-1+m/q} \hookrightarrow n_{q,0,\sigma}^{-1+m/q} \hookrightarrow n_{r,0,\sigma}^{-1+m/r} , \qquad r > q ,$$

by Corollary 3.11. Moreover, Proposition 2.4 and Remark 3.6 imply

$$\|v^0\|_{n_{r,0,\sigma}^{-1+m/r}} \le c \, \|v^0\|_{B_{r,\infty}^{-1+m/r}} .$$

Hence in this case (in particular, if $v^0 \in L_{m,\sigma}$) it follows that $v(\cdot, v^0)$ exists globally and satisfies (0.22) whenever the norm of v^0 is sufficiently small in $B_{r,\infty}^{-1+m/r}$ for some $r > m$ with $r \ge q$. Thus Theorem 0.11 is indeed the analogue to Cannone's result for the case where Ω is bounded. Note, however, that we can allow rather rough initial values.

(b) Theorem 0.11 remains true if the assumption that Ω be bounded is replaced by the hypothesis that the Stokes operator is well-defined and the Stokes semigroup is exponentially decaying.

(c) Suppose that $v^0 \in L_{2,\sigma} \cap L_{q,\sigma}$ for some $q \ge m$. Also suppose that either
(i) v^0 is small in $n_{r,\infty}^{-1+m/r}$ for some $r > m$ with $r \ge q$ and Ω is bounded,
or
(ii) v^0 is small in L_q,
or
(iii) Ω equals either $\mathbb{R}^3$ or a half-space in $\mathbb{R}^3$, $q = m$, and v^0 is small in $\dot{B}_{6,\infty}^{-1/2}$.
Then there exists exactly one Leray-Hopf weak solution and it is smooth for $t > 0$.

Proof. This follows from Theorems 0.8(ii) and 0.2(i) since $v(\cdot, v^0)$ exists globally. Indeed, in case (i) this is guaranteed by Theorem 0.11. If v^0 is small in L_q then $t^+ = \infty$ is a consequence of the results cited above and of Theorems 0.7 and 0.10. Lastly, if (iii) is satisfied then $t^+ = \infty$ is implied by the results of Cannone [13] and Cannone, Planchon, and Schonbek [16], respectively, since their solutions coincide with $v(\cdot, v^0)$ thanks to the fact that they belong to $C(\mathbb{R}^+, L_{3,\sigma})$. $\square$

Note that $\dot{B}_{r,\infty}^{-1+3/r} \hookrightarrow \dot{B}_{6,\infty}^{-1/2}$ for $3 < r \le 6$.

Finally, we give a brief outline of the contents of the following sections. In Section 1 we collect the basic facts on the interpolation-extrapolation theory which are fundamental for our approach. In contrast to all the other work, our results are neither based on fractional powers nor on L_p-L_q-estimates. In fact, it does not seem to be possible to obtain sharp results without using interpolation-extrapolation techniques.

In Section 2 we introduce the underlying function spaces which are related to the Dirichlet problem for the Laplace operator. These results are then used in Section 3 to find concrete realizations of the interpolation-extrapolation spaces for the Stokes operator. In the presence of a boundary this is rather complicated. We also prove natural extensions of Sobolev type embedding theorems for the spaces $H^s_{q,0,\sigma}$ and $n^s_{q,0,\sigma}$, being by no means obvious. This rather long section is basic for a good understanding of the occurring spaces of distributions.

In Section 4 we study continuity properties of the nonlinear convection term. In this investigation — as in other places also — Lemma 3.3 plays a crucial rôle since it allows to get rid of the non-local Helmholtz projection. By this way we obtain sharp results which improve on all continuity estimates known so far.

In Section 5 we develop a complete and self-contained existence, uniqueness, and regularity theory for abstract parabolic evolution equations with quadratic nonlinearities. By means of interpolation theory we get optimal results which cannot be obtained by the theory of fractional powers. Of course, the existence part rests on the contraction mapping principle and is, in this respect, close to but different from the work of von Wahl (see Theorem II.3.3 in [80]) and also Kato [45].

The basic existence, uniqueness and regularity result for strong q-solutions of the Navier-Stokes equations under minimal requirements on v^0 and f is Theorem 6.1. It is more or less a straightforward application of the results in Section 5, except for the regularity assertions. For those we rely on some general results of the author [5]. Theorem 6.2 guarantees global existence for small data and under appropriate assumptions on the Stokes operator. In the remainder of Section 6 it is shown that the maximal strong q-solution is essentially independent of the occurring parameters.

In Section 7 we discuss very weak q-solutions and show that they are the same as mild solutions. In the next section we prove a uniqueness theorem for very weak q-solutions (Theorem 8.2). Our proof is rather simple and different from related ones of Lions and Masmoudi [59] and Monniaux [63]. Whereas, up to this point, we only had to assume that the Stokes operator is well-defined and generates an analytic semigroup, now we have to rely on maximal regularity in $L_{m,\sigma}$ if $q = m$. For this it is sufficient to know that S_m has bounded imaginary powers. This assumption imposes (at our present state of knowledge) restrictions on Ω (which are met if Ω is a standard domain).

Section 9 is devoted to integrability properties of very weak q-solutions, and in Section 10 we study weak solutions. In particular, we show — given rather general hypotheses on Ω — that very weak q-solutions are weak solutions satisfying the strong energy equality. From this we obtain a (local) uniqueness theorem for Leray-Hopf weak solutions under rather general conditions (Theorem 10.3).

Finally, in the last section we prove the existence of a unique maximal strong solution and show that it possesses all the properties described in the theorems of this introduction. This is true for nonvanishing outer forces also. For simplicity, we do not give the most general hypotheses on f but restrict ourselves to relatively simple ones. Obvious generalizations are left to interested readers.

1. Interpolation-Extrapolation Scales

Throughout this paper all vector spaces are over the reals. If there occur, explicitly or implicitly, complex numbers then the corresponding statements always refer to the respective complexifications. Furthermore, c stands for various positive constants which may differ from occurrence to occurrence but are always independent of the free variables in a given equation. For $\xi, \eta \in \mathbb{R}$ we set $\xi \vee \eta := \max\{\xi, \eta\}$ and $\xi \wedge \eta := \min\{\xi, \eta\}$, as well as $\xi_+ := \xi \vee 0$.

Let E and F be Banach spaces. We write $E \hookrightarrow F$ if E is continuously injected in F, that is, E is a linear subspace of F and the natural injection $x \mapsto x$ from E into F is continuous. If E is also dense in F then we express this by $E \overset{d}{\hookrightarrow} F$. In this case (F, E) is said to be a densely injected Banach couple. We also write $E \doteq F$ if $E \hookrightarrow F$ and $F \hookrightarrow E$, that is, if E equals F except for equivalent norms.

By $\mathcal{L}(E, F)$ we mean the Banach space of all bounded linear operators from E into F, and $\mathcal{L}(E) := \mathcal{L}(E, E)$. If G is a third Banach space then $\mathcal{L}(E, F; G)$ is the Banach space of all continuous bilinear maps from $E \times F$ into G. We also set $\mathcal{L}^2(E, F) := \mathcal{L}(E, E; F)$.

We denote by $E' := \mathcal{L}(E, \mathbb{R})$ the dual of E and by $\langle \cdot, \cdot \rangle_E$ the duality pairing between E' and E, so that $\langle x', x \rangle_E$ is the value of $x' \in E'$ at $x \in E$.

Given $\omega \in \mathbb{R}$, we write $A \in \mathcal{G}(E, \omega)$ if $-A$ generates a strongly continuous semigroup $\{\, e^{-tA} \ ;\ t \geq 0 \,\}$ on E, that is, in $\mathcal{L}(E)$, such that there exists $M \geq 1$ satisfying

$$\|e^{-tA}\| \leq M e^{\omega t} , \qquad t \geq 0 .$$

The infimum of all such ω is the growth bound or the type, $\mathrm{type}(-A)$, of $-A$.

Let (E_0, E_1) be a densely injected Banach couple. Then $\mathcal{H}(E_1, E_0)$ is the set of all $A \in \mathcal{L}(E_1, E_0)$ such that $-A$, considered as a linear operator in E_0 with domain E_1, generates a strongly continuous analytic semigroup on E_0.

In the following we make free use of interpolation theory and refer to Section I.2 of [5] for a summary. As usual, we denote by $[\cdot, \cdot]_\theta$ the complex, by $(\cdot, \cdot)_{\theta, r}$, $1 \leq r \leq \infty$, the real, and by $(\cdot, \cdot)^0_{\theta, \infty}$ the continuous interpolation functors of exponent $\theta \in (0, 1)$.

For convenience, we recall that

$$(E_0, E_1)^0_{\theta, \infty} \text{ is the closure of } E_1 \text{ in } (E_0, E_1)_{\theta, \infty} . \tag{1.1}$$

We set $E_{\theta, r} := (E_0, E_1)_{\theta, r}$ and $E^0_{\theta, \infty} := (E_0, E_1)^0_{\theta, \infty}$. Then

$$E_1 \overset{d}{\hookrightarrow} E_{\theta,1} \overset{d}{\hookrightarrow} E_{\theta,r} \overset{d}{\hookrightarrow} E^0_{\theta,\infty} \hookrightarrow E_{\theta,\infty} \overset{d}{\hookrightarrow} E_{\vartheta,1} \overset{d}{\hookrightarrow} E_0 \tag{1.2}$$

for $0 < \vartheta < \theta < 1$ and $1 < r < \infty$, and

$$E_{\theta,1} \overset{d}{\hookrightarrow} E_{[\theta]} := [E_0, E_1]_\theta \overset{d}{\hookrightarrow} E^0_{\theta,\infty} \ , \qquad 0 < \theta < 1 \ . \tag{1.3}$$

Now we collect some basic facts about interpolation-extrapolation scales. Proofs and many more details are contained in [5, Chapter V].

Let (E_0, E_1) be a densely injected Banach couple such that $E := E_0$ is reflexive. Suppose that $A \in \mathcal{H}(E_1, E_0)$. Then $E_1 \doteq D(A)$, where $D(A)$ is the domain of A endowed with its graph norm. Set $E_k := D(A^k)$ for $k \in \mathbb{N}$ with $k \geq 2$. Also set $E^\sharp := E'$ and $A^\sharp := A'$, where A' is the dual of A in E in the sense of unbounded linear operators. Finally, let $E^\sharp_k := D\big((A^\sharp)^k\big)$ for $k \in \mathbb{N}$. Then we define E_{-k} for $k \in \mathbb{N}^\times := \mathbb{N}\backslash\{0\}$ by $E_{-k} := (E^\sharp_k)'$ with respect to the duality pairing (induced by) $\langle\cdot,\cdot\rangle_E$. This means the following: E_{-k} is the dual space of $E^\sharp_k$ and

$$\langle x^\sharp, x\rangle_{E_{-k}} = \langle x^\sharp, x\rangle_E \ , \qquad x \in E_k \ , \quad x^\sharp \in E^\sharp \ . \tag{1.4}$$

Since $E^\sharp_k \overset{d}{\hookrightarrow} E^\sharp$, it follows that $(E^\sharp)' = E \overset{d}{\hookrightarrow} (E^\sharp_k)'$ by reflexivity and the Hahn-Banach theorem. Thus, by density, $\langle\cdot,\cdot\rangle_{E_{-k}}$, and hence E_{-k}, is uniquely determined by (1.4).

For each $\theta \in (0,1)$ we fix

$$(\cdot,\cdot)_\theta \in \big\{\, [\cdot,\cdot]_\theta, (\cdot,\cdot)_{\theta,r}, (\cdot,\cdot)^0_{\theta,\infty} \ ; \ 1 \leq r < \infty \,\big\} \tag{1.5}$$

and put $E_{k+\theta} := (E_k, E_{k+1})_\theta$ for $k \in \mathbb{Z}$. It follows that

$$E_\alpha \overset{d}{\hookrightarrow} E_\beta \ , \qquad -\infty < \beta < \alpha < \infty \ . \tag{1.6}$$

If $\alpha \geq 0$ then we denote by A_α the maximal restriction of A to E_α whose domain equals $\{\, x \in E_\alpha \cap E_1 \ ; \ Ax \in E_\alpha \,\}$. If $\alpha < 0$ then A_α is the closure of A in E_α. It follows that A_α is well-defined for $\alpha \in \mathbb{R}$, and $A_0 = A$. The family $\big[\,(E_\alpha, A_\alpha) \ ; \ \alpha \in \mathbb{R}\,\big]$ is said to be the **interpolation-extrapolation scale generated by** (E, A) **and** $(\cdot,\cdot)_\theta$, $0 < \theta < 1$.

One shows that A_β is the closure of A_α in E_β if $\beta < \alpha$. Furthermore,

$$A_\alpha \in \mathcal{H}(E_{\alpha+1}, E_\alpha) \ , \qquad \alpha \in \mathbb{R} \ , \tag{1.7}$$

and $A \in \mathcal{G}(E,\omega)$ implies $A_\alpha \in \mathcal{G}(E_\alpha, \omega)$ for $\alpha \in \mathbb{R}$. In addition,

$$e^{-tA_\beta} \supset e^{-tA_\alpha} \ , \qquad t \geq 0 \ , \quad \beta < \alpha \ . \tag{1.8}$$

Now we define the dual interpolation functor $(\cdot,\cdot)^\sharp_\theta$ of $(\cdot,\cdot)_\theta$ by

$$(\cdot,\cdot)^\sharp_\theta := \begin{cases} [\cdot,\cdot]_\theta & \text{if } (\cdot,\cdot)_\theta = [\cdot,\cdot]_\theta \ , \\ (\cdot,\cdot)_{\theta,1} & \text{if } (\cdot,\cdot)_\theta = (\cdot,\cdot)^0_{\theta,\infty} \ , \\ (\cdot,\cdot)_{\theta,r'} & \text{if } (\cdot,\cdot)_\theta = (\cdot,\cdot)_{\theta,r} \ , \quad 1 < r < \infty \ , \end{cases} \tag{1.9}$$

where $r' := r/(r-1)$. Then we abbreviate the interpolation-extrapolation scale generated by $(E^\sharp, A^\sharp)$ and $(\cdot,\cdot)^\sharp_\theta$, $0<\theta<1$, by $\bigl[\,(E^\sharp_\alpha, A^\sharp_\alpha)\ ;\ \alpha\in\mathbb{R}\bigr]$ and call it **interpolation-extrapolation scale dual to** $\bigl[\,(E_\alpha, A_\alpha)\ ;\ \alpha\in\mathbb{R}\bigr]$.

If $(\cdot,\cdot)_\theta \neq (\cdot,\cdot)_{\theta,1}$ for $0<\theta<1$ then

$$(E_{-\alpha})' \doteq E^\sharp_\alpha\ ,\quad (A_{-\alpha})' = A^\sharp_\alpha\ ,\qquad \alpha\in\mathbb{R}\ , \tag{1.10}$$

with respect to the duality pairing $\langle\cdot,\cdot\rangle_{E_{-\alpha}}$ induced by $\langle\cdot,\cdot\rangle_E$.

We denote by

$$\bigl[\,(E_{\alpha,1}, A_{\alpha,1})\ ;\ \alpha\in\mathbb{R}\bigr] \quad\text{and}\quad \bigl[\,(E^0_{\alpha,\infty}, A^0_{\alpha,\infty})\ ;\ \alpha\in\mathbb{R}\bigr]$$

the interpolation-extrapolation scales generated by (E,A) and the functors $(\cdot,\cdot)_{\theta,1}$ and $(\cdot,\cdot)^0_{\theta,\infty}$, respectively, for $0<\theta<1$. (This notation is consistent with the one used in (1.2).)

Lemma 1.1. *Suppose that* $0\le\alpha\le 1$. *Then*

$$(E_{\alpha-1}, E_\alpha)_{\theta,1} \doteq E_{\alpha-1+\theta,1}\ ,\quad (E_{\alpha-1}, E_\alpha)^0_{\theta,\infty} \doteq E^0_{\alpha-1+\theta,\infty} \tag{1.11}$$

for $0<\theta<1$ *with* $\alpha+\theta\neq 1$. *Furthermore,*

$$(E_{\alpha-1}, E_\alpha)_{\theta,1} \overset{d}{\hookrightarrow} E_{\alpha-1+\theta} \overset{d}{\hookrightarrow} (E_{\alpha-1}, E_\alpha)^0_{\theta,\infty}\ ,\qquad 0<\theta<1\ . \tag{1.12}$$

Proof. It follows from Theorem V.1.5.7 and Corollary V.1.5.8 of [5] (if one sets $(\cdot,\cdot)_\theta := \{\cdot,\cdot\}_\theta := (\cdot,\cdot)_{\theta,1}$) that

$$(E_{\alpha-1,1}, E_{\alpha,1})_{\theta,1} \doteq E_{\alpha-1+\theta,1}\ ,\qquad 0<\theta<1\ ,\quad \alpha+\theta\neq 1\ . \tag{1.13}$$

Since

$$(E_{j-1}, E_j)_{\alpha,1} \hookrightarrow E_{\alpha+j-1} \hookrightarrow (E_{j-1}, E_j)^0_{\alpha,\infty}\ ,\qquad 0<\alpha<1\ , \tag{1.14}$$

for $j=0,1$, the reiteration theorem (cf. (I.2.8.7) in [5]) and (1.13) imply the first assertion in (1.11). The second one follows by replacing $(\cdot,\cdot)_{\theta,1}$ by $(\cdot,\cdot)^0_{\theta,\infty}$ in this argument.

Fix $\mu > \mathrm{type}(-A)$ and note that $E_j \doteq D\bigl((\mu+A_{-1})^{j+1}\bigr)$ for $j\in\{0,1\}$. Thus (cf. (I.2.9.6) in [5])

$$(E_{-1}, E_1)_{1/2,1} \overset{d}{\hookrightarrow} E_0 \overset{d}{\hookrightarrow} (E_{-1}, E_1)^0_{1/2,\infty}\ . \tag{1.15}$$

If $0<\alpha<1$ then, by (1.14), (1.15), and the reiteration theorem,

$$\begin{aligned}(E_{\alpha-1}, E_\alpha)^0_{1-\alpha,\infty} &\doteq \bigl((E_{-1}, E_0)^0_{\alpha,\infty}, (E_0, E_1)^0_{\alpha,\infty}\bigr)^0_{1-\alpha,\infty}\\ &\doteq \bigl((E_{-1}, E_1)^0_{\alpha/2,\infty}, (E_{-1}, E_1)^0_{(1+\alpha)/2,\infty}\bigr)^0_{1-\alpha,\infty}\\ &\doteq (E_{-1}, E_1)^0_{1/2,\infty}\ .\end{aligned} \tag{1.16}$$

By replacing in (1.16) the continuous interpolation functor everywhere by the functor $(\cdot,\cdot)_{\theta,1}$, we find that $(E_{\alpha-1},E_\alpha)_{1-\alpha,1} \doteq (E_{-1},E_1)_{1/2,1}$. This, combined with (1.15) and (1.16), implies (1.12) for $\theta := 1-\alpha$. If $\theta \neq 1-\alpha$ then (1.12) is an immediate consequence of (1.2), (1.3), and (1.11). $\square$

2. Dirichlet Scales

If $\Omega \neq \mathbb{R}^m$ then we assume throughout that $\partial\Omega$ is uniformly regular of class C^2 (in the sense of [9]). This guarantees the existence of suitable extension operators, so that all results on function spaces which we use below, and which are proven in Triebel's book [77] for bounded smooth domains, hold in this case also, provided the regularity indices are restricted to belong to $[-2,2]$ (cf. Section II of [3]). Of course, the extension of the results in [77] to the case of $\mathbb{R}^m$-valued distributions is trivial. Thus we simply refer to [77] and related work without further mention of this fact.

Throughout this paper

$$1 < q < \infty \quad \text{and} \quad -2 \leq s \leq 2 \ ,$$

unless further restrictions are explicitly mentioned.

We denote by

$$W_q^s := \big(W_q^s(\Omega,\mathbb{R}^m), \|\cdot\|_{W_q^s}\big)$$

the usual **Sobolev-Slobodeckii spaces**. Recall that $W_q^0 = L_q := L_q(\Omega,\mathbb{R}^m)$ and

$$\|u\|_{W_q^k} := \Big(\sum_{|\alpha|\leq k} \|\partial^\alpha u\|_{L_q}^q\Big)^{1/q} \ , \qquad k = 1,2 \ ,$$

whereas

$$\|u\|_{W_q^{k+\theta}} := \Big(\|u\|_{W_q^k}^q + \sum_{|\alpha|=k} \int_{\Omega\times\Omega} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^q}{|x-y|^{m+q\theta}}\, d(x,y)\Big)^{1/q} \tag{2.1}$$

for $k \in \{0,1\}$ and $0 < \theta < 1$.

We write $\mathcal{D} := \mathcal{D}(\Omega,\mathbb{R}^m)$ for the space of $\mathbb{R}^m$-valued smooth functions having compact supports in Ω, that is, $\mathcal{D}$ is the space of $\mathbb{R}^m$-valued test functions on Ω, and $\mathcal{D}'$ is its dual, the space of $\mathbb{R}^m$-valued distributions on Ω. Then $\mathring{W}_p^s$ is the closure of $\mathcal{D}$ in W_p^s, and

$$W_q^{-s} := (\mathring{W}_{q'}^s)' \ , \qquad 0 < s \leq 2 \ , \tag{2.2}$$

with respect to the L_q-duality pairing

$$\langle v,u\rangle := \int_\Omega v\cdot u\, dx \ , \qquad (v,u) \in L_{q'} \times L_q \ , \tag{2.3}$$

where $q' := q/(q-1)$.

We also put $\Lambda(\xi) := (1+|\xi|^2)^{1/2}$ for $\xi \in \mathbb{R}^m$ and denote by $\mathcal{F}$ the Fourier transform on $\mathcal{S}' := \mathcal{S}'(\mathbb{R}^m, \mathbb{R}^m)$, the space of $\mathbb{R}^m$-valued temperate distributions on $\mathbb{R}^m$. Then

$$(1-\Delta)^{s/2} := \Lambda^s(D) := \mathcal{F}^{-1}\Lambda^s\mathcal{F} \ ,$$

where Λ acts as a multiplication operator.

For $\Omega = \mathbb{R}^m$ the **Bessel potential spaces** are the Banach spaces defined by

$$H_q^s := H_q^s(\mathbb{R}^m, \mathbb{R}^m) := (1-\Delta)^{-s/2} L_q \ .$$

Fix any smooth ψ on $\mathbb{R}^m$ such that $\psi(\xi) = 1$ for $|\xi| \le 1$ and $\psi(\xi) = 0$ for $|\xi| \ge 2$. Put $\psi_0 := \psi$ and

$$\psi_k(\xi) := \psi(2^{-k}\xi) - \psi(2^{-k+1}\xi) \ , \qquad \xi \in \mathbb{R}^m \ , \quad k \in \mathbb{N}^\times := \mathbb{N}\backslash\{0\} \ . \tag{2.4}$$

Then

$$\operatorname{supp}(\psi_k) \subset \{\, \xi \in \mathbb{R}^m \ ; \ 2^{k-1} \le |\xi| \le 2^{k+1} \,\} \ , \qquad k \in \mathbb{N}^\times \ ,$$

and

$$\sum_{k=0}^\infty \psi_k(\xi) = 1 \ , \qquad \xi \in \mathbb{R}^m \ .$$

Thus (ψ_k) is a smooth dyadic resolution of the identity on $\mathbb{R}^m$.

The **Besov spaces**, $B_{q,r}^s := B_{q,r}^s(\mathbb{R}^m, \mathbb{R}^m)$, are the Banach spaces defined by

$$B_{q,r}^s := \big(\{\, u \in \mathcal{S}' \ ; \ \|u\|_{B_{q,r}^s} < \infty \,\},\ \|\cdot\|_{B_{q,r}^s}\big) \ ,$$

where

$$\|u\|_{B_{q,r}^s} := \begin{cases} \Big(\displaystyle\sum_{k=0}^\infty 2^{ksr} \, \|\psi_k(D)u\|_{L_q}^r\Big)^{1/r} \ , & 1 \le r < \infty \ , \\ \displaystyle\sup_{k\ge 0} 2^{ks} \, \|\psi_k(D)u\|_{L_q} \ , & r := \infty \ , \end{cases} \tag{2.5}$$

with $\psi_k(D) := \mathcal{F}^{-1}\psi_k\mathcal{F}$. These spaces are independent of the particular dyadic resolution of the identity, except for equivalent norms.

We denote by

$$r_\Omega : \mathcal{D}'(\mathbb{R}^m, \mathbb{R}^m) \to \mathcal{D}'(\Omega, \mathbb{R}^m)$$

the operator of restriction to Ω in the sense of distributions. If $\Omega \ne \mathbb{R}^m$ we set

$$H_q^s := H_q^s(\Omega, \mathbb{R}^m) := r_\Omega H_q^s(\mathbb{R}^m, \mathbb{R}^m)$$

and

$$B_{q,r}^s := B_{q,r}^s(\Omega, \mathbb{R}^m) := r_\Omega B_{q,r}^s(\mathbb{R}^m, \mathbb{R}^m) \ , \qquad 1 \le r \le \infty \ ,$$

where these spaces are given the obvious quotient norms. Hence they are Banach spaces as well. It is known that

$$H_p^s \doteq B_{q,r}^t \quad \text{iff} \quad s = t \text{ and } p = q = r = 2 \ , \tag{2.6}$$

and that

$$W_q^s \doteq \begin{cases} H_q^s \ , & s \in [-2,2] \cap \mathbb{Z} \ , \\ B_{q,q}^s \ , & s \in (-2,2) \backslash \mathbb{Z} \ . \end{cases} \tag{2.7}$$

It is also known that $B_{q,\infty}^s$ coincides for $s > 0$, except for equivalent norms, with the **Nikol'skii space** $N_q^s := N_q^s(\Omega, \mathbb{R}^m)$ (cf. [54], [77]).

For the reader's convenience, we recall the definition of the norm of N_q^s for $0 < s < 2$. We denote by $[s]^-$ the largest integer strictly smaller than s and put

$$[u]_{\vartheta,q,\infty} := \begin{cases} \sup_{h \neq 0} |h|^{-\vartheta} \, \|u(\cdot + h) - u\|_{L_q(\Omega_{|h|})} \ , & 0 < \vartheta < 1 \ , \\ \sup_{h \neq 0} |h|^{-1} \, \|u(\cdot + h) - 2u + u(\cdot - h)\|_{L_q(\Omega_{|h|})} \ , & \vartheta = 1 \ , \end{cases}$$

where $\Omega_{|h|} := \{ x \in \Omega \ ; \ \operatorname{dist}(x, \partial\Omega) > |h| \}$ and $L_q(\Omega_{|h|}) := L_q(\Omega_{|h|}, \mathbb{R}^m)$. Then

$$\|u\|_{N_q^s} := \|u\|_{W_q^{[s]^-}} + \max_{|\alpha| = [s]^-} [\partial^\alpha u]_{s - [s]^-, q, \infty} \ . \tag{2.8}$$

For convenience, we set

$$N_q^0 := L_q \ , \quad N_q^{-s} := B_{q,\infty}^{-s} \ , \qquad 0 < s \leq 2 \ . \tag{2.9}$$

We refer to [77] for equivalent intrinsic norms for $B_{q,r}^s$ with $1 \leq r < \infty$.

The following theorem gives another useful characterization (not contained in [77]) of the 'negative spaces' H_q^{-s} and $B_{q,r}^{-s}$ for $0 < s \leq 2$.

Theorem 2.1. *Suppose that $-1 \leq s \leq 2$ and $k \in \{1, 2\}$ with $s - k \geq -2$. Also suppose that $\mathcal{B}^s \in \{ H_q^s, B_{q,r}^s \ ; \ 1 \leq r \leq \infty \}$. Then u belongs to $\mathcal{B}^{s-k}$ iff there exist $u_\alpha \in \mathcal{B}^s$ for $|\alpha| \leq k$ such that*

$$u = \sum_{|\alpha| \leq k} \partial^\alpha u_\alpha \ . \tag{2.10}$$

Moreover,

$$u \mapsto \inf \Big(\sum_{|\alpha| \leq k} \|u_\alpha\|_{\mathcal{B}^s} \Big) \tag{2.11}$$

is an equivalent norm for $\mathcal{B}^{s-k}$, where the infimum is taken over all representations (2.10).

