Nonhomogeneous Linear and Quasilinear Elliptic and Parabolic Boundary Value Problems

Herbert Amann

It is the purpose of this paper to describe some of the recent developments in the mathematical theory of linear and quasilinear elliptic and parabolic systems with nonhomogeneous boundary conditions. For illustration we use the relatively simple set-up of reaction-diffusion systems which are — on the one hand — typical for the whole class of systems to which the general theory applies and — on the other hand — still simple enough to be easily described without too many technicalities. In addition, quasilinear reaction-diffusion equations are of great importance in applications and of actual mathematical and physical interest, as is witnessed by the examples we include.

In particular, we try to elucidate the rôles which are played in the theory of quasilinear parabolic systems by the modern theory of function spaces, interpolation and extrapolation techniques, and semigroup theory. Many of the proofs will be sketched only and we will be rather brief at times. However, we try to explain the basic underlying ideas and give references to the research literature where proofs can be found. A complete, detailed, and coherent presentation will be given in the forthcoming monograph [Ama94] which will also contain many additional results and extensions of the theory described in this paper.

1 Model Problems

Throughout this text Ω denotes a bounded domain in $\mathbb{R}^n$ with a smooth boundary so that $\overline{\Omega}$ is a compact n-dimensional C^∞–submanifold of $\mathbb{R}^n$ with boundary $\partial\Omega$. We denote by $\vec{\nu} = (\nu^1, \ldots, \nu^n)$ the outer unit normal vector field to $\partial\Omega$.

A great many physical (physico-chemical, biological, etc.) theories can be described mathematically by conservation laws of the form

$$\partial_t u_r + \operatorname{div} \vec{\jmath}_r = f_r \qquad \text{in } \Omega \ , \tag{1.1$_r$}$$

where $r = 1, \ldots, N$ and $u := (u_1, \ldots, u_N)$ is a vector of densities (concentrations, population densities, etc.), $\vec{\jmath}_r$ is the flux vector, and f_r is the production rate for the r^{th} species. With this interpretation $(1.1)_r$ is simply a mass balance law. Of course, other interpretations of $(1.1)_r$ are possible too. For example, u_r may describe (for some fixed r) the temperature of a system, $\vec{\jmath}_r$ the heat flux, and f_r the heat source density, etc.

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Generally, the production rate f_r will be a known function of the position $x \in \Omega$ and the vector u, that is,

$$f_r = f_r(x,u) \ . \tag{1.2$_r$}$$

(In general, f_r will depend on the time variable t too. However, for simplicity we restrict ourselves here to the autonomous case or simply suppress the variable t.) Note that $(1.2)_r$ represents the 'reaction rate' of the species r in the physico-chemical context.

The flux vector $\vec{\jmath}_r$ of the species r is specified by phenomenological laws. In many applications it is of the form

$$\vec{\jmath}_r = -\sum_s \bigl(a^{rs}(\cdot,u)\nabla u_s + \vec{a}^{rs}(\cdot,u)u_s\bigr) \ , \tag{1.3$_r$}$$

where a^{rs} and $\vec{a}^{rs}$are known functions of x and u (and t, in general). The first sum in $(1.3)_r$ is the so-called diffusion term, where a^{rs} for $s \neq r$ describe cross-diffusion effects occurring in multicomponent diffusion problems (e.g., [deGM84]). In the simplest case, where there are no cross-diffusion terms and $\vec{a}^{rs} = 0$ for all s, the flux vector reduces to

$$\vec{\jmath}_r = -a^{rr}(\cdot,u)\nabla u \ ,$$

which is nothing else than Fick's law (Fourier's law, Darcy's law, etc.) provided

$$a^{rr} > 0 \ . \tag{1.4$_r$}$$

In the general situation we shall impose a condition upon the matrix $[a^{rs}]_{1\leq r,s\leq N}$ generalizing $(1.4)_r$ suitably and guaranteeing the parabolicity of the resulting system.

The terms $\vec{a}^{rs}u_s$ in $(1.3)_r$ describe drifts in the direction of the vector fields $\vec{a}^{rs}$ proportional to the concentration of the species s. These drift vectors, which are often gradients of 'outer potentials', are sometimes specified by additional 'stationary equations'. In those cases the vector fields $\vec{a}_{rs}$ are nonlocal functions (i.e., nonlinear operators) of u.

By inserting $(1.2)_r$ and $(1.3)_r$ in $(1.1)_r$ we obtain a quasilinear system of partial differential equations for u in Ω, a quasilinear reaction-diffusion system. In order to describe the time evolution of the vector u completely we have to specify the behavior of u on the boundary, and initial conditions. In practice there occur two types of boundary conditions. Most prominently, we find 'prescribed (in-) flux conditions' for the species r of the form

$$-(\vec{\jmath}_r|\vec{\nu}) = g_r^1(\cdot,u) \qquad \text{on } \partial_{1,r}\Omega \ , \tag{1.5$_r$}$$

where g_r^1 is a known function of x and u (and t, in general), $(\cdot|\cdot)$ denotes the Euclidean inner product, and $\partial_{1,r}\Omega$ is a union of components of connectedness of $\partial\Omega$. Note that $(1.5)_r$ reduces to a 'no flux' boundary condition on $\partial_{1,r}\Omega$ if $g_r^1 = 0$. The second class of boundary conditions which are often considered, comprises Dirichlet boundary conditions where the value of u_r is prescribed on $\partial_{0,r}\Omega := \partial\Omega\backslash\partial_{1,r}\Omega$, that is,

$$u_r = g_r^0 \qquad \text{on } \partial_{0,r}\Omega \ , \tag{1.6$_r$}$$

where g_r^0 is a given function of x (and t, in general).

Note that the species r satisfies a prescribed flux boundary condition $(1.5)_r$ on one part of the boundary, namely on $\partial_{1,r}\Omega$, and a Dirichlet condition $(1.6)_r$ on the remaining part $\partial_{0,r}\Omega$.

In order to have a unified simple representation we introduce functions δ_r assuming the two values 0 and 1 only, such that $\delta_r|\partial_{1,r}\Omega = 1$ and $\delta_r|\partial_{0,r}\Omega = 0$. We also put

$$\mathcal{B}_r(u)u := -\delta_r(\vec{\jmath}_r|\vec{\nu}) + (1-\delta_r)u_r$$

and

$$g_r(\cdot,u) := \delta_r g_r^1(\cdot,u) + (1-\delta_r)g_r^0 \ .$$

Then the boundary conditions $(1.5)_r$ and $(1.6)_r$ can be rewritten in the concise form

$$\mathcal{B}_r(u)u = g_r(\cdot,u) \qquad \text{on } \partial\Omega \ . \tag{$1.7)_r$}$$

By introducing Euclidean coordinates we rewrite now the complete quasilinear reaction-diffusion system $(1.1)_r$–$(1.6)_r$, $r = 1,\ldots,N$, in the form

$$\begin{aligned} \partial_t u + \mathcal{A}(u)u &= f(u) \qquad && \text{in } \Omega \ , \\ \mathcal{B}(u)u &= g(u) \qquad && \text{on } \partial\Omega \ , \end{aligned} \tag{1.8}$$

where

$$\mathcal{A}(u)v := -\partial_j\big(a(\cdot,u)\partial_j v + a_j(\cdot,u)v\big) \tag{1.9}$$

and

$$\mathcal{B}(u)v := \delta\big[a(\cdot,u)\partial_\nu v + \nu^j a_j(\cdot,u)v\big] + (1-\delta)v \ . \tag{1.10}$$

Here and in the following we use the summation convention with j (and k) running from 1 to n. Moreover,

$$a(\cdot,u) := \big[a^{rs}(\cdot,u)\big]_{1\le r,s\le N}$$

is the 'diffusion matrix',

$$a_j(\cdot,u) := \big[\alpha_j^{rs}(\cdot,u)\big]_{1\le r,s\le N} \ , \qquad j = 1,\ldots,n \ ,$$

where $\vec{\alpha}^{rs} = (\alpha_1^{rs},\ldots,\alpha_n^{rs})$, and $\partial_\nu u := \nu^j\partial_j u$ is the normal derivative, and $\delta := \mathrm{diag}[\delta_r]_{1\le r\le N}$ is the diagonal matrix with diagonal elements δ_r. Of course, f and g have the obvious meanings.

