Nonhomogeneous Navier-Stokes equations with integrable low-regularity data

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Dedicated to O.A. Ladyzhenskaya on the occasion of her 80^{th} *birthday*

On the basis of semigroup and maximal regularity techniques, we derive optimal existence and uniqueness results for the Navier-Stokes equations in spaces of low regularity.

Introduction and main results

Throughout this paper, unless explicitly stated otherwise, Ω is a subdomain of $\mathbb{R}^3$ having a nonempty compact smooth boundary, Γ, lying locally on one side of Ω. We study solvability questions for the nonhomogeneous nonstationary incompressible Navier-Stokes equations

$$\left.\begin{aligned} \nabla \cdot v &= 0 && \\ \partial_t v + (v \cdot \nabla)v - \Delta v &= -\nabla \pi + f && \text{in } \Omega \times (0,\infty), \\ v &= g && \text{on } \Gamma \times (0,\infty), \\ v(\cdot,0) &= v^0 && \text{on } \Omega. \end{aligned}\right\} \tag{0.1}$$

The exterior force f, the boundary velocity g, and the initial velocity v^0 are the given data, and the velocity v and pressure π are the unknowns.

These equations have been studied by numerous authors under various hypotheses on the data, using many different concepts of solutions. So far, existence *and* uniqueness without restriction on the size of the data is known for local solutions only. The first such results are due to Kiselev and Ladyzhenskaya [**1**] (also see [**2**]). They have been considerably extended by Solonnikov [**3**] who obtained optimal results for the class of data considered by him. (We refer to the introduction in [**4**] for a survey of the literature on this subject.) In almost all publications devoted to solvability questions for (0.1) homogeneous boundary conditions, that is, $g = 0$, are considered only. Given sufficient regularity for g, this is justified since problem (0.1) can then be reduced to the case of homogeneous boundary conditions by choosing a suitable extension $\overline{g}$ of g and by studying the problem which has to be satisfied by $v - \overline{g}$. Noteworthy exceptions to this approach are contained in the papers by Grubb and Solonnikov [**5**], [**6**] and by Grubb [**7**]–[**10**]. These authors study nonhomogeneous Navier-Stokes equations, considering Neumann and other boundary conditions as well, by pseudodifferential techniques in anisotropic Sobolev spaces.

The solution spaces in all those papers are small enough to admit boundary traces, which seems necessary for a straightforward interpretation of (0.1). However, in [**11**] the case of less regular data has been investigated in a natural weak setting, requiring continuity with respect to the time variable. In this paper we are interested in situations in which f and g merely satisfy integrability conditions with respect to time and being, as well as v^0, possibly singular distributions in the space variables.

It should be mentioned that Navier-Stokes equations with nonhomogeneous data possessing little regularity are not only interesting for their own sake but also in relation to problems from control theory (e.g., [**12**], [**13**]), for example.

To give a precise meaning to a solution of (0.1) in a weak setting and to describe our main results we need some preparation.

We use standard notation and employ the following convention: If $\mathfrak{F}(\Omega, \mathbb{R}^3)$ is a vector space of $\mathbb{R}^3$-valued distributions on Ω then we simply denote it by $\mathfrak{F}$. If X is a subset of $\mathbb{R}^3$ different from Ω then we put $\mathfrak{F}(X) := \mathfrak{F}(X, \mathbb{R}^3)$. For example, $\mathcal{D}$, resp. $\mathcal{D}(\overline{\Omega})$, is the space of smooth $\mathbb{R}^3$-valued functions having compact support in Ω, resp. $\overline{\Omega}$, and $W_q^s(\Gamma)$ is the Sobolev-Slobodeckii space of $\mathbb{R}^3$-valued distributions on Γ.

We always assume that $q, r \in (1, \infty)$. Then H_q^s and $B_{q,\rho}^s$, $1 \le \rho \le \infty$, are the usual Bessel potential and Besov spaces, respectively, (of $\mathbb{R}^3$-valued distributions on Ω) for $s \in \mathbb{R}$. (See [**4**] for more detailed explanations.) We set

$$\langle v, w \rangle := \int_\Omega v \cdot w \, dx, \qquad v, w \in \mathcal{D}(\overline{\Omega}),$$

and, denoting by $d\sigma$ the volume measure of Γ,

$$\langle v, w \rangle_\Gamma := \int_\Gamma v \cdot w \, d\sigma, \qquad v, w \in C(\Gamma).$$

We also use $\langle\cdot,\cdot\rangle$ to denote the standard duality pairings between various spaces of (scalar- and vector-valued) distributions without fearing confusion. Similar conventions hold for $\langle\cdot,\cdot\rangle_\Gamma$. We write ∂_ν for the derivative on Γ with respect to the outer unit normal ν, denote by γ the trace, and by γ_ν the normal trace operator, that is, $\gamma_\nu u = \nu\cdot\gamma u$.

We set

$$\boldsymbol{H}_q^s := \begin{cases} \{\, u\in H_q^s \;;\; \gamma u = 0\,\}, & 1/q < s \le 2,\\ \{\, u\in H_q^{1/q}(\mathbb{R}^3) \;;\; \operatorname{supp}(u)\subset\overline{\Omega}\,\}, & s = 1/q,\\ \ H_q^s, & 0\le s<1/q,\\ \ (\boldsymbol{H}_{q'}^{-s})', & -2\le s<0, \end{cases} \tag{0.2}$$

where the dual space is determined by means of the duality pairing $\langle\cdot,\cdot\rangle$. It follows (cf. [**14**, Theorems 4.7.1(a) and 4.8.1]) that

$$\boldsymbol{H}_q^s = H_q^s, \qquad -2+1/q < s < 1/q. \tag{0.3}$$

(In [**14**] the case of a bounded Ω is considered only. However, it is easy to verify that all results in that book cited here and below continue to hold if it is only assumed that Γ is compact.) In [**11**, Remark 1.5] it is shown that

$$\mathcal{D}_0(\overline{\Omega}) := \{\, \varphi\in\mathcal{D}(\overline{\Omega},\mathbb{R}) \;;\; \gamma\varphi = 0\,\}$$

is dense in $\boldsymbol{H}_q^s$ for $|s|\le 2$.

We also set $\mathcal{D}_\sigma := \{\, u\in\mathcal{D} \;;\; \nabla\cdot u = 0\,\}$ and denote by $\mathbb{H}_q$ the closure of $\mathcal{D}_\sigma$ in L_q. Recall (e.g., [**2**], [**15**]–[**18**]) that

$$\mathbb{H}_q = \{\, u\in L_q \;;\; \nabla\cdot u = 0,\ \gamma_\nu u = 0\,\}.$$

We put

$$\mathbb{H}_q^s := \begin{cases} \boldsymbol{H}_q^s\cap\mathbb{H}_q, & 0\le s\le 2,\\ (\mathbb{H}_{q'}^{-s})', & -2\le s<0, \end{cases} \tag{0.4}$$

the dual spaces being determined by means of the duality pairing $\langle\cdot,\cdot\rangle_\sigma$, obtained by restricting $\langle\cdot,\cdot\rangle$ to $\mathbb{H}_{q'}\times\mathbb{H}_q$.

Similarly,

$$\boldsymbol{B}_{q,r}^s := \begin{cases} \{\, u\in B_{q,r}^s \;;\; \gamma u = 0\,\}, & 1/q < s \le 2,\\ \{\, u\in B_{q,r}^{1/q}(\mathbb{R}^3) \;;\; \operatorname{supp}(u)\subset\overline{\Omega}\,\}, & s = 1/q,\\ \ B_{q,r}^s, & 0\le s<1/q,\\ \ (\boldsymbol{B}_{q',r'}^{-s})', & -2\le s<0, \end{cases} \tag{0.5}$$

the dual space being determined by means of $\langle\cdot,\cdot\rangle$, and

$$\mathbb{B}^s_{q,r} := \begin{cases} \boldsymbol{B}^s_{q,r} \cap \mathbb{H}_q\,, & 0 < s \leq 2, \\ \text{the closure of } \mathcal{D}_\sigma \text{ in } B^0_{q,r}, & s = 0, \\ (\mathbb{B}^{-s}_{q',r'})', & -2 \leq s < 0, \end{cases} \tag{0.6}$$

where now the dual spaces are determined by the pairing $\langle\cdot,\cdot\rangle_\sigma$. Similarly as for the Bessel potential spaces,

$$\boldsymbol{B}^s_{q,r} = B^s_{q,r}, \qquad -2 + 1/q < s < 1/q. \tag{0.7}$$

In general (see (3.2), (3.7), and Proposition 3.4),

$$E^s_q \hookrightarrow E^t_q, \qquad E_q \in \{\, \boldsymbol{H}_q, \mathbb{H}_q, \boldsymbol{B}_{q,r}, \mathbb{B}_{q,r} \ ;\ 1 < r < \infty \,\}, \quad s > t.$$

Thus it follows from (0.3)–(0.5) and (0.7) that $\mathbb{H}^s_q$, resp. $\mathbb{B}^s_{q,r}$, is the closure of $\mathbb{H}_q$ in H^s_q, resp. $B^s_{q,r}$, for $-2 + 1/q < s \leq 0$.

Lastly,

$$\mathbb{G}_q := \big\{\, v \in L_q \ ;\ v = \nabla\pi,\ \pi \in L_{q,\mathrm{loc}}(\overline{\Omega}, \mathbb{R}) \,\big\}$$

and

$$\mathbb{G}^s_q := \boldsymbol{H}^s_q \cap \mathbb{G}_q\,, \qquad 0 \leq s \leq 2,$$

whereas

$$\mathbb{G}^s_q \text{ is the closure of } \mathbb{G}_q \text{ in } \boldsymbol{H}^s_q \text{ for } -2 \leq s < 0.$$

We put

$$s(r) := s(q,r) := \begin{cases} 1/r & \text{if } 1/r + 3/q \leq 1, \\ 2/r + 3/q - 1 & \text{otherwise.} \end{cases}$$

Then we suppose that

$$\left.\begin{array}{l} \bullet\ \ 2 \leq q < \infty,\ 1 < r < \infty,\ 2/r + 3/q \leq 3; \\ \bullet\ \ (f,g) \in L_{r,\mathrm{loc}}\big(\mathbb{R}^+, H^{s(r)-2}_q \times W^{s(r)-1/q}_q(\Gamma)\big) \\ \quad\ \text{with } g = 0 \text{ if } s(r) \geq 1/q; \\ \bullet\ \ v^0 \in \mathbb{B}^{s(r)-2/r}_{q,r}. \end{array}\right\} \tag{0.8}$$

We also assume that

$$\left.\begin{array}{l} \text{if } s(r) \leq 1/q \text{ then } f = f_0 + f_1 \text{ with} \\ \bullet\ \ \mathrm{dist}\big(\mathrm{supp}(f_0), \Gamma \times \mathbb{R}^+\big) > 0; \\ \bullet\ \ f_1 \in L_{r,\mathrm{loc}}(\mathbb{R}^+, H^{s_1}_q) \text{ for some } s_1 \in (-2 + 1/q, 0]. \end{array}\right\} \tag{0.9}$$

The assumptions on f guarantee that $\langle f, \varphi\rangle$, a priori making sense for φ belonging to $\mathcal{D}(\mathbb{R}^+, \mathcal{D})$ only, is well-defined for $\varphi \in \mathcal{D}\big(\mathbb{R}^+, \mathcal{D}_0(\overline{\Omega})\big)$. Indeed, this is clear for $\langle f_1, \varphi\rangle$ since $\mathcal{D} \subset \mathcal{D}_0(\overline{\Omega})$ and $\mathcal{D}$ is dense in $\boldsymbol{H}^{2-1/q}_{q'} = (H^{-2+1/q}_q)'$ (cf.

[**14**, Theorem 4.3.2.1]). Thanks to the presupposed condition on the support of the distribution

$$f_0 \in L_{r,\mathrm{loc}}(\mathbb{R}^+, H_q^{s(r)-2}) \subset \mathcal{D}(\mathbb{R}^4, \mathbb{R}^3),$$

this fact is also obvious for $\langle f_0, \varphi \rangle$.