Proof. By the usual extension and restriction procedure we can assume $\Omega = \mathbb{R}^m$. In this case the restriction $|s| \leq 2$ is not necessary.

Since

$$\partial^\alpha \in \mathcal{L}(\mathcal{B}^s, \mathcal{B}^{s - |\alpha|}) \ , \qquad \alpha \in \mathbb{N}^m \ , \tag{2.12}$$

it follows that $u \in \mathcal{B}^{s-k}$ if it is defined by the right-hand side of (2.10). Conversely, given $u \in \mathcal{B}^{s-k}$, it is known that $v := \Lambda^{-2k}(D)u \in \mathcal{B}^{s+k}$. Hence

$$u = \Lambda^{2k}(D)v = \sum_{|\alpha|\le k} (-1)^{|\alpha|} \binom{k}{\alpha} \partial^{2\alpha} v = \sum_{|\alpha|\le k} \partial^\alpha v_\alpha \ ,$$

where $v_\alpha := (-1)^{|\alpha|}\binom{k}{\alpha}\partial^\alpha v \in \mathcal{B}^s$ for $|\alpha| \le k$. Thus, putting $M(k) := \sum_{|\alpha|\le k} 1$, it follows that the linear map

$$T_k : (\mathcal{B}^s)^{M(k)} \to \mathcal{B}^{s-k} \ , \quad (u_\alpha)_{|\alpha|\le k} \mapsto \sum_{|\alpha|\le k} \partial^\alpha u_\alpha$$

is well-defined and surjective. Consequently, thanks to (2.12), it is a continuous linear surjection. Define $\widehat{T}_k$ by the commutativity of the diagram

$$\begin{tikzcd}
(\mathcal{B}^s)^{M(k)} \arrow[rr] \arrow[dr, "T_k"'] & & (\mathcal{B}^s)^{M(k)} / \ker(T_k) \arrow[dl, "\widehat{T}_k"] \\
& \mathcal{B}^{s-k} &
\end{tikzcd}$$

where the horizontal arrow denotes the canonical projection. Then $\widehat{T}_k$ is a toplinear isomorphism (that is, an isomorphism in the category of Banach spaces) by the open mapping theorem. This implies the assertion since (2.11) is the quotient norm of $(\mathcal{B}^s)^{M(k)} / \ker(T_k)$. $\square$

It should be remarked that Theorem 2.1 and (2.7) imply that definition (2.2) for $s \in \{1, 2\}$ is equivalent to the usual definition of the Sobolev spaces of negative orders (e.g., [1]).

Lastly, we define the **little Nikol'skii spaces** by

$$n_q^s := \text{closure of } H_q^2 \text{ in } N_q^s \ . \tag{2.13}$$

Since

$$\mathcal{D}_{\overline{\Omega}} := \mathcal{D}(\overline{\Omega}, \mathbb{R}^m) := \{\, u \,|\, \overline{\Omega} \ ; \ u \in \mathcal{D}(\mathbb{R}^m, \mathbb{R}^m) \,\}$$

is dense in W_q^s it follows that

$$n_q^s := \text{closure of } \mathcal{D}_{\overline{\Omega}} \text{ in } N_q^s \ . \tag{2.14}$$

We denote by γ_∂ the trace operator on $\partial\Omega$ if $\partial\Omega \ne \emptyset$ and put

$$H_{q,0}^2 := \begin{cases} H_q^2 & \text{if } \Omega = \mathbb{R}^m \ , \\ \{\, u \in H_q^2 \ ; \ \gamma_\partial u = 0 \,\} & \text{otherwise} \ . \end{cases}$$

Then $(\mathbf{E}_0, \mathbf{E}_1) := (L_q, H_{q,o}^2)$ is a densely injected Banach couple.

We write $\Delta_D := \Delta_{D,q}$ for the L_q-realization of the Dirichlet-Laplace operator defined by

$$\mathrm{dom}(\Delta_D) := H^2_{q,0} \ , \quad \Delta_D u := \Delta u \ .$$

It is well-known that

$$-\Delta_D \in \mathcal{H}(H^2_{q,0}, L_q) \ . \tag{2.15}$$

Thus the interpolation-extrapolation scale

$$\big[\,(\mathbf{E}_\alpha, \mathbf{A}_\alpha) \ ; \ \alpha \in \mathbb{R}\,\big] \ ,$$

the **Dirichlet scale**, generated by $(\mathbf{E}, \mathbf{A}) := (L_q, -\Delta_D)$ and $(\cdot,\cdot)_\theta, \ 0 < \theta < 1$, where $(\cdot,\cdot)_\theta$ satisfies (1.5), is well-defined.

In order to characterize this scale we set

$$F^{2j}_q := H^{2j}_q \ , \qquad j \in \{0, \pm 1\} \ , \tag{2.16}$$

and, for $s = 2j + 2\theta$ with $j \in \{-1, 0\}$ and $0 < \theta < 1$,

$$F^s_q := \begin{cases} H^s_q & \text{if } (\cdot,\cdot)_\theta = [\cdot,\cdot]_\theta \ , \\ B^s_{q,r} & \text{if } (\cdot,\cdot)_\theta = (\cdot,\cdot)_{\theta,r} \ , \quad 1 \le r < \infty \ , \\ n^s_q & \text{if } (\cdot,\cdot)_\theta = (\cdot,\cdot)^0_{\theta,\infty} \ . \end{cases} \tag{2.17}$$

We also set $F^{\#2j}_{q'} := H^{2j}_{q'}$ for $j \in \{0, 1\}$ and

$$F^{\#2\theta}_{q'} := \begin{cases} H^{2\theta}_{q'} & \text{if } (\cdot,\cdot)_\theta = [\cdot,\cdot]_\theta \ , \\ B^{2\theta}_{q',r'} & \text{if } (\cdot,\cdot)_\theta = (\cdot,\cdot)_{\theta,r} \ , \quad 1 < r < \infty \ , \\ B^{2\theta}_{q',1} & \text{if } (\cdot,\cdot)_\theta = (\cdot,\cdot)^0_{\theta,\infty} \end{cases} \tag{2.18}$$

for $0 < \theta < 1$. If $\Omega = \mathbb{R}^m$ then $F^s_{q,0} := F^s_q$ and $F^{\#s}_{q',0} := F^{\#s}_{q'}$. Otherwise, we put $\widetilde{F}^s_q := F^s_q(\mathbb{R}^m, \mathbb{R}^m)$ and

$$F^s_{q,0} := \begin{cases} \{\, u \in F^s_q \ ; \ \gamma_\partial u = 0 \,\} \ , & 1/q < s \le 2 \ , \\ \{\, u \in \widetilde{F}^{1/q}_q \ ; \ \mathrm{supp}(u) \subset \overline{\Omega} \,\} \ , & s = 1/q \ , \\ F^s_q \ , & 0 \le s < 1/q \ . \end{cases} \tag{2.19}$$

We also define $F^{\#s}_{q',0}$ for $s \ge 0$ by replacing F^s_q, resp. $\widetilde{F}^{1/q}_q$, in (2.19) by $F^{\#s}_{q'}$, resp. $\widetilde{F}^{\#1/q'}_{q'}$.

Now we define negative spaces

$$F^{-2}_{q,0} := (F^2_{q',0})' = (H^2_{q',0})' =: H^{-2}_{q,0} \tag{2.20}$$

by means of the duality pairing induced by (2.3), and

$$F^{-2+2\theta}_{q,0} := (F^{-2}_{q,0}, F^0_{q,0})_\theta = (H^{-2}_{q,0}, L_q)_\theta \ , \qquad 0 < \theta < 1 \ . \tag{2.21}$$

Theorem 2.2. $\mathbf{E}_\alpha \doteq F^{2\alpha}_{q,0}$ *for* $|\alpha| \le 1$.

Proof. Suppose that $0 < \alpha < 1$. If $\Omega = \mathbb{R}^m$ then the results of Section 2.4 in [77] imply

$$\mathbf{E}_\alpha = (\mathbf{E}_0, \mathbf{E}_1)_\alpha = (L_q, H_q^2)_\alpha \doteq F_q^{2\alpha} = F_{q,0}^{2\alpha} \ .$$

If $\Omega \neq \mathbb{R}^m$ then $E_\alpha \doteq F_{q,0}^{2\alpha}$ is a consequence of interpolation results due to Grisvard [36], Seeley [67], and Guidetti [39].

It is well-known that $(\mathbf{E}^\#, \mathbf{A}^\#) = (L_{q'}, -\Delta_{D,q'})$. Hence

$$\mathbf{E}_{-1} = (\mathbf{E}_1^\#)' = (H_{q',0}^2)' = H_{q,0}^{-2} = F_{q,0}^{-2}$$

by (1.10) and reflexivity. Hence (2.21) and known duality properties of the used interpolation functors imply $\mathbf{E}_{\alpha-1} \doteq F_{q,0}^{2\alpha-2}$. □

Corollary 2.3. *Suppose that* $(\cdot,\cdot)_\theta \neq (\cdot,\cdot)_{\theta,1}$ *for* $0 < \theta < 1$. *Then*

$$(F_{q,0}^{-s})' \doteq F_{q',0}^{\#s} \ , \qquad 0 < s \leq 2 \ , \tag{2.22}$$

with respect to the duality pairing (2.3). *Hence*

$$F_{q,0}^{-s} \doteq (F_{q',0}^{\#s})' \ , \qquad 0 < s \leq 2 \ , \tag{2.23}$$

provided $(\cdot,\cdot)_\theta \neq (\cdot,\cdot)_{\theta,\infty}^0$ *for* $0 < \theta < 1$.

Proof. The first assertion follows from Theorem 2.2 and (1.10). The second one is now a consequence of reflexivity. □

The next proposition shows that $F_{q,0}^s$ coincides with the simpler space F_q^s, provided s is suitably restricted.

Proposition 2.4. *If* $-2 + 1/q < s < 1/q$ *then* $F_{q,0}^s = F_q^s$.

Proof. It follows from Theorems 4.3.2.1 and 4.7.1 in [77], the definition of the little Nikol'skii spaces, and (1.2) that $\mathcal{D}$ is dense in $F_{q,0}^s$ for $0 \leq s < 1 + 1/q$ and in $F_{q',0}^{\#t}$ for $0 \leq t < 1 + 1/q' = 2 - 1/q$. Hence the assertion is entailed by (2.19) and (2.23) if either $s \geq 0$ or if $s < 0$ and $(\cdot,\cdot)_\theta \notin \{(\cdot,\cdot)_{\theta,1}, (\cdot,\cdot)_{\theta,\infty}^0\}$ for $0 < \theta < 1$.

Suppose that $-2 + 1/q < s_0 < s < s_1 = 0$. Set $\theta := (s_0 - s_1)/s_0$ and suppose that $(\cdot,\cdot)_\theta \in \{(\cdot,\cdot)_{\theta,1}, (\cdot,\cdot)_{\theta,\infty}^0\}$. Then it follows from what has already been shown and from Lemma 1.1 that

$$F_{q,0}^s \doteq (H_q^{s_0}, H_q^{s_1})_\theta \doteq F_q^s \ .$$

This covers the remaining cases. □

Recall that $\mathcal{D}$ is not dense in $F_{q',0}^{\#s}$ if $\Omega \neq \mathbb{R}^m$ and $1 + 1/q' \leq s \leq 2$. Thus, since $F_{q,0}^{-s} = (F_{q',0}^{\#s})'$ for $(\cdot,\cdot)_\theta \notin \{(\cdot,\cdot)_{\theta,1}, (\cdot,\cdot)_{\theta,\infty}^0\}$, it follows that $F_{q,0}^{-s}$ cannot be identified with a subspace of $\mathcal{D}'$ in this case.

We define $H^s_{q,0}$ etc. for $|s| \le 2$ by

$$F^s_{q,0} =: \begin{cases} H^s_{q,0} \\ B^s_{q,r,0} \\ n^s_{q,0} \end{cases} \quad \text{if} \quad F^s_q = \begin{cases} H^s_q \ , \\ B^s_{q,r} \ , \\ n^s_q \ , \end{cases}$$

respectively, and

$$W^s_{q,0} := \begin{cases} H^s_{q,0} \ , & s \in [-2,2] \cap \mathbb{Z} \ , \\ B^s_{q,q,0} \ , & s \in (-2,2) \backslash \mathbb{Z} \ . \end{cases}$$

Remarks 2.5. **(a)** If $\partial\Omega \neq \emptyset$ then we put

$$d(x) := 1 \wedge \operatorname{dist}(x, \partial\Omega) \ , \qquad x \in \Omega \ .$$

Then

$$u \mapsto \|u\|_{W^{1/q}_q} + \|d^{-1/q}u\|_{L_q} \tag{2.24}$$

is an equivalent norm for $W^{1/q}_{q,0}$. Similarly, set $\Omega^t := \{\, x \in \Omega \ ; \ d(x) < t \,\}$ for $t > 0$. Then

$$u \mapsto \Big(\|u\|^q_{N^{1/q}_q} + \sup_{0<t<1} t^{-1/q} \int_{\Omega^t} |u|^q \, dx \Big)^{1/q} \tag{2.25}$$

is an equivalent norm for $N^{1/q}_{q,0}$, hence for $n^{1/q}_{q,0}$.

Proof. See Remark 4.3.2.2 in [77]. □

(b) Suppose that $0 < s < 2 - 1/q$. Then $(B^s_{q',1,0})' = N^{-s}_q$.

Proof. Since $\mathcal{D}$ is dense in $B^s_{q',1,0}$ for $0 < s < 1 + 1/q' = 2 - 1/q$ this follows from Theorem 4.8.1 in [77] and from (2.9). □

(c) If $m < q < r < \infty$ then $N^{-1+m/q}_q \hookrightarrow N^{-1+m/r}_r$.

Proof. This is a consequence of (2.9) and the known embedding theorems for Besov spaces (e.g., [77]). □

3. Stokes Scales

We put $\mathcal{D}_\sigma := \{\, u \in \mathcal{D} \ ; \ \nabla \cdot u = 0 \,\}$ and let $L_{q,\sigma}$ be the closure of $\mathcal{D}_\sigma$ in L_q. We also set

$$L_{q,\pi} := \big\{\, v \in L_q \ ; \ \exists p \in L_{q,\mathrm{loc}}(\overline{\Omega}, \mathbb{R}) : v = \nabla p \,\big\} \ .$$

Then we **assume** that the topological direct sum decomposition, the *Helmholtz decomposition*,

$$L_q = L_{q,\sigma} \oplus L_{q,\pi} \tag{3.1}$$

is valid for each $q \in (1, \infty)$. Thus the *Helmholtz projector* P_q, that is, the projection of L_q onto $L_{q,\sigma}$ parallel to $L_{q,\pi}$, is well-defined for $1 < q < \infty$.

We also **assume** that

$$(P_q)' = P_{q'} \ , \qquad 1 < q < \infty \ . \tag{3.2}$$

We define the *Stokes operator* S_q in $L_{q,\sigma}$ by

$$\mathrm{dom}(S_q) := H^2_{q,0} \cap L_{q,\sigma} \ , \quad S_q u := -\nu P_q \Delta u \ ,$$

and **assume** that

$$\begin{aligned}&-S_q \text{ generates a strongly continuous}\\&\text{analytic semigroup on } L_{q,\sigma} \text{ for } 1 < q < \infty \ .\end{aligned} \tag{3.3}$$

Remark 3.1. Assumptions (3.1)–(3.3) are additional hypotheses on Ω, which are known to hold in a variety of situations, but not always (e.g., Remark III.1.3 in [28]). To be more precise, they are satisfied if Ω is

(i) $\mathbb{R}^m$;
(ii) a half-space [7];
(iii) an exterior domain ([8], [69], [79], also see [62], [74]);
(iv) a bounded domain ([25], [29], also see [61], [74], [80]);
(v) an aperture domain [20];
(vi) an infinite layer domain [84];
(vii) a compact perturbation of a half-space [19];

We also note that the Stokes semigroup is known to be bounded in each one of the cases (i)–(vi), as is shown in the above references. If Ω is bounded then this semigroup is even exponentially decaying, independently of q, that is, there exists $\omega > 0$ such that

$$\mathrm{type}(-S_q) \le -\omega \ , \quad 1 < q < \infty \ . \tag{3.4}$$

Proof. If Ω is bounded then S_q has a compact resolvent. Using this fact it is not difficult to see that the spectrum and the eigenfunctions of S_q are independent of $q \in (1, \infty)$. Since $\mathrm{type}(-S_q)$ equals the real part of the least eigenvalue of S_q, assertion (3.4) easily follows from the case $q = 2$ and Poincaré's inequality. $\square$

Henceforth Ω is said to be a **standard domain** if one of the following conditions is satisfied: Ω is

(i) $\mathbb{R}^m$;

(ii) a half-space;
(iii) an exterior domain and $m \geq 3$;
(iv) bounded.
(The reason for imposing the restriction $m \geq 3$ in case of an exterior domain will become clear in Remark 8.1).

Now we fix q and set $P := P_q$ and $S := S_q$. It is a consequence of (3.3) that the interpolation-extrapolation scale

$$\big[(\mathbb{E}_\alpha , \mathbb{A}_\alpha) \ ; \ \alpha \in \mathbb{R} \big] , \text{ generated by } (\mathbb{E}, \mathbb{A}) := (L_{q,\sigma}, S) \text{ and } (\cdot, \cdot)_\theta , \ 0 < \theta < 1 \ ,$$

where $(\cdot, \cdot)_\theta$ satisfies (1.5), the **Stokes scale**, is well-defined. The following lemma will be used to characterize $\mathbb{E}_\alpha$ for $0 \leq \alpha \leq 1$.

Lemma 3.2. *Given any interpolation functor $\mathfrak{F}$,*

$$\mathfrak{F}(\mathbb{E}_0 , \mathbb{E}_1) \doteq \mathfrak{F}(\mathbf{E}_0, \mathbf{E}_1) \cap \mathbb{E}_0 \ .$$

Proof. Fix $\mu > 0$ with $\mu > \mathrm{type}(-S)$ and recall that $\mathrm{type}(-\Delta_D) = 0$. Define the map $Q_1 \in \mathcal{L}(\mathbf{E}_1, \mathbb{E}_1)$ by the commutativity of the diagram

$$\begin{tikzcd} \mathbf{E}_1 \arrow[r, "Q_1"] \arrow[d, "\mu+\mathbf{A}"', "\cong"] & \mathbb{E}_1 \arrow[d, "\mu+\mathbb{A}", "\cong"'] \\ \mathbf{E}_0 \arrow[r, "P"] & \mathbb{E}_0 \end{tikzcd} \tag{3.5}$$

so that $Q_1 = (\mu + \mathbb{A})^{-1} P(\mu + \mathbf{A})$. Denote by $P^\top$ the dual of $P \in \mathcal{L}(\mathbf{E}_0, \mathbb{E}_0)$. Then $P^\top$ equals the injection $i^\sharp : \mathbb{E}_0^\sharp \hookrightarrow \mathbf{E}_0^\sharp$. Write Q if Q_1 is being considered as a densely defined linear operator from $\mathbf{E}_0$ into $\mathbb{E}_0$ with domain $\mathbf{E}_1$. Then, since $\mathrm{im}(\mu + \mathbb{A}^\sharp)^{-1} = \mathbb{E}_1^\sharp \subset \mathbf{E}_1^\sharp = \mathrm{dom}(\mu + \mathbf{A}^\sharp)$, it follows that

$$Q' = (\mu + \mathbf{A})' P^\top \big[(\mu + \mathbb{A})^{-1} \big]' = (\mu + \mathbf{A}^\sharp)(\mu + \mathbb{A}^\sharp)^{-1} \in \mathcal{L}(\mathbb{E}_0^\sharp , \mathbf{E}_0^\sharp) \ .$$

Hence $Q'' \in \mathcal{L}(\mathbf{E}_0, \mathbb{E}_0)$ which, thanks to $Q'' \supset Q$ and the density of $\mathbf{E}_1$ in $\mathbf{E}_0$, shows that Q_1 has a unique continuous extension $Q_0 \in \mathcal{L}(\mathbf{E}_0, \mathbb{E}_0)$. In other words: there exists a unique $Q_0 \in \mathcal{L}(\mathbf{E}_0, \mathbb{E}_0)$ for which the diagram

$$\begin{tikzcd} \mathbf{E}_1 \arrow[r, "Q_1"] \arrow[d, hook, "d"] & \mathbb{E}_1 \arrow[d, hook, "d"'] \\ \mathbf{E}_0 \arrow[r, "Q_0"] & \mathbb{E}_0 \end{tikzcd} \tag{3.6}$$

is commutative.

Owing to $\mathbb{E}_1 \hookrightarrow \mathbf{E}_1$, we can consider Q_1 as a bounded linear map in $\mathbf{E}_1$. Then

$$Q_1^2 = (\mu + \mathbb{A})^{-1} P(\mu + \mathbf{A})(\mu + \mathbb{A})^{-1} P(\mu + \mathbf{A}) = (\mu + \mathbb{A})^{-1} P(\mu + \mathbf{A}) = Q_1 \ ,$$

thanks to $P^2 = P$, which entails

$$P(\mu + \mathbf{A})(\mu + \mathbb{A})^{-1} P = P(\mu + P\mathbf{A})(\mu + \mathbb{A})^{-1} P = P$$

as a consequence of $P\mathbf{A} \,|\, \mathbb{E}_1 = \mathbb{A}$. Thus Q_1 is a continuous projection from $\mathbf{E}_1$ onto $\mathbb{E}_1$. From this and from (3.6) it follows that $Q_0 \in \mathcal{L}(\mathbf{E}_0)$ is also a projection from $\mathbf{E}_0$ onto $\mathbb{E}_0$. Now the assertion is a consequence of interpolation theory (e.g., Theorem 1.17.1 in [77]). □

The main idea of the preceding proof, namely the construction of projections Q_0 and Q_1, is due to Fujita and Morimoto [24] and has also been used by Giga (see Lemma 6 in [30]).

The next lemma, guaranteeing the existence of a unique extension of the Helmholtz projector to negative spaces, will be of fundamental importance for the proof of Theorem 4.2.

Lemma 3.3. *For $0 < \alpha \leq 1$ there exists a unique $P_{-\alpha} \in \mathcal{L}(\mathbf{E}_{-\alpha}, \mathbb{E}_{-\alpha})$ satisfying $P_{-\alpha} \supset P$.*

Proof. Define $P_{-1} \in \mathcal{L}(\mathbf{E}_{-1}, \mathbb{E}_{-1})$ by the commutativity of the diagram

$$\begin{tikzcd} \mathbf{E}_0 \arrow[r, "Q_0"] \arrow[d, "\mu + \mathbf{A}_{-1}"', "\cong"] & \mathbb{E}_0 \arrow[d, "\cong"', "\mu + \mathbb{A}_{-1}"] \\ \mathbf{E}_{-1} \arrow[r, "P_{-1}"] & \mathbb{E}_{-1} \end{tikzcd}$$

Then, thanks to (3.5) and (3.6),

$$P_{-1} = (\mu + \mathbb{A}_{-1}) Q_0 (\mu + \mathbf{A}_{-1})^{-1} \supset (\mu + \mathbb{A}) Q_1 (\mu + \mathbf{A})^{-1} = P \ .$$

Thus the diagram

$$\begin{tikzcd} \mathbf{E}_0 \arrow[r, "P"] \arrow[d, hook, "d"] & \mathbb{E}_0 \arrow[d, hook', "d"'] \\ \mathbf{E}_{-1} \arrow[r, "P_{-1}"] & \mathbb{E}_{-1} \end{tikzcd}$$

is commutative. Now the assertion follows by interpolation and the density of $\mathbf{E}_0$ in $\mathbf{E}_{-1}$. □

It is the purpose of the following considerations to characterize the negative spaces $\mathbb{E}_{-\alpha}$ for $0 < \alpha \leq 2$. For this we need some preparation.

Let M be a vector subspace of some Banach space E. Then its annihilator

$$M^{\perp} := \{\, e' \in E' \; ; \; \langle e', m \rangle = 0 \ \forall\, m \in M \,\}$$

is a closed linear subspace of E'. It is a well-known consequence of the Hahn-Banach theorem (e.g., Theorem I.6.4 in [35]) that the restriction map

$$E'/M^{\perp} \to M' \; , \quad [e'] \mapsto e' \,|\, M \tag{3.7}$$

is an isometric isomorphism.

Suppose that $E = M \oplus N$ and denote by Q the projection of E onto M parallel to N. Then $M = \operatorname{im}(Q) = \ker(1-Q)$ and $N = \ker(Q) = \operatorname{im}(1-Q)$. Since $Q' \in \mathcal{L}(E')$ is a projection as well,

$$E' = \ker(1-Q') \oplus \ker(Q') = \bigl[\operatorname{im}(1-Q)\bigr]^{\perp} \oplus \bigl[\operatorname{im}(Q)\bigr]^{\perp} = N^{\perp} \oplus M^{\perp} \; ,$$

since $\operatorname{im}(T)^{\perp} = \ker(T')$ for each $T \in \mathcal{L}(E)$. Thus it follows from (3.1) and (3.2) that

$$L_{p',\sigma} = (L_{p,\pi})^{\perp} \; , \quad L_{p',\pi} = (L_{p,\sigma})^{\perp} \; , \qquad 1 < p < \infty \; . \tag{3.8}$$

We define a continuous bilinear form $\langle \cdot , \cdot \rangle_{\sigma}$ on $L_{q',\sigma} \times L_{q,\sigma}$ by restriction of $\langle \cdot , \cdot \rangle$, that is,

$$\langle u, v \rangle_{\sigma} := \langle u, v \rangle \; , \qquad (u,v) \in L_{q',\sigma} \times L_{q,\sigma} \; .$$

It follows from (3.1) and (3.8) that $\langle \cdot , \cdot \rangle_{\sigma}$ is non-degenerate. Moreover, it is a consequence of (3.1) and (3.8) that

$$(L_{q,\sigma})' = L_{q',\sigma} \text{ by means of the duality pairing } \langle \cdot , \cdot \rangle_{\sigma} \; . \tag{3.9}$$

Now we put

$$F^{s}_{q,0,\sigma} := F^{s}_{q,0} \cap L_{q,\sigma} \; , \quad F^{\# s}_{q',0,\sigma} := F^{\# s}_{q',0} \cap L_{q',\sigma} \; , \qquad 0 \leq s \leq 2 \; .$$

Of course, $F^{s}_{q,0,\sigma}$ is called $H^{s}_{q,0,\sigma}$ if $F^{s}_{q,0} = H^{s}_{q,0}$, etc.

We also set

$$F^{-2\theta}_{q,0,\sigma} := (F^{\# 2\theta}_{q',0,\sigma})' \; , \qquad 0 < \theta < 1 \; , \quad (\cdot,\cdot)_{\theta} \notin \{ (\cdot,\cdot)_{\theta,1}, (\cdot,\cdot)^{0}_{\theta,\infty} \} \; ,$$

by means of the duality pairing $\langle \cdot , \cdot \rangle_{\sigma}$. This defines

$$[\, H^{s}_{q,0,\sigma} \; ; \; |s| \leq 2 \,] \quad \text{and} \quad [\, B^{s}_{q,r,0,\sigma} \; ; \; |s| \leq 2 \,] \; , \qquad 1 < r < \infty \; .$$

We put

$$N^{-s}_{q,0,\sigma} := (B^{s}_{q',1,0,\sigma})' \; , \qquad 0 < s < 2 \; ,$$

by means of $\langle\cdot,\cdot\rangle_\sigma$, and

$$n_{q,0,\sigma}^{-s} := \text{closure of } H_{q,0,\sigma}^2 \text{ in } N_{q,0,\sigma}^{-s}\ , \qquad 0 < s < 2\ . \tag{3.10}$$

Finally,

$$B_{q,1,0,\sigma}^{-2\theta} := (H_{q,0,\sigma}^{-2}, L_{q,\sigma})_{\theta,1}\ , \qquad 0 < \theta < 1\ . \tag{3.11}$$

Then the scales of Banach spaces

$$[\,B_{q,1,0,\sigma}^s\ ;\ |s| \le 2\,]\ ,\quad [\,n_{q,0,\sigma}^s\ ;\ |s| \le 2\,]\ ,\quad [\,W_{q,0,\sigma}^s\ ;\ |s| \le 2\,]$$

have also been well-defined, where

$$W_{q,0,\sigma}^s \doteq \begin{cases} H_{q,0,\sigma}^s\ , & s \in [-2,2] \cap \mathbb{Z}\ ,\\ B_{q,q,0,\sigma}^s\ , & s \in (-2,2) \backslash \mathbb{Z}\ . \end{cases}$$

The following theorem justifies the introduction of these spaces.