We include now a number of examples describing standard and not so standard reaction-diffusion systems, which are used to illustrate the scope and the generality of our approach. We will come back to these examples in Section 17 below and show how the general theory developed in this paper applies in these concrete instances.

1.1 Population Dynamics In the theoretical study of spacial segregation of interacting species Shigesada, Kawasaki, and Teramoto [ShKT79] introduced a two population model of the form

$$\begin{aligned} \partial_t u_r + \mathrm{div}\,\vec{\jmath}_r(u) &= u_r g_r(u) \qquad && \text{in } \Omega \ , \\ \big(\vec{\jmath}_r(u)|\vec{\nu}\big) &= 0 \qquad && \text{on } \partial\Omega \ , \end{aligned} \tag{1.11}$$

$r = 1,2$, where $u = (u_1,u_2)$ is the vector of population densities (i.e., $N = 2$) and Ω is two-dimensional (i.e., $n = 2$). Moreover, the 'flux vectors' $\vec{\jmath}_r(u)$ are of the form

$$\vec{\jmath}_r(u) := -\nabla\big[(\alpha_r + \beta_{r1}u_1 + \beta_{r2}u_2)u_r\big] - \gamma_r u_r \nabla\varphi \ ,$$

where $\alpha_r > 0$, $\beta_{rs} \geq 0$, $\gamma_r \geq 0$ are constants and φ is a known outer potential. Thus the terms $(\beta_{r1}u_1 + \beta_{r2}u_2)u_r$ model 'social friction and competition'. It is easy to rewrite $\vec{\jmath}_r$ in the form $(1.3)_r$. Then, letting

$$a(\cdot, u) := \begin{bmatrix} \alpha_1 + 2\beta_{11}u_1 + \beta_{12}u_2 & \beta_{12}u_1 \\ \beta_{21}u_2 & \alpha_2 + \beta_{21}u_1 + 2\beta_{22}u_2 \end{bmatrix} \tag{1.12}$$

and

$$a_j(\cdot, u) := \begin{bmatrix} \gamma_1 \partial_j \varphi & 0 \\ 0 & \gamma_2 \partial_j \varphi \end{bmatrix} , \tag{1.13}$$

it follows that (1.11) is of the form (1.8)–(1.10) with $\delta = 1$. Since the u_r represent densities, nonnegative solutions are of interest only. □

1.2 Phase Transition Models Penrose and Fife [PeF90] propose thermodynamical consistent models of phase-field type for the study of phase transitions. They involve an 'order parameter' φ and the absolute temperature T, which are supposed to satisfy the differential equations

$$\left.\begin{aligned} \partial_t \varphi &= \nabla \cdot \big(\kappa(T)\nabla\varphi\big) + f_1(\varphi, T) \\ \mu(\varphi)\partial_t T - \lambda(\varphi, T)\partial_t \varphi &= -\nabla \cdot \big(M(\varphi, T)\nabla(1/T)\big) \end{aligned}\right\} \quad \text{in } \Omega . \tag{1.14}$$

Here κ, λ, μ, M, and f_1 are given smooth functions such that κ, μ, and M are positive. In fact, various simplifying assumptions are given in [PeF90].

Let $u = (u_1, u_2) := (\varphi, T)$ and solve (1.14) algebraically for $\partial_t u = (\partial_t \varphi, \partial_t T)$. Then (1.14) takes the form $\partial_t u + \mathcal{A}(u)u = f(u)$, where $\mathcal{A}$ is given by (1.9) with

$$a(\cdot, u) := \begin{bmatrix} \kappa & 0 \\ \lambda\kappa/\mu & M/\mu T^2 \end{bmatrix} \tag{1.15}$$

and $a_j = 0$, and where

$$f(u) := \Big(f_1, (\lambda/\mu)f_1 - \big(\nabla(\lambda/\mu)|\kappa\nabla\varphi\big) - \big(\nabla(1/\mu)|(M/T^2)\nabla T\big)\Big) . \tag{1.16}$$

Note that

$$\big(\nabla(\lambda/\mu)|\kappa\nabla\varphi\big) = \partial_1(\lambda/\mu)\kappa\,|\nabla\varphi|^2 + \partial_2(\lambda/\mu)\kappa(\nabla T|\nabla\varphi) \tag{1.17}$$

and

$$\big(\nabla(1/\mu)|(M/T^2)\nabla T\big) = -(M/\mu T^2)\partial\mu(\nabla\varphi|\nabla T) \tag{1.18}$$

so that f is 'quadratic in the gradient ∇u'.

In [PeF90] there is no discussion of boundary conditions. However, Sprekels and Zheng [SpZ92] study the particular case of (1.14) with

$$\kappa := \text{const} , \quad M := \text{const} , \quad \mu := 1 , \quad \lambda(\varphi, T) := a\varphi \tag{1.19}$$

for some positive constant a, and with

$$f_1(\varphi, T) := s'(\varphi) + a\varphi/T , \tag{1.20}$$

where s is a given smooth function (such that $-s$ is a 'double-well potential'). These authors consider the Neumann boundary conditions

$$\partial_\nu \varphi = 0 \;, \quad \partial_\nu T + \alpha(T - T_\partial) = 0 \qquad \text{on } \partial\Omega \;, \tag{1.21}$$

with α a positive constant and T_∂ a positive function. Observe that these boundary conditions are of the form $\mathcal{B}(u)u = g(u)$, where $\mathcal{B}$ is given by (1.10) with $\delta = 1$, and

$$g(u) := \bigl(0, -(\alpha M / T^2 \mu)(T - T_\partial)\bigr) \;. \tag{1.22}$$

In the general case (1.14) we consider prescribed flux conditions too, that is, we assume that

$$\left.\begin{aligned} \kappa(T)\partial_\nu \varphi &= g_1(\varphi, T) \\ \bigl(M(\varphi,T)/T^2\bigr)\partial_\nu T + \lambda(\varphi,T)\kappa(T)\partial_\nu\varphi &= \mu(\varphi) g_2(\varphi, T) \end{aligned}\right\} \quad \text{on } \partial\Omega \;, \tag{1.23}$$

where $g := (g_1, g_2)$ is a given smooth function. Then the system (1.14), (1.23) is equivalent to a system of the form (1.8)–(1.10) where, in general, f depends quadratically on ∂u. Of course, solutions satisfying $T > 0$ are of interest only. □