For simplicity, we assume that $g = 0$ if $s(r) \geq 1/q$. If $s(r) > 1/q$ then the trace operator γ is well-defined on $\mathbb{H}_q^{s(r)}$ and, given sufficient smoothness with respect to the t-variable and suitable compatibility conditions, the case of nonhomogeneous boundary data can be reduced to the homogeneous problem as described above (cf. [**13**]).

Let J be a subinterval of $\mathbb{R}^+$ containing 0 such that $\dot{J} := J \backslash \{0\} \neq \emptyset$, and set $J^* := J \backslash \{\sup J\}$. The pair (v, w) is said to be a (*very weak*) $L_r(H_q^{s(r)})$-*solution* of the Navier-Stokes equations (0.1) on J if

$$(v, w) \in L_{r,\mathrm{loc}}(J^*, \mathbb{H}_q^{s(r)} \times \mathbb{G}_q^{s(r)-2}) \tag{0.10}$$

and

$$\begin{aligned} &\int_J \big\{ \langle \partial_t \varphi + \Delta \varphi, v \rangle + \langle \nabla \varphi, v \otimes v \rangle \big\}\, dt \\ &\quad = \int_J \big\{ \langle w, \varphi \rangle - \langle f, \varphi \rangle + \langle g, \partial_\nu \varphi \rangle_\Gamma \big\}\, dt - \langle v^0, \varphi(0) \rangle \end{aligned} \tag{0.11}$$

for all $\varphi \in \mathcal{D}\big(J^*, \mathcal{D}_0(\overline{\Omega})\big)$ and if all integrals and duality pairings occurring in (0.11) are well-defined.

Clearly, (0.11) is *formally* obtained from the second differential equation in (0.1), the momentum equation, by multiplying it by φ, integrating by parts, using Green's formula, the boundary and initial data, and setting $w := \nabla \pi$.

By admitting in (0.11) standard test functions φ only, that is, assuming that $\varphi \in \mathcal{D}(J^*, \mathcal{D}) = \mathcal{D}(\Omega \times J^*)$, it follows that a very weak solution is a distributional solution of the momentum equation.

If $s(r) < 1/q$ then it has been shown in [**11**] that

$$\boldsymbol{H}_q^{s(r)-2} \cong H_q^{s(r)-2} \times W_q^{s(r)-1/q}(\Gamma). \tag{0.12}$$

Thus, in this case, $\boldsymbol{H}_q^{s(r)-2}$ is not a space of distributions on Ω but contains also distributions being supported on Γ. This explains why there is no compatibility condition for g guaranteeing that g is a tangential vector field. Indeed, a possible nontrivial normal component of g is compensated by the 'boundary part' of the (generalized) pressure gradient w.

In Section 5 we see that (0.9) can be replaced by $f \in L_{r,\mathrm{loc}}(\mathbb{R}^+, \boldsymbol{H}_q^{s(r)-2})$. However, due to (0.12), in this case f 'contains a part on Γ' which should be covered by g. For this reason it is much more natural to assume that f takes values in $H_q^{s(r)-2}$. Since, in this case, $\langle f, \varphi \rangle$ is not defined for $\varphi \in \mathcal{D}\big(J^*, \mathcal{D}_0(\overline{\Omega})\big)$, in general, we decided to impose condition (0.9) in this introduction. The case of an arbitrary $f \in L_{r,\mathrm{loc}}(\mathbb{R}^+, H_q^{s(r)-2})$ with $s(r) < 1/q$ is considered in Remark 5.1(a).

Set

$$\mathcal{D}_{0,\sigma} := \{\, u \in \mathcal{D}_0(\overline{\Omega}) \ ;\ \nabla \cdot u = 0 \,\}.$$

Then v is said to be a (*very weak*) $L_r(\mathbb{H}_q^{s(r)})$*-solution* of the Navier-Stokes equations (0.1) if

$$v \in L_{r,\mathrm{loc}}(J^*, \mathbb{H}_q^{s(r)})$$

and

$$\begin{aligned}\int_J \bigl\{ \langle \partial_t \varphi + \Delta\varphi, v\rangle + \langle \nabla\varphi, v \otimes v\rangle \bigr\}\, dt \\ = \int_J \bigl\{ -\langle f, \varphi\rangle + \langle g, \partial_\nu \varphi\rangle_\Gamma \bigr\}\, dt - \bigl\langle v^0, \varphi(0)\bigr\rangle\end{aligned} \tag{0.13}$$

for all $\varphi \in \mathcal{D}(J^*, \mathcal{D}_{0,\sigma})$.

If $w \in L_{r,\mathrm{loc}}(J^*, \mathbb{G}_q^{s(r)-2})$ then $\langle \varphi, w\rangle = 0$ for each $\varphi \in \mathcal{D}(J^*, \mathcal{D}_{0,\sigma})$ (see [**11**, Theorem 2.2]). Thus, if (v, w) is an $L_r(H_q^{s(r)})$-solution on J of (0.1) then v is an $L_r(\mathbb{H}_q^{s(r)})$-solution on J of the Navier-Stokes equations. It follows from the considerations in Section 5 that the converse is also true: If v is an $L_r(\mathbb{H}_q^{s(r)})$-solution on J of (0.1) then there exists a unique $w \in L_{r,\mathrm{loc}}(J^*, \mathbb{G}_q^{s(r)-2})$ such that (v, w) is an $L_r(H_q^{s(r)})$-solution on J of the Navier-Stokes equations. Consequently, similarly as in the well-known regular case, it suffices to study the Navier-Stokes equations in its 'reduced' very weak setting (0.13) in which the pressure gradient is eliminated.

Clearly, each one of the above solutions is maximal if there does not exist another solution of the same type being a proper extension of it.

Now we can formulate the main result of this paper whose proof is given in Section 5. In the remainder of this section we write $\nabla\pi$ for w without fearing confusion.

Theorem. *Let assumptions* (0.8) *and* (0.9) *be satisfied. Then the Navier-Stokes equations* (0.1) *possess a unique maximal* $L_r(H_q^{s(r)})$*-solution,* $(v, \nabla\pi)$*. The interval of existence,* $J^+ := \mathrm{dom}(v, \nabla\pi)$*, is open in* $\mathbb{R}^+$*. Moreover,*

$$v \in C(J^+, \mathbb{B}_{q,r}^{s(r)-2/r}) \cap L_{r,\mathrm{loc}}(J^+, \mathbb{H}_q^{s(r)})$$

and

$$(\dot v, \nabla\pi) \in L_{r,\mathrm{loc}}(J^+, \mathbb{H}_q^{s(r)-2} \times \mathbb{G}_q^{s(r)-2}).$$

If $J^+ \neq \mathbb{R}^+$ *then* v *is not uniformly continuous on* J^+*.*

If Ω *is bounded and the norm of* $\bigl(v^0, (f, g)\bigr)$ *in*

$$\mathbb{B}_{q,r}^{s(r)-2/r} \times L_r\bigl(\mathbb{R}^+, H_q^{s(r)-2} \times W_q^{s(r)-1/q}(\Gamma)\bigr) \tag{0.14}$$

is sufficiently small then $(v, \nabla\pi)$ *is a global solution, that is,* $J^+ = \mathbb{R}^+$*.*

Remarks. (a) Of particular interest is the 'classical' case $q = r = 2$. Here, setting $d(x) := \mathrm{dist}(x, \Gamma) \wedge 1$,

$$\boldsymbol{B}^{1/2}_{2,2} \doteq \boldsymbol{H}^{1/2} := \{\, u \in H^{1/2} \;;\; d^{-1/2} u \in L_2 \,\} \tag{0.15}$$

with the obvious norm, where, of course, $H^s := H^s_2$. Here and below, $\doteq$ means 'equal, except for equivalent norms'. Thus the Navier-Stokes equations with homogeneous boundary conditions, that is, with $g = 0$, possess a unique maximal $L_2(H^{3/2})$-solution, $(v, \nabla\pi)$, provided

$$(v^0, f) \in \mathbb{H}^{1/2} \times L_{2,\mathrm{loc}}(\mathbb{R}^+, H^{-1/2}),$$

where $\mathbb{H}^{1/2} = \boldsymbol{H}^{1/2} \cap \mathbb{H}_2$. Moreover,

$$v \in C(J^+, \mathbb{H}^{1/2}) \cap L_{2,\mathrm{loc}}(J^+, H^{3/2})$$

and

$$(\dot v, \nabla\pi) \in L_{2,\mathrm{loc}}(J^+, \mathbb{H}^{-1/2} \times \mathbb{G}_2^{-1/2}).$$

Proof. First recall that $B^s_{2,2} \doteq H^s$ for $s \in \mathbb{R}$ (cf. [**14**, Theorem 4.6.1(b)]). Thus (0.15) is a consequence of [**14**, Remark 4.3.2.2]. Now the assertion follows from the Theorem. □

(b) Another 'classical' case of interest occurs for $q = 3$. Here the Theorem guarantees the existence of a unique maximal $L_2(H^1_3)$-solution, $(v, \nabla\pi)$, provided $g = 0$ and

$$(v^0, f) \in \mathbb{B}^0_{3,2} \times L_{2,\mathrm{loc}}(\mathbb{R}^+, H_3^{-1}).$$

Furthermore,

$$v \in C(J_2^+, \mathbb{B}^0_{3,2}) \times L_{2,\mathrm{loc}}(J_2^+, H_3^1)$$

and

$$(\dot v, \nabla\pi) \in L_{2,\mathrm{loc}}(J_2^+, \mathbb{H}^{-1} \times \mathbb{G}_3^{-1}).$$

If, instead,

$$(v^0, f) \in \mathbb{B}^0_{3,3} \times L_{3,\mathrm{loc}}(\mathbb{R}^+, H_3^{-4/3})$$

then

$$v \in C(J_3^+, \mathbb{B}^0_{3,3}) \times L_{3,\mathrm{loc}}(J_3^+, H_3^{2/3})$$

and

$$(\dot v, \nabla\pi) \in L_{3,\mathrm{loc}}(J_3^+, \mathbb{H}_3^{-4/3} \times \mathbb{G}_3^{-4/3}).$$

Observe that $\mathbb{B}^0_{3,2} \overset{d}{\hookrightarrow} \mathbb{H}_3 \overset{d}{\hookrightarrow} \mathbb{B}^0_{3,3}$.

Proof. This follows from the Theorem and $B^0_{3,2} \hookrightarrow L_3 \hookrightarrow B^0_{3,3}$. □

(c) For $1 < p < \infty$ and $1 \leq \rho \leq \infty$ we denote by $L_{p,\rho}$ the Lorentz spaces (e.g., [**14**], [**19**]). We also write $\mathbb{L}_{p,\rho}$ for the closure of $\mathcal{D}_\sigma$ in $L_{p,\rho}$. Then

$$\mathbb{H}_3 \overset{d}{\hookrightarrow} \mathbb{L}_{3,r} \overset{d}{\hookrightarrow} \mathbb{B}_{q,r}^{-1+3/q}, \qquad r \geq q > 3, \tag{0.16}$$

and

$$\mathbb{L}_p \overset{d}{\hookrightarrow} \mathbb{L}_{p,\infty} \overset{d}{\hookrightarrow} \mathbb{B}_{q,r}^{-1+3/q}, \qquad q > p > 3. \tag{0.17}$$

Hence, if $1/r + 3/q \geq 1$, that is, $s(r) = 2/r + 3/q - 1$, then the Theorem guarantees the unique solvability of the Navier-Stokes equations for initial values in Lorentz spaces, provided the additional restrictions for q and r in (0.16), and for q in (0.17), respectively, are satisfied.