Theorem 3.4. $\mathbb{E}_\alpha \doteq F_{q,0,\sigma}^{2\alpha}$ *for* $|\alpha| \le 1$.

Proof. (a) If $0 \le \alpha \le 1$ then this is an immediate consequence of Theorem 2.2 and Lemma 3.2.

(b) It follows from (3.9) that $\mathbb{E}^\# = L_{q',\sigma}$. Denote by $P^\top \in \mathcal{L}(L_{q',\sigma}, L_{q'})$ the dual of $P \in \mathcal{L}(L_q, L_{q,\sigma})$. Then it is easily verified that $P^\top = i^\# : L_{q',\sigma} \hookrightarrow L_{q'}$. Thus, given $(v,u) \in H_{q',0,\sigma}^2 \times H_{q,0,\sigma}^2$,

$$\begin{aligned} \langle v, \mathbb{A}u\rangle_\sigma &= -\nu\langle v, P_q\Delta_D u\rangle_\sigma = -\nu\big\langle i^\#(v), \Delta_D u\big\rangle \\ &= -\nu\big\langle \Delta_D v, i(u)\big\rangle = \langle -\nu P_{q'}\Delta_D v, u\rangle_\sigma = \langle S_{q'}v, u\rangle_\sigma\ . \end{aligned}$$

This shows that $\mathbb{A}' \supset S_{q'}$. Since the resolvent set of $\mathbb{A}'$ and the one of $S_{q'}$ have a nonempty intersection, it follows that $\mathbb{A}' = S_{q'}$, that is, $\mathbb{A}^\# = S_{q'}$. Hence (a) implies $\mathbb{E}_\alpha^\# \doteq F_{q',0,\sigma}^{\#2\alpha}$ for $0 \le \alpha \le 1$.

(c) Suppose that $(\cdot,\cdot)_\theta \notin \big\{(\cdot,\cdot)_{\theta,1}, (\cdot,\cdot)_{\theta,\infty}^0\big\}$. Then the reflexivity of $\mathbb{E} = L_{q,\sigma}$, which holds since $L_{q,\sigma}$ is a closed linear subspace of the reflexive space L_q, implies the reflexivity of $\mathbb{E}_\alpha$ for each $\alpha \in \mathbb{R}$ (cf. Theorem V.1.5.12 in [5]). Thus we infer $\mathbb{E}_{-\alpha} \doteq (\mathbb{E}_\alpha^\#)' = F_{q,0,\sigma}^{-2\alpha}$ for $0 \le \alpha \le 1$ from (b) and (1.10).

(d) Suppose that $(\cdot,\cdot)_\theta = (\cdot,\cdot)_{\theta,\infty}^0$ for $0 < \theta < 1$. Then, thanks to (1.9), (1.10), and (a),

$$(\mathbb{E}_{-\alpha})' = \mathbb{E}_\alpha^\# \doteq B_{q',1,0,\sigma}^{2\alpha}\ , \qquad 0 \le \alpha \le 1\ .$$

Consequently,

$$\mathbb{E}_{-\alpha} \hookrightarrow (\mathbb{E}_{-\alpha})'' = (B_{q',1,0,\sigma}^{2\alpha})' = N_{q,0,\sigma}^{-2\alpha}\ , \qquad 0 < \alpha < 1\ ,$$

where the first injection is the canonical injection of a Banach space into its bidual. Since $\mathbb{E}_{-\alpha} = (\mathbb{E}_{-1}, \mathbb{E}_0)_{1-\alpha,\theta}^0$, the assertion follows from the density of $\mathbb{E}_1 \doteq H_{q,0,\sigma}^2$ in $\mathbb{E}_{-\alpha}$ and from (3.10).

(e) If $(\cdot,\cdot)_\theta = (\cdot,\cdot)_{\theta,1}$ then the assertion concerning $\mathbb{E}_{-\alpha}$ follows immediately from (b), which gives $\mathbb{E}_{-1} \doteq H^{-2}_{q,0,\sigma}$, and from (3.11). $\square$

The next theorem shows that, in the reflexive case, the negative spaces $F^{-s}_{q,\sigma}$ possess further useful characterizations.

Theorem 3.5. *Suppose that $(\cdot,\cdot)_\theta \notin \{(\cdot,\cdot)_{\theta,1}, (\cdot,\cdot)^0_{\theta,\infty}\}$. Then, given $s \in (0,2]$,*

$$F^{-s}_{q,0} / \overline{L_{q,\pi}} \to F^{-s}_{q,0,\sigma} \ , \quad [u] \mapsto u \,|\, F^{\# s}_{q',0,\sigma}$$

is an isometric isomorphism, where $\overline{L_{q,\pi}}$ is the closure of $L_{q,\pi}$ in $F^{-s}_{q,0}$.

Proof. Note that

$$L_{q,\pi} \hookrightarrow L_q \hookrightarrow F^{-s}_{q,0} \ , \tag{3.12}$$

where the last injection follows from Theorem 2.2. Hence $\overline{L_{q,\pi}}$ is well-defined. Thanks to (3.8),

$$L_{q,\pi} = (L_{q',\sigma})^\perp \subset (F^{\# s}_{q',0} \cap L_{q',\sigma})^\perp = (F^{\# s}_{q',0,\sigma})^\perp \ . \tag{3.13}$$

The assumption implies that $F^{\# s}_{q',0}$ is reflexive. Thus we infer from (3.7) that

$$\big((F^{\# s}_{q',0,\sigma})^\perp\big)' \cong (F^{\# s}_{q',0})'' / (F^{\# s}_{q',0,\sigma})^{\perp\perp} = F^{\# s}_{q',0} / F^{\# s}_{q',0,\sigma} \ . \tag{3.14}$$

Suppose that $f \in F^{\# s}_{q',0} = (F^{-s}_{q,0})'$ and $f \,|\, L_{q,\pi} = 0$. Then $f \in (L_{q,\pi})^\perp = L_{q',\sigma}$, which shows that $f \in F^{\# s}_{q',0} \cap L_{q',\sigma} = F^{\#}_{q',0,\sigma}$. Hence we infer from (3.14) that there is no continuous linear form on $(F^{\# s}_{q',0,\sigma})^\perp$ vanishing on $L_{q,\pi}$. Thus $L_{q,\pi}$ is dense in $(F^{\# s}_{q',0,\sigma})^\perp$ by (3.13) and the Hahn-Banach theorem. Since $(F^{\# s}_{q',0,\sigma})^\perp$ is closed in $(F^{\# s}_{q',0,\sigma})' = F^{-s}_{q,0,\sigma}$, it follows that $\overline{L_{q,\pi}} = (F^{\# s}_{q',0,\sigma})^\perp$. This implies, together with Corollary 2.3, that

$$F^{-s}_{q,0} / \overline{L_{q,\pi}} = (F^{\# s}_{q',0})' / (F^{\# s}_{q',0,\sigma})^\perp \ .$$

Now the assertion is entailed by (3.7). $\square$

Remark 3.6. Suppose that $0 < s \le 2$. Then $n^{-s}_{q,0,\sigma}$ is isometrically isomorphic to the closure of $L_q / L_{q,\pi}$ in $N^{-s}_{q,0} / (B^s_{q',1,0,\sigma})^\perp$.

Proof. From (3.7) we deduce that $N^{-s}_{q,0} / (B^s_{q',1,0,\sigma})^\perp$ is isometrically isomorphic to $(B^s_{q',1,0,\sigma})' = N^{-s}_{q,0,\sigma}$. Again by (3.7), this isomorphism restricts to an isometric isomorphism

$$L_q / L_{q,\pi} = (L_{q'})' / (L_{q',\sigma})^\perp \cong (L_{q',\sigma})' = L_{q,\sigma} = \mathbb{E}_0 \ .$$

Since $\mathbb{E}_1 = H^2_{q,0,\sigma} \overset{d}{\hookrightarrow} \mathbb{E}_0$ the assertion follows from the definition of $n^{-s}_{q,0,\sigma}$. $\square$

The difficulty in treating the Navier-Stokes equations in a weak setting, which forces us to employ the somewhat complicated setting introduced above, stems from the fact that we have to characterize the negative spaces by duality. This is due to the fact that, in the presence of a boundary, we have no explicit representation either of $-\Delta_D$ or of the Helmholtz projection P. The situation is considerably simpler if $\Omega = \mathbb{R}^m$ (or if Ω is a torus, a case we do not consider here) as is explained below. In the full-space problem we have an explicit representation of P which commutes with Δ so that the Stokes operator reduces to $-\nu\Delta \,|\, H_q^2 \cap L_{q,\sigma}$.

Remarks 3.7. **(a)** Suppose that $\Omega = \mathbb{R}^m$. Then

$$\mathbb{E}_\alpha \doteq F_q^{2\alpha} \cap \ker(\nabla\cdot) = \{\, u \in F_q^{2\alpha} \ ; \ \nabla \cdot v = 0 \,\} \ , \qquad |\alpha| \leq 1 \ ,$$

where $\nabla\cdot$ denotes the divergence operator on $\mathcal{D}'$, of course.

Proof. It is well-known (and by means of the Fourier transform not difficult to verify) that

$$P = 1 - [R_j R_k]_{1 \leq j,k \leq m} \ ,$$

where $R_j := \mathcal{F}^{-1}(\xi^j / |\xi|)\mathcal{F}$ are the Riesz transforms for $1 \leq j \leq m$. Since R_j belongs to $\mathcal{L}(H_q^s)$ for $s \in \mathbb{R}$, as is a well-known consequence of Mikhlin's multiplier theorem, it follows that $P \in \mathcal{L}(H_q^s)$ with $P^2 = P$ and $P(H_q^s) = H_q^s \cap \ker(\nabla\cdot)$. Of course, P commutes with Δ. From this we infer that $1 + S_q$ is an isomorphism from $H_q^{s+2} \cap \ker(\nabla\cdot)$ onto $H_q^s \cap \ker(\nabla\cdot)$ for $s \in \mathbb{R}$. Consequently, if $(\cdot,\cdot)_\theta$ equals $[\cdot,\cdot]_\theta$ for $0 < \theta < 1$ then

$$\mathbb{E}_\alpha = H_q^{2\alpha} \cap \ker(\nabla\cdot) \ , \qquad \alpha \in \mathbb{R} \ .$$

Indeed, this follows from the general definition of interpolation-extrapolation scales given in Chapter V of [5], from Lemma 3.2, the fact that the Bessel potential spaces are invariant under complex interpolation, and from Theorem V.1.5.12 in [5].

Now the assertions for the other choices of $(\cdot,\cdot)_\theta$ follows by interpolation and by applying Lemma 3.2 once more. $\square$

(b) Let Ω be a standard domain with nonempty boundary. Then

$$F_{q,0,\sigma}^s = \begin{cases} \{\, u \in F_q^s \ ; \ \nabla \cdot u = 0, \ \gamma_\partial u = 0 \,\} \ , & 1/q < s \leq 2 \ , \\ \{\, u \in F_{q,0}^{1/q} \ ; \ \nabla \cdot u = 0, \ \gamma_{\vec{n}} u = 0 \,\} \ , & s = 1/q \ , \\ \{\, u \in F_q^s \ ; \ \nabla \cdot u = 0, \ \gamma_{\vec{n}} u = 0 \,\} \ , & 0 \leq s < 1/q \ , \end{cases}$$

where $\gamma_{\vec{n}}$ denotes the normal trace operator defined by $\gamma_{\vec{n}} u := (\gamma_\partial u) \cdot \vec{n}$ for $u \in L_q$ with $\nabla \cdot u \in L_q(\Omega, \mathbb{R})$.

Proof. This follows from $F_{q,0,\sigma}^s = F_{q,0}^s \cap L_{q,\sigma}$ for $0 \leq s \leq 2$, the definition of $F_{q,0}^s$, and the fact that

$$L_{q,\sigma} = \{\, u \in L_q \ ; \ \nabla \cdot u = 0, \ \gamma_{\vec{n}} u = 0 \,\}$$

(see [25] if Ω is bounded and [69] if Ω is unbounded; also cf. Section 5 in [19]). $\square$

Having found explicit representations for the Banach spaces $\mathbb{E}_\alpha$ for $|\alpha| \le 1$, we now turn to characterizations of the extrapolated Stokes operator.

Theorem 3.8. *Suppose that $0 \le \alpha < 1$. Then*

$$H^2_{q',0,\sigma} \times H^2_{q,0,\sigma} \to \mathbb{R} \ , \quad (v,u) \mapsto \langle \Delta v, u \rangle$$

extends to a continuous bilinear form over $F^{\#2-2\alpha}_{q',0,\sigma} \times F^{2\alpha}_{q,0,\sigma}$, again denoted by the same symbol, that is,

$$\big((v,u) \mapsto \langle \Delta v, u\rangle \big) \in \mathcal{L}(F^{\#2-2\alpha}_{q',0,\sigma}, F^{2\alpha}_{q,0,\sigma}; \mathbb{R}) \ .$$

Moreover,

$$\langle v, \mathbb{A}_{\alpha-1} u \rangle_{\mathbb{E}_{\alpha-1}} = -\nu \langle \Delta v, u \rangle \ , \qquad (v,u) \in F^{\#2-2\alpha}_{q',0,\sigma} \times F^{2\alpha}_{q,0,\sigma} \ . \tag{3.15}$$

Proof. From the proof of Theorem 3.4 we know that $(\mathbb{E}^\#, \mathbb{A}^\#) = (L_{q',\sigma}, S_{q'})$ with respect to $\langle \cdot, \cdot \rangle_\sigma$. Note that $\mathbb{A}_{\alpha-1} \in \mathcal{L}(\mathbb{E}_\alpha, \mathbb{E}_{\alpha-1})$ and (1.10) imply

$$\big((v,u) \mapsto \langle v, \mathbb{A}_{\alpha-1} u \rangle_{\mathbb{E}_{\alpha-1}} \big) \in \mathcal{L}(\mathbb{E}^\sharp_{1-\alpha}, \mathbb{E}_\alpha; \mathbb{R}) \ . \tag{3.16}$$

Hence, by Proposition V.1.5.14 in [5],

$$\langle v, \mathbb{A}_{\alpha-1} u \rangle_{\mathbb{E}_{\alpha-1}} = \langle \mathbb{A}^\sharp_{-\alpha} v, u \rangle_{\mathbb{E}_\alpha} \ , \qquad (v,u) \in \mathbb{E}^\sharp_{1-\alpha} \times \mathbb{E}_\alpha \ .$$

Given $(v,u) \in \mathbb{E}^\sharp_1 \times \mathbb{E}_1$,

$$\langle \mathbb{A}^\sharp_{-\alpha} v, u \rangle_{\mathbb{E}_\alpha} = \langle \mathbb{A}^\sharp v, u \rangle = -\nu \langle \Delta v, u \rangle \ . \tag{3.17}$$

Now the assertion follows from (3.16), (3.17), from $\mathbb{E}^\sharp_1 \times \mathbb{E}_1 \overset{d}{\hookrightarrow} \mathbb{E}^\sharp_{1-\alpha} \times \mathbb{E}_\alpha$, and from Theorem 3.4. $\square$

Suppose that $\Omega \ne \mathbb{R}^m$ and $0 \le 2\alpha < 1/q$. Then $F^{2\alpha-2}_{q,0}$ is not a space of distributions (since $\mathcal{D}$ is not dense in $F^{2-2\alpha}_{q',0}$). Thus Theorem 3.4 shows that $F^{2\alpha-2}_{q,0,\sigma}$ is not a (quotient) space of distributions either. Hence (3.17) is in this case not a distributional relation.

The following proposition gives distributional characterizations of $\mathbb{A}_{\alpha-1}$. Given $(A,B) \in L_{q'}(\Omega, \mathbb{R}^{m\times m}) \times L_q(\Omega, \mathbb{R}^{m \times m})$, we set

$$\langle A, B \rangle := \int_\Omega A : B \, dx \ ,$$

where $A : B := \operatorname{trace}(B^\top A)$.

Proposition 3.9. *Suppose that $0 \leq \alpha \leq 1$ and either $\Omega = \mathbb{R}^m$ or $2\alpha > 1/q$. Then*

$$\mathbb{A}_{\alpha-1} = -\nu P_{\alpha-1}\Delta \,|\, F^{2\alpha}_{q,0,\sigma} \ . \tag{3.18}$$

If $1/q < 2\alpha < 1 + 1/q$ then $\mathbb{A}_{\alpha-1}$ is also characterized by

$$\langle v, \mathbb{A}_{\alpha-1} u\rangle_{\mathbb{E}_{\alpha-1}} = \nu\langle \nabla v, \nabla u\rangle \ , \qquad (v,u) \in F^{\#2-2\alpha}_{q',0,\sigma} \times F^{2\alpha}_{q,0,\sigma} \ . \tag{3.19}$$

Proof. It is a consequence of Proposition 2.4 that $\Delta \in \mathcal{L}(F^{2\alpha}_{q,0}, F^{2\alpha-2}_{q,0})$, provided either $\Omega = \mathbb{R}^m$ or $2\alpha > 1/q$. Hence Lemma 3.3 and Theorem 3.4 imply

$$\mathbb{A} = -\nu P\Delta \,|\, H^2_{q,0,\sigma} \subset -\nu P_{\alpha-1}\Delta \,|\, F^{2\alpha}_{q,0,\sigma} \in \mathcal{L}(\mathbb{E}_\alpha, \mathbb{E}_{\alpha-1}) \ .$$

Since

$$\mathbb{A}_{\alpha-1} \in \mathcal{L}(\mathbb{E}_\alpha, \mathbb{E}_{\alpha-1}) \tag{3.20}$$

is the unique continuous extension of $\mathbb{A} \in \mathcal{L}(\mathbb{E}_1, \mathbb{E}_0)$, assertion (3.18) follows.

Now suppose that $1/q < 2\alpha < 1 + 1/q$. Then we infer from Proposition 2.4 that

$$\big((v,u) \mapsto \langle \nabla v, \nabla u\rangle\big) \in \mathcal{L}(F^{\#2-2\alpha}_{q',0,\sigma}, F^{2\alpha}_{q,0,\sigma}; \mathbb{R}) \ .$$

If $(v,u) \in H^2_{q',0,\sigma} \times H^2_{q,0,\sigma}$ then it is clear that $\langle v, \mathbb{A}u\rangle = \nu\langle \nabla v, \nabla u\rangle$. Hence a density argument and (3.20) prove (3.19). □

We close this section by proving some important embedding theorems. For this we recall that, given $-2 \leq t \leq s \leq 2$ and $1 < r < \infty$,

$$H^s_q \overset{d}{\hookrightarrow} H^t_r \ , \qquad 1/q \geq 1/r \geq 1/q - (s-t)/m \ . \tag{3.21}$$

The following proposition shows that a similar result is true for $H^s_{q,0,\sigma}$-spaces, at least if Ω is a standard domain.

Theorem 3.10. *Let Ω be a standard domain. Suppose that $s, t \in [-2, 2]$ and that $q, r \in (1, \infty)$ satisfy*

$$1/q \geq 1/r \geq 1/q - (s-t)/m \ . \tag{3.22}$$

Then

$$H^s_{q,0,\sigma} \overset{d}{\hookrightarrow} H^t_{r,0,\sigma} \ .$$

Proof. (a) First suppose that $t \geq 0$ and

$$s \neq 1/q \ , \quad t \neq 1/r \ . \tag{3.23}$$

Then, by (3.21) and the definition of $H^\tau_{p,0}$ (see (2.17) and (2.19)),

$$H^s_{q,0} \hookrightarrow H^t_{r,0} \ . \tag{3.24}$$

From this we obtain

$$H^s_{q,0,\sigma} \hookrightarrow H^t_{r,0,\sigma} \ , \tag{3.25}$$

thanks to Remark 3.7(b).

(b) Now suppose that $s \le 0$ and

$$-s \neq 1/q' \ , \quad -t \neq 1/r' \ . \tag{3.26}$$

Then the arguments leading to (3.24) show that

$$H^{-t}_{r',0} \hookrightarrow H^{-s}_{q',0} \ . \tag{3.27}$$

Suppose that $1 < p < \infty$ and $\tau \in [0,2]/\{1/p\}$. Then

$$C^2_{c,0} := \big\{\, u \in C^2(\overline{\Omega}, \mathbb{R}^m) \ ; \ \operatorname{supp}(u) \subset\subset \overline{\Omega},\ u \,|\, \partial\Omega = 0 \,\big\}$$

is dense in $H^\tau_{p,0}$. Indeed, if $\tau < 1/p$ then this follows from $\mathcal{D} \subset C^2_{c,0}$ and the density of $\mathcal{D}$ in $H^\tau_p = H^\tau_{p,0}$ (cf. Propostion 2.4). If $1/p < \tau \le 2$ then it is a consequence of the arguments of Section 5 in [2] (also see Appendix B in [4]). Now we infer that injection (3.27) is dense. Thus, by duality, reflexivity, Theorem 2.2 and (1.10),

$$H^s_{q,0} \overset{d}{\hookrightarrow} H^t_{r,0} \ .$$

This implies

$$H^s_{q,0} / \,\overline{L_{q,\pi}} \overset{d}{\hookrightarrow} H^t_{r,0} / \,\overline{L_{r,\pi}} \tag{3.28}$$

since $\overline{L_{p,\pi}} = (L_{p',\sigma})^\perp = \mathcal{D}^\perp_\sigma$ for $p \in \{q, r\}$, where the closure and the annihilator are taken in $H^\tau_{p,0}$ for $(\tau, p) \in \big\{(s,q), (t,r)\big\}$, respectively.

We infer from (3.27), similarly as in (a), that

$$H^{-t}_{r',0,\sigma} \hookrightarrow H^{-s}_{q',0,\sigma} \ . \tag{3.29}$$

Theorem 3.5 entails

$$T_{\tau,p} : H^\tau_{p,0} / \,\overline{L_{p,\pi}} \to H^\tau_{p,0,\sigma} \ , \quad [u] \mapsto u \,|\, H^{-\tau}_{p',0,\sigma}$$

is an isometric isomorphism for $(\tau, p) \in \big\{(t,r), (s,q)\big\}$. From (3.29) we deduce that $T_{t,r} \supset T_{s,q}$. Consequently, (3.28) guarantees that

$$H^s_{q,0,\sigma} \overset{d}{\hookrightarrow} H^t_{r,0,\sigma} \ . \tag{3.30}$$

(c) Suppose that $t \le 0 < s$ and (3.23) as well as (3.26) are true. Since

$$H^{s_1}_{q,0,\sigma} \hookrightarrow H^{s_2}_{q,0,\sigma} \ , \qquad -2 \le s_2 < s_1 \le 2 \ ,$$

we can assume, by decreasing s and increasing t, if necessary, that

$$1/r = 1/q - (s-t)/m \ . \tag{3.31}$$

This implies that $s-t<m/q$. Thus, since $t\le 0$, it follows that $s<m/q$. Hence $1/q>1/p:=1/q-s/m>0$. Consequently,

$$H^s_{q,0,\sigma}\hookrightarrow L_{p,\sigma} \tag{3.32}$$

by (3.25). Note that (3.31) entails $1/r=1/p+t/m$. Hence, by (3.30),

$$L_{p,\sigma}\hookrightarrow H^t_{r,0,\sigma}\ . \tag{3.33}$$

Now, by combining (3.32) and (3.33), we find

$$H^s_{q,0,\sigma}\hookrightarrow H^t_{r,0,\sigma} \tag{3.34}$$

in this case also.

(d) From (a)–(c) we know that (3.34) is true, provided (3.22), (3.23), and (3.26) are satisfied. Since Ω is a standard domain it is known that $\mathbb{A}$ has bounded imaginary powers (cf. Remark 8.2 below). Hence we infer from Theorem V.1.5.4 of [5] and from Theorem 3.4 that, given $-2\le t_0<t_1\le 2$,

$$[H^{t_0}_{p,0,\sigma},H^{t_1}_{p,0,\sigma}]_\theta\doteq H^{(1-\theta)t_0+\theta t_1}_{p,0,\sigma}\ ,\qquad 0<\theta<1\ ,\quad 1<p<\infty\ . \tag{3.35}$$

Now suppose that at least one of (3.23) and (3.26) is not satisfied. Then we can find $\varepsilon>0$ such that $s+\varepsilon,\ \ t+\varepsilon$ as well as $s-\varepsilon,\ \ t-\varepsilon$ satisfy (3.23) and (3.26). Similarly as in step (c), we can also assume that $-2\le t-\varepsilon<s+\varepsilon\le 2$. Thus, by what has already been shown,

$$H^{s+\varepsilon}_{q,0,\sigma}\hookrightarrow H^{t+\varepsilon}_{r,0,\sigma}\ ,\quad H^{s-\varepsilon}_{q,0,\sigma}\hookrightarrow H^{t-\varepsilon}_{r,0,\sigma}\ .$$

From this we obtain (3.34) by interpolation thanks to (3.35). Consequently, (3.34) has been verified whenever s, t, q and r satisfy the hypotheses of the proposition. Moreover, by (3.30) the asserted injection is dense if $s\le 0$.

(e) Suppose that $t\ge 0$. By replacing in step (b) injection (3.27) by (3.24) we deduce that

$$H^{-t}_{r',0,\sigma}\overset{d}{\hookrightarrow} H^{-s}_{q',0,\sigma}\ .$$

Hence duality, reflexivity, Theorem 3.4, and (1.10) give $H^s_{q,0,\sigma}\overset{d}{\hookrightarrow} H^t_{q,0,\sigma}$. Finally, if $s>0\ge t$ then the density of injection (3.34) is derived from (3.32) and (3.33) since we now know that these injections are dense. $\square$

Corollary 3.11. *Let Ω be a standard domain. Assume that $s,t\in[-2,2]$ and that $q,r\in(1,\infty)$. Then*

$$n^s_{q,0,\sigma}\overset{d}{\hookrightarrow} n^t_{r,0,\sigma}\ ,\qquad 1/q\ge 1/r\ge 1/q-(s-t)/m\ .$$

Proof. First suppose that $s,t \notin 2\mathbb{Z}$. Fix $\varepsilon > 0$ so that $-2 \le t-\varepsilon < s+\varepsilon \le 2$. Then

$$H^{s+\varepsilon}_{q,0,\sigma} \overset{d}{\hookrightarrow} H^{t+\varepsilon}_{r,0,\sigma} \ , \quad H^{s-\varepsilon}_{q,0,\sigma} \overset{d}{\hookrightarrow} H^{t-\varepsilon}_{r,0,\sigma} \tag{3.36}$$

by Theorem 3.10. Hence, thanks to the reiteration theorem and to (3.35),

$$\begin{aligned}(H^{s-\varepsilon}_{q,0,\sigma}, H^{s+\varepsilon}_{q,0,\sigma})^0_{1/2,\infty} & \\ &\doteq \big([H^{-2}_{q,0,\sigma}, H^2_{q,0,\sigma}]_{(s+2-\varepsilon)/4}, [H^{-2}_{q,0,\sigma}, H^2_{q,0,\sigma}]_{(s+2+\varepsilon)/4}\big)^0_{1/2,\infty} \\ &\doteq (H^{-2}_{q,0,\sigma}, H^2_{q,0,\sigma})^0_{(s+2)/4,\infty} \ .\end{aligned} \tag{3.37}$$

Let $\big[(\mathbb{E}_\alpha, \mathbb{A}_\alpha) \ ; \ \alpha \in \mathbb{R}\big]$ be the Stokes scale constructed with $[\cdot,\cdot]_\theta$, $0 < \theta < 1$. Then Theorem 3.4 and Theorem V.1.5.4 in [5] imply (see Remark 8.1 below)

$$H^{2k}_{q,0,\sigma} \doteq \mathbb{E}_k = [\mathbb{E}_{k-1}, \mathbb{E}_{k+1}]_{1/2} \ , \qquad k = \pm 1 \ .$$

Hence (1.3) and the reiteration theorem entail, together with Theorem V.1.5.9 in [5],

$$(H^{-2}_{q,0,\sigma}, H^2_{q,0,\sigma})^0_{\theta,\infty} \doteq \big((\mathbb{E}_{-2}, \mathbb{E}_0)^0_{1/2,\infty}, (\mathbb{E}_0, \mathbb{E}_2)^0_{1/2,\infty}\big)^0_{\theta,\infty} \doteq \mathbb{E}^0_{-1+2\theta,\infty}$$

if $\theta \ne 1/2$. Consequently, we infer from (3.37) that

$$(H^{s-\varepsilon}_{q,0,\sigma}, H^{s+\varepsilon}_{q,0,\sigma})^0_{1/2,\infty} \doteq n^s_{q,0,\sigma} \ .$$

Similarly,

$$(H^{t-\varepsilon}_{r,0,\sigma}, H^{t+\varepsilon}_{r,0,\sigma})^0_{1/2,\infty} \doteq n^t_{r,0,\sigma} \ .$$

Now the assertion follows from (3.36) by interpolating with $(\cdot,\cdot)^0_{1/2,\infty}$.