1.3 Flows in Porous Media In [CDL92] the authors consider the problem of the mixing of fresh and salt groundwater, due to mechanical dispersion and molecular diffusion. They are led to a single mass balance equation

$$\partial_t u + \operatorname{div} \vec{\jmath} = 0 \qquad \text{in } \Omega \;, \tag{1.24}$$

together with the no flux boundary condition

$$(\vec{\jmath} \,|\, \vec{\nu}) = 0 \qquad \text{on } \partial\Omega \;, \tag{1.25}$$

where

$$\vec{\jmath} = -M(\vec{q})\nabla u + \vec{q} u \tag{1.26}$$

and $n = 2$. Here $M(\vec{q})$ is the hydrodynamical dispersion matrix, given (in dimensionless form) by

$$M(\vec{q})_{jk} := \begin{cases} (a\,|\vec{q}| + m)\delta_{jk} + b q_j q_k / |\vec{q}| & \text{if } \vec{q} \neq 0 \;, \\ m\delta_{jk} & \text{if } \vec{q} = 0 \;, \end{cases} \tag{1.27}$$

where m is a positive and a and b are nonnegative constants. Here u is the (suitably normalized) specific weight of the fluid (the mixture of fresh and salt water) which is supposed to be incompressible. Lastly, $\vec{q}$ is the vector of specific discharge of the fluid. The functions u and $\vec{q}$ are related, via a (suitably normalized) stream function v, by

$$\vec{q} = \operatorname{curl} v \quad \bigl(= (\partial_2 v, -\partial_1 v)\bigr) \;. \tag{1.28}$$

Darcy's law and the particular geometry of the problem imply finally that u and v are connected by the elliptic boundary value problem

$$\begin{aligned} -\Delta v &= \partial_1 u && \text{in } \Omega \;, \\ v &= 0 && \text{on } \partial\Omega \;. \end{aligned} \tag{1.29}$$

Thus the model under consideration consists of the system (1.24)–(1.29) forming a parabolic-elliptic boundary value problem.

By solving (1.29) and substituting the solution via (1.27), (1.28) into (1.26) we arrive (at least formally) at a single quasilinear parabolic 'reaction-diffusion' system which is slightly more general than the form (1.8)–(1.10). More precisely, let

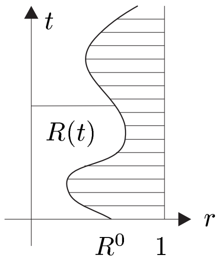
$$\vec{Q} := \operatorname{curl} \circ (-\Delta_D)^{-1} \circ \partial_1 \ , \qquad a := M \circ \vec{Q} \ , \qquad a_j := -Q_j \ , \quad j = 1, 2 \ , \tag{1.30}$$

where Δ_D denotes the Laplacian under zero Dirichlet boundary conditions, it follows that (1.24)–(1.29) is (formally) equivalent to the problem

$$\begin{aligned} \partial_t u - \partial_j\big(a_{jk}(u)\partial_k u + a_j(u)u\big) &= 0 \qquad \text{in } \Omega \ , \\ \nu^j\big(a_{jk}(u)\partial_k u + a_j(u)u\big) &= 0 \qquad \text{on } \partial\Omega \ . \end{aligned} \tag{1.31}$$

Note that in (1.31) the coefficients are nonlocal functions of u and that the dispersion matrix M is Lipschitz continuous but not continuously differentiable. Thus the coefficients a_{jk} have little regularity only. □

1.4 Moving Boundaries In [CoHS89] and [CoHS90] there is investigated a solid-liquid system of a small spherical grain exchanging matter with a liquid solution by a dissolution-growth process. Its mathematical formulation leads to a 'moving boundary problem'. To be more precise, denote by $c = c(r,t)$ the concentration in the liquid phase, where r is the distance from the center of the grain, and by $R = R(t)$ the radius of the grain. The authors study the following (rescaled) equations which describe the growth of the spherical grain.



$$\begin{aligned} c_t - r^{-2}(r^2 c_r)_r &= F_1(c) && \text{in } \big(R(t), 1\big) \ , \\ c_r &= (1-c)H(r,c) && \text{for } r = R(t) \ , \\ c &= \varphi && \text{for } r = 1 \ , \\ \dot{R} &= H\big(R, c(R,\cdot)\big) \ , \end{aligned} \tag{1.32}$$

where F_1, H, and φ are given functions.

We transform (1.32) to a 'fixed domain' problem by introducing $x := (1-r)/(1-R)$ as new independent space variable. We also put

$$u := (u_1, u_2) := (c - \varphi, R) \ .$$

Then (1.32) goes over into the following equivalent system:

$$\begin{aligned} \partial_t u_1 + \mathcal{A}_{11}(u)u_1 &= f_1(u) \qquad \text{in } \Omega \ , \\ \mathcal{B}_{11}(u)u_1 &= g_1(u) \qquad \text{on } \partial\Omega \ , \\ \dot{u}_2 &= f_2(u) \ , \end{aligned} \tag{1.33}$$

where

$$\mathcal{A}_{11}(u)v := -\big(a(u)v_x\big)_x + b(\cdot,u)v_x \tag{1.34}$$

and

$$\mathcal{B}_{11}(u)v := \delta a(u)\partial_\nu v + (1-\delta)v$$

with $\Omega := (0,1)$, $\delta(0) := 0$, $\delta(1) := 1$,

$$a(u) := (u_2 - 1)^{-2} \ , \tag{1.35}$$

and

$$b(x,u) := (1-u_2)^{-1}\bigl\{2\bigl[(u_2-1)x+1\bigr]^{-1} - xH(u_2, \gamma_1 u_1 + \varphi)\bigr\} \ , \tag{1.36}$$

with γ_1 being the restriction operator to the boundary point $x = 1$, and where

$$f_1(u) := F_1(u_1 + \varphi) - \dot{\varphi} \ , \quad f_2(u) := H(u_2, \gamma_1 u_1 + \varphi) \ , \tag{1.37}$$

and

$$g_1(u) := \delta(u_2 - 1)^{-1}(1 - u_1 - \varphi)H(u_2, \gamma_1 u_1 + \varphi) \ . \tag{1.38}$$

Note that b and f_2 are nonlocal operators and that $\mathcal{A}_{11}$ differs from an operator of the form (1.9) by the additive term $b\partial_x$ only.

We introduce now matrices of operators

$$\mathcal{A}(u) := \begin{bmatrix} \mathcal{A}_{11}(u) & 0 \\ 0 & 0 \end{bmatrix} \ , \quad \mathcal{B}(u) := \begin{bmatrix} \mathcal{B}_{11}(u) & 0 \\ 0 & 0 \end{bmatrix} \tag{1.39}$$

and put

$$f(u) := \begin{bmatrix} f_1(u) \\ f_2(u) \end{bmatrix} \ , \quad g(u) := \begin{bmatrix} g_1(u) \\ 0 \end{bmatrix} \ . \tag{1.40}$$

Then problem (1.33) has *formally* the appearance (1.8). Note that the operator matrices (1.39) are degenerate. □

1.5 Diffusion in Polymers In the study of diffusion and viscoelastic relaxation in polymers D.S. Cohen introduced a class of nonstandard reaction-diffusion models which he analyzed — together with coworkers — by means of formal asymptotic techniques and numerical methods (cf. [ChC89], [ChW89], [ChW91], for example). There are two unknown functions, the density c of a quantity (a fluid, for example) penetrating a polymer entanglement network, and the stress σ induced by the penetrating moleculs. These quantities satisfy the system of differential equations

$$\left.\begin{aligned} c_t + j_x &= 0 \\ \sigma_t - \varphi(c)c_t &= \rho(c)c - \beta(c)\sigma \end{aligned}\right. \qquad \text{in } \Omega \ , \tag{1.41}$$

where $\Omega \subset \mathbb{R}$ is a bounded open interval (i.e., $n = 1$), subject to the Dirichlet boundary condition

$$c = \psi \qquad \text{on } \partial\Omega \ . \tag{1.42}$$

The flux 'vector' is given by

$$j := -D(c)c_x - H(c)\sigma_x - M(c,\sigma)c \ ,$$

where $D > 0$ and H, φ, ρ, and β are nonnegative. Moreover, all these functions, as well as M and ψ, are smooth.