PROOF. Suppose that $1 < p < q$ and fix $p_j \in (1, \infty)$ and $\theta \in (0, 1)$ such that

$$1/p = (1-\theta)/p_0 + \theta/p_1, \qquad s_j := 3(1/q - 1/p_j) < 0, \quad j = 0, 1.$$

Then

$$L_{p_j} \hookrightarrow B^0_{p_j,\infty} \hookrightarrow B^{s_j}_{q,\infty}, \qquad j = 0, 1,$$

and, consequently,

$$L_{p,\rho} \doteq (L_{p_0}, L_{p_1})_{\theta,\rho} \hookrightarrow (B^{s_0}_{q,1}, B^{s_1}_{q,1})_{\theta,\rho} \doteq B^{3(1/q-1/p)}_{q,\rho} \tag{0.18}$$

for $1 \leq \rho \leq \infty$, where the injection is dense if $\rho < \infty$. Recall that

$$L_{p,1} \overset{d}{\hookrightarrow} L_{p,\rho_0} \overset{d}{\hookrightarrow} L_{p,\rho_1} \hookrightarrow L_{p,\infty}, \qquad 1 < \rho_0 < \rho_1 < \infty,$$

and $L_p \doteq L_{p,p}$. Hence we infer from (0.18) that

$$L_3 \overset{d}{\hookrightarrow} L_{q,r} \overset{d}{\hookrightarrow} B^{-1+3/q}_{q,r}, \qquad r \geq q > 3,$$

as well as

$$L_p \hookrightarrow L_{p,\infty} \hookrightarrow B^{3(1/q-1/p)}_{q,\infty} \hookrightarrow B^{-1+3/q}_{q,r}, \qquad q > p > 3,$$

thanks to $3(1/p - 1/q) > -1 + 3/q$ and well-known embedding properties of Besov spaces. Now the assertion is obvious. $\square$

(d) Suppose that $\Omega = \mathbb{R}^3$ and neglect any reference to Γ in the definition of spaces and solutions given above. In particular, $\boldsymbol{H}^s_q = H^s_q$ and $\boldsymbol{B}^s_{q,r} = B^s_{q,r}$ for all s, and $\mathbb{H}^s_q = \{\, u \in H^s_q \ ;\ \nabla \cdot u = 0 \,\}$ as well as $\mathbb{B}^s_{q,r} = \{\, u \in B^s_{q,r} \ ;\ \nabla \cdot u = 0 \,\}$. Then the Cauchy problem for the Navier-Stokes equations

$$\left.\begin{aligned} \nabla \cdot v &= 0 \\ \partial_t v + (v \cdot \nabla) v - \Delta v &= -\nabla \pi + f \end{aligned} \quad \text{in } \mathbb{R}^3 \times (0,\infty), \atop v(\cdot, 0) = v^0 \qquad \text{on } \mathbb{R}^3, \right\} \tag{0.19}$$

possesses a unique (very weak) $L_r(H^{s(r)}_q)$-solution, $(v, \nabla\pi)$, provided

$$2 \leq q < \infty, \quad 1 < r < \infty, \quad 2/r + 3/q \leq 3$$

and

$$(v^0, f) \in B_{q,r}^{s(r)-2/r} \times L_{r,\mathrm{loc}}(\mathbb{R}^+, H_q^{s(r)-2}), \quad \nabla \cdot v^0 = 0.$$

The maximal interval of existence, J^+, is open in $\mathbb{R}^+$. Moreover,

$$v \in C(J^+, B_{q,r}^{s(r)-2/r}) \cap L_{r,\mathrm{loc}}(J^+, H_q^{s(r)})$$

and

$$(\dot{v}, \nabla\pi) \in L_{r,\mathrm{loc}}(J^+, H_q^{s(r)-2} \times H_q^{s(r)-2}).$$

If $J \neq \mathbb{R}^+$ then v is not uniformly continuous.

In particular, given

$$(v^0, f) \in H^{1/2} \times L_{2,\mathrm{loc}}(\mathbb{R}, H^{-1/2}), \quad \nabla \cdot v^0 = 0,$$

problem (0.19) has a unique maximal $L_2(H^{3/2})$-solution, $(v, \nabla\pi)$, and

$$v \in C(J^+, H^{1/2}) \cap L_{2,\mathrm{loc}}(J^+, H^{3/2})$$

as well as

$$(\dot{v}, \nabla\pi) \in L_{2,\mathrm{loc}}(J^+, H^{-1/2} \times H^{-1/2}).$$

Proof. This follows by modifying the arguments of this paper in the obvious way omitting all references to Γ. □

(e) Let v be the unique maximal $L_r(\mathbb{H}_q^{s(r)})$-solution of (0.1). Then v belongs to $L_{r_0,\mathrm{loc}}(J^+, L_{q_0})$ for all $q_0 \geq q$ and $r_0 \geq r$ satisfying $2/r_0 + 3/q_0 = 1$.

Note that this means that v satisfies a Serrin-type condition. Consequently, v (and hence $\nabla\pi$) is smooth, provided f is smooth and $g = 0$ (or g is smooth and satisfies appropriate compatibility conditions).

Proof. The first assertion follows from Theorem 3.3. For details concerning the regularity statements we refer to [**4**, Sections 9–11]. □

(f) By employing Remarks 2.3 it is easy to give blow-up estimates for the maximal $L_r(\mathbb{H}_q^{s(r)})$-solution, v, provided v^0 and (f, g) possess appropriate additional regularity. □

Navier-Stokes equations with low-regularity data have been thoroughly investigated in [**4**]. In contrast to the present paper, in that one it is assumed that f is continuous with respect to the time-variable (except possibly for $t = 0$), and that v^0 belongs to the little Nikolskii space $n_{q,0,\sigma}^{-1+3/q}$ of solenoidal vector fields (see [**4**] for a precise definition of the latter space and observe that in that paper $\mathbb{H}_q^s$ is denoted by $H_{q,0,\sigma}^s$). The results obtained there are optimal in that setting. Thus it is interesting to compare the Theorem with the results in [**4**] in a case where both theories apply. To reduce technicalities we restrict ourselves to the 'classical' case $q = r = 2$.

Hence suppose that $g = 0$ and

$$(v^0, f) \in \mathbb{H}^{1/2} \times C(\mathbb{R}, H^{-1/2}).$$

Then we know from Remark (a) and the observations after (0.13) that the Navier-Stokes equations (with $g = 0$) have a unique maximal $L_2(\mathbb{H}^{3/2})$-solution, v, and

$$v \in C(J^+, \mathbb{H}^{1/2}) \cap L_{2,\mathrm{loc}}(J^+, \mathbb{H}^{3/2}) \cap W^1_{2,\mathrm{loc}}(J^+, \mathbb{H}^{-1/2}).$$

From this and Theorem 3.3 it follows that $v \in L_{4,\mathrm{loc}}(J^+, \mathbb{H}^1)$. On the other hand, we infer from [**4**, Theorems 6.1, 7.2, and 8.2] that there exists a unique maximal

$$u \in C(I^+, \mathbb{H}^{1/2}) \cap C(\dot{I}^+, \mathbb{H}^1) \cap C^1(\dot{I}^+, \mathbb{H}^{-1})$$

satisfying

$$\lim_{t \to 0} t^{1/4} \, \|u(t)\|_{\mathbb{H}^1} = 0$$

and (0.13). Consequently, $u \in L_{\rho,\mathrm{loc}}(I^+, \mathbb{H}^1)$ for $1 \le \rho < 4$. The results in [**4**] also imply that $u \supset v$. Thus we see that, even in the case where f is continuous in the t-variable, the Theorem implies better integrability properties of the solution than the earlier approach, which is not applicable if f is not continuous in t.

If $g = 0$ then the Theorem complements, similarly as above, the results in [**11**], where continuity of (f, g) with respect to t is assumed.

As mentioned in the beginning, nonhomogeneous Navier-Stokes equations with integrable low-regularity data have been investigated by Grubb and Solonnikov and, in particular, by Grubb, in anisotropic Bessel potential and Besov spaces. In her most recent paper, Grubb [**10**] concentrates on Dirichlet boundary conditions. Then, given suitable regularity hypotheses on (f, g), she proves a local solvability result, provided $v^0 \in \mathbb{B}^{\sigma}_{q,q}$, where $\sigma > 0$ and $\sigma > -1 + 3/q$ (we refer to that paper for precise statements and definitions of the anisotropic spaces). Observe that we admit initial values in $\mathbb{B}^{s(r)-2/r}_{q,r}$, where

$$s(r) - 2/r = -1 + 3/q \qquad \text{if} \quad 1/r + 3/q \ge 1.$$

Also note that $s(r) - 2/r < 0$ if $q > 3$. The main advantage of our approach, leading to the better results, lies in the fact that we can separate space and time regularity, whereas in the work of Grubb and Solonnikov $r = q$.

Similarly as in our previous papers on Navier-Stokes equations, we employ a semigroup approach. This has been done by many authors, almost always by taking into consideration the singular behavior of the Stokes semigroup at $t = 0$, using a device which can be traced back to the work of Kato [**20**]. That approach has become standard in this field (see the references in [**4**]).

In contrast to this, the proofs in this paper are different. Namely, we employ the maximal regularity property in Lebesgue spaces of the Stokes semigroup. This allows us to develop the above $L_r(H^{s(r)}_q)$-results which are optimal in the sense that all terms, including the nonlinearity, are of the same strength as the Stokes operator, so that we cannot take advantage of any smoothing properties. (See [**21**], [**22**] for related, though different, uses of maximal regularity theory.) It should be observed that, even in the well-studied case of homogeneous boundary conditions, our approach gives new results, for example, the ones exhibited in Remarks (a), (b), and (d).

Navier-Stokes equations with initial data in Lorentz spaces have been studied by Kozono and Yamazaki (cf. [**23**]–[**25**] and the references therein; also see [**21**]). These authors are mainly interested in the space $L_{3,\infty}$ which is excluded in Remark (c), due to the fact that we are dealing with solutions belonging to L_r with respect to t.

In the next section we collect some basic facts from interpolation and semigroup theory. In particular, we present the fundamental maximal regularity result and prove an important abstract embedding theorem (Theorem 1.3). In Section 2 we derive a general existence and uniqueness theorem for semilinear evolution equations by means of maximal regularity techniques. Motivated by our applications in this paper, we restrict ourselves to the case of quadratic nonlinearities. However, as pointed out at the end of that section, the proof extends straightforwardly to locally Lipschitzian nonlinearities. These abstract results are of independent interest.

In Section 3 we investigate interpolation and embedding properties of the relevant function spaces. Section 4 is devoted to the study of mapping properties of the nonlinearity, and Section 5 contains the proof of the Theorem.

1. Interpolation spaces, analytic semigroups, and maximal regularity

Let E and F be Banach spaces. Then $\mathcal{L}(E,F)$ is the Banach space of all bounded linear operators from E into F, and $\mathcal{L}(E) := \mathcal{L}(E,E)$. Moreover, $\mathcal{L}\mathrm{is}(E,F)$ is the set of all isomorphisms in $\mathcal{L}(E,F)$. If E is continuously embedded in F then this is expressed by $E \hookrightarrow F$, and $E \overset{d}{\hookrightarrow} F$ means that E is also dense in F. (These notations are used for locally convex spaces as well.)

We write $[\cdot,\cdot]_\theta$ for the complex and $(\cdot,\cdot)_{\theta,p}$, $1 \le p \le \infty$, for the real interpolation functors of exponent $\theta \in (0,1)$ (see [**26**, Section I.2], for example, for a summary of interpolation theory, and [**19**] or [**14**] for proofs).

Let E_0 and E_1 be Banach spaces with $E_1 \overset{d}{\hookrightarrow} E_0$. Then $E_{[\theta]} := [E_0,E_1]_\theta$ and $E_{\theta,p} := (E_0,E_1)_{\theta,p}$ for $1 \le p \le \infty$ and $0 < \theta < 1$. Recall that

$$E_1 \overset{d}{\hookrightarrow} E_{\theta,1} \overset{d}{\hookrightarrow} E_{\theta,q} \overset{d}{\hookrightarrow} E_{\theta,r} \hookrightarrow E_{\theta,\infty} \overset{d}{\hookrightarrow} E_{\vartheta,1} \overset{d}{\hookrightarrow} E_0 \tag{1.1}$$

for $1 < q < r < \infty$ and $0 < \vartheta < \theta < 1$. Moreover,

$$E_{\theta,1} \overset{d}{\hookrightarrow} E_{[\theta]} \hookrightarrow E_{\theta,\infty}, \qquad 0 < \theta < 1. \tag{1.2}$$

For convenience, we set $E_{[j]} := E_{j,p} := E_j$ for $j = 0, 1$ and $1 \leq p \leq \infty$.

For $T > 0$ we put $J_T := [0, T]$, and we use I to denote a subinterval of $\mathbb{R}^+$ containing 0 such that $\dot{I} \neq \emptyset$.