If $s \in 2\mathbb{Z}$ then we deduce from Theorem 3.10 that

$$n^s_{q,0,\sigma} = H^s_{q,0,\sigma} \overset{d}{\hookrightarrow} H^t_{q,0,\sigma} \overset{d}{\hookrightarrow} n^t_{q,0,\sigma} \ ,$$

where the last injection is a consequence of (1.2) and Lemma 1.1. This proves everything. $\square$

Remark 3.12. Instead of assuming in Theorem 3.10 that Ω is a standard domain it suffices to presuppose that

$$L_{p,\sigma} = \{ u \in L_p \ ; \ \nabla \cdot u = 0, \ \gamma_{\vec{n}} u = 0 \} \ , \qquad 1 < p < \infty \ ,$$

and that the Stokes operator has bounded imaginary powers. In fact, this last assumption is not needed if

$$1/r \notin \{ 1/qm', \ m'(1/q - 2/m), \ m'/q', \ 1 - 2/m - 1/q'm' \} \ .$$

Proof. Suppose that $s = 1/q$ and

$$1/r = 1/q - (1/q - t)/m = 1/qm' + t/m$$

with $t \neq 0$. Then we can choose $\varepsilon \in (0, |t| \wedge 1/r)$ and proceed by interpolation as in step (d) of the proof of Theorem 3.10. Since in this case 'we do not interpolate across 0', it is true that

$$[H^{\tau-\varepsilon}_{p,0,\sigma}, H^{\tau+\varepsilon}_{p,0,\sigma}]_{1/2} \doteq H^{\tau}_{p,0,\sigma} \ , \qquad (\tau, p) \in \{(s,q),(r,t)\} \ ,$$

without the hypothesis that $\mathbb{A}$ has bounded imaginary powers, thanks to Theorem 3.4 with $F := H$. The exceptional value $1/r = 1/qm'$ corresponds to $t = 0$ where we would have to interpolate across 0. The other values correspond to $s = 2$, $s = 0$, and $t = -2$, respectively. $\square$

4. The Convection Term

Of course, it is most important to have a good understanding of the continuity properties of the nonlinear convection term. It is the purpose of this section to derive results giving such information.

Recall that

$$\partial_j \in \mathcal{L}(H^s_q, H^{s-1}_q) \ , \qquad 1 \leq j \leq m \ , \quad -1 \leq s \leq 2 \ . \tag{4.1}$$

Moreover, given $p_1, p_2 \in [1, \infty]$, Hölder's inequality implies

$$\big((a,b) \mapsto ab\big) \in \mathcal{L}\big(L_{p_1}(\Omega, \mathbb{R}), L_{p_2}(\Omega, \mathbb{R}); L_{p_0}(\Omega, \mathbb{R})\big) \ , \tag{4.2}$$

where $1/p_0 := 1/p_1 + 1/p_2$. From this it easily follows that

$$Q := \big((u,v) \mapsto u \cdot \nabla v\big) \in \mathcal{L}^2(H^2_q, L_q) \ , \tag{4.3}$$

provided $q \geq m/3$. Thus, setting

$$b(u,v) := -P(u \cdot \nabla v) = -P\nabla \cdot (u \otimes v) \ , \qquad u, v \in H^2_{q,0,\sigma} \ , \tag{4.4}$$

we obtain $b \in \mathcal{L}^2(H^2_{q,0,\sigma}, L_{q,\sigma})$. In the following we establish continuity properties of b on various other spaces as well.

Proposition 4.1. *Suppose that* $1 < p < \infty$ *and* $0 \leq t \leq 2$.
Let one of conditions (i)–(iii) *be satisfied*:
(i) $-1 + m/p \leq t \leq m/p$, $m(1/p + 1/q - 1) < t < m/q$.
(ii) $1 \leq t \leq 1 + m/p$, $1 + m(1/p + 1/q - 1) < t < 1 + m/q$.

(iii) $p \geq q$ *and*

$$-1+m/p \leq t \leq 1+m/p \ , \quad m(1/p+1/q-1) < t < 1+m/q \ ,$$

where $m(1/p+1/q-1) < 1$ *if* $t > m/p$.

Then

$$b \in \mathcal{L}(L_{p,\sigma}, H^t_{q,0,\sigma}; H^{t-1-m/p}_{q,0,\sigma}) \ . \tag{4.5}$$

Proof. (i) Set $R(u,v) := \nabla \cdot (u \otimes v)$. Since $t < m/q$ it follows from (3.21) and (4.2) that

$$\big((u,v) \mapsto u \otimes v\big) \in \mathcal{L}\big(L_p, H^t_q; L_r(\Omega, \mathbb{R}^m \times \mathbb{R}^m)\big) \ ,$$

where $1 > 1/r := 1/p + 1/q - t/m > 0$. Thus, by (4.1),

$$R \in \mathcal{L}(L_p, H^t_q, H^{-1}_r) \ . \tag{4.6}$$

Put $s := 1 + m/p - t \in [1,2]$. Note that $1/r' = 1/q' - (s-1)/m$. Hence we obtain $H^s_{q'} \hookrightarrow H^1_{r'}$ from (3.21). Thanks to

$$\mathcal{D} \hookrightarrow H^s_{q',0} \hookrightarrow H^s_{q'} \quad \text{and} \quad \mathcal{D} \overset{d}{\hookrightarrow} H^1_{r',0} \hookrightarrow H^1_{r'} \ ,$$

this implies $H^s_{q',0} \overset{d}{\hookrightarrow} H^1_{r',0}$. Consequently, by Proposition 2.4, the Hahn-Banach theorem, and Theorem 2.2,

$$H^{-1}_r = H^{-1}_{r,0} \hookrightarrow H^{-s}_{q,0} \ .$$

By composing this injection with (4.6) and using $H^t_{q,0} \hookrightarrow H^t_q$, we see that R belongs to $\mathcal{L}(L_p, H^t_q; H^{t-1-m/p}_{q,0})$. Now the assertion follows in this case from

$$L_{p,\sigma} \times H^t_{q,0,\sigma} \hookrightarrow L_p \times H^t_q \tag{4.7}$$

and from Lemma 3.3 and Theorem 3.4, since $b = -PR \,|\, L_{p,\sigma} \times H^t_{q,0,\sigma}$.

(ii) By (4.1)

$$(v \mapsto \nabla v) \in \mathcal{L}\big(H^t_q, H^{t-1}_q(\Omega, \mathbb{R}^{m\times m})\big) \ .$$

Thus, as a consequence of (3.21) and (4.2),

$$Q \in \mathcal{L}(L_p, H^t_q; L_\rho) \ , \tag{4.8}$$

where $1 > 1/\rho := 1/p + 1/q - (t-1)/m > 0$. Since $H^s_{q',0} \overset{d}{\hookrightarrow} L_{\rho'}$, we obtain by duality that $L_\rho \hookrightarrow H^{-s}_{q,0}$. By combining this with (4.8) and using (4.7) we see that

$$Q \in \mathcal{L}(L_{p,\sigma}, H^t_q; H^{t-1-m/p}_{q,0}) \ .$$

Now again the assertion follows from Lemma 3.3 and Theorem 3.4.

(iii) First suppose that $p > q$. If $t \leq m/p$ then (4.5) is entailed by (i). Thus suppose that $m/p < t \leq 1 + m/p$. Set $t_0 := m/p$ and $t_1 := (1+m/p) \wedge 2$. Note that $t_1 > 1 + m(1/p + 1/q - 1)$. Consequently, thanks to (i) and (ii),

$$b \in \mathcal{L}(L_{p,\sigma}, H^{t_j}_{q,0,\sigma}; H^{t_j-1-m/p}_{q,0,\sigma}) , \qquad j = 0,1 .$$

Thus, given $u \in L_{p,\sigma}$,

$$b(u,\cdot) \in \mathcal{L}(H^{t_j}_{q,0,\sigma}; H^{t_j-1-m/p}_{q,0,\sigma}) , \qquad j = 0,1 ,$$

and the norms of these linear operators are bounded by $c \|u\|_{L_p}$. Theorem 3.4 and the reiteration theorem for the complex interpolation functor entail

$$H^t_{q,0,\sigma} = [H^{t_0}_{q,0,\sigma}, H^{t_1}_{q,0,\sigma}]_\theta , \quad H^{t-1-m/p}_{q,0,\sigma} = [H^{t_0-1-m/p}_{q,0,\sigma}, H^{t_1-1-m/p}_{q,0,\sigma}]_\theta$$

for $\theta := (t-t_0)/(t_1-t_0)$. Hence the validity of (4.5) follows in this case by interpolation.

Lastly, suppose that $p = q$. Then (4.5) is a consequence of (i) if $t < m/q$. If $m/q \leq t < 1 + m/q$ then the above argument gives the desired result, provided we choose $t_0 < m/q$ and $t_1 < 1 + m/q$ sufficiently close to m/q and to $(1+m/q) \wedge 2$, respectively. $\square$

It is now easy to deduce from this proposition the following basic continuity result.

Theorem 4.2. *Suppose that* $s, t \in [0,2]$ *satisfy*

$$s < m/q , \quad t < 1 + m/q ,$$

and

$$-1 + m/q \leq s + t \leq 1 + m/q \tag{4.9}$$

as well as

$$s + t > m(2/q - 1) . \tag{4.10}$$

Furthermore, assume that

$$s + 1 > m(2/q - 1) \qquad \textit{if} \quad s + t > m/q . \tag{4.11}$$

Then

$$b \in \mathcal{L}(H^s_{q,0,\sigma}, H^t_{q,0,\sigma}; H^{s+t-1-m/q}_{q,0,\sigma}) . \tag{4.12}$$

Proof. Set $1/p := 1/q - s/m \leq 1/q$ and note that $H^s_{q,0,\sigma} \hookrightarrow L_{p,\sigma}$. Then (4.9) is equivalent to

$$-1 + m/p \leq t \leq 1 + m/p ,$$

and (4.10) is equivalent to $t > m(1/p + 1/q - 1)$. Furthermore, (4.11) says that

$$m(1/p + 1/q - 1) < 1 \qquad \text{if} \quad t > m/p \; .$$

Hence the assertion follows from Proposition 4.1(iii). $\square$

It should be noted that Proposition 4.1 and Theorem 4.2 are sharp results. Weaker statements can be obtained, of course, by combining those statements with embeddings of the form $H^{\overline{s}}_{q,0,\sigma} \hookrightarrow H^{\overline{t}}_{q,0,\sigma}$ for $-2 \leq \overline{t} < \overline{s} \leq 2$.

Bilinear estimates for the convection term are fundamental for the study of the Navier-Stokes equations and have been derived — given various hypotheses — by several authors. Most of these papers contain estimates involving fractional powers of the Stokes operator. We refer, in particular, to Lemma 2.2 of Giga and Miyakawa [32], where Ω is supposed to be bounded. Since in that case the Stokes operator has bounded imaginary powers, it follows (cf. Theorem V.1.5.4 in [5]) that Lemma 2.2 of [32] guarantees the validity of (4.12), provided (4.9) and the additional assumptions

$$s > 0 \; , \quad t > 0 \; , \quad s + t > 2 + m(1/q - 1/q') \tag{4.13}$$

are satisfied. (Observe that the second inequality in (4.9) and the last inequality in (4.13) entail the restriction $q > m'$.)

More recently, Kobayashi and Muramatu (see Lemma 5.1 in [49]) have been able to drop the additional hypotheses (4.13) at the price of using certain abstract Besov spaces. Their result is weaker than Theorem 4.2 (since it involves abstract Besov spaces of the type $B^s_{q,1}$ as domains and of the type $B^s_{q,\infty}$ as image spaces). In addition, there are no concrete characterizations of these spaces in [49].

Assertion (4.12) has also been obtained by Grubb (see in Theorem 2.1 of [38]), provided the additional assumptions

$$s + t > 1 + m(2/q - 1)_+ \; , \quad s + t - (1 + m/q) > -1 + 1/q$$

are satisfied.

Remark 4.3. Suppose that $m/q \leq s < 2$. Then there exists $t \in (s-2, s)$ such that $b \in \mathcal{L}^2(H^s_{q,0,\sigma}, H^t_{q,0,\sigma})$.

Proof. This follows by obvious modifications of the proof of Proposition 4.1. $\square$

5. Evolution Equations With Quadratic Nonlinearities

Let X be a metric space. Then $BC(X, E)$ is the Banach space of all bounded and continuous functions from X into E, endowed with the supremum norm.

Given $\mu, \sigma \in \mathbb{R}$ and a perfect subinterval J of $\mathbb{R}^+$ containing 0, we denote by $BC_{\mu,\sigma}(\dot{J}, E)$ the Banach space of all $u : \dot{J} \to E$ such that

$$\bigl(t \mapsto t^\mu e^{\sigma t} u(t)\bigr) \in BC(\dot{J}, E) \ ,$$

equipped with the norm

$$u \mapsto \|u\|_{C_{\mu,\sigma}} := \|u\|_{C_{\mu,\sigma}(\dot{J},E)} := \sup_{t\in\dot{J}} t^\mu e^{\sigma t} \|u(t)\| \ .$$

We write $C_{\mu,\sigma}(\dot{J}, E)$ for the closed linear subspace thereof, consisting of all u satisfying $t^\mu u(t) \to 0$ as $t \to 0$. We also set $C_\mu := C_{\mu,0}$.

Throughout this section (E_0, E_1) is a densely injected Banach couple, and $A \in \mathcal{H}(E_1, E_0)$. We put $U(t) := e^{-tA}$ for $t \ge 0$ and fix $\overline{\omega}$ such that

$$\|U(t)\|_{\mathcal{L}(E_0,E_j)} \le ct^{-j} e^{-\overline{\omega} t} \ , \qquad j \in \{0,1\} \ , \quad t > 0 \ .$$

Moreover,

$$\omega := \begin{cases} \overline{\omega} & \text{if } J = \mathbb{R}^+ \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

It is well-known that

$$U \in C\bigl(J, \mathcal{L}_s(E_j)\bigr) \cap BC_{j-k,\omega}\bigl(\dot{J}, \mathcal{L}(E_k, E_j)\bigr) \ , \qquad j,k \in \{0,1\} \ , \quad k \le j \ , \tag{5.1}$$

where $\mathcal{L}_s(E)$ denotes $\mathcal{L}(E)$ endowed with the strong topology. We also set

$$E_{[j]} := E_{j,p} := E^0_{j,\infty} := E_j \ , \qquad j \in \{0,1\} \ , \quad 1 \le p \le \infty \ .$$

Using these facts we can easily prove the following continuity properties of U.

Lemma 5.1.
(i) *If* $0 \le \alpha \le 1$ *then*

$$U \in C\bigl(J, \mathcal{L}_s(E_\alpha)\bigr) \cap BC_{0,\omega}\bigl(\dot{J}, \mathcal{L}(E_\alpha)\bigr) \cap BC_{0,\omega}\bigl(\dot{J}, \mathcal{L}(E_{\alpha,\infty})\bigr)$$

for $E_\alpha \in \{\, E_{[\alpha]}, E_{\alpha,p}, E^0_{\alpha,\infty} \ ; \ 1 \le p < \infty \,\}$.
(ii) *If* $0 \le \beta < \alpha \le 1$ *then*

$$U \in BC_{\alpha-\beta,\omega}\bigl(\dot{J}, \mathcal{L}(E_{\beta,\infty}, E_{\alpha,1})\bigr) \ . \tag{5.2}$$

Proof. (i) Thanks to (5.1) we can assume that $0 < \alpha < 1$. Then the assertion follows easily by interpolation from (5.1) and the density of E_1 in E_α (also cf. Lemma V.2.1.2 of [5]).

(ii) By interpolation we infer from (5.1) that (5.2) is true if either $\beta = 0$ and $\alpha < 1$ or $\beta > 0$ and $\alpha = 1$. If $0 < \beta < \alpha < 1$ then, by the reiteration theorem, $(E_{\beta,\infty}, E_1)_{\theta,1} \doteq E_{\alpha,1}$ for $\theta := (\alpha - \beta)/(1-\beta)$. Thus (5.2) follows in this case from

$$U \in BC_{1-\beta,\omega}\big(\dot{J}, \mathcal{L}(E_{\beta,\infty}, E_1)\big) \cap BC_{0,\omega}\big(\dot{J}, \mathcal{L}(E_{\beta,\infty})\big)$$

by interpolating with $(\cdot,\cdot)_{\theta,1}$. $\square$

Given $u \in L_1(\dot{J}, E_0)$, we put

$$U \star u(t) := \int_0^t U(t-\tau)u(\tau)\, d\tau \ , \qquad t \in \dot{J} \ ,$$

whenever these integrals exist.

Lemma 5.2. *Suppose that* $\beta < 1$.
(i) *If* $0 < \alpha \leq 1$ *then* $u \mapsto U \star u$ *belongs to*

$$\mathcal{L}\big(C_{\beta,\omega}(\dot{J}, E_\alpha), C_{\beta-1,\omega}(\dot{J}, E_\alpha)\big)$$

and to

$$\mathcal{L}\big(C\big([0,T], E_\alpha\big), BC_{-\alpha}\big((0,T], E_1\big)\big) \ , \qquad T > 0 \ ,$$

for $E_\alpha \in \{\, E_{[\alpha]}, E_{\alpha,p}, E^0_{\alpha,\infty} \ ; \ 1 \leq p < \infty \,\}$.
(ii) *If* $0 < \gamma < \alpha \leq 1$ *then*

$$(u \mapsto U \star u) \in \mathcal{L}\big(C_{\beta,\omega}(\dot{J}, E_{\gamma,\infty}), C_{\alpha+\beta-\gamma-1,\omega}(\dot{J}, E_{\alpha,1})\big) \ .$$

The norms of these linear maps are bounded by an increasing function of the length of J *for* $J \neq \mathbb{R}^+$.

Proof. (i) From Lemma 5.1(i) we obtain the estimate

$$\|U \star u(t)\|_{E_\alpha} \leq c \int_0^t e^{-\omega(t-\tau)}\, \|u(\tau)\|_{E_\alpha}\, d\tau \leq c e^{-\omega t} \int_0^t \tau^{-\beta}\, d\tau\, \|u\|_{C_{\beta,\omega}((0,t),E_\alpha)}$$

for $t \in \dot{J}$. Now the assertion is obvious.

(ii) Similarly as in (i) we infer from Lemma 5.1(ii) that

$$\begin{aligned}\|U \star u(t)\|_{E_{\alpha,1}} &\leq c \int_0^t (t-\tau)^{\gamma-\alpha} e^{-\omega(t-\tau)}\, \|u(\tau)\|_{E_{\gamma,\infty}}\, d\tau \\ &\leq c t^{\gamma-\alpha-\beta+1} e^{-\omega t} \mathsf{B}(1+\gamma-\alpha, 1-\beta)\, \|u\|_{C_{\beta,\omega}((0,t),E_{\gamma,\infty})}\end{aligned}$$

for $t \in \dot{J}$, where B denotes the beta function. This implies that the map

$$t \mapsto t^{\alpha+\beta-\gamma-1} e^{\omega t} U \star u(t)$$

is bounded from $\dot{J}$ into $E_{\alpha,1}$. It also shows that $t^{\alpha+\beta-\gamma-1}U \star u(t) \to 0$ in $E_{\alpha,1}$ as $t \to 0$. Finally, it is not difficult to see that $U \star u$ belongs to $C(\dot{J}, E_{\alpha,1})$. Now the assertion is obvious. ☐

Let X, Y, and Z be nonempty sets and $f : X \times Y \to Z$. Then

$$f^\natural(u)(x) := f\big(x, u(x)\big) \ , \qquad x \in X \ , \quad u : X \to Y \ .$$

Thus $f^\natural : Y^X \to Z^X$ is the Nemyt'skii operator induced by f.

Lemma 5.3. *Suppose that* $0 < \gamma \le \alpha \le 1, \ \ 0 \le \beta < 1/2$, *and* $\omega \ge 0$. *Also assume that* $Q \in \mathcal{L}^2(E_1, E_{\gamma,\infty})$. *Then*

$$(u, v) \mapsto U \star Q^\natural(u, v)$$

belongs to

$$\mathcal{L}^2\big(C_{\beta,\omega}(\dot{J}, E_1), C_{\alpha+2\beta-\gamma-1,\omega}(\dot{J}, E_{\alpha,1})\big)$$

and to

$$\mathcal{L}^2\Big(C\big([0,T], E_1\big), BC_{-\gamma}\big((0,T], E_1\big)\Big) \ , \qquad T > 0 \ .$$

Proof. Since $\omega \ge 0$ it follows that

$$t^{2\beta} e^{\omega t} \, \|Q^\natural(u,v)(t)\|_{E_{\gamma,\infty}} \le \|Q\| \left(t^\beta e^{\omega t} \, \|u(t)\|_{E_1}\right)\left(t^\beta e^{\omega t} \, \|v(t)\|_{E_1}\right)$$

for $u, v : \dot{J} \to E_1$ and $t \in \dot{J}$. This shows that

$$Q^\natural \in \mathcal{L}^2\big(C_{\beta,\omega}(\dot{J}, E_1), C_{2\beta,\omega}(\dot{J}, E_{\gamma,\infty})\big) \ .$$

Now Lemma 5.2 implies the assertion. ☐

Remark 5.4. It is obvious that the analogues of Lemmas 5.2 and 5.3 are valid, which are obtained by replacing $C_{\cdot,\omega}$ by $BC_{\cdot,\omega}$ everywhere. ☐

In the following we denote by Uu^0 the function $t \mapsto U(t)u^0$ for $u^0 \in E_0$.

Lemma 5.5. *Suppose that* $0 < \alpha < 1$ *and let* $F_{1-\alpha}$ *be a Banach space such that*

$$E_{1-\alpha,1} \stackrel{d}{\hookrightarrow} F_{1-\alpha} \hookrightarrow E_{1-\alpha,\infty} \ .$$

If $u^0 \in F_{1-\alpha}$ *then* $Uu^0 \in C_{\alpha,\omega}(\dot{J}, E_1)$.

Proof. Lemma 5.1 and $F_{1-\alpha} \hookrightarrow E_{1-\alpha,\infty}$ imply

$$U \in BC_{\alpha,\omega}\big(\dot{J}, \mathcal{L}(F_{1-\alpha}, E_1)\big) \ ,$$

thus $Uu^0 \in BC_{\alpha,\omega}(\dot J, E_1)$. Hence it remains to show that $t^\alpha \|U(t)u^0\|_{E_1} \to 0$ as $t \to 0$. Fix $T > 0$ and $\varepsilon > 0$. Since E_1 is dense in $E_{1-\alpha,1}$, hence in $F_{1-\alpha}$, there exists $v \in E_1$ such that

$$\|v - u^0\|_{F_{1-\alpha}} < \varepsilon \Big/ \sup_{0<t\le T} t^\alpha \|U(t)\|_{\mathcal{L}(F_{1-\alpha},E_1)} \ .$$

The strong continuity of U on E_1 implies $t^\alpha U(t)v \to 0$ in E_1 as $t \to 0$. Thus

$$\begin{aligned} t^\alpha \|U(t)u^0\|_{E_1} &\le t^\alpha \|U(t)v\|_{E_1} + t^\alpha \|U(t)\|_{\mathcal{L}(F_{1-\alpha},E_1)} \|u^0 - v\|_{F_{1-\alpha}} \\ &\le t^\alpha \|U(t)v\|_{E_1} + \varepsilon \end{aligned}$$

for $0 < t \le T$. Hence the assertion follows. $\square$

Now we suppose that

$$0 < \gamma < 1/2 \quad \text{and} \quad Q \in \mathcal{L}^2(E_1, E_{\gamma,\infty}) \ . \tag{5.3}$$

Then we consider the Cauchy problem

$$\dot u + Au = Q(u,u) + g(t) \ , \quad t \in \dot J \ , \qquad u(0) = u^0 \ , \tag{5.4}$$

where $g : \dot J \to E_0$. By a **solution** u of (5.4) on J' we mean a function

$$u \in C(J', E_0) \cap C(\dot J', E_1) \cap C^1(\dot J', E_0)$$

satisfying (5.4) point-wise on J', where J' is a perfect subinterval of J containing 0. It is **maximal** if there does not exist a solution $\widetilde u \supset u$ with $\widetilde u \ne u$. If it is defined on all of J then it is **global**. Each function $u \in C(J', E_0)$ satisfying $u = Uu^0 + U \star \left[Q^\natural(u,u) + g\right]$ on J' is a **mild** solution of (5.4) on J'.

Theorem 5.6. *Suppose that* (5.3) *is satisfied and* $0 \le \alpha \le \gamma$. *Let* $F_{1-\alpha}$ *be a Banach space with*

$$E_{1-\alpha,1} \overset{d}{\hookrightarrow} F_{1-\alpha} \hookrightarrow E_{1-\alpha,\infty} \tag{5.5}$$

and

$$U \,|\, F_{1-\alpha} \text{ is strongly continuous.} \tag{5.6}$$

Also suppose that

$$(u^0, g) \in F_{1-\alpha} \times C_{\alpha+\gamma}(\dot J, E^0_{\gamma,\infty}) \ . \tag{5.7}$$

Then problem (5.4) *possesses a unique maximal solution* $u := u(\cdot, u^0, g)$ *such that*

$$t^\gamma \|u(t)\|_{E_1} \to 0 \qquad \text{as } t \to 0 \ . \tag{5.8}$$

The maximal interval of existence, $J^+ := J^+(u^0, g) := \mathrm{dom}(u)$, *is open in* J, *and*

$$u \in C(J^+, F_{1-\alpha}) \ . \tag{5.9}$$

If $\alpha > 0$ *then*

$$t^{\alpha}\,\|u(t)\|_{E_1} \to 0 \qquad \textit{as } t \to 0\ . \tag{5.10}$$

Remark. Of course, (5.10) implies (5.8). However, this theorem guarantees uniqueness among all perspective solutions satisfying (5.8) only.

Proof. (a) Fix $T_* \in \dot{J}$ and set $J_* := [0, T_*]$. By assumption (5.6),

$$Uu^0 \in C(J_*, F_{1-\alpha})\ . \tag{5.11}$$

Hypothesis (5.5) and Lemma 5.5 imply

$$Uu^0 \in C_{\alpha}(\dot{J}_*, E_1) \qquad \text{if } \alpha > 0\ . \tag{5.12}$$

From Lemma 5.2 and assumptions (5.5) and (5.7) we infer that

$$U \star g \in C_0(\dot{J}_*, F_{1-\alpha}) \cap C_{\alpha}(\dot{J}_*, E_1)\ . \tag{5.13}$$

Put $X_T := C_{\gamma}\big((0,T], E_1\big)$ for $T \in \dot{J}$ and $a := Uu^0 + U \star g$. Then (5.11)–(5.13) and $\alpha \leq \gamma$ imply that

$$a \in X_T\ , \qquad 0 < T \leq T_*\ . \tag{5.14}$$

Set

$$\varphi(u) := a + U \star Q^{\natural}(u,u)\ , \qquad u \in X_T\ .$$

Lemma 5.3 guarantees the existence of $\mu := \mu(T_*) \in \mathbb{R}^+$ such that

$$\|\varphi(u) - \varphi(v)\|_{X_T} \leq \mu(\|u\|_{X_T} + \|v\|_{X_T})\,\|u - v\|_{X_T} \tag{5.15}$$

and

$$\|\varphi(u) - a\|_{X_T} \leq \mu\,\|u\|_{X_T}^2 \tag{5.16}$$

for $u, v \in X_T$ and $0 < T \leq T_*$.