Let Φ be the primitive of φ satisfying $\Phi(0) = 0$ and assume without loss of generality that $\Omega = (0,1)$. Put

$$\Psi(x) := (1-x)\psi(0) + x\psi(1) \ , \qquad 0 \le x \le 1 \ ,$$

and

$$u_1 := c - \Psi \ , \quad u_2 := \sigma - \Phi(c) \ , \quad u := (u_1, u_2) \ .$$

Then (1.41), (1.42) is transformed into the equivalent system

$$\begin{aligned} \partial_t u_1 + \mathcal{A}_{11}(u)u_1 + \mathcal{A}_{12}(u)u_2 &= f_1(u) \\ \partial_t u_2 &= f_2(u) \end{aligned} \quad \text{in } \Omega \ , \qquad u_1 = 0 \quad \text{on } \partial\Omega \ , \tag{1.43}$$

where $\mathcal{A}_{11}$ and $\mathcal{A}_{12}$ are of the form (1.9), the top order coefficient of $\mathcal{A}_{11}$ being

$$(D + \varphi H) \circ c(u_1) \ge D\big(c(u_1)\big) \ . \tag{1.44}$$

Also note that

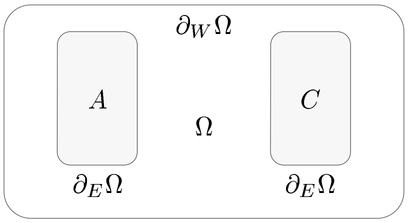
$$f_1(u) = \big(h(u)\big)_x \tag{1.45}$$

with a smooth function h. By introducing the operator matrices

$$\mathcal{A}(u) := \begin{bmatrix} \mathcal{A}_{11}(u) & \mathcal{A}_{12}(u) \\ 0 & 0 \end{bmatrix} \ , \quad \mathcal{B} := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

the system (1.43) takes (formally) the form (1.8) with $g = 0$. Of course, this problem is again degenerate. In fact, it is even more degenerate than (1.33) since, in general, the differential expression $\mathcal{A}_{12}$ operating on u_2 is of the same strength — namely a second order operator — as $\mathcal{A}_{11}$ operating on u_1. Of course, only solutions satisfying $c = u_1 + \Psi \ge 0$ are of interest. □

1.6 Reaction and Diffusion in Electrolysis Consider a vessel Ω, whose boundary consists of the outer wall $\partial_W\Omega$ and the surfaces $\partial_E\Omega$ of electrodes, an anode A and a cathode C, such that $\partial\Omega$ is the disjoint union of $\partial_W\Omega$ and $\partial_E\Omega$. In Ω there is contained a solute of N dissolved species, which can be ions or electrically neutral, whose nonnegative concentrations are denoted by $u_1, \ldots, u_N$. Each one of the species r satisfies the mass balance law $(1.1)_r$. The flux $\vec{\jmath}_r$ is due to the concentration gradient of species r, to convection, and to migration in an electric field φ. Thus



$$\vec{\jmath}_r := -a_r(\cdot, u)\nabla u_r + u_r\vec{\omega} - m_r u_r \nabla\varphi \ ,$$

where the 'diffusion coefficients' a_r are positive, $\vec{\omega}$ is a given velocity vector, and m_r is the 'mobility' of species r. On the outer wall there are the no flux conditions

$$(\vec{\jmath}_r|\vec{\nu}) = 0 \qquad \text{on } \partial_W\Omega \ , \quad r = 1, \ldots, N \ . \tag{1.46$_r$}$$

On the surface of the electrodes electrochemical reactions occur which are modelled by nonlinear boundary conditions of the form

$$-(\vec{\jmath}_r|\vec{\nu}) = g_r(\cdot,u,\varphi) \qquad \text{on } \partial_E\Omega\ , \quad 1 \le r \le N\ , \tag{1.47$_r$}$$

where the g_r are highly nonlinear functions of u and φ (e.g., [RMK86], [New68]). All occurring functions are smooth.

Letting

$$a(\cdot,u) := \operatorname{diag}\bigl[a_r(\cdot,u)\bigr]\ , \quad a_j(\cdot,u,\varphi) := \operatorname{diag}[-\omega_j + m_r\partial_j\varphi]\ ,$$

the resulting reaction-diffusion system $(1.1)_r$, $(1.46)_r$, $(1.47)_r$ is of the form (1.8)–(1.10) with $\delta = 1$ (and $g|\partial_W\Omega = 0$, of course). Note that φ, the electric potential in the solute, is an unknown too, so that the system is underdetermined as it stands. However, there is an additional algebraic condition which has to be satisfied. Indeed, let z_r be the charge of species r so that z_r is a given real number which equals zero if species r is electrically neutral. Then $\sum z_r u_r$ is the total charge in the solute, and the 'electroneutrality condition'

$$\textstyle\sum_r z_r u_r = 0 \tag{1.48}$$

requires that the total charge be zero at any place and at any time. It is an important information — as we shall see in Section 15 — that

$$z_r m_r \ge 0\ , \qquad r = 1,\ldots,N\ , \tag{1.49}$$

that is, the mobility of each species, has the same sign as the corresponding charge.

Thus the full system, describing the mathematical model for reactions and diffusion in the electrolytical model envisaged, is given by $(1.1)_r$, $(1.46)_r$, $(1.47)_r$, and (1.48). Note again that only solutions satisfying $u_r \ge 0$ are of interest. $\square$

1.7 Stefan Problems Consider a one-dimensional heat conducting medium occupying the interval $[0,1]$ and suppose that a phase change takes place, where the 'interface' between the two phases (a solid and a liquid one, for example) is at time t at the point $s(t) \in (0,1)$. Also suppose that in each phase the temperature T satisfies a nonlinear heat equation

$$\partial_t T - \partial_x\bigl(k_i(\cdot,T)\partial_x T\bigr) = g_i(\cdot,T) \quad \text{in } \Omega_i(t)\ , \qquad i = 0,1\ , \tag{1.50}$$

where

$$\Omega_i(t) := \begin{cases} \bigl(0,s(t)\bigr)\ , & i = 0\ , \\ \bigl(s(t),1\bigr)\ , & i = 1\ , \end{cases}$$

and the boundary conditions

$$\begin{aligned} -\delta(0)k_0(0,T)\partial_x T + \bigl(1-\delta(0)\bigr)T &= 0 \qquad \text{at } x = 0\ , \\ \delta(1)k_1(1,T)\partial_x T + \bigl(1-\delta(1)\bigr)T &= 0 \qquad \text{at } x = 1\ . \end{aligned} \tag{1.51}$$

Of course, we suppose that the k_j are everywhere positive and — for simplicity — that all occurring functions are smooth.