We set

$$\mathbb{W}_r^1\bigl(I, (E_0, E_1)\bigr) := L_r(I, E_1) \cap W_r^1(I, E_0).$$

Note that

$$\mathbb{W}_r^1\bigl(I, (E_0, E_1)\bigr) \doteq \Bigl(\bigl\{\, u \in L_r(I, E_1) \ ; \ \partial u \in L_r(I, E_0) \,\bigr\},\ \|\cdot\|_{\mathbb{W}_r^1}\Bigr),$$

where ∂ is the distributional derivative, and

$$\|u\|_{\mathbb{W}_r^1} := \|u\|_{L_r(I,E_1)} + \|\partial u\|_{L_r(I,E_0)}.$$

It is well-known (e.g., [**27**, Section I.2.2] that $u \in W_r^1(I, E_0)$ iff u is locally absolutely continuous (in the sense specified in [**26**, Subsection III.1.2]) and u as well as the point-wise derivative $\dot{u}$ belong to $L_r(I, E_0)$. (We adopt the usual convention not to distinguish (notationally) between a measurable function and its equivalence class modulo functions vanishing almost everywhere.) In this case $\partial u = \dot{u}$. If $u \in \mathbb{W}_r^1\bigl(I, (E_0, E_1)\bigr)$ then interpolation theory guarantees a better continuity result, namely

$$\mathbb{W}_r^1\bigl(I, (E_0, E_1)\bigr) \hookrightarrow BUC(I, E_{1/r',r}) \tag{1.3}$$

(see [**26**, Theorem III.4.10.2]).

We assume that $A \in \mathcal{H}(E_1, E_0)$, that is, $A \in \mathcal{L}(E_1, E_0)$ and $-A$, considered as a linear operator in E_0 with domain E_1, generates a strongly continuous analytic semigroup, denoted by $\{\, U(t) \ ; \ t \geq 0 \,\}$ or $\{\, e^{-tA} \ ; \ t \geq 0 \,\}$, on E_0, that is, in $\mathcal{L}(E_0)$. We set $Ux := U(\cdot)x$ for $x \in E_0$. Recall (e.g., [**26**, Proposition III.4.10.3]) that

$$(x \mapsto Ux) \in \mathcal{L}\Bigl(E_{1/r',r}, \mathbb{W}_r^1\bigl(I, (E_0, E_1)\bigr)\Bigr), \tag{1.4}$$

provided U is exponentially decaying if $I = \mathbb{R}^+$.

We also put

$$U \star u(t) := \int_0^t U(t - \tau) u(\tau)\, d\tau, \qquad u \in L_{1,\mathrm{loc}}(I, E_0), \quad t \in I,$$

provided these integrals exist. Then $U\star$ is the map $u \mapsto U \star u$.

Fix a real number $\mathsf{T} > 0$ and put $\mathsf{J} := J_{\mathsf{T}}$. Then we use the following conventions: For $T \in \dot{\mathsf{J}}$ let X_T and Y_T be Banach spaces of E_0-valued distributions on J_T, being continuously embedded in $L_1(J_T, E_0)$. Also suppose that $B \in \mathcal{L}(X_{\mathsf{T}}, Y_{\mathsf{T}})$ with $B(X_T) \subset B(Y_T)$. Then, given $\kappa \in \mathbb{R}$, we write

$$T^\kappa B \in \mathcal{L}(X_T, Y_T) \qquad \mathsf{J}\text{-uniformly},$$

if $T^\kappa \|B\|_{\mathcal{L}(X_T,Y_T)} \le c$ for all $T \in \dot{\mathsf{J}}$. Here and below, we denote by c generic positive constants, perhaps differing from occurrence to occurrence, but being always independent of the free variables in a given equation or inequality.

In [**28**, Lemma 4] it is shown that, given $\alpha \in (0,1)$,

$$U\star \in \mathcal{L}\big(L_r(J_T, E_{\alpha,\infty}), \mathbb{W}_r^1\big(J_T,(E_0,E_1)\big)\big) \qquad \mathsf{J}\text{-uniformly}. \tag{1.5}$$

Furthermore, (1.5) remains valid if J is replaced by $\mathbb{R}^+$, provided the semigroup is exponentially decaying.

We denote by $\mathcal{BIP}(E)$ the set of all closed and densely defined linear operators A in E for which there exist constants $\omega \ge 0$, $N \ge 1$, and $\vartheta \in [0,\pi/2)$ such that $\{\, z \in \mathbb{C}\ ;\ \operatorname{Re} z \ge 0 \,\}$ belongs to the resolvent set of $A_\omega := \omega + A$ and

$$\|A_\omega^{it}\|_{\mathcal{L}(E)} \le N e^{\vartheta\,|t|}, \qquad t \in \mathbb{R},$$

(see [**26**, Section III.4.7] for more details).

Now we introduce the following assumption:

$$\left.\begin{aligned} &E_0 \text{ is a UMD space,}\\ &A \in \mathcal{H}(E_1,E_0) \cap \mathcal{BIP}(E_0); \end{aligned}\right\} \tag{1.6}$$

where we refer to [**26**, Section III.4.5] for precise definitions and examples.

The reason for this hypothesis is the following 'maximal regularity' result, a consequence of the Dore-Venni theorem [**29**]. Here

$$\gamma_0 :\ \mathbb{W}_r^1\big(\mathsf{J},(E_0,E_1)\big) \to E_{1/r',r}, \quad u \mapsto u(0)$$

is the trace operator (at $t = 0$), being well-defined by (1.3).

Theorem 1.1. *Let* (1.6) *be satisfied. Then*

$$(\partial + A, \gamma_0) \in \mathcal{L}\mathrm{is}\big(\mathbb{W}_r^1\big(\mathsf{J},(E_0,E_1)\big), L_r(\mathsf{J},E_0) \times E_{1/r',r}\big). \tag{1.7}$$

Furthermore,

$$(\partial + A, \gamma_0)^{-1}(x,g) = Ux + U \star g \tag{1.8}$$

and

$$U\star \in \mathcal{L}\big(L_r(J_T,E_0), \mathbb{W}_r^1\big(J_T,(E_0,E_1)\big)\big) \qquad \mathsf{J}\text{-}\textit{uniformly}. \tag{1.9}$$

Proof. Assertion (1.7) follows from [**26**, Theorem III.4.10.8], and (1.8) is a consequence of [**26**, Theorem III.1.5.2]. Hence there exists a constant κ such that

$$\|U \star g\|_{\mathbb{W}_r^1(\mathsf{J},(E_0,E_1))} \le \kappa \|g\|_{L_r(\mathsf{J},E_0)}, \qquad g \in L_r(\mathsf{J},E_0). \tag{1.10}$$

For $g \in L_r(J_T,E_0)$ let $\widetilde{g} \in L_r(\mathsf{J},E_0)$ be the extension of g by zero. Then

$$\|U \star g\|_{\mathbb{W}_r^1(J_T,(E_0,E_1))} \le \|U \star \widetilde{g}\|_{\mathbb{W}_r^1(\mathsf{J},(E_0,E_1))}, \quad \|\widetilde{g}\|_{L_r(\mathsf{J},E_0)} = \|g\|_{L_r(J_T,E_0)}$$

which, together with (1.10), imply (1.9). ∎

Remark 1.2. Suppose, in addition to (1.6), that U is exponentially decaying. Then Theorem 1.1 remains valid if J is replaced by $\mathbb{R}^+$.

PROOF. This follows from [**26**, Theorem III.4.10.7]. □

Next we prove a basic embedding theorem.

Theorem 1.3. *Let* (1.6) *be satisfied and suppose that* $1/r' < \alpha < 1$. *Then*

$$\mathbb{W}_r^1\big(I,(E_0,E_1)\big) \overset{d}{\hookrightarrow} L_p(I,E_{[\alpha]}), \qquad 1/r \geq 1/p \geq \alpha - 1/r'.$$

PROOF. (i) Suppose that $1/p = \alpha - 1/r'$ and assume, without loss of generality, that U is exponentially decaying. Then it follows from (1.7), (1.8), and Remark 1.2 that

$$u = U\gamma_0 u + U \star (\partial + A)u, \qquad u \in \mathbb{W}_r^1\big(I,(E_0,E_1)\big).$$

Thus, since

$$(\partial + A, \gamma_0) \in \mathcal{L}\big(\mathbb{W}_r^1\big(I,(E_0,E_1)\big), L_r(I,E_0) \times E_{1/r',r}\big),$$

it remains to show that

$$U \in \mathcal{L}\big(E_{1/r',r}, L_p(I,E_{[\alpha]})\big), \quad U\star \in \mathcal{L}\big(L_r(I,E_0), L_p(I,E_{[\alpha]})\big). \tag{1.11}$$

We denote by $\big[\,(E_s,A_s)\ ;\ s\in\mathbb{R}\,\big]$ the interpolation-extrapolation scale generated by (E_0,A) and $[\cdot,\cdot]_\theta$, $0<\theta<1$ (see [**4**, Section 1] for a short summary of the theory of interpolation-extrapolation spaces, and [**26**, Chapter V] for details). Then [**26**, Theorem V.1.5.4] implies, thanks to hypothesis (1.6), that

$$E_{\alpha-1} \doteq [E_{-1},E_1]_{\alpha/2}, \quad E_\alpha \doteq [E_{-1},E_1]_{(1+\alpha)/2}.$$

From this we obtain by the reiteration theorem (e.g., [**26**, (I.2.8.5)]) that

$$\begin{aligned}(E_{\alpha-1},E_\alpha)_{\theta,r} &\doteq \big([E_{-1},E_1]_{\alpha/2},[E_{-1},E_1]_{(1+\alpha)/2}\big)_{\theta,r}\\ &\doteq (E_{-1},E_1)_{(\alpha+\theta)/2,r}\end{aligned} \tag{1.12}$$

for $0<\theta<1$. Similarly,

$$E_{1/r',r} = (E_0,E_1)_{1/r',r} \doteq \big([E_{-1},E_1]_{1/2},E_1\big)_{1/r',r} \doteq (E_{-1},E_1)_{(1+1/r')/2,r}.$$

Hence it follows from (1.12) that

$$E_{1/r',r} \doteq (E_{\alpha-1},E_\alpha)_{1/p',r} \hookrightarrow (E_{\alpha-1},E_\alpha)_{1/p',p}, \tag{1.13}$$

where the continuous injection is a consequence of $p>r$ and (1.1).

From [**26**, Theorem V.2.1.3] we know that

$$A_{\alpha-1} \in \mathcal{H}(E_\alpha,E_{\alpha-1}), \qquad e^{-tA_{\alpha-1}} \supset U(t), \quad t\geq 0.$$

Thus interpolation theory guarantees that

$$(t \mapsto e^{-tA_{\alpha-1}}) \in \mathcal{L}\big((E_{\alpha-1},E_\alpha)_{1/p',p}, \mathbb{W}_p^1\big(I,(E_{\alpha-1},E_\alpha)\big)\big)$$

(cf. [**26**, Proposition III.4.10.3]). So the first part of (1.11) is implied by (1.13), thanks to $E_\alpha = E_{[\alpha]}$. Since $E_0 \doteq [E_{\alpha-1}, E_\alpha]_{1-\alpha}$, the second part is a consequence of (1.2) and (1.5).

(ii) If $1/p > \alpha - 1/r'$ then we fix $\beta > \alpha$ with $1/p > \beta - 1/r'$. It follows from [**28**, Theorem 3] that

$$\mathbb{W}_r^1\big(I, (E_0, E_1)\big) \hookrightarrow L_p(I, E_{\beta,\infty}).$$

Now the assertion is a consequence of (1.1) and (1.2). □

2. Evolution equations with quadratic nonlinearities

For a Banach space F we denote by $\mathcal{L}^2(E, F)$ the Banach space of all continuous bilinear maps from $E \times E$ into F, endowed with its usual norm. Given $B \in \mathcal{L}^2(E, F)$, we set $B(x) := B(x, x)$ for $x \in E$.

For convenience, we include the short proof for the following simple fixed point and continuity theorem which is implicit in the proof of [**4**, Theorem 5.6]. Here $\mathbb{B}$ is the closed unit ball in E.