Set $R := \big(\sqrt{3} - 1\big)/4\mu$ and $r := \big(2 - \sqrt{3}\big)/4\mu$. Thanks to (5.14) we can find $\overline{T} \in (0, T_*]$ such that $\|a\|_{X_{\overline{T}}} \leq R$. Using this fact it is not difficult to verify that the sequence (u_j), defined by $u_0 := a$ and $u_{j+1} := \varphi(u_j)$ for $j \in \mathbb{N}$, lies in

$$M := M_X := \{\, u \in X_{\overline{T}}\ ;\ \|u - a\|_{X_{\overline{T}}} \leq r \,\}$$

and that $\varphi | M$ is a contraction. From this we infer that (u_j) converges to a fixed point $\overline{u}$ of φ in M and that this is the only one in M. Thus $\overline{u} \in C_{\gamma}\big((0,\overline{T}], E_1\big)$ and

$$\overline{u} = Uu^0 + U \star \big(Q^{\natural}(\overline{u},\overline{u}) + g\big) \tag{5.17}$$

on $(0, \overline{T}]$. From (1.2) and Lemma 5.3 we deduce that

$$U \star Q^{\natural}(\overline{u},\overline{u}) \in C_0\big((0,\overline{T}], E_{1-\gamma,\infty}\big)\ . \tag{5.18}$$

Hence (5.11) and (5.13) entail

$$\overline{u} \in C\big([0,\overline{T}], E_{1-\gamma,\infty}\big) \hookrightarrow C\big([0,\overline{T}], E_0\big)\ .$$

Thus $\overline{u}$ is a mild solution of (5.4) on $[0, \overline{T}]$.

(b) Thanks to (5.3), (5.7), and (1.2),

$$h := Q^\natural(\overline{u},\overline{u}) + g \in C\big((0,\overline{T}],E_{\beta,1}\big)$$

for $0<\beta<\gamma$. Thus, given $0<\varepsilon<\overline{T}$, it follows from $\overline{u}(\varepsilon)\in E_1$, Lemma 5.1, and Theorem IV.1.2.1 in [5] that the Cauchy problem

$$\dot v + Av = h(\varepsilon+t)\ , \quad 0<t\le \overline{T}-\varepsilon\ , \qquad v(0)=\overline{u}(\varepsilon) \tag{5.19}$$

has a unique solution $\overline{v}\in C\big([0,\overline{T}-\varepsilon],E_1\big)\cap C^1\big([0,\overline{T}-\varepsilon],E_0\big)$. Clearly, $\overline{v}$ is a mild solution of (5.19) on $[0,\overline{T}-\varepsilon]$. Since mild solutions of linear Cauchy problems are unique and $\overline{u}(\varepsilon+\cdot)$ is also a mild solution of (5.19) on $[0,\overline{T}-\varepsilon]$ we find $\overline{u}(\varepsilon+t)=\overline{v}(t)$ for $0\le t\le T-\varepsilon$. This being true for every $\varepsilon\in(0,T)$, it follows that $\overline{u}$ is a solution of (5.4) on $[0,\overline{T}]$.

(c) For $T\in\dot J$ set

$$Y_T := \left\{\begin{array}{ll} C\big([0,T],E_1\big) & \text{if } \alpha=0\ ,\\ C_\alpha\big((0,T],E_1\big) & \text{if } \alpha>0\ .\end{array}\right.$$

Then (5.11)–(5.13) imply $a\in Y_T$ for $0<T\le T_*$. It also follows from Lemma 5.3 and $\alpha\le\gamma$ that φ satisfies (5.15) and (5.16) with X_T replaced by Y_T. If $\alpha>0$ then the above arguments imply, by replacing M by M_Y and making $\overline{T}$ smaller if necessary, the existence of a unique solution v of (5.4) on $[0,\overline{T}]$ belonging to $C_\alpha\big((0,\overline{T}],E_1\big)$. If $\alpha=0$ then, by Lemma 5.3,

$$\|U\star Q^\natural(u,u)\|_{C([0,T],E_1)} \le cT^\gamma\,\|u\|^2_{C([0,T],E_1)}$$

for $T\in\dot J$ and $u\in C\big([0,T],E_1\big)$. Thus, by making $\overline{T}$ smaller if necessary, we see that in this case $\varphi(M_Y)\subset M_Y$ and $\varphi\,|\,M_Y$ is a contraction. Hence Banach's fixed point theorem implies that φ has a unique fixed point v in M_Y. Clearly, v is a mild solution of (5.4) on $[0,T]$. Since $Y_{\overline{T}}\hookrightarrow X_{\overline{T}}$, uniqueness implies $v=\overline{u}$. Consequently, $\overline{u}\in C\big([0,\overline{T}],E_1\big)$ if $\alpha=0$, and $\overline{u}\in C_\alpha\big((0,\overline{T}],E_1\big)$ otherwise.

(d) Since $\overline{u}(\overline{T})\in E_1$, the argument in (c) with $\alpha=0$, combined with a standard continuation procedure, shows that the Cauchy problem

$$\dot u + Au = Q(u,u) + g(\overline{T}+t)\ , \quad t\in(\dot J-\overline{T})\cap(0,\infty)\ , \qquad u(0)=\overline{u}(\overline{T})$$

possesses a unique maximal solution $\overline{w}\in C(J^+_{\overline{T}},E_1)$. Hence, setting

$$u(t) := \left\{\begin{array}{ll} \overline{u}(t)\ , & 0\le t\le\overline{T}\ ,\\ \overline{w}(t-\overline{T})\ , & t-\overline{T}\in J^+_{\overline{T}}\ ,\end{array}\right.$$

it follows that u is the unique maximal solution of (5.4) satisfying (5.8). Since (5.10) has already been proven, as well as (5.9) for $\alpha=0$, it remains to verify that (5.9) holds if $\alpha > 0$. But this is obvious from (5.12), (5.13), $\overline{u} \in C_\alpha\big((0,\overline{T}], E_1\big)$, and

$$U \star Q^\natural(\overline{u},\overline{u}) \in C_{\alpha-\gamma}\big((0,\overline{T}], E_{1-\alpha,1}\big) \hookrightarrow C_0\big((0,\overline{T}], F_{1-\alpha}\big) \ ,$$

which is a consequence of Lemma 5.3 and $\alpha \le \gamma$. $\square$

Remarks 5.7. Let conditions (5.3) be satisfied.

(a) Suppose that

$$(u^0, g) \in E^0_{1-\gamma,\infty} \times C_{2\gamma}(\dot{J}, E^0_{\gamma,\infty}) \quad \text{and} \quad u \in C(J', E^0_{1-\gamma,\infty}) \cap C_\gamma(\dot{J}', E_1)$$

for some perfect subinterval J' of J containing 0. Then u is a mild solution of (5.4) on J' iff it is a solution on J'.

Proof. Clearly, every solution on J' is a mild solution on J'. The converse has been shown in step (b) of the preceding proof. $\square$

(b) Suppose that $0 \le \alpha \le \gamma$ and

$$(u^0_j, g_j) \in E^0_{1-\alpha,\infty} \times C_{\alpha+\gamma}(\dot{J}, E^0_{\gamma,\infty}) \ , \qquad j = 1,2 \ ,$$

and $T \in \dot{J}^+(u^0_1, g_1) \cap \dot{J}^+(u^0_2, g_2)$. Then there exists a constant $\kappa := \kappa(T)$ such that, letting $u_j := u(\cdot, u^0_j, g_j)$,

$$\|u_1 - u_2\|_{C_\alpha((0,T],E_1)} \le \kappa\big(\|u^0_1 - u^0_2\|_{E^0_{1-\alpha,\infty}} + \|g_1 - g_2\|_{C_{\alpha+\gamma}((0,T],E^0_{\gamma,\infty})}\big) \ ,$$

where $C_\alpha\big((0,T], E_1\big)$ is replaced by $C\big([0,T], E_1\big)$ if $\alpha = 0$.

Proof. Note that

$$u_1 - u_2 = a_1 - a_2 + U \star \big(Q^\natural(u_1,u_1) - Q^\natural(u_2,u_2)\big) \ ,$$

where $a_j := Uu^0_j + U \star g_j$. Fix $T_0 > 0$ such that $\|a_j\|_{Y_{T_0}} \le R$ for $j = 0,1$. Then parts (a) and (c) of the proof of Theorem 5.6 show that $\|u_j - a_j\|_{Y_{T_0}} \le r$. Thus

$$\|u_j\|_{Y_{T_0}} \le R + r = 1/4\mu \ , \qquad j = 1,2 \ .$$

Hence we infer from the analogue of (5.15) that

$$\|u_1 - u_2\|_{Y_{T_0}} \le 2\,\|a_1 - a_2\|_{Y_{T_0}} \ . \tag{5.20}$$

Now suppose that $\tau \in [T_0, T]$ and put

$$a^\tau_j := U\big(u_j(\tau)\big) + U \star g_j(\tau + \cdot) \ , \qquad j = 1,2 \ .$$

Note that $(t,w) \mapsto U(t)w$ is continuous from $\mathbb{R}^+ \times E_1$ to E_1. Hence

$$[0,T] \times [T_0,T] \to E_1 \ , \quad (t,\tau) \mapsto U\big(u_j(\tau)\big)(t)$$

is uniformly continuous. Clearly, $(t,\tau) \mapsto U \star g_j(\tau + \cdot)(t)$ is also uniformly continuous from $[0,T] \times [T_0,T]$ into E_1. Thus there exists $\tau_1 > 0$ with $\|a_j^\tau\|_{Y_{\tau_1}} \le R$ for $j = 1,2$ and $\tau \in [T_0,T]$. Now, similarly as above, we find that

$$\|u_1 - u_2\|_{C([\tau,\tau+\tau_1],E_1)} \le 2\,\|a_1^\tau - a_2^\tau\|_{C([\tau,\tau+\tau_1],E_1)} \ , \qquad T_0 \le \tau \le T \ .$$

This implies the existence of $T_0 < T_1 < \cdots < T_{m-1} < T_m := T$ such that

$$\|u_1(t) - u_2(t)\|_{E_1} \le c\big(\|u_1(T_k) - u_2(T_k)\|_{E_1} + \|g_1 - g_2\|_{C([T_k,T_{k+1}],E_1)}\big)$$

for $T_k \le t \le T_{k+1}$ and $0 \le k \le m-1$. Now the assertion follows by finite induction starting with (5.20).

(c) Given any $T^* \in \dot{J}$, there exists $R > 0$ such that $u(\cdot,u^0,g)$ exists on $[0,T_*]$ whenever (u^0,g) satisfies

$$\|Uu^0 + U \star g\|_{C_\gamma((0,T_*],E_1)} \le R \ .$$

Proof. This is obvious by the proof of Theorem 5.6. $\square$

Besides the foregoing local existence theorem we obtain — given the additional hypothesis that $\overline{\omega} \ge 0$ if $J = \mathbb{R}^+$ — global existence for small data.

Theorem 5.8. *Let the assumptions of Theorem* 5.6 *be satisfied and suppose that* $\omega \ge 0$. *Then there exists* $R > 0$ *such that* $u(\cdot,u^0,g)$ *is a global solution of* (5.4) *and belongs to* $C_{\gamma,\omega}(\dot{J},E_1)$, *provided*

$$\|Uu^0 + U \star g\|_{C_{\gamma,\omega}(\dot{J},E_1)} \le R \ . \tag{5.21}$$

Proof. This follows from part (a) of the proof of Theorem 5.6 by replacing X_T by $X := C_{\gamma,\omega}(\dot{J},E_1)$ and by denoting by μ the norm of the bilinear map $U \star Q^\natural$ on X. $\square$

The following remarks give further sufficient criteria for $u(\cdot,u^0,g)$ to be global.

Remarks 5.9. (a) Note that (5.21) is estimated above by

$$c\big(\|Uu^0\|_{C_{\gamma,\omega}(\dot{J},E_1)} + \|g\|_{C_{2\gamma,\omega}(\dot{J},E_{\gamma,\infty})}\big) \ ,$$

which, in turn, is majorized by

$$c\big(\|u^0\|_{E_{1-\gamma,\infty}} + \|g\|_{C_{2\gamma,\omega}(\dot{J},E_{\gamma,\infty})}\big) \ .$$

Proof. This follows from Lemmas 5.1 and 5.2. $\square$

(b) Let the hypotheses of Theorem 5.6 be satisfied and put $u := u(\cdot, u^0, g)$. Suppose that either
(i) $u(J^+)$ is relatively compact in $E^0_{1-\gamma,\infty}$
or
(ii) $u : J^+ \to E^0_{1-\gamma,\infty}$ is bounded and uniformly continuous.
Then $J^+ = J$.

Proof. (i) Suppose that $u(J^+)$ is relatively compact in $E^0_{1-\gamma,\infty}$ and that $J^+ \neq J$. Fix $T_0 \in J^+$ and $T \in J \backslash J^+$. Since $g \in C\big([T_0, T], E^0_{\gamma,\infty}\big)$, Lemma 5.2(i) implies

$$\|U \star g_\tau\|_{C_\gamma((0,t],E_1)} \leq c \, \|U \star g_\tau\|_{C([0,t],E_1)} \leq ct^\gamma \, \|g\|_{C([T_0,T],E^0_{\gamma,\infty})}$$

for $\tau \in [T_0, t^+]$ and $t \in (0, T - t^+]$, where $t^+ := \sup J^+$ and $g_\tau := g(\tau + \cdot)$. Thus there exists T' such that

$$\|U \star g_\tau\|_{C_\gamma((0,T'],E_1)} \leq R/2 \ , \qquad T_0 \leq \tau \leq t^+ \ . \tag{5.22}$$

It follows from Lemma 5.1 that

$$\|Ue\|_{C_\gamma((0,T],E_1)} \leq \|U u_\tau\|_{C_\gamma((0,T],E_1)} + c \, \|e - u_\tau\|_{E^0_{1-\gamma,\infty}}$$

for $\tau \in J^+$ and $e \in E^0_{1-\gamma,\infty}$, where $u_\tau := u(\tau)$. Thus, given $\tau \in [T_0, t^+)$, there exist $T_\tau \in (0, T - \tau]$ and $r_\tau > 0$ such that

$$\|Ue\|_{C_\gamma((0,T_\tau],E_1)} \leq R/2 \ , \qquad e \in \mathbb{B}(u_\tau, r_\tau) \ , \tag{5.23}$$

where $\mathbb{B}(e, r)$ is the open ball in $E^0_{1-\gamma,\infty}$ centered at e with radius r. Since $u(J^+)$ is relatively compact in $E^0_{1-\gamma,\infty}$ there exist $\tau_j \in J^+$ and $e_{\tau_j} \in u(J^+)$ for $0 \leq j \leq m$ such that $u(J^+)$ is contained in $\bigcup \{ \mathbb{B}(e_{\tau_j}, r_{\tau_j}) \ ; \ 0 \leq j \leq m \}$. Set

$$\overline{T} := \min\{ T_{\tau_j} \ ; \ 0 \leq j \leq m \} \wedge T'$$

and fix $\overline{t} \in J^+$ satisfying $\overline{t} > t^+ - \overline{T}/2$. Then it follows from (5.22) and (5.23) (with $e := u(\overline{t})$) that

$$\|U u_{\overline{t}} + U \star g_{\overline{t}}\|_{C_\gamma((0,\overline{T}],E_1)} \leq R \ .$$

Thus, since $u(\overline{t}) \in E_1$, (the proof of) Theorem 5.6 implies that the problem

$$\dot{v} + Av = Q(v, v) + g_{\overline{t}} \ , \quad t \in J - \overline{t} \ , \qquad v(0) = u(\overline{t})$$

has a unique solution $v \in C\big([0, \overline{T}], E_1\big)$. Consequently, u has been extended to the interval $[0, \overline{t} + \overline{T}]$ with $\overline{t} + \overline{T} > t^+$, which contradicts the maximality of t^+.

(ii) Suppose that $u : J^+ \to E^0_{1-\gamma,\infty}$ is bounded and uniformly continuous and, without loss of generality, that J is bounded. Then u possesses an extension

$\overline{u} \in C(\overline{J}^+, E^0_{1-\gamma,\infty})$. Since $\overline{J}^+$ is compact, $\overline{u}(\overline{J}^+)$ is compact in $E^0_{1-\gamma,\infty}$. Hence $u(J^+)$ is relatively compact in $E^0_{1-\gamma,\infty}$ and (i) implies the assertion. $\square$

(c) Let the hypotheses of Theorem 5.6 be satisfied and suppose that $\alpha < \gamma$. Fix $T_* \in \dot{J}$. Then

$$t^+(u^0, g) > T_* \wedge c\big(\|u^0\|_{F_{1-\alpha}} + \|g\|_{C_{\alpha+\gamma}(\dot{J}, E_{\gamma,\infty})}\big)^{-1/(\gamma-\alpha)} ,$$

where $t^+(u^0, g) := \sup J^+(u^0, g)$ and $c > 0$ is independent of (u^0, g).

Proof. Note that

$$\begin{aligned} \|Uu^0 + U \star g\|_{C_\gamma((0,T],E_1)} &\leq T^{\gamma-\alpha}\, \|Uu^0 + U \star g\|_{C_\alpha((0,T],E_1)} \\ &\leq cT^{\gamma-\alpha}\big(\|u^0\|_{F_{1-\alpha}} + \|g\|_{C_{\alpha+\gamma}((0,T],E_{\gamma,\infty})}\big) \end{aligned}$$

for $T \in \dot{J}$ with $T \leq T_*$. Hence $u(\cdot, u^0, g)$ exists on $[0, \overline{T}]$ at least if $\overline{T} \leq T_*$ and

$$c\overline{T}^{\gamma-\alpha}\big(\|u^0\|_{F_{1-\alpha}} + \|g\|_{C_{\alpha+\gamma}(\dot{J}, E_{\gamma,\infty})}\big) \leq R .$$

Thus the assertion follows. $\square$

(d) Let the hypotheses of Theorem 5.6 be satisfied and suppose that $\alpha < \gamma$. If $t^+ := t^+(u^0, g) < \sup J$ then

$$\lim_{t \to t^+} \|u(t)\|_{E_{\beta,\infty}} = \infty$$

for each $\beta \in (1-\gamma, 1]$.

Proof. Suppose that there are $\beta \in (1-\gamma, 1]$ and a sequence (t_j) in J^+ with $t_j \to t^+$ and

$$\sup_j \|u(t_j)\|_{E_{\beta,\infty}} < \infty .$$

Then, fixing $\alpha < \gamma$ such that $\beta > 1 - \alpha$, it follows from $E_{\beta,\infty} \hookrightarrow E^0_{1-\alpha,\infty}$ and (c) (with $F_{1-\alpha} := E^0_{1-\alpha,\infty}$) that there exists $\tau > 0$ such that $t^+\big(u(t_j), g\big) \geq \tau$ for $j \in \mathbb{N}$. This implies that u can be continued beyond t^+, contradicting its maximality. $\square$

In this paper we are interested in the Navier-Stokes equations. For this reason we have restricted our considerations to quadratic nonlinearities. However, it is not too difficult to extend the results of this section to other cases and to non-autonomous situations as well. This and applications to parabolic systems will be done elsewhere.

6. The Navier-Stokes Evolution Equation

Throughout this section we suppose that

$$(3.1)\text{–}(3.3) \text{ are satisfied }. \tag{6.1}$$

We consider the Navier-Stokes evolution equation

$$\dot{v} + Sv = b(v,v) + Pf(t) \ , \quad t > 0 \ , \qquad v(0) = v^0 \tag{6.2}$$

in $L_{q,\sigma}$ and in suitable superspaces thereof. We suppose that $(\cdot,\cdot)_\theta$ satisfies (1.5) for $0 < \theta < 1$ and employ the notations $F^s_{q,0}$ and $F^s_{q,0,\sigma}$, etc., introduced in Sections 2 and 3, respectively.

We begin by proving the following fundamental existence, regularity, and blow-up theorem concerning maximal solutions of (6.1).

Theorem 6.1. *Suppose that $q > m/3$. Fix $s \in [0,2)$ satisfying*

$$-1 + m/q < s < (m/q) \wedge (1 + m/q)/2 \ . \tag{6.3}$$

Also suppose that

$$-1 + m/q \le r \le s \tag{6.4}$$

and

$$(v^0, f) \in n^{-1+m/q}_{q,0,\sigma} \times C_{(2s-r+1-m/q)/2}\big((0,T], n^{2s-1-m/q}_{q,0}\big) \tag{6.5}$$

for each $T > 0$.
Then:

(i) *Problem (6.2) possesses (in $H^{s-2}_{q,0,\sigma}$) a unique maximal solution*

$$v := v(\cdot, v^0, f) \in C(J^+, n^{-1+m/q}_{q,0,\sigma}) \cap C(\dot{J}^+, H^s_{q,0,\sigma}) \cap C^1(\dot{J}^+, H^{s-2}_{q,0,\sigma}) \tag{6.6}$$

satisfying

$$\lim_{t\to 0} t^{(s+1-m/q)/2} \, \|v(t)\|_{H^s_q} = 0 \ . \tag{6.7}$$

The maximal interval of existence of v, that is,

$$J^+ := J^+(v^0, f) := \mathrm{dom}\big(v(\cdot, v^0, f)\big) \ , \tag{6.8}$$

is open in $\mathbb{R}^+$, and

$$v \in C(J^+, F^r_{q,0,\sigma}) \qquad \text{if} \quad v^0 \in F^r_{q,0,\sigma} \ . \tag{6.9}$$

If $r < s$ and $v^0 \in F^r_{q,0,\sigma}$ then

$$\lim_{t\to 0} t^{(s-r)/2} \, \|v(t)\|_{H^s_q} = 0 \ . \tag{6.10}$$

(ii) *For each $T>0$ there exists $R>0$ such that $J^+ \supset [0,T]$ whenever*

$$\|v^0\|_{N_{q,0,\sigma}^{-1+m/q}} + \|f\|_{C_{s+1-m/q}((0,T],N_{q,0}^{2s-1-m/q})} \le R \ .$$

(iii) *Put $t^+(v^0,f) := t^+ := \sup J^+$. If $r > -1+m/q$ then t^+ exceeds*

$$1 \wedge c\Big(\|v^0\|_{F_{q,0,\sigma}^r} + \sup_{t>0} t^{(2s-r+1-m/q)/2}\|f(t)\|_{N_{q,0}^{2s-1-m/q}}\Big)^{-2/(r+1-m/q)} \ ,$$

where $c>0$ is independent of (v^0,f).

(iv) *Suppose that $t^+ < \infty$. Then either*

$$\overline{\lim_{t\to t^+}} \|v(t)\|_{N_{q,0}^{-1+m/q}} = \infty$$

or $v : J^+ \to n_{q,0,\sigma}^{-1+m/q}$ is not uniformly continuous. Furthermore,

$$\lim_{t\to t^+} \|v(t)\|_{H_{q,0,\sigma}^\tau} = \infty$$

for each $\tau > -1+m/q$.

(v) *If*

$$f \in C^\rho(\mathbb{R}^+, L_q) + C(\mathbb{R}^+, H_q^\rho) \tag{6.11}$$

for some $\rho \in (0,1)$ then v is a **strong** q**-solution** *on $\dot{J}^+$, that is,*

$$v \in C(\dot{J}^+, H_{q,0,\sigma}^2) \cap C^1(\dot{J}^+, L_q) \ ,$$

and $v^0 \in F_{q,0,\sigma}^r$ implies $v \in C(J^+, F_{q,0,\sigma}^r)$, provided $-1+m/q \le r \le 2$.

(vi) *Suppose that $\partial\Omega$ is uniformly regular of class C^∞ (that is, of class C^k for every $k \in \mathbb{N}$) if $\Omega \ne \mathbb{R}^m$ and that, in addition to (6.11), $f \in C^\infty\big((0,\infty)\times\overline{\Omega}, \mathbb{R}^m\big)$. Then*

$$v \in C^\infty(\dot{J}^+ \times \overline{\Omega}, \mathbb{R}^m) \ .$$

Proof. (a) First we note that $(1+m/q)/2 \le m/q$ iff $q \le m$. Furthermore,

$$(1+m/q)/2 > -1+m/q \text{ and } -1+m/q < 2 \qquad \text{iff} \quad q > m/3 \ .$$

Thus there exists $s \in [0,2)$ satisfying (6.3).

(b) Denote by $\big[(\mathbb{E}_\alpha, \mathbb{A}_\alpha) \ ; \ \alpha \in \mathbb{R}\big]$ the Stokes scale constructed with $[\cdot,\cdot]_\theta$ and by $\big[(\mathbb{F}_\alpha, \mathbb{B}_\alpha) \ ; \ \alpha \in \mathbb{R}\big]$ the one constructed with $(\cdot,\cdot)_\theta$ for $0<\theta<1$. Set

$$(E_0, E_1) := (\mathbb{E}_{s/2-1}, \mathbb{E}_{s/2}) \ , \quad A := \mathbb{A}_{s/2-1} \ .$$

By Lemma 1.1

$$E_{\theta,1} \overset{d}{\hookrightarrow} \mathbb{F}_{s/2-1+\theta} \overset{d}{\hookrightarrow} E_{\theta,\infty}^0 \doteq n_{q,0,\sigma}^{s-2+2\theta} \ , \qquad 0<\theta<1 \ . \tag{6.12}$$

Theorem 4.2 gives $b \in \mathcal{L}^2(\mathbb{E}_{s/2}, \mathbb{E}_{s-(1+m/q)/2})$. Setting $2\gamma := s + 1 - m/q$, we infer from (6.3) that $0 < \gamma < 1/2$. Consequently, thanks to (6.12) and (1.2),

$$b \in \mathcal{L}^2(E_1, E_{\gamma,\infty}) \ . \tag{6.13}$$

Put $2\alpha := s - r$ and $F_{1-\alpha} := F^r_{q,0,\sigma} = \mathbb{F}_{r/2}$. Then (6.4) entails $0 \le \alpha \le \gamma$, and (6.12) implies

$$E_{1-\alpha,1} \overset{d}{\hookrightarrow} F_{1-\alpha} \overset{d}{\hookrightarrow} E_{1-\alpha,\infty} \ .$$

Moreover, $U \,|\, F_{1-\alpha}$ is strongly continuous. (Here and below we use the same symbol for U and any one of its continuous extensions over superspaces of $\mathbb{E}_0$.) Since $-2 < 2s - 1 - m/q < 0$ by (6.3), it follows, by invoking Lemma 1.1 once more, that

$$n^{2s-1-m/q}_{q,0} \doteq (\mathbf{E}_{-1}, \mathbf{E}_0)^0_{\gamma+s/2,\infty} = \mathbf{E}^0_{\gamma-1+s/2,\infty} \ ,$$

where $\big[(\mathbf{E}_\alpha, \mathbf{A}_\alpha) \ ; \ \alpha \in \mathbb{R}\big]$ denotes the Dirichlet scale constructed with $[\cdot,\cdot]_\theta$ for $0 < \theta < 1$. Hence Theorem 2.2, Lemma 3.3, and (6.5) imply

$$g := Pf \in C_{\alpha+\gamma}\big((0,T), \mathbb{E}^0_{\gamma-1+s/2,\infty}\big) \ , \qquad T > 0 \ .$$

Since

$$\mathbb{E}^0_{\gamma-1+s/2,\infty} \doteq (\mathbb{E}_{s/2-1}, \mathbb{E}_s)^0_{\gamma,\infty} = E^0_{\gamma,\infty}$$

by Lemma 1.1, it follows that

$$g \in C_{\alpha+\gamma}\big((0,T], E^0_{\gamma,\infty}\big) \ , \qquad T > 0 \ .$$

Consequently, setting $Q := b$ and invoking Theorem 3.4, assertion (i) follows from Theorem 5.6.

(c) Assertion (ii) is entailed by the above considerations, Remark 5.9(a), and Theorem 5.8 by applying the latter Theorem to (6.2) on $[0,T]$ for each fixed $T > 0$.

(d) Remarks 5.9(b)–(d) entail, by invoking (1.2), assertions (iii) and (iv).