At the interface the temperature must be equal to the phase change temperature, which we normalize to be zero. Hence

$$T\bigl(s(t)-\bigr) = T\bigl(s(t)+\bigr) = 0\ . \tag{1.52}$$

Let L be the heat absorbed or released for melting or solidifying (that is, for the mechanism of the phase change) per unit volume of the material. Then $L\dot{s}$ is the local rate of heat absorption or release in the process. The heat balance states that $L\dot{s}$ must equal the difference between the heat flux coming to the interphase from each side, that is, the 'Stefan condition'

$$L\dot{s} = -k_0\partial_x T|_{x=s(t)-} + k_1\partial_x T|_{x=s(t)+} \tag{1.53}$$

has to be satisfied (e.g., [Cra84], [Rub71]). The one-dimensional two phase Stefan problem consists in finding the temperature distribution T and the interface s such that (1.50)–(1.53) are satisfied, provided, of course, initial conditions are specified.

Observe that the coordinate transformations

$$\varphi_i(x,t) := (y,\tau) := \begin{cases} \big(x/s(t),t\big) & \text{if } i=0 \ , \\ \big((1-x)/(1-s(t)),t\big) & \text{if } i=1 \ , \end{cases}$$

transform each

$$D_i := \big\{\, (x,t) \ ; \ x \in \Omega_i(t) \,\big\} \ , \qquad i=0,1 \ ,$$

onto $\Omega \times \mathbb{R}^+$, where $\Omega := (0,1)$, such that

$$\varphi_0(0,t) = (0,t) \ , \quad \varphi_i\big(s(t),t\big) = (1,t) \ , \quad \varphi_1(1,t) = (0,t) \ ,$$

(provided, of course, $0 < s(t) < 1$ for $t \in \mathbb{R}^+$).

Let

$$u_i := T \circ \varphi_i^{-1}$$

and note that

$$\partial_x = -(i-s)^{-1}\partial_y \ , \quad \partial_t = \partial_\tau + \big(y\dot{s}/(i-s)\big)\partial_y \ .$$

Put

$$a_i(\cdot,u_i,s) := (i-s)^{-2}k_i(\cdot,T) \circ \varphi_i^{-1}$$

and

$$f_i(\cdot,u_i) := g_i(\cdot,T) \circ \varphi_i^{-1} \ .$$

Then (1.50) transforms into

$$\partial_\tau u_i - \partial_y\big(a_i(\cdot,u_i,s)\partial_y u_i\big) + \big(y\dot{s}/(i-s)\big)\partial_y u_i = f_i(u_i) \qquad \text{in } \Omega \ . \tag{1.54}$$

The boundary conditions (1.51) and (1.52) are equivalent to

$$\begin{aligned} -\delta_i a_i(\cdot,u_i,s)\partial_y u_i + (1-\delta_i)u_i = 0 \qquad & \text{at } x=0 \ , \\ u_i = 0 \qquad & \text{at } x=1 \ , \end{aligned} \tag{1.55}$$

where $\delta_0 := \delta(0)$ and $\delta_1 := \delta(1)$. Lastly, the Stefan condition (1.53) transforms into

$$L\dot{s} = -sa_0\big(1,u_0(1,\cdot),s\big)\partial_y u_0(1,\cdot) - (1-s)a_1\big(1,u_1(1,\cdot),s\big)\partial_y u_1(1,\cdot) \ . \tag{1.56}$$

Define a nonlocal map h by

$$h(s,u) := (1/L)\sum_i (-1)^i(i-s)a_i(1,\gamma_1 u_i,s)\gamma_1\partial_y u_i$$

and put $u := (u_0, u_1)$,

$$a(\cdot, u, s) := \operatorname{diag}\bigl[a_0(\cdot, u_0, s), a_1(\cdot, u_1, s)\bigr] \tag{1.57}$$

and

$$b(\cdot, u, s) := y \operatorname{diag}\bigl[-h(s, u)/s, h(s, u)/(1 - s)\bigr] \ . \tag{1.58}$$

Then we can rewrite (1.54)–(1.56) as (writing again t for τ)

$$\begin{aligned} \partial_t u + \mathcal{A}(u, s)u &= f(u) && \text{in } \Omega \ , \\ \mathcal{B}(u, s)u &= 0 && \text{on } \partial\Omega \ , \\ \dot{s} &= h(s, u) \ , \end{aligned} \tag{1.59}$$

where $\mathcal{A}$ and $\mathcal{B}$ are of the form (1.9) (or rather of the form (1.34)) and (1.10), respectively. Thus we have transformed the Stefan problem into an equivalent form which has — superficially — the same appearance as (1.33). $\square$

1.8 Dynamic Boundary Conditions In the mathematical description of chemical kinetics and chemical technology there occur also 'dynamic boundary conditions' for the reaction-diffusion systems $(1.1)_r$, which are of the form

$$\partial_t u_r - (\vec{\jmath}_r | \vec{\nu}) = g_r \qquad \text{on } \partial\Omega \ , \quad 1 \le r \le N \ , \tag*{$(1.60)_r$}$$

(e.g., [Cra56], [SlH72]). Although the reaction-diffusion problem $(1.1)_r$, $(1.60)_r$ does not fit immediately into the framework discussed so far, the more general abstract setting described in this paper encompasses these problems too. $\square$

2 Informal Discussion

We return to problem (1.8)–(1.10) and consider first the particular case where

$$g = 0 \ . \tag{2.1}$$

If u is a sufficiently regular fixed function, $\mathcal{B}(u)$ is a well defined linear operator (on a suitable space of distributions). Thus it makes sense to put

$$A(u) := \mathcal{A}(u) | \ker \mathcal{B}(u) \ . \tag{2.2}$$

With this definition the system (1.8) with $g = 0$ can be written as an abstract quasilinear ordinary differential equation of the form

$$\dot{u} + A(u)u = f(u) \tag{2.3}$$

in a suitable Banach space. To be more specific, we fix $p \in (1, \infty)$ and put

$$E_0 := L_p(\Omega, \mathbb{R}^N) \ .$$

We also put

$$E_1(u) := W^2_{p,\mathcal{B}(u)} := \bigl\{ v \in W^2_p(\Omega, \mathbb{R}^N) \ ; \ \mathcal{B}(u)v = 0 \bigr\} = \ker \mathcal{B}(u) \ ,$$

where $\mathcal{B}(u)$ has to be interpreted in the sense of traces, of course. Then $A(u)$, defined by (2.2), is a densely defined linear operator in E_0 with domain $E_1(u)$, and equation (2.3) is meaningful if f is sufficiently nice.

In order to study equation (2.3) we have to impose regularity conditions and structural assumptions upon $A(u)$. In the most elementary case, where $N = 1$ and $\vec{\jmath} := -\nabla u$, where f and g are zero, and where $\delta = 1$ (resp. $\delta = 0$) on $\partial\Omega$, the system (1.8) reduces to the linear heat equation

$$\partial_t u - \Delta u = 0 \qquad \text{in } \Omega \tag{2.4}$$

under homogeneous Neumann (resp. Dirichlet) boundary conditions. In this case it is well known that the operator A, that is, $-\Delta$ with Neumann (resp. Dirichlet) boundary conditions, is the negative infinitesimal generator of a strongly continuous analytic semigroup on E_0 and that the domain of A, equipped with its graph norm, equals (except for equivalent norms) $E_1 = W^2_{p,\mathcal{B}}$. Of course, the general results should contain (2.4) as a special case. For this reason we impose the 'abstract parabolicity condition' that $A(u)$ be, for each u which may occur at any given instant of time as a solution to the system (1.8), the negative infinitesimal generator of an analytic semigroup on E_0 with domain $E_1(u)$; in symbols:

$$A(u) \in \mathcal{H}\big(E_1(u), E_0\big)\ . \tag{2.5}$$

The formulation of the reaction-diffusion system (1.8) in the form (2.3) has the advantage of being very flexible and of comprising a great many different problems like higher order systems, problems with nonlocal nonlinearities, and problems with dynamic boundary conditions, to name a few.