Lemma 2.1. *Suppose that $\beta > 0$ and $B \in \mathcal{L}^2(E, E)$ satisfies $\|B\| \le \beta$. Set $\rho := \big(2 - \sqrt{3}\big)/4\beta$ and $\mathsf{P} := \big(\sqrt{3} - 1\big)/4\beta$. Then, given $a \in \mathsf{P}\mathbb{B}$, the equation $x = a + B(x)$ has a unique solution, $x(a)$, in $a + \rho\mathbb{B}$. Moreover,*

$$\|x(a) - x(b)\| \le 2\,\|a - b\|, \qquad a, b \in \mathsf{P}\mathbb{B}. \tag{2.1}$$

Proof. Suppose that $a \in \mathsf{P}\mathbb{B}$. Then

$$M := a + \rho\mathbb{B} \subset (\mathsf{P} + \rho)\mathbb{B} = (1/4\beta)\mathbb{B}.$$

Hence $B(x) - B(y) = B(x - y, x) + B(y, x - y)$ implies

$$\|B(x) - B(y)\| \le \beta\big(\|x\| + \|y\|\big)\,\|x - y\| \le \|x - y\|/2, \qquad x, y \in M.$$

Thus $x \mapsto \varphi(x) := a + B(x)$ is a contraction on M. Furthermore,

$$\|\varphi(x) - a\| \le \beta\,\|x\|^2 \le 1/16\beta \le \rho, \qquad x \in M,$$

so that $\varphi(M) \subset M$. Hence Banach's fixed point theorem guarantees the existence of a unique fixed point $x(a)$ of φ in M. If $b \in \mathsf{P}\mathbb{B}$ then

$$\begin{aligned}\|x(a) - x(b)\| &= \big\|\varphi\big(x(a)\big) - \varphi\big(x(b)\big)\big\| \le \|a - b\| + \big\|B\big(x(a)\big) - B\big(x(b)\big)\big\| \\ &\le \|a - b\| + \|x(a) - x(b)\|/2,\end{aligned}$$

from which we obtain (2.1). □

Of course, fixed point lemmas of this type have been used (implicitly and explicitly) by many authors in connection with Navier-Stokes equations (cf. the proof of Theorem 10.1 in [**3**], for example, for an early reference).

Suppose that

$$b \in \mathcal{L}^2\Big(\mathbb{W}_r^1\big(J_T, (E_0, E_1)\big), L_r(J_T, E_0)\Big) \qquad \mathsf{J}\text{-uniformly}, \tag{2.2}$$

and consider the semilinear evolution equation

$$\dot{u} + Au = b(u) + h \qquad \text{in } \dot{\mathsf{J}}, \tag{2.3}$$

where $h \in L_r(\mathsf{J}, E_0)$ is given. Let I be a subinterval of J containing 0 such that $\dot{I} \neq \emptyset$. By a $\mathbb{W}_r^1$*-solution* (more precisely, a $\mathbb{W}_r^1(E_0, E_1)$-solution) on I of (2.3) we mean an element $u \in \mathbb{W}_{r,\mathrm{loc}}^1\bigl(I, (E_0, E_1)\bigr)$ satisfying (2.3) on I. It is *maximal* if there does not exist another such solution being a proper extension of it. If $\mathrm{dom}(u) = \mathsf{J}$ then u is *global*. If u is a $\mathbb{W}_r^1$-solution of (2.3) on I then $u \in C(I, E_{1/r',r})$ by (1.3). Thus, if $x \in E_{1/r',r}$, then by a $\mathbb{W}_r^1$*-solution* on I of the initial value problem

$$\dot{u} + Au = b(u) + h \quad \text{in } \dot{J}, \qquad u(0) = x \tag{2.4}$$

we mean a $\mathbb{W}_r^1$-solution u of (2.3) on I such that $u(0) = x$.

Theorem 2.2. *Let assumptions* (1.6) *and* (2.2) *be satisfied. Then:*

(i) *Problem* (2.4) *possesses for each*

$$(x, h) \in E_{1/r',r} \times L_r(\mathsf{J}, E_0) \tag{2.5}$$

a unique maximal $\mathbb{W}_r^1$*-solution,* $u := u(x, h)$*, and* $J(x, h) := \mathrm{dom}(u)$ *is an open subinterval of* J.

If $J^+ := J(x, h) \neq \mathsf{J}$ *then* $u \notin \mathbb{W}_r^1\bigl(J^+, (E_0, E_1)\bigr)$ *and, equivalently,* $u \notin BUC(J^+, E_{1/r',r})$.

(ii) *For each* $T \in \dot{\mathsf{J}}$ *there exists* $R > 0$ *such that* $J(x, h) \supset J_T$ *whenever* (x, h) *satisfies*

$$\|x\|_{E_{1/r',r}} + \|h\|_{L_r(J_T, E_0)} \leq R.$$

(iii) *If* U *is exponentially decaying and* $\mathsf{J} = \mathbb{R}^+$ *then there exists* R *such that* $J(x, h) = \mathbb{R}^+$ *whenever*

$$\|x\|_{E_{1/r',r}} + \|h\|_{L_r(\mathbb{R}^+, E_0)} \leq R. \tag{2.6}$$

Proof. For abbreviation, we set $\mathbb{W}_r^1(I) := \mathbb{W}_r^1\bigl(I, (E_0, E_1)\bigr)$.

(1) Set $B := U \star b$. Then it follows from (1.9) and (2.2) that

$$B \in \mathcal{L}^2\bigl(\mathbb{W}_r^1(J_T), \mathbb{W}_r^1(J_T)\bigr) \qquad \mathsf{J}\text{-uniformly}.$$

Also put $a := Ux + U \star h$. Then Theorem 1.1 and (2.5) imply that $a \in \mathbb{W}_r^1(\mathsf{J})$. Furthermore, u is a $\mathbb{W}_r^1$-solution of (2.4) on I iff $u = a + B(u)$ in $\mathbb{W}_r^1(I)$.

Fix $\beta > 0$ such that

$$\|B(u, v)\|_{\mathbb{W}_r^1(J_T)} \leq \beta \, \|u\|_{\mathbb{W}_r^1(J_T)} \, \|v\|_{\mathbb{W}_r^1(J_T)}, \qquad u, v \in \mathbb{W}_r^1(J_T), \quad T \in \dot{\mathsf{J}}.$$

Given $T \in \dot{\mathsf{J}}$, Lemma 2.1 implies the existence of a unique $\mathbb{W}_r^1$-solution u on J_T satisfying $\|u - a\|_{\mathbb{W}_r^1(J_T)} \leq \rho$, provided $\|a\|_{\mathbb{W}_r^1(J_T)} \leq \mathsf{P}$.

(2) Since $\|a\|_{\mathbb{W}_r^1(J_T)} \to 0$ as $T \to 0$ we can fix $T_0 \in \dot{\mathsf{J}}$ with $\|a\|_{\mathbb{W}_r^1(J_{T_0})} \le \mathsf{P}$. Let u_0 be the unique $\mathbb{W}_r^1$-solution of (2.4) on J_{T_0} satisfying

$$\|u_0 - a\|_{\mathbb{W}_r^1(J_{T_0})} \le \rho.$$

Suppose that $\mathsf{J} \backslash J_{T_0} \neq \emptyset$ and set $J_1 := (\mathsf{J} - T_0) \cap \mathbb{R}^+$. Define $h_1 \in L_r(J_1, E_0)$ by $h_1(t) := h(t + T_0)$. Set $y := u(T_0)$ and note that $y \in E_{1/r',r}$ by (1.3). Lastly, put $a_1 := Uy + U \star h_1$. Then $a_1 \in \mathbb{W}_r^1(J_1)$ by Theorem 1.1. Hence there exists $T_1 \in \dot{J}_1$ such that $\|a_1\|_{\mathbb{W}_r^1(J_{T_1})} \le \mathsf{P}$. Thus, similarly as above, there is a unique $\mathbb{W}_r^1$-solution v_1 on J_{T_1} of

$$\dot{v} + Av = b(v) + h_1 \quad \text{in } \dot{J}_1, \qquad v(0) = y$$

satisfying $\|v_1 - a_1\|_{\mathbb{W}_r^1(J_{T_1})} \le \rho$. Define u_1 on $J_{T_0+T_1}$ by $u_1 \,|\, J_{T_0} := u_0$ and $u_1(t) := v_1(t - T_0)$ for $T_0 < t \le T_0 + T_1$. Then we infer from (1.3) that

$$u_1 \in C(J_{T_0+T_1}, E_{1/r',r}).$$

Using this and $u \,|\, I \in \mathbb{W}_r^1(I)$ for $I \in \big\{ J_{T_0}, [T_0, T_0 + T_1] \big\}$ it is not difficult to verify that $u_1 \in \mathbb{W}_r^1(J_{T_0+T_1})$. Moreover, u_1 solves (2.4) on $J_{T_0+T_1}$. By iterating this argument we arrive at a maximal extension $u := u(x, h)$ of u_0, defined on J^+, such that $u \in \mathbb{W}_{r,\mathrm{loc}}^1(J^+)$ and u is a $\mathbb{W}_r^1$-solution of (2.4).

(3) If $J^+ \neq \mathsf{J}$ then $u \notin \mathbb{W}_r^1(J^+)$. Indeed, otherwise $u \in BUC(J^+, E_{1/r',r})$ by (1.3), so that u has an extension $\overline{u} \in C\big(\overline{J^+}, E_{1/r',r}\big)$. Thus we can apply the above continuation argument with the initial value $\overline{u}(t^+)$, where t^+ is the right endpoint of J^+, to obtain a contradiction to the maximality of u. This argument shows that J^+ is open in J, that $u \notin \mathbb{W}_r^1(J^+)$, and that this is equivalent to $u \notin BUC(J^+, E_{1/r',r})$ if $J^+ \neq \mathsf{J}$.

(4) Suppose that v is a $\mathbb{W}_r^1$-solution of (2.4) on some interval $I \subset \mathsf{J}$ such that $u \not\supset v$ Then

$$T' := \max\big\{ t \in I \ ; \ u(t) = v(t) \text{ in } E_{1/r',r} \big\}$$

is well-defined and $I' := (I - T') \cap \mathbb{R}^+$ is a nontrivial subinterval of $\mathbb{R}^+$ containing 0. Set $z := u(T')$ and $k(t) := h(t + T')$ and consider the initial value problem

$$\dot{w} + Aw = b(w) + k \quad \text{in } \dot{I}', \qquad w(0) = z.$$

It has the two distinct $\mathbb{W}_r^1$-solutions $w_1 := u(\cdot + T')$ and $w_2 := v(\cdot + T')$ on I'. Put $a' := Uz + U \star k$ and note that $\|w_j - a'\|_{\mathbb{W}_r^1(J_T)} \to 0$ as $T \to 0$. Hence there exists $T_0' \in \dot{I}'$ such that $\|w_j - a'\|_{\mathbb{W}_r^1(J_{T_0'})} \le \rho$. Now the uniqueness assertion of (1) implies $w_1 = w_2$ in $C(J_{T_0'}, E_{1/r',r})$ which contradicts the definition of T'. This proves the uniqueness of u. Thus assertion (i) has been shown.

(5) Since

$$\|a\|_{\mathbb{W}_r^1(\mathsf{J})} \le c\big(\|x\|_{E_{1/r',r}} + \|h\|_{L_r(\mathsf{J},E_0)}\big)$$

by Theorem 1.1, assertion (ii) is an immediate consequence of (1).

(6) Suppose that U is exponentially decaying and $\mathsf{J} = \mathbb{R}^+$. Then Remark 1.2 and step (1) show that there exists a unique $\mathbb{W}_r^1$-solution on $\mathbb{R}^+$, provided $\|a\|_{\mathbb{W}_r^1(\mathbb{R}^+)} \le \mathsf{P}$. Since the norm of a in $\mathbb{W}_r^1(\mathbb{R}^+)$ is majorized by a constant multiple of the left-hand side of (2.6), assertion (iii) follows. □

For completeness, we add some further properties of the $\mathbb{W}_r^1$-solution u of (2.4), although they are not used in this paper.