(e) Let (6.11) be true. Fix $T \in \dot{J}^+$ and set $\varphi_\varepsilon(t) := \varphi(\varepsilon + t)$ for $0 \le t \le T - \varepsilon$ whenever φ is defined on $[0,T]$. Since $v \in C\big((0,T], \mathbb{E}_{s/2}\big)$, it follows from Theorem 4.2 that

$$h := b^\natural(v,v) + g \in C\big((0,T], \mathbb{E}_{s-\delta}\big) \ , \tag{6.14}$$

where $2\delta := 1 + m/q$.

Observe that

$$v_\varepsilon(t) = U(t)v(\varepsilon) + U \star h_\varepsilon(t) \ , \qquad 0 \le t \le T - \varepsilon \ . \tag{6.15}$$

Since this holds for every $\varepsilon \in (0,T)$, we infer from Lemmas 5.1 and 5.2 that

$$v \in C\big((0,T], \mathbb{E}_{s-\delta+\xi}\big) \ , \qquad 0 \le \xi < 1 \ . \tag{6.16}$$

Fix $\xi \in (0,1)$ such that $\kappa := \kappa(\xi) := s/2 + \xi - \delta > 0$ and $2\kappa(\xi) < m/q$, and that that there exists $N \in \mathbb{N}$ with

$$s/2 + N\kappa < \delta < s/2 + (N+1)\kappa \ .$$

Define $\xi_N \in (0,\xi)$ by

$$s/2 + N\kappa(\xi) + \kappa(\xi_N) = \delta \ .$$

Also set

$$\beta_0 := s/2 \ , \quad \beta_j := \beta_{j-1} + \kappa = s/2 + j\kappa \ , \qquad 1 \le j \le N \ .$$

Since $v \in C\big((0,T], \mathbb{E}_{\beta_1}\big)$ by (6.16) we deduce from Theorem 4.2 that

$$h \in C\big((0,T], \mathbb{E}_{s/2+\beta_1-\delta}\big) \ .$$

Thus (6.15) and

$$s/2 + \beta_1 - \delta + \xi = \beta_1 + \kappa = \beta_2$$

imply (similarly as (6.14) entailed (6.16)) that $v \in C\big((0,T], \mathbb{E}_{\beta_2}\big)$. This bootstrapping argument leads by induction to $v \in C\big((0,T], \mathbb{E}_{\beta_N}\big)$.

Note that $\kappa(\xi_N) < \kappa(\xi)$ so that $2\kappa(\xi_N) < m/q$. Hence, by invoking Theorem 4.2 once more (with $\alpha := \kappa(\xi_N)$ and $\beta = \beta_N$), we get from $\kappa(\xi_N) + \beta_N = \delta$ that h belongs to $C\big((0,T], \mathbb{E}_0\big)$. Now (6.15) entails

$$v \in C\big((0,T], \mathbb{E}_\eta\big) \cap C^1\big((0,T], \mathbb{E}_{\eta-1}\big) \ , \qquad 0 < \eta < 1 \ . \tag{6.17}$$

Fix $\overline{\alpha}, \overline{\beta}, \eta \in (0,1)$ satisfying $\eta > \overline{\alpha} \vee \overline{\beta}$ and

$$b \in \mathcal{L}(\mathbb{E}_{\overline{\alpha}}, \mathbb{E}_{\overline{\beta}}; \mathbb{E}_0) \ . \tag{6.18}$$

We deduce from (6.17), (1.2), and Proposition II.1.1.2 in [5] that v belongs to $C^\theta\big((0,T], \mathbb{E}_{\overline{\alpha}\vee\overline{\beta}}\big)$ for some $\theta \in (0,\rho)$. From this, from (6.18), from (6.11), and from (1.2) we infer that

$$h \in C^\theta\big((0,T], \mathbb{E}_0\big) + C\big((0,T], \mathbb{E}_\theta\big) \ .$$

Now Theorems II.1.2.1 and II.1.2.2 of [5] imply

$$v \in C\big((0,T], \mathbb{E}_1\big) \cap C^1\big((0,T], \mathbb{E}_0\big) \ .$$

Since this is true for every $T \in J^+$ it follows that v is a strong q-solution on J^+.

The fact that v is continuous at $t = 0$ in $F^r_{q,0,\sigma}$ if

$$(m/q) \wedge (1 + m/q)/2 \le r \le 2$$

follows by the arguments used in the above bootstrapping procedure and by the assertions contained in Theorems II.1.2.1, II.1.2.2, and IV.1.2.1, IV.1.2.2 of [5] concerning strict solutions. This proves (v).

(f) Assertion (vi) follows by a standard but tedious bootstrapping argument (cf. [32] and [49]) which we leave to the reader. $\square$

Next we prove a global existence theorem for small data in the case where Ω is bounded.

Theorem 6.2. *Let the hypotheses of Theorem* 6.1 *be satisfied and let Ω be bounded. Denote by λ_0 the smallest eigenvalue of the Stokes operator. Then, given $\omega \in [0, \lambda_0)$, there exists $R > 0$ such that $t^+(v^0, f) = \infty$ and*

$$\sup_{t>0} t^{(s+1-m/q)/2} e^{\omega t} \, \|v(t, v^0, f)\|_{H_q^s} < \infty \tag{6.19}$$

whenever

$$\|v^0\|_{N_{q,0,\sigma}^{-1+m/q}} + \sup_{t>0} t^{s+1-m/q} e^{\omega t} \, \|f(t)\|_{N_{q,0}^{2s-1-m/q}} \leq R \ . \tag{6.20}$$

Proof. This is an easy consequence of Theorem 5.8, Remark 5.9(a), and the proof of Theorem 6.1, thanks to $\mathrm{type}(-S) = -\lambda_0$ (see Remark 3.1). $\square$

Remarks 6.3. (a) Theorem 6.2 shows that $v(\cdot, v^0, f)$ is exponentially decaying in H_q^s whenever this is the case for f in $N_{q,0}^{2s-1-m/q}$, provided (6.19) is true.

(b) Suppose that $f = 0$ and $0 \leq \omega < \lambda_0$. Then Theorem 5.8 implies the existence of a constant $R' > 0$ such that $t^+(v^0) = \infty$ and (6.19) is true, provided

$$\sup_{t>0} t^{\gamma} e^{\omega t} \, \|e^{-tS} v^0\|_{H_q^s} \leq R' \ , \tag{6.21}$$

where $2\gamma := s + 1 - m/q$. However, this condition is equivalent to the requirement that $\|v^0\|_{N_{q,0,\sigma}^{-1+m/q}}$ be small.

Proof. It suffices to show that (6.21) defines an equivalent norm on $N_{q,0,\sigma}^{-1+m/q}$. Using the notations of the proof of Theorem 6.1 we see that, denoting by $\sim$ equivalent norms,

$$\begin{aligned} t^{\gamma} e^{\omega t} \, \|e^{-tS} v^0\|_{H_q^s} &= t^{\gamma-1} \, \|(A-\omega)^{-1} t (A-\omega) e^{-(A-\omega)t} v^0\|_{E_1} \\ &\sim t^{\gamma-1} \, \|tBe^{-tB} v^0\|_{E_0} \ , \end{aligned}$$

since $B := A - \omega \in \mathcal{H}(E_1, E_0)$ and B is an isomorphism from E_1 onto E_0, thanks to $\mathrm{type}(-B) = \omega + \mathrm{type}(-A) = \omega - \lambda_0 < 0$. It is known (cf. Section I.2.10 in [5]) that

$$\sup_{t>0} t^{\gamma-1} \, \|tBe^{-tB} v^0\|_{E_0} \sim \|v^0\|_{E_{1-\gamma,\infty}} \ .$$

Since, thanks to (6.12),

$$E^0_{1-\gamma,\infty} = n^{s-2\gamma}_{q,0,\sigma} = n^{-1+m/q}_{q,0,\sigma} ,$$

the assertion follows. $\square$

(c) Theorem 6.2 and the above remarks remain valid if the assumption that Ω be bounded is replaced by the hypothesis that type$(-S) < 0$. $\square$

A priori Theorem 6.1 guarantees a unique maximal solution v_s for each choice of s. The following proposition implies, however, that v_s is independent of s if f is sufficiently regular.

Proposition 6.4. *Suppose that $q > m/3$ and $s, \overline{s} \in [0,2)$ satisfy*

$$-1 + m/q < s < \overline{s} < (m/q) \wedge (1 + m/q)/2$$

and

$$-1 + m/q < 2s - \overline{s} < 1 + m/q . \tag{6.22}$$

Also suppose that

$$(v^0, f) \in n^{-1+m/q}_{q,0,\sigma} \times C_{s+1-m/q}\big((0,T], n^{2\overline{s}-1-m/q}_{q,0}\big) \tag{6.23}$$

for each $T > 0$. Let

$$v \in C(J^+, n^{-1+m/q}_{q,0,\sigma}) \cap C(\dot{J}^+, H^s_{q,0,\sigma}) \cap C^1(\dot{J}^+, H^{s-2}_{q,0,\sigma}) \tag{6.24}$$

be the unique maximal solution of (6.2) *satisfying* (6.7), *guaranteed by Theorem* 6.1(i). *Then*

$$v \in C_{(\overline{s}+1-m/q)/2}\big((0,T], H^{\overline{s}}_{q,0,\sigma}\big) \cap C^1\big((0,T], H^{\overline{s}-2}_{q,0,\sigma}\big) , \qquad 0 < T < t^+ .$$

Proof. First we note that (6.23) implies

$$f \in C_{s+1-m/q}\big((0,T], n^{2s-1-m/q}_{q,0}\big) , \qquad T > 0 .$$

Thus Theorem 6.1 guarantees that v is well-defined.

Fix $T \in \dot{J}^+$ and set $(\overline{E}_0, \overline{E}_1) := (H^{\overline{s}-2}_{q,0,\sigma}, H^{\overline{s}}_{q,0,\sigma})$. Also set

$$2\overline{\gamma} := 2s - 1 - m/q - (\overline{s} - 2) = 2s - \overline{s} + 1 - m/q .$$

Then (6.22) entails $0 < \overline{\gamma} < 1$. Moreover, setting $(\cdot,\cdot)_\theta = (\cdot,\cdot)^0_{\theta,\infty}$,

$$\overline{E}^0_{\overline{\gamma},\infty} \doteq n^{2s-1-m/q}_{q,0,\sigma} \doteq E^0_{\gamma,\infty} ,$$

where, here and below, we use the notations of the proof of Theorem 6.1. Thus we infer from (6.7), (6.13), and (6.24) (cf. the proof of Lemma 5.3) that

$$b^\natural(v,v) \in C_{s+1-m/q}\big((0,T],\overline{E}^0_{\overline{\gamma},\infty}\big) \ .$$

Since $\overline{s} > s$ it follows from (6.23), Theorem 2.2, and Lemma 3.3 that

$$Pf \in C_{s+1-m/q}\big((0,T],\overline{E}^0_{\overline{\gamma},\infty}\big) \ .$$

Thus, recalling that $h := b^\natural(v,v) + Pf$, Lemma 5.2(ii) entails

$$U \star h \in C_{(\overline{s}+1-m/q)/2}\big((0,T],H^{\overline{s}}_{q,0,\sigma}\big) \ .$$

Set $2\overline{\alpha} := \overline{s} + 1 - m/q$ so that $n^{-1+m/q}_{q,0,\sigma} \doteq \overline{E}_{1-\overline{\alpha}}$. Then Lemma 5.5 implies

$$Uv^0 \in C_{(\overline{s}+1-m/q)/2}\big((0,T],H^{\overline{s}}_{q,0,\sigma}\big) \ .$$

Consequently,

$$v = Uv^0 + U \star h \in C_{(\overline{s}+1-m/q)/2}\big((0,T],H^{\overline{s}}_{q,0,\sigma}\big) \ . \tag{6.25}$$

Fix $\varepsilon \in (0,T)$ and note that

$$v_\varepsilon = Uv(\varepsilon) + U \star h_\varepsilon$$

on $[\varepsilon, T-\varepsilon]$. Thanks to $v(\varepsilon) \in \overline{E}_1$ and $h_\varepsilon \in C\big([\varepsilon,T-\varepsilon],\overline{E}^0_{\gamma,\infty}\big)$ it is an easy consequence of Theorem II.1.2.2 in [5] that $v_\varepsilon \in C^1\big([\varepsilon,T],\overline{E}_0\big)$. Since this holds for every $\varepsilon \in (0,T)$ it follows that

$$v \in C^1\big((0,T],H^{\overline{s}-2}_{q,0,\sigma}\big) \ . \tag{6.26}$$

This proves the assertion. $\square$

Our next proposition shows that v is also independent of q in a suitable sense if the data are regular enough.

Proposition 6.5. *Suppose that Ω is a standard domain and $m < q < r < \infty$. Also suppose that*

$$(v^0,f) \in n^{-1+m/q}_{q,0,\sigma} \times \big[C^\rho(\mathbb{R}^+,L_q \cap L_r) + C(\mathbb{R}^+,H^\rho_q \cap H^\rho_r)\big] \tag{6.27}$$

for some $\rho \in (0,1)$. Given $p \in \{q,r\}$, let

$$v_p \in C(J_p^+,n^{-1+m/q}_{q,0,\sigma}) \cap C(\dot{J}_p^+,L_{p,\sigma}) \cap C^1(\dot{J}_p^+,H^{-2}_{q,0,\sigma}) \tag{6.28}$$

be the unique maximal solution of (6.2) in $L_{p,\sigma}$ satisfying

$$\lim_{t\to 0} t^{(1-m/p)/2}\,\|v_p(t)\|_{L_p} = 0 \ . \tag{6.29}$$

Then $v_r \supset v_q$.

Proof. First note that $L_q \cap L_r \hookrightarrow L_p$ and $H_q^\rho \cap H_r^\rho \hookrightarrow H_p^\rho$. Furthermore, Corollary 3.11 implies

$$n_{q,0,\sigma}^{-1+m/q} \hookrightarrow n_{r,0,\sigma}^{-1+m/r} \ . \tag{6.30}$$

Hence it follows from Theorem 6.1 that the unique maximal solution v_p of (6.2) satisfying (6.28) and (6.29) is well-defined.

(a) Suppose that $1/r > 2/q - 1/m$, that is, $-1 + m/q < -m(1/q - 1/r)$. Put $s := 0$ and $\overline{s} := m(1/q - 1/r)$. Since $\overline{s} < m/q = (m/q) \wedge (1 + m/q)/2$, it follows from (6.27) that

$$(v^0, f) \in n_{q,0,\sigma}^{-1+m/q} \times C_{s+1-m/q}\big((0,T], n_{q,0}^{2\overline{s}-1-m/q}\big)$$

for $T > 0$. Thus Proposition 6.4 implies

$$v_q \in C_{(1-m/r)/2}\big((0,T], H_{q,0,\sigma}^{\overline{s}}\big) \cap C^1\big((0,T], H_{q,0,\sigma}^{\overline{s}-2}\big) \ , \qquad T \in \dot{J}_q^+ \ .$$

From Theorem 3.10 we infer that

$$H_{q,0,\sigma}^{\overline{s}-2j} \overset{d}{\hookrightarrow} H_{r,0,\sigma}^{-2j} \ , \qquad j = 0, 1 \ .$$

Hence

$$v_q \in C(\dot{J}_q^+, L_{r,\sigma}) \cap C^1(\dot{J}_q^+, H_{r,0,\sigma}^{-2}) \tag{6.31}$$

and

$$\lim_{t \to 0} t^{(1-m/r)/2} \left\| v_q(t) \right\|_{L_r} = 0 \ . \tag{6.32}$$

From (6.30) it also follows that

$$v_q \in C(J_q^+, n_{r,0,\sigma}^{-1+m/r}) \ . \tag{6.33}$$

Theorem 6.1 implies, thanks to (6.27), that

$$v_q \in C(\dot{J}_q^+, H_{q,0,\sigma}^2) \cap C^1(\dot{J}_q^+, L_{q,\sigma}) \ . \tag{6.34}$$

By Theorem 3.10,

$$H_{q,0,\sigma}^2 \hookrightarrow L_{r,\sigma} \ . \tag{6.35}$$

Observe that

$$S_q \,|\, H_{q,0,\sigma}^2 \cap H_{r,0,\sigma}^2 = S_r \,|\, H_{q,0,\sigma}^2 \cap H_{r,0,\sigma}^2 \ . \tag{6.36}$$

Denote by $\big[(\mathbb{E}_{p,\alpha}, \mathbb{A}_{p,\alpha} \ ; \ \alpha \in \mathbb{R}\big]$ the interpolation-extrapolation scale generated by $(L_{p,\sigma}, S_p)$ and $[\cdot,\cdot]_\theta$, $0 < \theta < 1$. Since $\mathcal{D}_\sigma \subset L_{q,\sigma} \cap L_{r,\sigma}$ and $\mathcal{D}_\sigma$ is dense in $L_{p,\sigma}$ it follows that $L_{q,\sigma} \cap L_{r,\sigma}$ is dense in $L_{p,\sigma}$. Hence, $1 + \mathbb{A}_{p,0}$ being an isomorphism from $H^2_{p,0,\sigma}$ onto $L_{p,\sigma}$, we see that $H^2_{q,0,\sigma} \cap H^2_{r,0,\sigma}$ is dense in $H^2_{p,0,\sigma}$. Thus we infer from (6.36) and

$$S_p = \mathbb{A}_{p,0} \subset \mathbb{A}_{p,-1} \in \mathcal{L}(L_{p,\sigma}, H^{-2}_{p,0,\sigma})$$

that

$$\mathbb{A}_{q,-1} \,|\, L_{q,\sigma} \cap L_{r,\sigma} = \mathbb{A}_{r,-1} \,|\, L_{q,\sigma} \cap L_{r,\sigma} \ .$$

Consequently, (6.34) and (6.35) imply

$$\mathbb{A}_{q,-1} v_q(t) = \mathbb{A}_{r,-1} v_q(t) \ , \qquad t \in \dot{J}^+_q \ . \tag{6.37}$$

Therefore we deduce, by taking into account (6.31) and Theorem 4.2, that

$$\dot{v}_q(t) + \mathbb{A}_{r,-1} v_q(t) = b\big(v_q(t), v_q(t)\big) + Pf(t) \ , \qquad t \in \dot{J}^+_q \ . \tag{6.38}$$

Now the assertion follows from (6.31)–(6.33) and (6.38).

(b) Suppose $1/r \le 2/q - 1/m$. Set $r_0 := q$ and define $r_k \in \mathbb{R} \cup \{-\infty\}$ by

$$\frac{1}{r_k} := \frac{3}{2}\Big(\frac{1}{r_{k-1}} - \frac{1}{m}\Big) + \frac{1}{m} = \Big(\frac{3}{2}\Big)^k \Big(\frac{1}{q} - \frac{1}{m}\Big) + \frac{1}{m}$$

for $k \in \mathbb{N}^\times$. Then there exists $\ell \in \mathbb{N}^\times$ such that $1/r_{\ell+1} < 1/r \le 1/r_\ell$. Note that $r_k > r_{k-1}$ and $1/r_k > 2/r_{k-1} - 1/m$ for $1 \le k \le \ell$ as well as $r \ge r_\ell$ and $1/r > 2/r_\ell - 1/m$. Thus (a) implies $v_q \subset v_{r_1} \subset v_{r_2} \subset \cdots \subset v_{r_\ell} \subset v_r$, from which the assertion follows. □

Remark 6.6. Throughout this section — as well as in the remainder of this paper — we always impose regularity hypotheses for f although only Pf occurs in (6.2). This is done for convenience since f is the quantity given in the original equations. Of course, if $f(t) \in L_q$ for $t \in \mathbb{R}^+$ then it is no loss of generality to assume that $f = Pf$ since a term of the form $(1-P)f$ can always be subsumed in the pressure term ∇p of (0.1). (Note that this argument does not work if $f(t)$ belongs to a negative space since then P is defined by continuous extension and we did not prove that it is a projection.) □

7. Very Weak Solutions

Throughout this section we suppose that

$$\begin{aligned} &\text{(3.1)–(3.3) are satisfied and} \\ &\text{either } q > m \text{ or } 1 \le m/3 \le q \le m \ . \end{aligned} \tag{7.1}$$

We set $s(q) := (-1 + m/q)_+$ and assume that

$$(v^0, f) \in H^{s(q)}_{q,0,\sigma} \times C(\mathbb{R}^+, H^{s(q)-2}_{q,0}) \ . \tag{7.2}$$

Note that $0 \le s(q) \le 2$.

By a **very weak q-solution on** J of the Navier-Stokes equations we mean a function

$$v \in C(J, H^{s(q)}_{q,0,\sigma}) \tag{7.3}$$

satisfying

$$-\int_J \bigl\{\bigl\langle(\partial_t + \nu\Delta)w, v\bigr\rangle + \langle \nabla w, v \otimes v\rangle\bigr\}\, dt = \int_J \langle w, f\rangle\, dt + \bigl\langle w(0), v^0\bigr\rangle \tag{7.4}$$

for all

$$w \in L_1(J, H^{2-s(q)}_{q',0,\sigma}) \cap W^1_1(J, H^{-s(q)}_{q',0,\sigma}) \tag{7.5}$$

having compact supports in $J^* := J \backslash \sup J$.

Let (7.3) and (7.5) be satisfied. Theorem 3.8 implies that

$$\int_J \bigl\langle(\partial_t + \nu\Delta)w, v\bigr\rangle\, dt$$

is well-defined. It follows from Theorem 4.2 that

$$b \in \mathcal{L}^2(H^{s(q)}_{q,0,\sigma}, H^{2s(q)-1-m/q}_{q,0,\sigma}) \ .$$

Thus, since $2s(q) - 1 - m/q \geq s(q) - 2$, we see that (7.3) and (7.5) imply that $\bigl\langle w, b(v,v)\bigr\rangle$ is integrable over J. If $v \in H^2_{q,0,\sigma}$ and $w \in H^2_{q',0,\sigma}$ then it is clear that

$$\bigl\langle w, b(v,v)\bigr\rangle = \langle \nabla w, v \otimes v\rangle \ . \tag{7.6}$$

Hence, by the density of $H^2_{q,0,\sigma}$ in $H^{s(q)}_{q,0,\sigma}$ and the one of $H^2_{q',0,\sigma}$ in $H^{2-s(q)}_{q',0,\sigma}$, we infer from (7.6) that

$$\int_J \langle \nabla w, v \otimes v\rangle\, dt$$

is meaningful if (7.3) and (7.5) are satisfied. It is obvious from (7.2) that the integral on the right-hand side of (7.4) is well-defined. Lastly, since

$$W^1_1(J, H^{-s(q)}_{q',0}) \hookrightarrow C(J, H^{-s(q)}_{q',0})$$

(e.g., Theorem III.1.2.2 in [5]), also the term $\bigl\langle w(0), v^0\bigr\rangle$ makes sense. Consequently, the concept of a very weak q-solution is meaningful.

Of course, a very weak q-solution is **maximal** if there does not exist another such solution being a proper extension of the former.

Remarks 7.1. **(a)** Let v be a very weak q-solution on J. Then v is a **distributional solution on** J of the Navier-Stokes equations in class (7.3). This means that v satisfies (7.3) and (7.4) for all

$$w \in \bigl\{\, \varphi \in \mathcal{D}(\Omega \times J^*, \mathbb{R}^m) \ ; \ \nabla \cdot \varphi(\cdot, t) = 0,\ t \in J^* \,\bigr\} =: \mathcal{D}_\sigma(\Omega \times J^*) \ .$$

Proof. Using obvious identifications it is known (e.g., Theorem 40.1 in [76]) that $\mathcal{D}(\Omega \times \mathbb{R}) = \mathcal{D}(\mathbb{R}, \mathcal{D})$. Thus, by restriction, $\mathcal{D}(\Omega \times J^*) = \mathcal{D}(J^*, \mathcal{D})$. From this we infer that

$$\mathcal{D}_\sigma(\Omega \times J^*) = \mathcal{D}(J^*, \mathcal{D}_\sigma) \ . \tag{7.7}$$

Since each $w \in \mathcal{D}(J^*, \mathcal{D}_\sigma)$ satisfies (7.5) and has compact support in J^*, the assertion is obvious. $\square$

(b) Suppose that

$$\mathcal{D}_\sigma \overset{d}{\subset} H^{2-s(q)}_{q',0,\sigma} \ . \tag{7.8}$$

Then v is a very weak q-solution on J iff it is a distributional solution on J satisfying (7.3).

Proof. Thanks to (a) it suffices to show that a distributional solution in class (7.3) is a very weak q-solution. It is not difficult to verify that (7.8) implies

$$\mathcal{D}(J^*, \mathcal{D}_\sigma) \overset{d}{\subset} L_1(J, H^{2-s(q)}_{q',0,\sigma}) \cap W^1_1(J, H^{-s(q)}_{q',0}) \ .$$

Now the assertion is obvious. $\square$

(c) Suppose that Ω is a standard domain. Then (7.8) is true for
(i) $m/3 \le q < \infty$ if $\Omega = \mathbb{R}^m$,
(ii) $m/3 \le q < m-1$ otherwise.
Thus in each of these cases very weak q-solutions are distributional solutions belonging to class (7.3), and vice versa. However, if $\Omega \ne \mathbb{R}^m$ and $q > m-1$ then $2 - s(q) > 1 + 1/q'$. Consequently, each w satisfying (7.5) has a normal derivative on $\partial\Omega$ which does not vanish, in general. Thus in this case $\mathcal{D}(J^*, \mathcal{D}_\sigma)$ is not dense in class (7.5) of test functions being admissible for very weak q-solutions. It follows that in this situation the class of very weak q-solutions is a proper subset of the class of distributional solutions satisfying (7.3).

Proof. Assertions (i) and (ii) are known and can be shown by the techniques exposed in Chapter III of [28], for example. $\square$

(d) The considerations following the definition of a very weak q-solution show that $s(q)$ is the smallest $s \in \mathbb{R}$ such that the term $\langle \nabla w, v \otimes v \rangle$ is well-defined for all $v \in H^s_{q,0,\sigma}$. $\square$

(e) A function v is a very weak q-solution on J iff v satisfies (7.3) and (7.4) for all $w = \varphi u$ with $\varphi \in \mathcal{D}(J^*, \mathbb{R})$ and $u \in H^{2-s(q)}_{q,0,\sigma}$.

Proof. Let (E_0, E_1) be a densely injected Banach couple. Suppose that

$$u \in L_1(J^*, E_1) \cap W_1^1(J^*, E_0)$$

with compact support. By a standard extension-by-reflexion procedure we may assume that

$$u \in L_1(\mathbb{R}, E_1) \cap W_1^1(\mathbb{R}, E_0)$$

with compact support. Let $\{\varphi_\varepsilon ; \varepsilon > 0\}$ be a mollifier. Then $\varphi_\varepsilon * u \in \mathcal{D}(\mathbb{R}, E_1)$ and

$$\varphi_\varepsilon * u \to u \quad \text{in } L_1(\mathbb{R}, E_1) \cap W_1^1(\mathbb{R}, E_0)$$

as $\varepsilon \to 0$. Hence $\mathcal{D}(\mathbb{R}, E_1)$ is dense in $L_1(\mathbb{R}, E_1) \cap W_1^1(\mathbb{R}, E_0)$. Since the tensor product $\mathcal{D}(\mathbb{R}, \mathbb{R}) \otimes E_1$ is dense in $\mathcal{D}(\mathbb{R}, E_1)$ by Proposition V.2.4.1 of [5] it follows, by restriction, that

$$\mathcal{D}(J^*, \mathbb{R}) \otimes E_1 \overset{d}{\subset} L_1(J^*, E_1) \cap W_1^1(J^*, E_0) \ .$$

Now the assertion is obvious. □

(f) Let Ω be a standard domain. Suppose that $1 \leq m/3 \leq q < r \leq m$ and that

$$(v^0, f) \in H_{q,0,\sigma}^{-1+m/q} \times C(J, H_{q,0,\sigma}^{-3+m/q}) \ .$$

If v is a very weak q-solution on J of the Navier-Stokes equations then it is a very weak r-solution on J.