On the other hand, it has serious shortcomings. First, it requires condition (2.1) and there is no easy way to incorporate nonlinearities on the boundary of the form $(1.7)_r$ with $\delta_r \neq 0$. Second, in general, the domains of the operators $A(u)$, namely $E_1(u)$, depend on the sought-for solution u itself. At present, there is no good theory which is sufficiently flexible and well developed to treat abstract evolution equations of the form (2.3) in the case of 'variable domains' of the negative generators $A(u)$.

It is an important feature of the reaction-diffusion system (1.8) that it is of 'divergence form'. This suggests a weak formulation of (1.8). For this we introduce a (generalized) 'Dirichlet form' by

$$\boldsymbol{a}(u)(w,v) := \big\langle \partial_j w, a(\cdot,u)\partial_j v + a_j(\cdot,u)v \big\rangle\ , \qquad w \in W^1_{p'}(\Omega,\mathbb{R}^N)\ , \quad v \in W^1_p(\Omega,\mathbb{R}^N)\ , \tag{2.6}$$

where

$$\langle w,v\rangle := \int_\Omega \big\langle w(x), v(x)\big\rangle_{\mathbb{K}^N}\, dx \tag{2.7}$$

is the standard duality pairing

$$\langle\cdot,\cdot\rangle : L_{p'}(\Omega,\mathbb{K}^N) \times L_p(\Omega,\mathbb{K}^N) \to \mathbb{K}\ ,$$

with $p' := p/(p-1)$ and $\langle \xi,\eta\rangle_{\mathbb{K}^N} := \sum_{r=1}^N \xi_r\eta_r$ for $\xi,\eta \in \mathbb{K}^N$, and with $\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$.

We also put

$$E_{1/2} := W^1_{p,\mathcal{B}} := \big\{\, u \in W^1_p(\Omega,\mathbb{R}^N)\ ;\ (1-\delta)\gamma_\partial u = 0 \,\big\}\ ,$$

where γ_∂ denotes the trace operator. Note that $W^1_{p,\mathcal{B}} = W^1_p(\Omega, \mathbb{R}^N)$ if $\delta = 1 = 1_{\mathbb{R}^N}$, that is, if the Neumann type boundary condition $(1.5)_r$ holds for each r and everywhere on $\partial\Omega$.

Lastly, we assume instead of (2.1) that $g^0 = 0$, that is,

$$(1-\delta)g = 0 \ . \tag{2.8}$$

In other words, we assume that the homogeneous Dirichlet boundary conditions $u_r = 0$ on $\partial_{0,r}\Omega$ are satisfied if $\partial_{0,r}\Omega \neq \emptyset$. Thus (2.8) is no restriction at all if $\delta = 1$, that is, if the prescribed flux condition $(1.5)_r$ is satisfied for each species r and on all of $\partial\Omega$. Putting

$$\langle w, v\rangle_\partial := \int_{\partial\Omega} \bigl\langle w(x), v(x)\bigr\rangle_{\mathbb{K}^N} \, d\sigma(x) \ ,$$

where $d\sigma$ is the standard Riemann-Lebesgue volume measure on $\partial\Omega$, we associate with (1.8) the following 'weak W^1_p-formulation':

$$\begin{aligned}
&\text{find } u : \mathbb{R}^+ \to W^1_{p,\mathcal{B}} \text{ such that} \\
&\langle w, \partial_t u\rangle + \boldsymbol{a}(u)(w,u) = \bigl\langle w, f(u)\bigr\rangle + \bigl\langle \gamma_\partial w, g(u)\bigr\rangle_\partial \\
&\text{for each } w \in W^1_{p',\mathcal{B}}.
\end{aligned} \tag{2.9}$$

Of course, $\partial_t u$ has to exist and be sufficiently regular so that $\langle w, \partial_t u\rangle$ makes sense.

If $a(\cdot, u)$ and $a_j(\cdot, u)$ belong to $L_\infty\bigl(\Omega, \mathcal{L}(\mathbb{R}^N)\bigr)$, the Dirichlet form (2.6) is a continuous bilinear form on $W^1_{p',\mathcal{B}} \times W^1_{p,\mathcal{B}}$. Thus it defines a bounded linear operator

$$A_{-1/2}(u) \in \mathcal{L}(E_{1/2}, E_{-1/2}) \ , \qquad \text{where} \quad E_{-1/2} := (W^1_{p',\mathcal{B}})' \ , \tag{2.10}$$

by

$$\bigl\langle w, A_{-1/2}(u)v\bigr\rangle_{E_{-1/2}} := \boldsymbol{a}(u)(w,v) \ , \qquad w \in W^1_{p',\mathcal{B}} \ , \quad v \in W^1_{p,\mathcal{B}} \ . \tag{2.11}$$

Here and in the following $\langle\cdot,\cdot\rangle_E$ denotes the duality pairing

$$\langle\cdot,\cdot,\rangle_E : E' \times E \to \mathbb{K}$$

between the dual E' of a Banach space E and E itself. Defining $F(u) \in E_{-1/2}$ by

$$\bigl\langle w, F(u)\bigr\rangle_{E_{-1/2}} := \bigl\langle w, f(u)\bigr\rangle + \bigl\langle \gamma_\partial w, g(u)\bigr\rangle_\partial \ , \qquad w \in W^1_{p',\mathcal{B}} \ , \tag{2.12}$$

we can rewrite (2.9), formally, as an abstract evolution equation in the Banach space $E_{-1/2}$:

$$\dot{u} + A_{-1/2}(u)u = F(u) \ . \tag{2.13}$$

The weak formulation (2.9), resp. its abstract formulation (2.13), has two important advantages over its 'strong' counterpart (2.3). First, the domains of the operators $A_{-1/2}(u)$ are independent of u since they equal the Banach space $E_{1/2} = W^1_{p,\mathcal{B}}$. Second, the weak formulation handles with relative ease nonlinear boundary conditions satisfying (2.8), that is, nonlinear prescribed flux conditions. Of course, in order to treat (2.13) efficiently one has to show that $A_{-1/2}(u)$ is the negative infinitesimal generator of a strongly continuous analytic semigroup on $E_{-1/2}$, that is,

$$A_{-1/2}(u) \in \mathcal{H}(E_{1/2}, E_{-1/2}) \ .$$

Traditionally, weak formulations of partial differential equations are W_2^1-formulations, that is, one considers the case $p = p' = 2$. In addition, in this connection it is always assumed that the quadratic form $\boldsymbol{a}(u)(\cdot,\cdot)$ is coercive in the sense that

$$\boldsymbol{a}(u)(v,v) \geq \alpha \|v\|_{1,2}^2 - \beta \|v\|_2^2 \ , \qquad v \in W_{2,\mathcal{B}}^1 \ , \tag{2.14}$$

where $\alpha > 0$ and $\beta \geq 0$, and where $\|\cdot\|_{s,q}$ is the norm in the Sobolev space W_q^s with $\|\cdot\|_q := \|\cdot\|_{0,q}$. Condition (2.14) amounts to an ellipticity condition for the differential operator which is too restrictive and unnatural in many applications. Moreover, the W_2^1-setting requires severe growth restrictions for the nonlinearities a, a_j, f, and g in order that the problem be well-posed.