Remarks 2.3. **(a)** Suppose that $r < p < \infty$. Then there exists κ such that $J^+(x,h) \supset J_T$ whenever $T \in \dot{\mathsf{J}}$ and $(x,h) \in E_{1/p',\infty} \times L_p(\mathsf{J}, E_0)$ satisfy

$$T^{1/r-1/p}\big(\|x\|_{E_{1/p',\infty}} + \|h\|_{L_p(J_T,E_0)}\big) \le \kappa.$$

Proof. We infer from [**28**, Lemma 4(ii)] that

$$\|Ux\|_{\mathbb{W}_r^1(J_T)} \le cT^{1/r-1/p}\,\|x\|_{E_{1/p',\infty}} \qquad \mathsf{J}\text{-uniformly}.$$

From Hölder's inequality it follows that

$$\|h\|_{L_r(J_T,E_0)} \le T^{1/r-1/p}\,\|h\|_{L_p(J_T,E_0)}, \qquad T \in \dot{\mathsf{J}}.$$

Hence we deduce from Theorem 1.1 that

$$\|U \star h\|_{\mathbb{W}_r^1(J_T)} \le cT^{1/r-1/p}\,\|h\|_{L_p(J_T,E_0)} \qquad \mathsf{J}\text{-uniformly}.$$

Consequently,

$$\|a\|_{\mathbb{W}_r^1(J_T)} \le cT^{1/r-1/p}\big(\|x\|_{E_{1/p',\infty}} + \|h\|_{L_p(J_T,E_0)}\big), \qquad T \in \dot{\mathsf{J}},$$

and the assertion is also a consequence of part (i) of the preceding proof. □

(b) If the hypotheses of (a) are satisfied and $t^+ := \sup J^+ < \mathsf{T}$ then

$$\lim_{t \uparrow t^+} \|u(t)\|_{E_{1/p',\infty}} = \infty.$$

Proof. Suppose that there is a sequence (t_j) in $(0,t^+)$ converging towards t^+ such that $\|u(t_j)\|_{E_{1/p',\infty}} \le c < \infty$ for $j \in \mathbb{N}$. Set $h_j(t) := h(t+t_j)$ for $t \in \mathsf{J}$ with $t + t_j \in \mathsf{J}$, and $h_j(t) := 0$ for the remaining $t \in \mathsf{J}$. Then h_j belongs to $L_p(\mathsf{J}, E_0)$ and

$$\|h_j\|_{L_p(\mathsf{J},E_0)} \le \|h\|_{L_p(\mathsf{J},E_0)}.$$

Hence we infer from (a) and the translation argument used in step (2) of the proof of Theorem 2.2 that there exists $T > 0$ such that $J^+ \supset [0, t_j + T] \cap J$ for $j \in \mathbb{N}$. This contradicts $t^+ < \mathsf{T}$ and proves the stated property. □

(c) Using the continuity assertion (2.1) it is not difficult to prove that $u(x,h)$ depends continuously on (x,h).

(d) Suppose that b is a function mapping $\mathbb{W}_r^1\big(\mathsf{J}, (E_0,E_1)\big)$ into $L_r(\mathsf{J}, E_0)$ such that, given any $T \in \dot{\mathsf{J}}$, its restriction to $\mathbb{W}_r^1\big(J_T, (E_0,E_1)\big)$ maps into the space $L_r(J_T, E_0)$ and is uniformly Lipschitz continuous on bounded sets. Then Theorem 2.2 and the preceding remarks (a)–(c) continue to hold.

PROOF. This follows by an obvious modification of the above proofs. $\square$

3. Interpolation-extrapolation scales

We put $\boldsymbol{A}_0 := -\Delta \,|\, \boldsymbol{H}_q^2$, that is, $\boldsymbol{A}_0$ is the negative Dirichlet-Laplacian, considered as an unbounded linear operator in L_q. It is known that

$$\boldsymbol{A}_0 \in \mathcal{H}(\boldsymbol{H}_q^2, L_q) \cap \mathcal{BIP}(L_q) \tag{3.1}$$

(see [**30**], for example).

Let $\big[\,(F_\alpha, A_\alpha)\ ;\ \alpha \in \mathbb{R}\,\big]$ be the interpolation-extrapolation scale generated by $(L_q, \boldsymbol{A}_0)$ and $[\cdot,\cdot]_\theta,\ 0<\theta<1$. Then it follows from [**4**, Theorem 2.2] that $F_\alpha \doteq \boldsymbol{H}_q^{2\alpha}$ for $|\alpha| \le 1$. This implies, in particular, that

$$\boldsymbol{H}_q^s \overset{d}{\hookrightarrow} \boldsymbol{H}_q^t, \qquad -2 \le t < s \le 2\ . \tag{3.2}$$

Moreover, we know from [**26**, Theorem V.1.5.4], thanks to (3.1), that

$$[\boldsymbol{H}_q^{s_0}, \boldsymbol{H}_q^{s_1}]_\theta \doteq \boldsymbol{H}_q^{s_\theta}, \qquad -2 \le s_0 < s_1 \le 2, \quad 0<\theta<1, \tag{3.3}$$

where $s_\theta := (1-\theta)s_0 + \theta s_1$.

Now we put $\boldsymbol{A}_s := A_{s/2}$ for $-2 \le s \le 0$. Then (see Theorem V.2.1.3 and Proposition V.1.5.5 in [**26**])

$$\boldsymbol{A}_s \in \mathcal{H}(\boldsymbol{H}_q^{s+2}, \boldsymbol{H}_q^s) \cap \mathcal{BIP}(\boldsymbol{H}_q^s), \qquad -2 \le s \le 0,$$

and

$$\boldsymbol{A}_{s_0} \supset \boldsymbol{A}_{s_1}, \quad e^{-t\boldsymbol{A}_{s_0}} \supset e^{-t\boldsymbol{A}_{s_1}}, \qquad -2 \le s_0 < s_1 \le 0. \tag{3.4}$$

Recall that $\mathcal{D}_0(\overline{\Omega})$ is dense in $\boldsymbol{H}_q^2$ for $|s| \le 2$. Since $\mathcal{D}_0(\overline{\Omega})$ is a core for $\boldsymbol{A}_0$, that is, $\boldsymbol{A}_0$ is the closure of its restriction to $\mathcal{D}_0(\overline{\Omega})$, it follows from (3.2) and (3.4) that $\mathcal{D}_0(\overline{\Omega})$ is also a core for $\boldsymbol{A}_s$ for $-2 \le s < 0$. This shows that $\boldsymbol{A}_s$ is 'essentially' independent of q in the sense that $\boldsymbol{A}_s$ is uniquely determined by $\boldsymbol{A}_0 \,|\, \mathcal{D}_0(\overline{\Omega})$. For this reason we do not indicate the q-dependence of $\boldsymbol{A}_s$. (Of course, for similar reasons we could omit the index s. However, for the sake of clarity we continue to indicate the s-dependence.)

Lemma 3.1. *Suppose that $-2 \le s_0 \le s_1 \le 2$ and*

$$s_1 - 3/q_1 \ge s_0 - 3/q_0, \quad 1 > 1/q_1 \ge 1/q_0 > 0.$$

Then $\boldsymbol{H}_{q_1}^{s_1} \overset{d}{\hookrightarrow} \boldsymbol{H}_{q_0}^{s_0}$.

PROOF. (Cf. the proof of [**4**, Theorem 3.10].)

(i) If $s_0 \ge 0$ and $s_j \ne 1/q_j$ then $\boldsymbol{H}_{q_j}^{s_j}$ is a closed linear subspace of $H_{q_j}^{s_j}$. Thus the assertion follows in this case from the corresponding embedding theorem for Bessel potential spaces.

(ii) If $s_1 \le 0$ then we obtain the assertion from what has just been shown by duality (and reflexivity).

(iii) If $s_0 \le 0 < s_1$ with $s_1 \ne 1/q$ then we obtain the desired result from (i) and (ii) by embedding $\boldsymbol{H}_{q_1}^{s_1}$ in L_r and L_r in $\boldsymbol{H}_{q_0}^{s_0}$.

(iv) The remaining case is now covered by interpolation, thanks to (3.3). □

It has been proved by Solonnikov [**3**] (also see [**2**], [**15**], [**17**], [**18**], [**31**]) that the Helmholtz decomposition

$$L_q = \mathbb{H}_q \oplus \mathbb{G}_q \tag{3.5}$$

is valid. We denote by $P := P_q$ the Helmholtz projection, that is,

$$P = P^2 \in \mathcal{L}(L_q), \quad \mathrm{im}(P) = \mathbb{H}_q, \quad \ker(P) = \mathbb{G}_q,$$

and recall that $(P_q)' = P_{q'}$.

We write $\mathbb{A}_0$ for the Stokes operator in L_q, that is,

$$\mathbb{A}_0 := P\boldsymbol{A}_0 \,|\, \mathbb{H}_q^2. \tag{3.6}$$

Then

$$\mathbb{A}_0 \in \mathcal{H}(\mathbb{H}_q^2, \mathbb{H}_q) \cap \mathcal{BIP}(\mathbb{H}_q).$$

In fact, it follows from [**3**] that $-\mathbb{A}_0$ generates an analytic semigroup (also see [**31**]–[**34**]). Giga [**35**] proved that $\mathbb{A}_0$ has bounded imaginary powers if Ω is bounded; in [**36**] this result is extended to exterior domains.

Let $\big[\,(G_\alpha, B_\alpha)\ ;\ \alpha \in \mathbb{R}\,\big]$ be the interpolation-extrapolation scale generated by $(\mathbb{H}_q, \mathbb{A}_0)$ and $[\cdot,\cdot]_\theta$, $0 < \theta < 1$. Then [**4**, Theorem 3.4] guarantees that $G_\alpha \doteq \mathbb{H}_q^{2\alpha}$ for $|\alpha| \le 1$. Thus, similarly as above,

$$\mathbb{H}_q^s \overset{d}{\hookrightarrow} \mathbb{H}_q^t, \qquad -2 \le t < s \le 2, \tag{3.7}$$

and

$$[\mathbb{H}_q^{s_0}, \mathbb{H}_q^{s_1}]_\theta \doteq \mathbb{H}_q^{s_\theta}, \qquad -2 \le s_0 < s_1 \le 2, \quad 0 < \theta < 1. \tag{3.8}$$

Furthermore, setting $\mathbb{A}_s := B_{s/2}$ for $-2 \le s \le 0$,

$$\mathbb{A}_s \in \mathcal{H}(\mathbb{H}_q^{s+2}, \mathbb{H}_q^s) \cap \mathcal{BIP}(\mathbb{H}_q^s), \qquad -2 \le s \le 0, \tag{3.9}$$

and

$$\mathbb{A}_{s_0} \supset \mathbb{A}_{s_1}, \quad e^{-t\mathbb{A}_{s_0}} \supset e^{-t\mathbb{A}_{s_1}}, \qquad -2 \le s_0 \le s_1 \le 0. \tag{3.10}$$

From Theorem 2.2 in [**11**] we know that $\boldsymbol{H}_q^s$ possesses the direct sum decomposition

$$\boldsymbol{H}_q^s = \mathbb{H}_q^s \oplus \mathbb{G}_q^s, \qquad |s| \le 2. \tag{3.11}$$

We denote by $\boldsymbol{P}_s$ the projection onto $\mathbb{H}_q^s$ parallel to $\mathbb{G}_q^s$, that is,

$$\boldsymbol{P}_s = (\boldsymbol{P}_s)^2 \in \mathcal{L}(\boldsymbol{H}_q^s), \quad \mathrm{im}(\boldsymbol{P}_s) = \mathbb{H}_q^s, \quad \ker(\boldsymbol{P}_s) = \mathbb{G}_q^s. \tag{3.12}$$

Theorem 2.2 and Corollary 2.3 of [**11**] imply

$$\boldsymbol{P} := \boldsymbol{P}_0 = P, \quad \boldsymbol{P}_t \supset \boldsymbol{P}_s, \qquad -2 \le t < s \le 2. \tag{3.13}$$

Note that the first equality in (3.13) means that (3.11) coincides for $s=0$ with the Helmholtz decomposition (3.5). Thus it follows from (3.6), (3.7) and (3.9), (3.10) that

$$\mathbb{A}_s = \boldsymbol{P}_s \boldsymbol{A}_s, \qquad -2 \le s \le 0. \tag{3.14}$$

Lemma 3.2. *Suppose that* $-2 \le s_0 \le s_1 \le 2$ *and*

$$s_1 - 3/q_1 \ge s_0 - 3/q_0, \quad 1 > 1/q_1 \ge 1/q_0 > 0.$$

Then $\mathbb{H}_{q_1}^{s_1} \overset{d}{\hookrightarrow} \mathbb{H}_{q_0}^{s_0}$.