Proof. Theorem 3.10 implies that

$$(v^0, f) \in H_{r,0,\sigma}^{-1+m/r} \times C(J, H_{r,0,\sigma}^{-3+m/r}) \ .$$

Let v be a very weak q-solution on J. Then, again by Theorem 3.10,

$$v \in C(J, H_{q,0,\sigma}^{-1+m/q}) \hookrightarrow C(J, H_{r,0,\sigma}^{-1+m/r}) \ .$$

By invoking Theorem 3.10 once more we see that

$$H_{r',0,\sigma}^{2j-s(r)} \hookrightarrow H_{q',0,\sigma}^{2j-s(q)} \ , \qquad j = 0, 1 \ .$$

Now the assertion is an easy consequence of (e). □

The following theorem shows that v is a very weak q-solution on J of the Navier-Stokes equations iff v solves the integral equation

$$v(t) = e^{-tS} v^0 + \int_0^t e^{-(t-\tau)S} P\big(-\nabla \cdot (v(\tau) \otimes v(\tau)) + f(\tau)\big)\, d\tau \ , \qquad t \in J \ ,$$

in $C(J, H_{q,0,\sigma}^{s(q)})$.

Below we denote by $\left[\,(\mathbb{E}_\alpha, \mathbb{A}_\alpha)\ ;\ \alpha \in \mathbb{R}\right]$ the Stokes scale constructed with $[\cdot,\cdot]_\theta$ for $0 < \theta < 1$.

Theorem 7.2. *Let (7.1) be satisfied and suppose that $v \in C(J, H^{s(q)}_{q,0,\sigma})$. Then v is a very weak q-solution of the Navier-Stokes equations on J iff v is a mild solution of the evolution equation*

$$\dot{u} + \mathbb{A}_{\alpha-1} u = b(u,u) + g(t)\ , \quad t \in J\ , \qquad u(0) = v^0 \tag{7.9}$$

in $\mathbb{E}_{\alpha-1}$, where $2\alpha := s(q)$ and $g := Pf$.

Proof. It follows from (7.1) and Theorem 4.2 that

$$h := b(v,v) + g \in C(J, \mathbb{E}_{\alpha-1}) \subset L_{\infty,\mathrm{loc}}(J, \mathbb{E}_{\alpha-1})\ .$$

Let $(E_0, E_1) := (\mathbb{E}_{\alpha-1}, \mathbb{E}_\alpha)$ and $A := \mathbb{A}_{\alpha-1}$. Then (E_0, E_1) is a densely injected Banach couple and $A \in \mathcal{H}(E_1, E_0)$. Thus Theorem V.2.8.3 of [5] guarantees that there exists a unique $u \in L_{\infty,\mathrm{loc}}(J, E_0)$ satisfying

$$\int_J \big\langle (-\partial_t + A^\sharp_{-1}) u^\sharp, u \big\rangle\, dt = \int_J \langle u^\sharp, h\rangle\, dt + \big\langle u^\sharp(0), v(0)\big\rangle \tag{7.10}$$

for all $u^\sharp \in L_{1,\mathrm{loc}}(J, E^\sharp_0) \cap W^1_{1,\mathrm{loc}}(J, E^\sharp_{-1})$ having compact support in J^*, and that this unique solution is given by

$$u(t) = e^{-tA} v(0) + \int_0^t e^{-(t-\tau)A} h(\tau)\, d\tau\ , \qquad t \in J\ . \tag{7.11}$$

Thanks to Theorem 3.4,

$$E^\sharp_0 \doteq (E_0)' = (\mathbb{E}_{\alpha-1})' \doteq \mathbb{E}^\sharp_{1-\alpha} \doteq H^{2-s(q)}_{q',0,\sigma}$$

and

$$E^\sharp_{-1} \doteq (E_1)' = (\mathbb{E}_\alpha)' \doteq \mathbb{E}^\sharp_{-\alpha} \doteq H^{-s(q)}_{q',0,\sigma}\ .$$

It also follows from (1.10) that

$$A^\sharp_{-1} = (A_1)' = (\mathbb{A}_\alpha)' = \mathbb{A}^\sharp_{-\alpha}$$

with respect to the duality pairing $\langle\cdot,\cdot\rangle_{E_0} := \langle\cdot,\cdot\rangle_{\mathbb{E}_{\alpha-1}}$. Thus we infer from (3.17) that

$$\langle -\partial_t u^\sharp + A^\sharp_{-1} u^\sharp, u\rangle = \langle -\partial_t u^\sharp - \nu\Delta u^\sharp, u\rangle\ . \tag{7.12}$$

From (7.6) and the density of $\mathbb{E}^\sharp_1 \times \mathbb{E}_1$ in $\mathbb{E}^\sharp_{1-\alpha} \times \mathbb{E}_\alpha$ it follows that

$$\langle u^\sharp, h\rangle = \langle \nabla u^\sharp, v \otimes v\rangle + \langle u^\sharp, f\rangle\ . \tag{7.13}$$

Now suppose that v is a very weak q-solution of (0.1) on J. Then we deduce from (7.12), (7.13), and (7.10) that $u := v$ satisfies (7.10). Thus

$$v = Uv^0 + U \star \bigl(b(v,v) + g\bigr) \qquad \text{on } J \tag{7.14}$$

in $\mathbb{E}_{\alpha-1}$, that is, v is a mild solution of (7.9) on J. Conversely, if v satisfies (7.14) then the above considerations show that v is a very weak q-solution of (0.1) on J. $\square$

The only result known to the author which is related to Theorem 7.2 is due to Fabes, Jones, and Rivière [18]. These authors show that, in the case $\Omega = \mathbb{R}^m$ with $m \geq 3$, a function $v \in L_r\bigl((0,T), L_{q,\sigma}\bigr)$, where $m < q < \infty$ and $m/q + 2/r = 1$, is a distributional solution of (0.1) iff it satisfies an integral equation involving, besides the heat kernel, an m-dimensional generalization of Oseen's divergence free matrix fundamental solution. Thanks to Remark 3.7(a), it is not difficult to see that their integral equation is a representation of (7.10).

Remark 7.3. Suppose that

$$f \in L_{r,\mathrm{loc}}(\mathbb{R}^+, H_{q,0}^{s(q)-2})$$

for some $r \in [1,\infty)$. Then Theorem 7.2 remains valid, provided we define a very weak solution v on J to be a function satisfying (7.3) and (7.4), where (7.5) is replaced by

$$w \in L_{r'}(J, H_{q',0,\sigma}^{2-s(q)}) \cap W_{r'}^{1}(J, H_{q',0}^{-s(q)}) , \tag{7.15}$$

having compact support in J^*.

Proof. This follows from Theorem V.2.8.3 of [5]. $\square$

8. Uniqueness

Let (E_0, E_1) be a densely injected Banach couple and $A \in \mathcal{H}(E_1, E_0)$. Given $\theta \geq 0$, we write $A \in \mathcal{BIP}(E_0, \theta)$ if $\mathrm{type}(-A) < 0$ and there exists $N > 0$ such that

$$\|A^{it}\|_{\mathcal{L}(E_0)} \leq Ne^{\theta\,|t|} , \qquad t \in \mathbb{R} , \tag{8.1}$$

that is, if A has a bounded inverse and bounded imaginary powers.

Remark 8.1. Suppose that Ω is a standard domain. Then, given $p \in (1,\infty)$ and $\omega, \theta > 0$,

$$\omega + S_p \in \mathcal{BIP}(L_{p,\sigma}, \theta) . \tag{8.2}$$

Proof. It has been shown by Giga [30] that (8.2) is true with $\omega = 0$ if Ω is bounded. Giga and Sohr [33] proved that $A := S_p$ satisfies (8.1) with $E_0 := L_{p,\sigma}$ and p belonging to $(1,\infty)$ if Ω is an exterior domain and $m \geq 3$. This estimate is contained in Appendix A of [34] if Ω is a half-space. It is well-known and not difficult to derive by Fourier analysis if $\Omega = \mathbb{R}^m$.

Note that $\mathrm{type}(-S_p) \le 0$. Hence $\mathrm{type}(-\omega + S_p) \le -\omega$ for each $\omega > 0$. Now the assertion follows from Theorem 3 in [65] (if Ω is bounded also see Corollary III.4.8.6 in [5]). □

We again assume that

$$\begin{aligned}&(3.1)\text{–}(3.3) \text{ are satisfied and}\\ &\text{either } q > m \text{ or } 1 \le m/3 \le q \le m\ .\end{aligned} \tag{8.3}$$

In the proof of the following theorem $\big[\,(\mathbb{E}_\alpha\,, \mathbb{A}_\alpha)\ ;\ \alpha \in \mathbb{R}\big]$ denotes the Stokes scale constructed with $[\cdot,\cdot]_\theta,\ 0 < \theta < 1$. Recall that $s(q) := (-1 + m/q)_+$.

Theorem 8.2. *Suppose that*

$$(v^0, f) \in H^{s(q)}_{q,0,\sigma} \times C(\mathbb{R}^+, H^{s(q)-2}_{q,0})\ .$$

Also suppose that there are $\omega \ge 0$ and $\theta \in [0, \pi/2)$ such that

$$\omega + S_q \in \mathcal{BIP}(L_{q,\sigma}; \theta)\ . \tag{8.4}$$

Then there exists at most one maximal very weak q-solution of the Navier-Stokes equations.

Proof. Thanks to Theorem 7.2 we have to show that (7.9) possesses at most one maximal mild solution v in $H^{s(q)-2}_{q,0,\sigma}$. For this it suffices, by obvious arguments, to prove that, given any sufficiently small $T > 0$, equation (7.9) has at most one mild solution $v \in C\big([0,T], \mathbb{E}_\alpha\big)$, where $2\alpha := s(q)$.

From (8.4) and Proposition V.1.5.5 in [5] we infer that $\omega + \mathbb{A}_{\alpha-1}$ belongs to $\mathcal{BIP}(\mathbb{E}_{\alpha-1}; \theta)$. Since $\mathbb{E}_0$ is a closed linear subspace of L_q it follows that $\mathbb{E}_0$ is a UMD space. Thus $\mathbb{E}_{\alpha-1}$ is a UMD space as well (cf. Theorem III.4.5.2 in [5]). Hence the Dore-Venni theorem (see [5], Theorems III.1.5.2 and III.4.10.8) entails that, given any $r \in (1, \infty)$, there exists $\kappa_r > 0$ such that

$$\|U \star u\|_{L_r((0,T),\mathbb{E}_\alpha)} \le \kappa_r\, \|u\|_{L_r((0,T),\mathbb{E}_{\alpha-1})} \tag{8.5}$$

for $u \in L_r\big((0,T), \mathbb{E}_{\alpha-1}\big)$ and $T > 0$.

Theorem 4.2 guarantees the existence of $\lambda \in \mathbb{R}^+$ such that

$$\|b(u,v)\|_{\mathbb{E}_{\alpha-1}} \le \lambda\, \|u\|_{\mathbb{E}_\alpha}\, \|v\|_{\mathbb{E}_\alpha}\ , \qquad u, v \in \mathbb{E}_\alpha\ . \tag{8.6}$$

Fix $2\beta \in (2\alpha, 2 \wedge m/q)$ and $r > 1/(\beta - \alpha)$ and set $\kappa := \kappa_r$. Since $\mathbb{E}_\beta \overset{d}{\hookrightarrow} \mathbb{E}_\alpha$ there is $v^* \in \mathbb{E}_\beta$ such that

$$\|v^0 - v^*\|_{\mathbb{E}_\alpha} < 1/8\lambda\kappa\ . \tag{8.7}$$

Let $v_1, v_2 \in C\big([0,T], \mathbb{E}_\alpha\big)$ be mild solutions of (7.9). Set $w := v_1 - v_2$ and note that

$$w = U \star \big(b(v_1,v_1) - b(v_2,v_2)\big) = U \star \big(a_0(w) + a(w)\big) \tag{8.8}$$

with

$$a_0(w) := b(w, v_1 - v^0) + b(v_2 - v^0, w) + b(w, v^0 - v^*) + b(v^0 - v^*, w)$$

and

$$a(w) := b(w, v^*) + b(v^*, w) \ .$$

Then

$$\|a_0(w)(t)\|_{\mathbb{E}_{\alpha-1}} \le \varphi(T)\, \|w(t)\|_{\mathbb{E}_\alpha} \ , \qquad 0 \le t \le T \ ,$$

where

$$\varphi(T) := \lambda\big(\|v_1 - v^0\|_{C([0,T],\mathbb{E}_\alpha)} + \|v_2 - v^0\|_{C([0,T],\mathbb{E}_\alpha)}\big) + 1/4\kappa \ .$$

Thus there exists $T_0 > 0$ such that $\varphi(T) \le 1/2\kappa$ for $0 < T \le T_0$, and, consequently,

$$\|a_0(w)(t)\|_{\mathbb{E}_{\alpha-1}} \le \|w(t)\|_{\mathbb{E}_\alpha}/2\kappa \ , \qquad 0 \le t \le T \le T_0 \ .$$

Now we infer from (8.5) and (8.8) that

$$\|w\|_{L_r((0,T),\mathbb{E}_\alpha)} \le 2\, \|U \star a(w)\|_{L_r((0,T),\mathbb{E}_\alpha)} \ , \qquad 0 < T \le T_0 \ . \tag{8.9}$$

Observe that $b \in \mathcal{L}(\mathbb{E}_\alpha, \mathbb{E}_\beta; \mathbb{E}_{\beta-1})$ by Theorem 4.2. Hence it follows from Lemma 5.1 and Hölder's inequality that

$$\begin{aligned} \|U \star a(w)\|_{L_r((0,T),\mathbb{E}_\alpha)} &\le c\Big(\int_0^T \Big(\int_0^t (t-\tau)^{-1+\beta-\alpha}\, \|w(\tau)\|_{\mathbb{E}_\alpha}\, d\tau\Big)^r dt\Big)^{1/r} \\ &\le c_0 T^{\beta-\alpha}\, \|w\|_{L_r((0,T),\mathbb{E}_\alpha)} \ . \end{aligned} \tag{8.10}$$

Finally, we deduce from (8.9) and (8.10) that $w(t) = 0$ for $0 \le t \le T$ and any $T \in (0, T_0]$ satisfying $2c_0 T^{\beta-\alpha} < 1$. ☐

Remarks 8.3. **(a)** If $q > m$ then assumption (8.4) is not needed for Theorem 8.2 to be valid.

Proof. Indeed, in this case Theorem 4.2 implies that b belongs to $\mathcal{L}^2(\mathbb{E}_0, \mathbb{E}_{-\gamma})$, where $2\gamma = 1 + m/q < 2$. Thus maximal regularity is not needed and the assertion follows by an easy modification of the above proof based on the singular Gronwall inequality (cf. Theorem II.3.3.1 in [5]). Details are left to the reader. ☐

(b) By replacing in the definition of a very weak solution condition (7.5) by (7.15) it follows that Theorem 8.2 is also true if

$$f \in L_{r,\mathrm{loc}}(\mathbb{R}^+, H_{q,0}^{s(q)-2}) \ .$$

for some $r \in [1, \infty)$.

Proof. This is a consequence of Remark 7.3 and the above proof. $\square$

Recently, Furioli et al. [26], Lions and Masmoudi [59] and Monniaux [63] published uniqueness proofs for mild solutions in $C([0,T],L_m)$ of the Navier-Stokes equations. More precisely, in [26] and [63] there is studied the case $\Omega = \mathbb{R}^3$. Monniaux's proof is also based on maximal L_r-regularity, but is different from ours and — even in that simpler situation — more complicated. In [59] there is given a sketch of an idea for a uniqueness proof in the case where $m \geq 3$ and either $\Omega = \mathbb{R}^m$, or Ω is an m-dimensional torus, or Ω is regular, relying on completely different techniques. We also mention that, by means of probabilistic methods, Le Jan and Sznitman [56] established uniqueness and existence results for a class of generalized solutions if $\Omega = \mathbb{R}^3$.

By combining Theorems 6.1, 7.2, and 8.2 we obtain the following existence and uniqueness theorem. For simplicity, we impose more restrictive hypotheses on f than actually needed and leave it to the reader to formulate weaker assumptions.

Theorem 8.4. *Let assumptions* (8.3) *be satisfied and suppose that*

$$(v^0, f) \in H^{s(q)}_{q,0,\sigma} \times C(\mathbb{R}^+, L_q) \ . \tag{8.11}$$

If $q \leq m$ then also suppose that (8.4) *is true. Then the Navier-Stokes equations possess a unique maximal very weak q-solution.*

Proof. Fix $s \in [0,2)$ satisfying (6.3). Then (8.11) and Theorem 6.1 imply the existence of a unique maximal solution v_q of (6.2) satisfying (6.5) and belonging to $C(J^+, H^{s(q)}_{q,0,\sigma})$. Using the notations of Theorem 7.2, it follows that

$$\mathbb{A}_{(s-2)/2} \subset \mathbb{A}_{(s(q)-2)/2}$$

thanks to $s > s(q)$. Hence, taking into account Theorem 4.2, we see that v_q is a mild solution of (7.9) on J^+. Now Theorem 8.2 and Remark 8.3(a) entail the assertion. $\square$

Remarks 8.5. Let the hypotheses of Theorem 8.4 be satisfied.

(a) Fix any $s \in [0,2)$ satisfying (6.3). Then the unique maximal very weak q-solution v coincides with the unique maximal solution v_q of (6.2) in $H^{s-2}_{q,0,\sigma}$ whose existence is guaranteed by Theorem 6.1(i). In particular, v_q is independent of the choice of s.

Proof. The proof of Theorem 8.4 shows that $v \supset v_q$. The converse relation, $v_q \supset v$, follows from Theorem 7.2, Remark 5.7(a), and the maximality of v_q as a solution of (6.2) in $H^{s(q)-2}_{q,0,\sigma}$. $\square$

(b) Also suppose that

$$f \in C^\rho(\mathbb{R}^+, L_q) + C(\mathbb{R}^+, H^\rho_q)$$

for some $\rho \in (0,1)$. Then the Navier-Stokes equations possess a unique maximal strong q-solution

$$v \in C(J^+, H^{s(q)}_{q,0,\sigma}) \cap C(\dot{J}^+, H^2_{q,0}) \cap C(\dot{J}^+, L_q) \ .$$

It coincides with the maximal very weak q-solution.

Proof. By (a) and Theorem 6.1(v) the maximal very weak q-solution is a strong q-solution. Since every strong q-solution belonging to $C(J, H^{s(q)}_{q,0,\sigma})$ is a very weak q-solution on J, the assertion is obvious. □

9. Integrability Properties

In this section we show that very weak solutions possess additional integrability properties. For this we need the following embedding result.

Lemma 9.1. *Let Ω be a standard domain. If $0 < \tau < 2$ and $2 \le p \le r < \infty$ then*

$$L_{p,\sigma} \hookrightarrow (H^{\tau-2}_{p,0,\sigma}, H^{\tau}_{p,0,\sigma})_{1-\tau/2,r} \ .$$

Proof. It follows from Remark 8.1 and Theorem V.1.5.4 of [5] that

$$[H^{s_0}_{p,0,\sigma}, H^{s_1}_{p,0,\sigma}]_\theta \doteq H^{s_\theta}_{p,0,\sigma} \tag{9.1}$$

for $-2 \le s_0 < s_1 \le 2$ and $0 < \theta < 1$, where $s_\theta := (1-\theta)s_0 + \theta s_1$. Hence

$$H^{\tau-2j}_{p,0,\sigma} \doteq [H^{-2}_{p,0,\sigma}, H^{2}_{p,0,\sigma}]_{(\tau-2j+2)/4} \ , \qquad j = 0,1 \ . \tag{9.2}$$

By the reiteration theorem and (9.1)

$$\begin{aligned} B^{\tau}_{p,r,0,\sigma} \doteq (L_{p,\sigma}, H^2_{p,0,\sigma})_{\tau/2,r} &\doteq \big([H^{-2}_{p,0,\sigma}, H^{2}_{p,0,\sigma}]_{1/2}, H^2_{p,0,\sigma}\big)_{\tau/2,r} \\ &= (H^{-2}_{p,0,\sigma}, H^{2}_{p,0,\sigma})_{(\tau+2)/4,r} \end{aligned} \tag{9.3}$$

and, similarly,

$$B^{\tau-2}_{p,r,0,\sigma} \doteq (H^{-2}_{p,0,\sigma}, H^{2}_{p,0,\sigma})_{\tau/4,r} \ . \tag{9.4}$$

Thus we infer from (9.2)–(9.4) that

$$(H^{-2}_{p,0,\sigma}, H^{2}_{p,0,\sigma})_{\theta_j,1} \hookrightarrow F_j \hookrightarrow (H^{-2}_{p,0,\sigma}, H^{2}_{p,0,\sigma})_{\theta_j,\infty} \ ,$$

where $F_j \in \{H^{\tau-2j}_{p,0,\sigma}, B^{\tau-2j}_{p,r,0,\sigma}\}$ and $\theta_j := (\tau - 2j + 2)/4$ for $j = 0,1$. Consequently, the reiteration theorem implies

$$(H^{\tau-2}_{p,0,\sigma}, H^{\tau}_{p,0,\sigma})_{1-\tau/2,r} \doteq (B^{\tau-2}_{p,r,0,\sigma}, B^{\tau}_{p,r,0,\sigma})_{1-\tau/2,r} \ . \tag{9.5}$$

Since $2 \le p \le r$, it is known (cf. Theorem 4.6.1 in [77]) that

$$H_p^s \hookrightarrow B_{p,r}^s , \quad B_{p',r'}^s \hookrightarrow H_{p'}^s , \qquad 0 \le s \le 2 .$$

From this and the characterization of the Stokes scales contained in Theorem 3.4 we deduce that

$$H_{p,0,\sigma}^\tau \overset{d}{\hookrightarrow} B_{p,r,0,\sigma}^\tau , \quad B_{p',r',0,\sigma}^{2-\tau} \overset{d}{\hookrightarrow} H_{p',0,\sigma}^{2-\tau} .$$

Thus, by duality,

$$H_{p,0,\sigma}^{\tau-2j} \hookrightarrow B_{p,r,0,\sigma}^{\tau-2j} , \qquad j = 0,1 .$$

Hence

$$L_{p,\sigma} \doteq [H_{p,0,\sigma}^{\tau-2}, H_{p,0,\sigma}^\tau]_{1-\tau/2} \hookrightarrow [B_{p,r,0,\sigma}^{\tau-2}, B_{p,r,0,\sigma}^\tau]_{1-\tau/2} . \tag{9.6}$$

Finally, Theorems V.1.5.4, V.1.5.9, and V.1.5.10 of [5] imply that

$$[B_{p,r,0,\sigma}^{\tau-2}, B_{p,r,0,\sigma}^\tau]_{1-\tau/2} \doteq (B_{p,r,0,\sigma}^{\tau-2}, B_{p,r,0,\sigma}^\tau)_{1-\tau/2,r} .$$

Now the assertion follows from (9.5) and (9.6). □

Remark 9.2. It should be noted that Lemma 9.1 remains valid if the hypothesis that Ω is a standard domain is replaced by the assumption that the Stokes operator is well-defined and condition (8.2) is satisfied. □

After these preparations we can prove the following integrability result.

Theorem 9.3. *Suppose that*
(i) *Ω is a standard domain;*
(ii) *$q \ge m$ and $m \ge 3$ if $q = m$;*
(iii) *$2 \le p \le r < \infty$ and $p \le q$;*
(iv) *$(v^0, f) \in (L_{p,\sigma} \cap L_{q,\sigma}) \times C(\mathbb{R}^+, L_p \cap L_q)$.*
Then the unique maximal very weak q-solution $v_q \in C(J_q^+, L_{q,\sigma})$ of the Navier-Stokes equations satisfies

$$v_q \in L_r\big((0,T), H_{p,0,\sigma}^{2/r}\big) , \qquad T \in \dot{J}_q^+ .$$

Proof. First we note that $v := v_q$ is well-defined, thanks to Theorem 8.4 and Remark 8.1.

Proposition 4.1(iii) and assumptions (ii) and (iii) imply

$$b \in \mathcal{L}(L_{q,\sigma}, H_{p,0,\sigma}^{2/r}; H_{p,0,\sigma}^{2/r-1-m/q}) . \tag{9.7}$$

Fix $T \in \dot{J}_q^+$. Then, by (9.7),

$$\|b^\natural(v - v^0, v)\|_{L_r((0,T_0),H_{p,0,\sigma}^{2/r-2})} \le c\, \|v - v^0\|_{C([0,T_0],L_q)}\, \|v\|_{L_r((0,T_0),H_{p,0,\sigma}^{2/r})} \tag{9.8}$$

and, given any $v^* \in \mathcal{D}_\sigma$,

$$\|b^\natural(v^0 - v^*, v)\|_{L_r((0,T_0),H^{2/r-2}_{p,0,\sigma})} \le c\,\|v^0 - v^*\|_{L_q}\,\|v\|_{L_r((0,T_0),H^{2/r}_{p,0,\sigma})} \tag{9.9}$$

for $0 < T_0 \le T$.

Denote by $\big[(\mathbb{E}_\alpha, \mathbb{A}_\alpha)\ ;\ \alpha \in \mathbb{R}\big]$ the Stokes scale generated by $(L_{p,\sigma}, S_p)$ and $[\cdot,\cdot]_\theta$, $0 < \theta < 1$. Set $U(t) := e^{-t\mathbb{A}_{-1+1/r}}$ for $t \ge 0$. Since (8.2) is satisfied it follows from Proposition V.1.5.5 and the maximal regularity theorem III.4.10.7 in [5] that

$$\|U \star g\|_{L_r((0,T_0),H^{2/r}_{p,0,\sigma})} \le c\,\|g\|_{L_r((0,T_0),H^{2/r-2}_{p,0,\sigma})}\ , \qquad 0 < T_0 \le T\ . \tag{9.10}$$

Thus (9.8) and (9.9) entail

$$\begin{aligned}\big\|U \star \big[b^\natural(v - v^0, v) + b^\natural(v^0 - v^*, v)\big]\big\|_{L_r((0,T_0),H^{2/r}_{p,0,\sigma})}&\\ \le \|v\|_{L_r((0,T_0),H^{2/r}_{p,0,\sigma})}\big/2\ ,&\end{aligned} \tag{9.11}$$

provided v^* is chosen sufficiently close to v^0 and $T_0 \in (0,T]$ is sufficiently small.

From Theorem 7.2, formula (9.7), Remark 8.5(a), and (6.37) we infer that

$$\begin{aligned}v &= a + U \star b^\natural(v,v)\\ &= a + U \star \big[b^\natural(v - v^0, v) + b^\natural(v^0 - v^*, v)\big] + U \star b^\natural(v^*, v)\ ,\end{aligned}$$

where $a := Uv^0 + U \star Pf$. Hence, thanks to (9.11),

$$\begin{aligned}&\|v\|_{L_r((0,T_0),H^{2/r}_{p,0,\sigma})}\\ &\qquad \le 2\,\|a\|_{L_r((0,T_0),H^{2/r}_{p,0,\sigma})} + 2\,\|U \star b^\natural(v^*, v)\|_{L_r((0,T_0),H^{2/r}_{p,0,\sigma})}\ .\end{aligned} \tag{9.12}$$

Since $\mathcal{D}_\sigma \subset L_{s,\sigma}$ for $1 < s < \infty$, it follows from Proposition 4.1 that

$$b(v^*, \cdot) \in \mathcal{L}(H^{2/r}_{p,0,\sigma}, H^{2/r-2+1/r}_{p,0,\sigma})\ .$$

Hence, setting

$$k(t) := t^{-1+2/r}\chi_{(0,T_0]}(t)\ , \quad w(t) := \|v(t)\|_{H^{2/r}_{p,0,\sigma}}\,\chi_{(0,T_0]}(t)$$

for $t \in \mathbb{R}$, where $\chi_{(0,T_0]}$ is the characteristic function of the interval $(0,T_0]$, we obtain

$$\|U \star b^\natural(v^*, v)\|_{L_r((0,T_0),H^{2/r}_{p,0,\sigma})} \le c\,\|k * w\|_{L_r(\mathbb{R},\mathbb{R})} \le cT_0^{1/2r}\,\|v\|_{L_r((0,T_0),H^{2/r}_{p,0,\sigma})}\ ,$$

where the last inequality is a consequence of Young's inequality for convolutions. Thus, by making T_0 smaller if necessary, we infer from (9.12) that

$$\|v\|_{L_r((0,T_0),H^{2/r}_{p,0,\sigma})} \le c\,\|a\|_{L_r((0,T_0),H^{2/r}_{p,0,\sigma})}\ .$$

Note that (9.10) and $f \in C(\mathbb{R}, L_p)$ imply $U \star Pf \in L_r\big((0,T_0), H^{2/r}_{p,0,\sigma}\big)$. From the maximal regularity theorem III.4.10.7 of [5] and from Lemma 9.1 we deduce that

$$\|Uv^0\|_{L_r((0,T_0),H^{2/r}_{p,0,\sigma})} \le c\,\|v^0\|_{(H^{2/r-2}_{p,0,\sigma},H^{2/r}_{p,0,\sigma})_{1-1/r,r}} \le c\,\|v^0\|_{L_p} < \infty \ .$$

Hence a belongs to $L_r\big((0,T_0), H^{2/r}_{p,0,\sigma}\big)$, consequently v also.