It is the purpose of the following considerations

- to show that the weak W_p^1-formulation (2.9) is meaningful in the sense that the corresponding evolution equation (2.13) is well-posed and generates a local semiflow provided, of course, suitable regularity conditions are satisfied. Since the coerciveness condition (2.14) is not available in the general case, we have to develop a method which is completely different from the standard Galerkin approximation technique usually employed in the weak W_2^1-setting (e.g., [DauL90, Chapter XVIII, § 3]). Our method is based upon semigroup theory, abstract evolution equations, and extrapolation techniques and will be explained in the following.

- to have a unified treatment of weak and strong solutions which provides an 'automatic regularity theory'. In particular, it will be shown that weak solutions are automatically classical, in fact C^∞–smooth, provided all data are in this regularity class. This is by no means obvious since it is well known that weak W_2^1-solutions to *systems* are — in general — not even Hölder continuous. The reason why we succeed in proving smoothness stems from the fact that we can study weak W_p^1-solutions for $p > n$, which are automatically Hölder continuous.

- to be able to treat nonlinear boundary conditions which occur rather naturally in many applications.

- to have a general framework which allows us to handle, with the same ease, general problems which are not parabolic systems in the narrow sense but are 'abstractly parabolic', be it in the sense that they involve nonlocal operators or that they are degenerate in the sense of some of the model problems of Section 1. The general theory presented below is applicable whenever the problem can be cast in the form (2.3), where $-A(u)$ is the infinitesimal generator of a strongly continuous analytic semigroup such that $\operatorname{dom}\big(A(u)\big)$ is independent of u, and if A and F satisfy certain mild continuity hypotheses. Thus the whole discussion will essentially be reduced to the task of finding the correct function space setting and proving generation and continuity theorems.

- to present a 'dynamic theory' of quasilinear parabolic evolution equations which is the basis for further qualitative studies of the semiflow, like questions of stability, global behavior, bifurcation phenomena, periodic orbits, etc. To prevent this rather long paper from being even longer and more demanding, we restrict ourselves here to the basic existence, uniqueness, and continuity questions for maximal solutions. We give only a few references to these qualitative questions. More of this will be found in [Ama94].

Although we are ultimately interested in nonlinear problems, large parts of the theory belong to the 'linear field'. For this reason a considerable amount of this paper is devoted to the study of function spaces and linear differential operators. Even in the case of linear elliptic and parabolic boundary value problems the results about weak and 'very weak' solvability given below are new.

3 Notations and Conventions

In general, we use standard notation. Nevertheless, a few explanations are in order.

Let X and Y be topological spaces. If $X \subset Y$ and the topology of X is stronger than the one of Y, that is, if the natural injection $i\colon X \subset Y$ is continuous, we denote this by $X \hookrightarrow Y$. We write $X \overset{d}{\hookrightarrow} Y$ if X is also dense in Y. If X and Y are topological vector spaces, $X \hookrightarrow Y$ also means that X is a vector subspace of Y.

Suppose $X \hookrightarrow Y$ and let Z be a nonempty subset of Y. Then $Z \cap X$ is given the topology induced by X, unless explicitly stated otherwise.

Let E and F be Banach spaces over $\mathbb{K}$. Then $\mathcal{L}(E,F)$ is the Banach space of all bounded linear operators from E into F, and $\mathcal{L}(E) := \mathcal{L}(E,E)$. We denote by $\mathcal{L}\mathrm{is}(E,F)$ the open set of isomorphisms in $\mathcal{L}(E,F)$. If G is a third Banach space, $\mathcal{L}(E,F;G)$ is the Banach space of all continuous bilinear maps $E \times F \to G$ with its usual norm.

If $E \hookrightarrow F$ and $F \hookrightarrow E$, we write $E \doteq F$, that is, $E \doteq F$ iff E and F coincide as vector spaces and carry equivalent norms. If $E \hookrightarrow F$ and the natural injection is compact, we denote this by $E \overset{c}{\hookrightarrow} F$.

We write $A\colon \mathrm{dom}(A) \subset E \to F$ if A is a linear operator with domain, $\mathrm{dom}(A)$, in E and image, $\mathrm{im}(A)$, in F. We denote by $D(A)$ the domain of A endowed with its graph norm. Then $D(A) \hookrightarrow E$ and $A \in \mathcal{L}\big(D(A),F\big)$. Recall that $D(A)$ is a Banach space iff A is closed.

If $E \hookrightarrow F$ and $A\colon \mathrm{dom}(A) \subset F \to F$ is a linear operator in F, the **E-realization** (the maximal restriction of A to E, or the part of A in E) is the linear operator A_E in E defined by

$$\mathrm{dom}(A_E) := \big\{\, x \in E \cap \mathrm{dom}(A) \ ; \ Ax \in E \,\big\} \ , \qquad A_E x := Ax \ .$$

If no confusion seems likely, we denote A_E again by A.

If $\mathbb{K} = \mathbb{R}$ and in a given formula there occur (implicitly or explicitly) complex numbers, it is always understood that the corresponding formula refers to its complexified version. For example, if A is a linear operator in E then $\sigma(A)$ and $\rho(A)$ denote the spectrum and the resolvent set, respectively, of A if $\mathbb{K} = \mathbb{C}$ and of the complexification of A if $\mathbb{K} = \mathbb{R}$.

If no confusion seems likely, 1 denotes the identity 1_E in $\mathcal{L}(E)$. In general, $\mathcal{L}(\mathbb{K}^n,\mathbb{K}^m)$ is canonically identified with the space $\mathbb{K}^{m\times n}$ of $m \times n$-matrices with entries in $\mathbb{K}$ by means of the canonical bases of $\mathbb{K}^m$ and $\mathbb{K}^n$, respectively.

Let E_0 and E_1 be Banach spaces with $E_1 \overset{d}{\hookrightarrow} E_0$. We denote by

$$\mathcal{H}(E_1,E_0)$$

the set of all $A \in \mathcal{L}(E_1,E_0)$ such that A, considered as a linear operator *in* E_0 with domain E_1, is the negative infinitesimal generator of an analytic semigroup $\{\, e^{-tA} \ ; \ t \geq 0 \,\}$

on E_0, that is, in $\mathcal{L}(E_0)$. (We refer to [CH*87], [Dav80], [Fat83], [Gol85], or [Paz83], for example, for the basic theory of analytic semigroups. However, a deeper understanding of this theory is not required for reading this paper.) It is an important fact that

$$\begin{aligned}&A \in \mathcal{H}(E_1, E_0) \text{ iff there exist constants } \kappa \geq 1 \text{ and } \omega > 0\\ &\text{such that } [\mathrm{Re}\, z \geq \omega] \subset \rho(-A) \text{ and}\\ &|\lambda|\, \|u\|_0 + \|u\|_1 \leq \kappa \|(\lambda + A)u\|_0 \ , \qquad u \in E_1 \ , \quad \mathrm{Re}\,\lambda \geq \omega \ ,\\ &\text{where } \|\cdot\|_j := \|\cdot\|_{E_j} .\end{aligned} \tag{3.1}$$

Moreover,

$$\mathcal{H}(E_1, E_0) \text{ is open in } \mathcal{L}(E_1, E_0) \tag{3.2}$$

(cf. [Ama91b, Section 1] for (3.1) and (3.2), and [Ama88a, Lemma 4.1] or [CH*87, Theorem 5.3] for (3.2)).