Proof. This is an immediate consequence of (3.13) and Lemma 3.1. □

It should be remarked that Lemma 3.2 coincides with [**4**, Theorem 3.10]. However, the proof in that paper is rather more complicated since we had to work with a quotient space representation of $\mathbb{H}_q^s$ for $-2 \le s < 0$. For this reason we decided to include the elementary proof of Lemma 3.2 which is now possible thanks to the *generalized Helmholtz decomposition* (3.11).

Next we observe that

$$\mathbb{H}_q^s \text{ is a UMD space for } |s| \le 2. \tag{3.15}$$

Indeed, $\mathbb{H}_q^s$ is isomorphic to the closed linear subspace $\mathbb{H}_q$ of the UMD space L_q. Thus (3.15) follows from basic facts on UMD spaces (see Theorem III.4.5.2 in [**26**]).

After these preparations we can prove the following embedding theorem which is important for our approach. To simplify the writing, we henceforth set

$$\mathcal{W}_r^1(I, \mathbb{H}_q^s) := \mathbb{W}_r^1\big(I, (\mathbb{H}_q^{s-2}, \mathbb{H}_q^s)\big).$$

Theorem 3.3. *Let I be a nontrivial subinterval of $\mathbb{R}^+$ containing 0. Suppose that $0 \le s_1 \le 2$ and $s_1 - 2 \le s_0 \le s_1$ with*

$$s_1 - 2/r_1 - 3/q_1 \ge s_0 - 2/r_0 - 3/q_0, \tag{3.16}$$

and

$$1 > 1/q_1 \ge 1/q_0 > 0, \quad 1 > 1/r_1 \ge 1/r_0 > 0.$$

Then

$$\mathcal{W}_{r_1}^1(I, \mathbb{H}_{q_1}^{s_1}) \overset{d}{\hookrightarrow} L_{r_0}(I, \mathbb{H}_{q_0}^{s_0}).$$

Proof. Set $(E_0, E_1) := (\mathbb{H}_{q_1}^{s_1-2}, \mathbb{H}_{q_1}^{s_1})$ and $A := \mathbb{A}_{s_1-2}$. Then (3.9) and (3.15) guarantee that (1.6) is satisfied. Put

$$s := s_1 - 2(1/r_1 - 1/r_0), \quad \theta := 1 - (s_1 - s)/2.$$

Then (3.8) implies

$$\mathbb{H}_{q_1}^s \doteq [\mathbb{H}_{q_1}^{s_1-2}, \mathbb{H}_{q_1}^{s_1}]_\theta = [E_0, E_1]_\theta = E_{[\theta]}.$$

Since $1/r_0 = \theta - 1/r_1' > 0$, it follows from Theorem 1.3 that

$$\mathcal{W}^1_{r_1}(I, \mathbb{H}^{s_1}_{q_1}) \overset{d}{\hookrightarrow} L_{r_0}(I, \mathbb{H}^{s}_{q_1}). \tag{3.17}$$

From (3.16) we see that $s - 3/q_1 \geq s_0 - 3/q_0$. Thus $\mathbb{H}^s_{q_1} \overset{d}{\hookrightarrow} \mathbb{H}^{s_0}_{q_0}$ by Lemma 3.2. This and (3.17) imply the assertion. □

In the next proposition we collect the relevant properties of the spaces $\boldsymbol{B}^s_{q,r}$ and $\mathbb{B}^s_{q,r}$.

Proposition 3.4. (i) *If* $-2 \leq s_0 < s_1 \leq 2$ *and* $0 < \theta < 1$ *then*

$$\boldsymbol{B}^{s_\theta}_{q,r} \doteq (\boldsymbol{H}^{s_0}_q, \boldsymbol{H}^{s_1}_q)_{\theta,r}, \quad \mathbb{B}^{s_\theta}_{q,r} \doteq (\mathbb{H}^{s_0}_q, \mathbb{H}^{s_1}_q)_{\theta,r}. \tag{3.18}$$

(ii) *If* $s < 1$ *then* $\mathcal{D}_\sigma$ *is dense in* $\mathbb{B}^s_{q,r}$.

(iii) *Suppose that* $-2 < s_0 \leq s_1 < 2$ *and*

$$s_1 - 3/q_1 \geq s_0 - 3/q_0, \quad 1 > 1/q_1 \geq 1/q_0 > 0, \quad 1 > 1/r_1 \geq 1/r_0 > 0.$$

Then

$$\boldsymbol{B}^{s_1}_{q_1,r_1} \overset{d}{\hookrightarrow} \boldsymbol{B}^{s_0}_{q_0,r_0}, \quad \mathbb{B}^{s_1}_{q_1,r_1} \overset{d}{\hookrightarrow} \mathbb{B}^{s_0}_{q_0,r_0}.$$

Proof. (i) Denote by $\big[(E_\alpha, A_\alpha)\ ;\ \alpha \in \mathbb{R}\big]$ the interpolation-extrapolation scale generated by $(L_q, \boldsymbol{A}_0)$ and $(\cdot,\cdot)_{\theta,r}$ for $0 < \theta < 1$. Then we infer from [**4**, Theorem 2.2] that

$$\boldsymbol{B}^{-2+2\theta}_{q,r} \doteq (\boldsymbol{H}^{-2}_q, \boldsymbol{H}^0_q)_{\theta,r}, \quad \boldsymbol{B}^{2\theta}_{q,r} \doteq (\boldsymbol{H}^0_q, \boldsymbol{H}^2_q)_{\theta,r}.$$

Since $\boldsymbol{H}^0_q \doteq [\boldsymbol{H}^{-2}_q, \boldsymbol{H}^2_q]_{1/2}$ by (3.3), the reiteration theorem implies

$$\boldsymbol{B}^{-2+2\theta}_{q,r} \doteq \big(\boldsymbol{H}^{-2}_q, [\boldsymbol{H}^{-2}_q, \boldsymbol{H}^2_q]_{1/2}\big)_{\theta,r} \doteq (\boldsymbol{H}^{-2}_q, \boldsymbol{H}^2_q)_{\theta/2,r}$$

and

$$\boldsymbol{B}^{2\theta}_{q,r} \doteq \big([\boldsymbol{H}^{-2}_q, \boldsymbol{H}^2_q]_{1/2}, \boldsymbol{H}^2_q\big)_{\theta,r} \doteq (\boldsymbol{H}^{-2}_q, \boldsymbol{H}^2_q)_{(1+\theta)/2,r}$$

for $0 < \theta < 1$. Furthermore, thanks to the interpolation characterization of Bessel potential spaces (e.g., [**14**, Theorem 4.3.1]), we deduce from (0.3) that, given $\tau \in (0, 1/q \wedge 1/q')$,

$$\boldsymbol{B}^0_{q,r} = B^0_{q,r} = (H^{-\tau}_q, H^\tau_q)_{1/2,r} = (\boldsymbol{H}^{-\tau}_q, \boldsymbol{H}^\tau_q)_{1/2,r}.$$

Thus, using (3.3) and the reiteration theorem once more,

$$\begin{aligned} \boldsymbol{B}^0_{q,r} &\doteq \big([\boldsymbol{H}^{-2}_q, \boldsymbol{H}^2_q]_{(2-\tau)/4}, [\boldsymbol{H}^{-2}_q, \boldsymbol{H}^2_q]_{(2+\tau)/4}\big)_{1/2,r} \\ &\doteq (\boldsymbol{H}^{-2}_q, \boldsymbol{H}^2_q)_{1/2,r}. \end{aligned} \tag{3.19}$$

This shows that

$$\boldsymbol{B}^s_{q,r} \doteq (\boldsymbol{H}^{-2}_q, \boldsymbol{H}^2_q)_{(2+s)/4,r}, \qquad -2 < s < 2. \tag{3.20}$$

Now suppose that $-2 < s_0 < s_1 < 2$. Then we infer from (3.3) and the reiteration theorem that

$$\begin{aligned}(\boldsymbol{H}_q^{s_0}, \boldsymbol{H}_q^{s_1})_{\theta,r} &\doteq \big([\boldsymbol{H}_q^{-2}, \boldsymbol{H}_q^2]_{(2+s_0)/4}, [\boldsymbol{H}_q^{-2}, \boldsymbol{H}_q^2]_{(2+s_1)/4}\big)_{\theta,r} \\ &\doteq (\boldsymbol{H}_q^{-2}, \boldsymbol{H}_q^2)_{(2+s_\theta)/4,r}\end{aligned}$$

for $0 < \theta < 1$. By comparing this to (3.20) we obtain the first relation in (3.18).

Suppose that $0 \le s_0 < s_1 \le 2$. Then, by (0.6) and by what has just been shown,

$$\mathbb{B}_{q,r}^{s_\theta} = \boldsymbol{B}_{q,r}^{s_\theta} \cap \mathbb{H}_q = (\boldsymbol{H}_q^{s_0}, \boldsymbol{H}_q^{s_1})_{\theta,r} \cap \mathbb{H}_q = (\mathbb{H}_q^{s_0}, \mathbb{H}_q^{s_1})_{\theta,r}$$

for $0 < \theta < 1$, thanks to $\mathbb{H}_q^s = \boldsymbol{H}_q^s \cap \mathbb{H}_q$ for $0 \le s \le 2$ and to (3.11) (cf. [**14**, Theorem 1.17.1.1]). Due to (0.2) and (0.6) we now obtain the second relation in (3.18) for $-2 \le s_0 < s_1 \le 0$ by duality. Suppose that $0 < \tau < 1/q \wedge 1/q'$. Then (3.11) and (3.19) imply, thanks to [**26**, Proposition I.2.3.3], that

$$B_{q,r}^0 = \boldsymbol{B}_{q,r}^0 \doteq (\boldsymbol{H}_q^{-\tau}, \boldsymbol{H}_q^\tau)_{1/2,r} \doteq (\mathbb{H}_q^{-\tau}, \mathbb{H}_q^\tau)_{1/2,r} \oplus (\mathbb{G}_q^{-\tau}, \mathbb{G}_q^\tau)_{1/2,r}.$$

Hence

$$B_{q,r}^0 \cap \mathbb{H}_q^\tau \doteq (\mathbb{H}_q^{-\tau}, \mathbb{H}_q^\tau)_{1/2,r} \cap \mathbb{H}_q^\tau.$$

Since $\mathbb{H}_q^\tau$ is dense in $(\mathbb{H}_q^{-\tau}, \mathbb{H}_q^\tau)_{1/2,r}$ and $\mathbb{H}_q^\tau$ is contained in $B_{q,r}^0$, it follows that $(\mathbb{H}_q^{-\tau}, \mathbb{H}_q^\tau)_{1/2,r}$ is the closure of $\mathbb{H}_q^\tau$ in $B_{q,r}^0$. Hence the density of $\mathcal{D}_\sigma$ in $\mathbb{H}_q^\tau$ and (0.6) imply that

$$\mathbb{B}_{q,r}^0 \doteq (\mathbb{H}_q^{-\tau}, \mathbb{H}_q^\tau)_{1/2,r}.$$

Now we obtain the second relation in (3.18) from (3.8) and the reiteration theorem, by the arguments of the first part of this proof.

(ii) It is well-known that $\mathcal{D}_\sigma$ is dense in $\mathbb{H}_q^1$ (e.g., [**37**, Section III.4] and the references therein). Since $\mathbb{H}^1$ is dense in $(\mathbb{H}_q^{-2}, \mathbb{H}_q^1)_{\theta,r}$ for $0 < \theta < 1$ the assertion follows from (i).

(iii) Thanks to (i), we obtain these assertions from Lemmas 3.1 and 3.2, respectively, by interpolation. □

4. Bilinear estimates

We put

$$B(u,v) := -(u \cdot \nabla)v = \nabla \cdot (u \otimes v), \qquad u,v \in \mathbb{H}_q^2, \tag{4.1}$$

and study some mapping properties of this bilinear operator. Here and below, we use the same letter to denote various continuous extensions of B over superspaces of $\mathbb{H}_q^2$.

Lemma 4.1. *Suppose that $q \ge 2$ and $\sigma \in [0,2]$ satisfy*

$$\sigma \le 3/q, \quad -1 + 3/q \le 2\sigma \le 1 + 3/q.$$

Then $B \in \mathcal{L}^2(\mathbb{H}_q^\sigma, \boldsymbol{H}_q^{2\sigma-1-3/q})$.