Now we replace v^0 by $v(s)$ for any $s \in (0,T]$. Since v is uniformly continuous on $[0,T]$ the preceding arguments guarantee the existence of $T_1 > 0$, being independent of $s \in [0,T]$, such that $\big(t \mapsto v(t+s)\big) \in L_r\big((0,T_1), H^{2/r}_{p,0,\sigma}\big)$. Now the assertion is obvious. □

Corollary 9.4. *Let hypotheses* (i), (ii), *and* (iv) *of Theorem* 9.3 *be satisfied. Then*

$$v_q \in L_r\big((0,T), L_s\big) \ , \qquad T \in J_q^+ \ ,$$

provided

$$\frac{2}{r} + \frac{m}{s} = \frac{m}{p} > \frac{2}{r} \ , \qquad r,s \ge p \ge 2 \ , \quad q \ge p \ . \tag{9.13}$$

Proof. Since condition (9.13) guarantees $H^{2/r}_{p,0,\sigma} \hookrightarrow L_{s,\sigma} \hookrightarrow L_s$, thanks to Theorem 3.10, the assertion follows. □

Remarks 9.5. (a) Suppose that Ω is a standard domain and $q \ge m$ with $m \ge 3$ if $q = m$. Also suppose that

$$(v^0, f) \in L_{q,\sigma} \times C(\mathbb{R}^+, L_q) \ .$$

Then

$$v_q \in L_r\big((0,T), L_s\big) \ , \qquad T \in J_q^+ \ , \tag{9.14}$$

provided

$$\frac{2}{r} + \frac{m}{s} = \frac{m}{q} \ , \qquad q \le r \le \infty \ , \quad q \le s < \infty \ , \tag{9.15}$$

and

$$\lim_{t\to 0} t^{1/r} \, \|v_q(t)\|_{L_s} = 0 \ . \tag{9.16}$$

Proof. Assertion (9.14) follows from Corollary 9.4 by setting $p = q$ and by observing that the case $(r,s) = (\infty, q)$ is covered by the continuity of v_q from J_q^+ into L_q.

By Corollary 3.11

$$L_{q,\sigma} \hookrightarrow n^{-\tau}_{s,0,\sigma} \ , \qquad \tau := m(1/q - 1/s) = 2/r \ .$$

Since $-\tau \ge -1 + m/s$ thanks to $q \ge m$, assertion (9.16) follows from (6.10) and Theorem 7.2. □

(b) Given the hypotheses of (a), it follows that $v_q \in L_r\big((0,T),L_s\big)$ for $T \in J_q^+$, where r and s satisfy Serrin's condition

$$2/r + m/s = 1$$

with $q \le s < \infty$ and $1 - m/q \le 2/r \le 1 + (2-m)/q$.

Proof. From (a) we know that $v_q \in L_{\overline{r}}\big((0,T),L_s\big)$ whenever $2/\overline{r} + m/s = m/q$. Since

$$L_{\overline{r}}\big((0,T),L_s\big) \hookrightarrow L_r\big((0,T),L_s\big)\ , \qquad 2/r := 2/\overline{r} + 1 - m/q\ ,$$

the assertion follows. □

(c) Theorem 9.3 and, hence, Corollary 9.4 and (a) remain valid if the hypothesis that Ω be a standard domain is replaced by the condition that (3.1)–(3.3) should be true and assumption (8.2) should hold for the particular p under consideration and for $p := q$ if $q = m$.

Proof. This follows from the proof of Theorem 9.3, Remark 9.2, and Theorem 8.4. □

It is easily verified that the regularity hypotheses for f in the preceding theorems can be relaxed. We leave this to the interested reader.

10. Weak Solutions

Suppose that

$$(v^0, f) \in L_{2,\sigma} \times L_{1,\mathrm{loc}}(\mathbb{R}^+, L_2)\ .$$

Recall that u is a **weak solution** on J of the Navier-Stokes equations (0.1), provided

$$u \in L_\infty(J, L_{2,\sigma}) \cap L_2(J, H_2^1)$$

and

$$\int_J \big\{-\langle\dot{\varphi}, u\rangle + \nu\langle\nabla\varphi, \nabla u\rangle + \big\langle\varphi, (u\cdot\nabla)u\big\rangle\big\}\,dt = \int_J \langle\varphi, f\rangle\,dt + \big\langle\varphi(0), v^0\big\rangle \qquad (10.1)$$

for all $\varphi \in \mathcal{D}(J^*, \mathcal{D}_\sigma)$. It is a **global** weak solution if it is a weak solution on $[0,T]$ for every $T > 0$.

Theorem 10.1. *Let* (3.1)–(3.3) *be satisfied and suppose that* $q \ge m$*, where* $m \ge 3$ *if* $q = m$*. Also suppose that* (8.4) *is satisfied if* $q = m$*, and that*

$$(v^0, f) \in (L_{2,\sigma} \cap L_{q,\sigma}) \times C(\mathbb{R}^+, L_2 \cap L_q)\ . \qquad (10.2)$$

Denote by $v_q \in C(J_q^+, L_{q,\sigma})$ the unique maximal very weak q-solution of (0.1). *Then v_q is a weak solution on $[0,T]$ for every $T \in \dot{J}_q^+$. Furthermore,*

$$v_q \in C(J_q^+, L_{2,\sigma}) \ . \tag{10.3}$$

Proof. Since S_2 is self-adjoint and positive semi-definite it follows that (8.2) is satisfied for $p = 2$. Thus, thanks to Remark 9.5(c), we infer from Theorem 9.3 with $p = r = 2$ that

$$v_q \in L_2\big((0,T), H^1_{2,0,\sigma}\big) \ , \qquad T \in \dot{J}_q^+ \ . \tag{10.4}$$

Proposition 4.1 entails

$$b \in \mathcal{L}(L_{q,\sigma}, H^1_{2,0,\sigma}; H^{-m/q}_{2,0,\sigma}) \ .$$

Hence $v := v_q \in C(J_q^+, L_{q,\sigma})$ and (10.4) imply

$$b^\natural(v,v) \in L_2\big((0,T), H^{-1}_{2,0,\sigma}\big) \ , \qquad T \in \dot{J}_q^+ \ .$$

Thus the maximal regularity theorem III.4.10.7 of [5] entails that, given $T \in \dot{J}_q^+$, the linear Cauchy problem

$$\dot{u} + \mathbb{A}_{-1/2} u = b^\natural(v,v) + Pf \ , \quad 0 < t \le T \ , \qquad u(0) = v^0 \tag{10.5}$$

possesses a unique solution

$$u \in L_2\big((0,T), H^1_{2,0,\sigma}\big) \cap W^1_2\big((0,T), H^{-1}_{2,0,\sigma}\big) \tag{10.6}$$

and it is given by

$$Uv^0 + U \star \big(b^\natural(v,v) + Pf\big) \ .$$

Hence $u = v$. By Theorem III.4.10.2 of [5] the intersection space in (10.6) embeddes continuously in

$$C\big([0,T], (H^{-1}_{2,0,\sigma}, H^1_{2,0,\sigma})_{1/2,2}\big) \ .$$

Since $\mathbb{A}_0$ is self-adjoint it follows from Theorem V.1.5.15 in [5] that

$$(H^{-1}_{2,0,\sigma}, H^1_{2,0,\sigma})_{1/2,2} \doteq [H^{-1}_{2,0,\sigma}, H^1_{2,0,\sigma}]_{1/2} \doteq L_{2,\sigma} \ .$$

This implies (10.3) and, thanks to (10.4),

$$u \in L_\infty\big((0,T), L_{2,\sigma}\big) \cap L_2\big((0,T), H^1_2\big) \ , \qquad T \in J_q^+ \ .$$

Fix $T \in \dot{J}_q^+$. Since v is a very weak q-solution on $[0,T]$, relation (7.4) holds, in particular, for each $w \in \mathcal{D}\big([0,T), \mathcal{D}_\sigma\big)$. For such a w it follows from (10.4) that

$$-\int_0^T \langle \Delta w, v \rangle \, dt = \int_0^T \langle \nabla w, \nabla v \rangle \, dt \ .$$

It is also not difficult to verify that

$$-\int_0^T \langle \nabla w, v \otimes v \rangle \, dt = \int_0^T \big\langle w, (v \cdot \nabla) v \big\rangle \, dt \ .$$

Hence v is a weak solution of (10.1) on $[0, T]$. $\square$

Remarks 10.2. **(a)** Let the hypotheses of Theorem 10.1 be satisfied. Then the strong energy equality

$$\|v_q(t)\|_{L_2}^2 + 2\nu \int_{t'}^t \|\nabla v_q(\tau)\|_{L_2}^2 \, d\tau = \|v_q(t')\|_{L_2}^2 + 2 \int_{t'}^t \big\langle v_q(\tau), f(\tau) \big\rangle \, d\tau$$

is valid for $t, t' \in J_q^+$ with $t' < t$.

Proof. By Hölder's inequality $(u, v, w) \mapsto \big\langle u, b(v, w) \big\rangle$ is a continuous trilinear form on $L_q \times H_2^1 \times H_2^1$, as is well-known. It is also well-known and easily seen that $\big\langle u, b(u, u) \big\rangle = 0$ for $u \in \mathcal{D}_\sigma$. Hence we infer from Theorem 10.1 by a density argument that $\big\langle v_q, b^\natural(v_q, v_q) \big\rangle = 0$ on $\dot{J}_q^+$. From the preceding proof we know that

$$v_q \in L_2\big((0, T), H_{2,0,\sigma}^1\big) \cap W_2^1\big((0, T), H_{2,0,\sigma}^{-1}\big)$$

and

$$\dot{v}_q + \mathbb{A}_{-1/2} v_q = b^\natural(v_q, v_q) + Pf \ , \qquad 0 < t \leq T \ , \tag{10.7}$$

for any $T \in J_q^+$. Hence we can apply $\langle v_q, \cdot \rangle$ to (10.7) and integrate from t' to t. Then it follows from Proposition 3.9 that

$$\int_{t'}^t \big\{ \langle v_q, \dot{v}_q \rangle + \nu \, \|\nabla v_q\|_{L_2}^2 \big\} \, d\tau = \int_{t'}^t \langle v_q, f \rangle \, d\tau$$

which implies the assertion (cf. Proposition V.2.4.7 in [5]). $\square$

(b) Suppose that Ω is a standard domain, $q \geq m$ with $m \geq 3$ if $q = m$, and $f = 0$. If $v^0 \in L_{2,\sigma} \cap L_{q,\sigma}$ then v_q is a weak solution belonging to a Serrin class on $[0, T]$ for every $T \in J_q^+$, as we know from Theorem 10.1 and Remark 9.5(b). Hence we can invoke regularity results due to Heywood [40], Sohr [72], and others (see Theorem 5.2 in Galdi's survey [27]) to obtain another proof for $v_q \in C^\infty(\dot{J}_q^+ \times \overline{\Omega}, \mathbb{R}^m)$. $\square$

Recall that, thanks to results due to Leray [57] and Hopf [41], it is known that, given any Ω (without any restriction on $\partial\Omega$), there exists a global weak solution v to the Navier-Stokes equations satisfying the energy inequality

$$\|v(t)\|_{L_2}^2 + 2\nu \int_0^t \|\nabla v(\tau)\|_{L_2}^2 \, d\tau \leq \|v^0\|_{L_2}^2 + 2 \int_0^t \big\langle v(\tau), f(\tau) \big\rangle \, d\tau \tag{10.8}$$

for $t > 0$. But neither uniqueness nor smoothness (if f and Ω are smooth) is known if $m \geq 3$. (We refer to Galdi [27] and Wiegner [85] for more information on the present state of the art of these problems, as well as for additional references.) A global weak solution satisfying (10.8) is called **Leray-Hopf weak solution**.

The results derived above have the following implications on uniqueness and regularity for weak solutions.

Theorem 10.3. *Let conditions* (3.1)–(3.3) *be satisfied and suppose that* $q \geq m$ *with* $m \geq 3$ *if* $q = m$. *Also suppose that* (8.4) *is true if* $q = m$, *that*

$$(v^0, f) \in (L_{2,\sigma} \cap L_{q,\sigma}) \times C(\mathbb{R}^+, L_2 \cap L_q) \ ,$$

and denote by $v_q \in C(J_q^+, L_{q,\sigma})$ *the unique maximal very weak* q*-solution of the Navier-Stokes equations. If* u *is any Leray-Hopf weak solution of* (0.1) *then* $u \supset v_q$.

Proof. Recall that assumption (8.2) holds for $p = 2$. Hence it follows from Remarks 9.5(b) and (c) that v_q is well-defined and

$$v_q \in L_r\big((0,T), L_s\big) \ , \qquad T \in \dot{J}_q^+ \ ,$$

where $r > q$ and $s > 2$ satisfy $2/r + m/s = 1$. Since v_q is also a weak solution by Theorem 10.1, the assertion follows from Serrin's uniqueness theorem [68, Theorem 6] (also see Galdi's proof [27, Theorem 4.2] which is based on Masuda's paper [60] and does not need Serrin's restriction $m \leq 4$). □

Remark 10.4. Suppose, in addition to the hypotheses of Theorem 10.3, that

$$f \in C^\rho(\mathbb{R}^+, L_q) + C(\mathbb{R}^+, H_q^\rho)$$

for some $\rho \in (0,1)$. Then

$$v_q \in C(\dot{J}_q^+, H_q^2) \cap C^1(\dot{J}_q^+, L_q)$$

by Remark 8.5(b). If, moreover, $f \in C^\infty\big((0,\infty) \times \overline{\Omega}, \mathbb{R}^m\big)$ and $\partial\Omega$ is uniformly regular of class C^∞ if $\Omega \neq \mathbb{R}^m$, then $v_q \in C^\infty(\dot{J}_q^+ \times \overline{\Omega}, \mathbb{R}^m)$ by Theorem 6.1(vi). Hence Theorem 10.3 entails that there exists exactly one Leray-Hopf weak solution on J_q^+ and that it enjoys the regularity properties just described. □

Of course, the technique for proving smoothness of weak solutions by identifying them (if possible) with strong ones is standard and has been used by many authors starting with Leray [57] (e.g., Galdi's survey [27]).

11. Strong Solutions

In this section we prove our main results concerning the strong solvability of the Navier-Stokes equations. For the sake of obtaining simple statements we impose conditions which are more restrictive than necessary. We leave it to the reader to weaken those assumptions by employing the more general theorems of the preceding sections.

Recall that
$$n^{-1}_{\infty,0,\sigma} := \varinjlim_{r} n^{-1+m/r}_{r,0,\sigma} \ .$$
Also note that the definition of a maximal strong solution given in the Introduction carries over to the case where $f \neq 0$.

Theorem 11.1. *Let the following assumptions be satisfied*:
(i) Ω *is a standard domain;*
(ii) *either* $1 \leq m/3 < q \leq m$ *or* $q > m \geq 2$*;*
(iii) $f \in C^{\rho}(\mathbb{R}^+, L_q \cap L_\infty)$ *for some* $\rho \in (0,1)$*;*
(iv) $v^0 \in H^{-1+m/q}_{q,0,\sigma}$.
Then there exists a unique maximal strong solution $v := v(\cdot, v^0, f)$ *of the Navier-Stokes equations satisfying*
$$\lim_{t\to 0} v(t) = v^0 \qquad \text{in } H^{-1+m/q}_q \tag{11.1}$$
and, if $q > m$,
$$\lim_{t\to 0} t^{(1-m/q)/2} v(t) = 0 \qquad \text{in } L_q \ . \tag{11.2}$$

Proof. Hypotheses (i) and (ii) and Remarks 3.1 and 8.1 imply that assumptions (6.1), (8.3), and (8.4) are satisfied.

(a) Suppose that $q > m$. Given $r \geq q$, denote by v_r the unique maximal solution of (6.2) in $L_{r,\sigma}$ satisfying (6.6) and (6.7) with $s := 0$ and q replaced by r. Since $L_q \cap L_\infty \hookrightarrow L_r$ we see, as in the beginning of the proof of Proposition 6.5, that v_r is well-defined. That proposition also guarantees that $v_{r_1} \supset v_{r_0}$ if $q \leq r_0 < r_1 < \infty$. Set $t^+ := \sup_{r\geq q} t^+_r$ and define $v \in C\big([0,t^+), n^{-1}_{\infty,0,\sigma}\big)$ by setting $v \,|\, [0,t^+_r) := v_r$ for $r \geq q$. Then v is well-defined and well-adapted to $n^{-1}_{\infty,0,\sigma}$. Moreover, Theorem 6.1(v) implies that $v \,|\, [0,t^+_r)$ is a strong r-solution on $(0,t^+_r)$. Thus we infer from (6.36) that v satisfies (0.2) on $(0,t^+)$. Hence v is a maximal strong solution. Since $v \,|\, [0,t^+_q) := v_q$ it follows from Theorem 6.1 that v satisfies (11.1) and (11.2). Lastly, v is uniquely determined since this is true for every v_r.

(b) Suppose that $1 \leq m/3 < q \leq m$. Fix $r > m$ such that $2m/r > -1 + m/q$, which is possible. Set $s := m(1/q - 1/r)$. Then Theorem 6.1 implies the existence of a unique maximal solution
$$v_q \in C(J^+_q, H^{-1+m/q}_{q,0,\sigma}) \cap C(\dot{J}^+_q, H^s_{q,0,\sigma}) \cap C^1(\dot{J}^+_q, H^{s-2}_{q,0,\sigma}) \tag{11.3}$$
satisfying
$$\lim_{t\to 0} t^{(1-m/r)/2} \, \|v_q(t)\|_{H^s_{q,0,\sigma}} = 0 \ . \tag{11.4}$$

Moreover,

$$v_q \in C(\dot{J}_q^+, H^2_{q,0,\sigma}) \cap C^1(\dot{J}_q^+, L_{q,\sigma}) \ . \tag{11.5}$$

By Remark 5.7(a), v_q is a mild solution on J_q^+ of (7.9) in $C(J_q^+, H^s_{q,0,\sigma})$, hence in $C(J_q^+, H^{-1+m/q}_{q,0,\sigma})$, since $s > -1 + m/q$. Thus v_q is a very weak q-solution on J_q^+ of the Navier-Stokes equations by Theorem 7.2. Hence, by Theorem 8.2, it is the only one in $C(J_q^+, H^{-1+m/q}_{q,0,\sigma})$.

Theorem 3.10 implies

$$H^{s-2j}_{q,0,\sigma} \hookrightarrow H^{-2j}_{r,0,\sigma} \ , \qquad j = 0, 1 \ .$$

Since, by (1.3) and Corollary 3.11,

$$H^{-1+m/q}_{q,0,\sigma} \hookrightarrow n^{-1+m/q}_{q,0,\sigma} \hookrightarrow n^{-1+m/r}_{r,0,\sigma} \ ,$$

we deduce from (11.3) and (11.4) that

$$v_q \in C(J_q^+, n^{-1+m/r}_{r,0,\sigma}) \cap C(\dot{J}_q^+, L_{r,\sigma}) \cap C^1(\dot{J}_q^+, H^{-2}_{r,0,\sigma}) \tag{11.6}$$

and

$$\lim_{t\to 0} t^{(1-m/r)/2} \, \|v_q(t)\|_{L_r} = 0 \ .$$

From this it follows (cf. the proof of Proposition 6.5) that $v_q \subset v_r$, where v_r is defined as in (a). Now the assertion is an obvious consequence of (11.5) and (a). □

Corollary 11.2. *Let the hypotheses of Theorem* 11.1 *be satisfied. If*

$$v^0 \in H^{(-1+m/q)_+}_{q,0,\sigma} \tag{11.7}$$

then the maximal strong solution $v := v(\cdot, v^0, f)$ *satisfies*

$$\lim_{t\to 0} v(t) = v^0 \qquad \text{in } H^{(-1+m/q)_+}_{q,0,\sigma} \tag{11.8}$$

and it is unique in this class. If, in addition,

$$v^0 \in F^s_{q,0,\sigma} \in \{\, H^s_{q,0,\sigma}, B^s_{q,r,0,\sigma}, n^s_{q,0,\sigma} \ ; \ 1 \le r < \infty \,\}$$

for some $s \in \big((-1 + m/q)_+, 2\big]$ *then*

$$\lim_{t\to 0} v(t) = v^0 \qquad \text{in } F^s_{q,0,\sigma} \ . \tag{11.9}$$

Proof. Using the notations of Theorem 11.1, we know that $v \supset v_q$. From (11.7) and Theorem 6.1 we infer that $v_q \in C(J_q^+, H^{(-1+m/q)_+}_{q,0,\sigma})$. Hence Theorems 7.2 and 8.2 imply the uniqueness of v_q in this class. Now the proof of Theorem 11.1 shows that v is the only maximal strong solution satisfying (11.8). If $v^0 \in F^s_{q,0,\sigma}$ then $v_q \in C(J_q^+, F^s_{q,0,\sigma})$ by Theorem 6.1. Hence (11.9) follows from $v \supset v_q$. $\square$

Remarks 11.3. Let the hypotheses of Theorem 11.1 be satisfied and denote by $v := v(\cdot, v^0, f)$ the unique maximal strong solution of the Navier-Stokes equations.

(a) If there exist $s \in (-1, 2]$ and $r > m/(s+1)$ with $r \geq q$ such that

$$\sup_{0<t<t_r^+} \|v(t)\|_{H^s_{r,0,\sigma}} < \infty$$

then $t^+ := t^+(v^0, f) = \infty$.

Proof. Using the notations of Theorem 11.1, it follows from Theorem 6.1(iv) that $t_r^+ = \infty$. Hence $t^+ = \sup_{p>q} t_p^+ = \infty$. $\square$

(b) Suppose that $r > m$ with $r \geq q$ and $-1 + m/r < s \leq 0$. Then

$$t^+ - t > 1 \wedge c\Big(\|v(t)\|_{H^s_{r,0,\sigma}} + \sup_{\tau>t}(\tau - t)^\alpha \, \|f(\tau)\|_{N^{-1-m/r}_{r,0}}\Big)^{-1/\beta}$$

for $0 \leq t < t^+$, where $\alpha := (-s + 1 - m/r)/2$ and $\beta := (s + 1 - m/r)/2$, and where $c > 0$ is independent of v and f.

Proof. Note that (11.6) and (6.30) imply that $v(t) \in L_{r,\sigma}$. Thus the assertion follows by applying Theorem 6.1(iii) (with $s := 0$, $r := s$, and $q := r$) to v_r on the interval $[t, \infty)$. $\square$

(c) Suppose that $v^0 \in H^{(-1+m/q)_+}_{q,0,\sigma}$. Then

$$v \in L_r\big((0,T), L_s\big) \ , \qquad 0 < T < t^+_{q\vee m} \ ,$$

if $(1 - m/q)_+ \leq 2/r \leq (1 - m/q)_+ + 2/m$ with $r \geq 2$ and $m \vee q \leq s < \infty$ such that $2/r + m/s = 1$.

Proof. If $q \geq m$ then this follows from $v \supset v_q$ and Remark 9.5(b) since $v^0 \in L_{q,\sigma}$. If $q < m$ then $v^0 \in H^{-1+m/q}_{q,0,\sigma} \hookrightarrow L_{m,\sigma}$ and the assertion follows once more from Remark 9.5(b) and $v \supset v_m$. $\square$

(d) Suppose that $v^0 \in L_{2,\sigma} \cap H^{(-1+m/q)_+}_{q,0,\sigma}$. Then v is a weak solution on $[0, T]$ for every $T < t^+$ and $v \in C\big([0, t^+), L_{2,\sigma}\big)$.

Proof. The embedding

$$H^{(-1+m/q)_+}_{q,0,\sigma} \hookrightarrow L_{q\vee m,\sigma} \text{ entails } v^0 \in L_{2,\sigma} \cap L_{q\vee m,\sigma} \ . \tag{11.10}$$

Hence Theorem 10.1 implies the assertion for all $T < t^+_{q\vee m}$. Fix $t_0 \in (0, t^+_{q\vee m})$ and note that

$$v(t_0) \in H^2_{q\vee m,0,\sigma} \hookrightarrow L_{p,\sigma}\ , \qquad p > q \vee m\ . \tag{11.11}$$

Thus, by applying Theorem 10.1 to the Navier-Stokes equations on the interval (t_0, ∞) with initial value $v(t_0)$, we find that $t \mapsto v(t + t_0)$ is a weak solution on $[0, T]$ for every positive $T < t^+_p - t_0$. Consequently, v is a weak solution on $[0, T]$ for every $T < t^+_p$ and every $p \geq q \vee m$, which, thanks to (10.3), proves the assertion. □

(e) Suppose that $f \in C^\infty\big((0,\infty) \times \overline{\Omega}, \mathbb{R}^m\big)$ and $\partial\Omega$ is uniformly regular of class C^∞. Then $v \in C^\infty\big((0,t^+) \times \overline{\Omega}, \mathbb{R}^m\big)$.

Proof. This is a consequence of Theorem 6.1(vi). □

Our next theorem shows that each global weak solution of (0.1) satisfying the energy inequality coincides on $[0, t^+)$ with v, provided v^0 is suitably regular.

Theorem 11.4. *Let hypotheses* (i)–(iii) *of Theorem* 11.1 *be satisfied and suppose that*

$$v^0 \in L_{2,\sigma} \cap H^{(-1+m/q)_+}_{q,0,\sigma}\ .$$

Denote by v the unique maximal strong solution of the Navier-Stokes equations satisfying

$$\lim_{t\to 0} v(t) = v^0 \qquad \text{in } H^{(-1+m/q)_+}_q\ .$$

If u is any Leray-Hopf weak solution then $u \supset v$.

Proof. Fix $T \in (0, t^+)$ and $t_0 \in (0, t^+_{q\vee m})$. Then (11.10) and Theorem 10.3 imply that u and v coincide on $[0, t_0]$. From the last part of Theorem 10.1, that is, from (10.3), and from (11.11) we infer that $v(t_0) \in L_{2,\sigma} \cap L_{p,\sigma}$ for any given $p > q \vee m$. Thus, fixing such a p with $t^+_p > T$, we find that u and v coincide on $[t_0, T]$ by applying Theorem 10.3 to the Navier-Stokes equations on $[t_0, \infty)$ with initial value $v(t_0)$. Since this is true for every $T \in (0, t^+)$ it follows that $u = v$ on $[0, t^+)$. □

Corollary 11.5. *Given the hypotheses of Theorem* 11.4*, suppose that v is global. Then there exists exactly one Leray-Hopf weak solution. It is smooth for $t > 0$ if f is smooth for $t > 0$.*

Next we prove that the unique maximal strong solution is global if Ω is bounded and the data are sufficiently small.

Theorem 11.6. *Let hypotheses* (ii)–(iv) *of Theorem* 11.1 *be satisfied and suppose that Ω is bounded. Denote by λ_0 the smallest eigenvalue of the Stokes operator and* *fix* $\omega \in [0, \lambda_0)$. *Then, given any* $r > m$ *with* $r \geq q$, *there exists a constant* R *such that* $v(\cdot, v^0, f)$ *exists globally and satisfies*

$$\sup_{t>0} t^{(1-m/r)/2} e^{\omega t} \, \|v(t, v^0, f)\|_{L_r} < \infty$$

whenever

$$\|v^0\|_{N^{-1+m/r}_{r,0,\sigma}} + \sup_{t>0} t^{1-m/r} e^{\omega t} \, \|f(t)\|_{N^{-1-m/r}_{r,0}} \leq R \, .$$

Proof. Thanks to (6.30) and $v \supset v_r$ the assertion follows by applying Theorem 6.2 (with $q := r$ and $s := 0$) to v_r. □

Acknowledgment I am indebted to G.P. Galdi and H. Sohr for interesting and useful discussions.

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