Proof. This follows from the proof of Theorem 4.2 in [**4**] by omitting there the projection $\boldsymbol{P}$. $\square$

Henceforth, we denote the Nemyts'kii operator induced by B again by B, that is, $B(u,v)(t) = B\big(u(t),v(t)\big)$ for $t \in \mathsf{J}$, etc.

Proposition 4.2. *Suppose that $q \in [2,\infty)$, $r \in (1,\infty)$, and $s \in [0,2]$ are such that*

$$-1 + 2/r + 3/q \le s \le 1/r + 3/q \tag{4.2}$$

and

$$2/r \le 2s \le 1 + 2/r + 3/q. \tag{4.3}$$

Then

$$B \in \mathcal{L}^2\big(\mathcal{W}_r^1(J_T,\mathbb{H}_q^s), L_r(J_T,\boldsymbol{H}_q^{s-2+\alpha(s)})\big) \qquad \mathsf{J}\text{-uniformly},$$

where $\alpha(s) := s + 1 - 2/r - 3/q$ and $\mathsf{J} = \mathbb{R}^+$ is admitted.

Proof. Set $\sigma := s - 1/r$. Then (4.2) and (4.3) imply $0 \le \sigma \le 3/q$ and $-1 + 3/q \le 2\sigma \le 1 + 3/q$. Also note that $2\sigma - 1 - 3/q = s - 2 + \alpha(s)$. Hence, by Lemma 4.1,

$$B \in \mathcal{L}^2(\mathbb{H}_q^\sigma, \boldsymbol{H}_q^{s-2+\alpha(s)}). \tag{4.4}$$

Since $s - 2/r - 3/q = \sigma - 2/(2r) - 3/q$ we infer from Theorem 3.3 that

$$\mathcal{W}_r^1(\mathbb{R}^+,\mathbb{H}_q^s) \hookrightarrow L_{2r}(\mathbb{R}^+,\mathbb{H}_q^\sigma). \tag{4.5}$$

Denoting by κ the norm of the map (4.4), it follows from Hölder's inequality and (4.5) that there exists $\kappa_0 > 0$ such that

$$\begin{aligned}\|B(u,v)\|_{L_r(\mathbb{R}^+,\boldsymbol{H}_q^{s-2+\alpha(s)})} &\le \kappa\, \|u\|_{L_{2r}(\mathbb{R}^+,\mathbb{H}_q^\sigma)}\, \|v\|_{L_{2r}(\mathbb{R}^+,\mathbb{H}_q^\sigma)} \\ &\le \kappa_0\, \|u\|_{\mathcal{W}_r^1(\mathbb{R}^+,\mathbb{H}_q^s)}\, \|v\|_{\mathcal{W}_r^1(\mathbb{R}^+,\mathbb{H}_q^s)}\end{aligned} \tag{4.6}$$

for $u,v \in \mathcal{W}_r^1(\mathbb{R}^+,\mathbb{H}_q^s)$.

Now suppose that $0 < T < \mathsf{T}$. Given $u \in \mathcal{W}_r^1(J_T,\mathbb{H}_q^s)$, define $\overline{u}$ by

$$\overline{u} := \begin{cases} u & \text{on } J_T, \\ V(\cdot)u(T) & \text{on } \mathsf{J}\backslash J_T, \end{cases}$$

where $V(t) := e^{-(t-T)(1+\mathbb{A}_{s-2})}$ for $t \ge T$. Then $\overline{u} \in \mathcal{W}_r^1(\mathbb{R}^+,\mathbb{H}_q^s)$ by (1.4). Thus, given $u,v \in \mathcal{W}_r^1(J_T,\mathbb{H}_q^s)$, it follows that $\overline{u}$ and $\overline{v}$ satisfy (4.6), where κ_0 is independent of the particular extensions $\overline{u}$ and $\overline{v}$ of u and v, respectively. Hence, by restriction, we see that

$$\|B(u,v)\|_{L_r(J_T,\boldsymbol{H}_q^{s-2+\alpha(s)})} \le \kappa_0\, \|u\|_{\mathcal{W}_r^1(J_T,\mathbb{H}_q^s)}\, \|v\|_{\mathcal{W}_r^1(J_T,\mathbb{H}_q^s)}$$

for $u,v \in \mathcal{W}_r^1(J_T,\mathbb{H}_q^s)$ and $0 < T < \mathsf{T}$. $\square$

It should be noted that the restriction $q \ge 2$ has been imposed for simplicity. By employing [**4**, Theorem 4.2] in its full strength, the case $q < 2$ can be handled as well.

5. Proof of the Theorem

Let assumptions (0.8) and (0.9) be satisfied.

If $s(r) > 1/q$ then it follows from (0.3) and (0.8) that

$$f \in L_{r,\mathrm{loc}}(\mathbb{R}^+, \boldsymbol{H}_q^{s(r)-2}). \tag{5.1}$$

Thus suppose that $s(r) \leq 1/q$. Then (0.9) and the remarks following it imply that (5.1) is true in this case also.

By the trace theorem

$$\partial_\nu \in \mathcal{L}\big(\boldsymbol{H}_{q'}^{2-s(r)}, W_{q'}^{1/q-s(r)}(\Gamma)\big).$$

Hence

$$\mathcal{R}_{s(r)} := -(\partial_\nu)' \in \mathcal{L}\big(W_q^{s(r)-1/q}(\Gamma), \boldsymbol{H}_q^{s(r)-2}\big),$$

thanks to the duality properties of the Sobolev-Slobodeckii spaces on Γ and to (0.2). Consequently, (5.1) and (0.8) guarantee that

$$f + \mathcal{R}_{s(r)}g \in L_{r,\mathrm{loc}}(\mathbb{R}^+, \boldsymbol{H}_q^{s(r)-2}). \tag{5.2}$$

From Proposition 4.2 we infer that

$$B \in \mathcal{L}^2\big(\mathcal{W}_r^1(J_T, \mathbb{H}_q^{s(r)}), L_r(J_T, \boldsymbol{H}_q^{s(r)-2})\big) \qquad \mathsf{J}\text{-uniformly}. \tag{5.3}$$

Thus we can consider

$$\dot{v} + \boldsymbol{A}_{s(r)-2}v = -w + B(v) + f + \mathcal{R}_{s(r)}g \quad \text{in } \dot{J}, \qquad v(0) = v^0. \tag{5.4}$$

Then (v, w) is said to be an $L_r(H_q^{s(r)})$**-solution** of (5.4) on J iff

$$(v, w) \in \mathcal{W}_{r,\mathrm{loc}}^1(J, \mathbb{H}_q^{s(r)}) \times L_{r,\mathrm{loc}}(J^*, \mathbb{G}_q^{s(r)-2}) \tag{5.5}$$

and (v, w) satisfies (5.4). Set

$$b := \boldsymbol{P}B, \quad h := \boldsymbol{P}(f + \mathcal{R}_{s(r)}g).$$

Then (3.12) and (5.3) imply

$$b \in \mathcal{L}^2\big(\mathcal{W}_r^1(J_T, \mathbb{H}_q^{s(r)}), L_r(J_T, \mathbb{H}_q^{s(r)-2})\big) \qquad \mathsf{J}\text{-uniformly}, \tag{5.6}$$

and it follows from (3.12) and (5.2) that

$$h \in L_r(\mathsf{J}, \mathbb{H}_q^{s(r)-2}). \tag{5.7}$$

Furthermore, (3.11)–(3.14) show that (5.4) is equivalent to

$$\left.\begin{aligned} \dot{v} + \mathbb{A}_{s(r)-2}v = b(v) + h, \qquad t \in \dot{\mathsf{J}}, \quad v(0) = v^0, \\ w = (1 - \boldsymbol{P})\big(-\boldsymbol{A}_{s(r)-2}v + B(v) + f + \mathcal{R}_{s(r)}g\big). \end{aligned}\right\} \tag{5.8}$$

Thanks to (5.6) and (5.7) we can apply Theorem 2.2 to the first equation in (5.8). Then it follows that this equation has a unique maximal solution $u \in \mathcal{W}_{r,\mathrm{loc}}^1(J^+, \mathbb{H}_q^{s(r)})$. Consequently, w, defined by the second equation in (5.8), belongs to $L_{r,\mathrm{loc}}(J^+, \mathbb{G}_q^{s(r)-2})$. Hence (v, w) satisfies (5.5) with $J := J^+$. This shows that (5.4) has a unique maximal $L_r(H_q^{s(r)})$-solution. Furthermore, it has the properties stated in the Theorem.

If Ω is bounded then the Stokes semigroup $\{\, e^{-t\mathbb{A}_0}\ ;\ t \geq 0\,\}$ is well-known to be exponentially decaying. It follows from [**26**, Theorem V.2.1.3] that this is also true for $\{\, e^{-t\mathbb{A}_{s(r)-2}}\ ;\ t \geq 0\,\}$. Hence we deduce from Theorem 2.2 that (v, w) is a global solution if Ω is bounded and the norm of $\bigl(v^0, (f, g)\bigr)$ in the space (0.14) is sufficiently small.

On the basis of [**26**, Theorem V.2.8.3] we can modify the arguments of the last part of [**11**, Section 3] in the obvious way to show that (v, w) is an $L_r(H_q^{s(r)})$-solution of (5.4) on J iff it satisfies (0.10) and (0.11). These arguments and (5.8) show that this is the case iff v is an $L_r(\mathbb{H}_q^{s(r)})$-solution of the Navier-Stokes equations (0.1). Then w is determined by the second equation in (5.8). This proves the Theorem.

Remarks 5.1. (a) Suppose that $s := s(r) < 1/q$. Set $\delta := -\partial_\nu \,|\, \boldsymbol{H}_{q'}^{2-s}$ and fix $\delta^c \in \mathcal{L}\bigl(W_q^{-s+1/q}(\Gamma), H_{q'}^{2-s}\bigr)$ satisfying $\delta\delta^c = 1$ and $\gamma\delta^c = 0$. Then the map $p_\Omega := 1 - \delta^c\delta$ belongs to $\mathcal{L}(\boldsymbol{H}_{q'}^{2-s})$ and is a projection onto $\mathring{H}_{q'}^{2-s}$, parallel to $\mathrm{im}(\delta^c)$ (cf. [**11**, Proof of Theorem 1.1]). Setting $R(f, g) := p'_\Omega f + \mathcal{R}_{s(r)} g$, it follows from [**11**, Corollary 1.2] that R is an isomorphism from the space $H_q^{s-2} \times W_q^{s-1/q}(\Gamma)$ onto $\boldsymbol{H}_q^{s-2}$. Hence the Theorem remains true if assumption (0.9) is omitted, provided $\langle f, \varphi\rangle$ in (0.11) and (0.13), respectively, is replaced by $\langle f, p_\Omega\varphi\rangle$.

Assumption (0.9) guarantees that we can choose the coretraction δ^c for δ in such a way that $\langle f, p_\Omega\varphi\rangle = \langle f, \varphi\rangle$.

Proof. It suffices to replace $f + \mathcal{R}_{s(r)} g$ in (5.2) by $R(f, g)$. □

(b) Suppose that $1/r + 3/q < 1$ so that $s(r) = 1/r$. Then Proposition 4.2 implies that

$$b \in \mathcal{L}^2\bigl(\mathcal{W}_r^1(J_T, \mathbb{H}_q^{1/r}), L_r(J_T, \mathbb{H}_q^{-1-3/q})\bigr) \qquad \text{J-uniformly.}$$

Thus, since $-1 - 3/q > s(r) - 2$, maximal regularity is not needed, provided (f, g) has values in $H_q^{s(r)+\beta-2} \times W_q^{s(r)+\beta-1/q}(\Gamma)$ for some $\beta > 0$. In this case the right-hand side of the first equation in (5.8) is subordinate to the operator $\mathbb{A}_{s(r)-2}$ so that the smoothing property of the semigroup $\{\, e^{-t\mathbb{A}_{s(r)-2}}\ ;\ t \geq 0\,\}$ implies better regularity properties of the $L_r(H_q^{s(r)})$-solution for $t > 0$.

Proof. This follows by replacing (1.9) in the proof of Theorem 2.2 by [**28**, Lemma 4]. Details are left to the interested reader. □

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