Let $X := (X, d)$ be a metric space. Then $u : X \to E$ is uniformly ρ-Hölder continuous, where $0 < \rho \leq 1$, if

$$[u]_\rho := [u]_{\rho, X} := \sup_{x \neq y} \frac{\|u(x) - u(y)\|}{\big[d(x,y)\big]^\rho} < \infty \ .$$

It is ρ-Hölder continuous if each point has a neighborhood Y such that $u|Y$ is uniformly ρ-Hölder continuous. We put

$$C^\rho(X, E) := \{\, u \in E^X \ ; \ u \text{ is } \rho\text{-Hölder continuous} \,\} \ , \qquad 0 < \rho < 1 \ ,$$

and $C^0(X, E) := C(X, E)$ is the set of all continuous maps from X to E. A 1-Hölder continuous function is Lipschitz continuous, and

$$C^{1\text{-}}(X, E) := \{\, u \in E^X \ ; \ u \text{ is Lipschitz continuous} \,\} \ .$$

In numerical calculations the symbol 1- will be identified with the real number 1.

We denote by $B(X, E)$ the Banach space of all bounded $u \in E^X$, endowed with the supremum norm $\|\cdot\|_\infty$, and $BUC(X, E)$ is the closed subspace of all bounded and uniformly continuous functions. If $\rho \in (0,1) \cup \{1\text{-}\}$ then $BUC^\rho(X, E)$ is the Banach space of all uniformly ρ-Hölder continuous functions equipped with the norm

$$\|\cdot\|_{C^\rho} := \|\cdot\|_\infty + [\cdot]_\rho \ .$$

Recall that a [ρ-Hölder] continuous function is uniformly [ρ-Hölder] continuous on compact sets. Hence $C^\rho(X, E) = BUC^\rho(X, E)$ for $\rho \in [0, 1) \cup \{1\text{-}\}$, if X is compact.

Lastly, if X is an open subset of some Banach space or a perfect subset of $\mathbb{K}$, the spaces $C^s(X, E)$ and $BUC^s(X, E)$ have the obvious meanings and topologies for $s \in \mathbb{R}^+$ and $s \in \{\, k\text{-} \ ; \ k \in \dot{\mathbb{N}} \,\}$, where $\dot{\mathbb{N}} := \mathbb{N} \backslash \{0\}$ and $(k+1)\text{-} := k + 1\text{-}$ for $k \in \mathbb{N}$.

By an ordered Banach space, $E := (E, \leq) := (E, P)$, we mean a real Banach space E together with a closed, proper, and convex cone P, the positive cone of E, such that $x \leq y$ iff $y - x \in P$. If E and F are ordered Banach spaces, $T \in \mathcal{L}(E, F)$ is said to be positive iff $x \geq 0$ implies $Tx \geq 0$. The subset of all positive elements of $\mathcal{L}(E, F)$ is denoted by $\mathcal{L}^+(E, F)$, and we write $T \geq 0$ iff $T \in \mathcal{L}^+(E, F)$. Note that $\mathcal{L}^+(E, F)$ is a closed convex cone in $\mathcal{L}(E, F)$, which is proper iff P is total, that is, $\overline{P - P} = E$. Of course, $\mathcal{L}^+(E) := \mathcal{L}^+(E, E)$. Moreover,

$$(P')^+ := (P^+)' := \mathcal{L}^+(E, \mathbb{R})$$

is the dual positive cone of E'. Thus $\big(E',(P')^+\big)$ is an ordered Banach space if P is total, and the order induced by $(P')^+$ is the canonical order of E'.

Note that, given $x' \in E'$,

$$x' \geq 0 \quad \text{iff} \quad \langle x', x\rangle \geq 0 \ , \qquad x \in P \ . \tag{3.3}$$

Moreover, given $T \in \mathcal{L}(E)$, it follows that $T \geq 0$ iff $T' \geq 0$. (We refer to [Jam70] or [Sch71], for example, for the elementary facts about ordered Banach spaces which are being used in this paper.)

Let $E_0 := (E_0, P_0)$ be an ordered Banach space. If $i : E_1 \hookrightarrow E_0$ then E_1 is given the order whose positive cone equals $i^{-1}(P) = E_1 \cap P_0$. Thus E_1 is an ordered Banach space, whose order is said to be canonically induced by P_0, and $i \in \mathcal{L}^+(E_1, E_0)$. Suppose now that E_1 is dense in E_0 and $A \in \mathcal{H}(E_1, E_0)$. Then A is resolvent positive if there exists $\omega_0 \in \mathbb{R}$ such that $(\lambda + A)^{-1} \geq 0$ for $\lambda > \omega_0$. If $A \in \mathcal{H}(E_1, E_0)$, it is resolvent positive iff

$$e^{-tA} \geq 0 \ , \qquad t \geq 0 \ , \tag{3.4}$$

that is, iff the semigroup generated by $-A$ is positive. In this case

$$\omega_0 = \sup\big\{ \operatorname{Re}\lambda \ ; \ \lambda \in \sigma(-A) \big\} \tag{3.5}$$

(e.g., [CH*87, Propositions 7.1 and 9.2]).

4 Normally Elliptic Boundary Value Problems

In the following,

$$\mathcal{A}u := -\partial_j(a_{jk}\partial_k u + a_j u) + b_j \partial_j u + a_0 u \tag{4.1}$$

denotes a general second order linear differential operator on Ω 'in divergence form' with $\mathcal{L}(\mathbb{K}^N)$-valued coefficients, that is, operating on $\mathbb{K}^N$-valued functions on Ω. To be precise, we assume that

$$a_{jk}, a_j, b_j \in C(\overline{\Omega}, X) \ , \quad a_0 \in L_\infty(\Omega, X) \ ,$$

where

$$X := \mathcal{L}(\mathbb{K}^N) \ .$$

We fix a diagonal-matrix-valued function

$$\delta := \operatorname{diag}[\delta_r]_{1 \leq r \leq N} : \partial\Omega \to X \ , \tag{4.2}$$

a **boundary characterization map**, such that

$$\delta_r \in C\big(\partial\Omega, \{0,1\}\big) \ , \qquad 1 \leq r \leq N \ . \tag{4.3}$$

Note that each δ_r is constant on each component of $\partial\Omega$. Then we associate with $\mathcal{A}$ a linear boundary operator of order at most one, defined by

$$\mathcal{B}u := \delta\big\{ \nu^j \gamma_\partial(a_{jk}\partial_k u + a_j u) + c\gamma_\partial u \big\} + (1-\delta)\gamma_\partial u \ , \tag{4.4}$$

where γ_∂ denotes the trace operator for $\partial\Omega$ and

$$c \in C(\partial\Omega, X) \; .$$

Then $(\mathcal{A},\mathcal{B})$ is said to be a **linear boundary value problem** on Ω of order at most two in divergence form. We topologize the set of these boundary value problems by identifying them with the Banach space

$$\mathbb{E}(\Omega) := \mathbb{E}(\Omega, N) := C(\overline{\Omega}, X)^{n^2+n+n} \times L_\infty(\Omega, X) \times C(\partial\Omega, X)$$

by means of the identification

$$(\mathcal{A},\mathcal{B}) \longleftrightarrow (a_{jk}, a_j, b_j, a_0, c) \; . \tag{4.5}$$

With the boundary value problem $(\mathcal{A},\mathcal{B})$ we associate its **principal symbol**

$$(a_\pi, b_\pi) \in C(\overline{\Omega} \times \mathbb{R}^N, X) \times C(\partial\Omega \times \mathbb{R}^N, X)$$

defined by

$$a_\pi(x,\xi) := a_{jk}(x)\xi^j\xi^k \; , \qquad (x,\xi) \in \overline{\Omega} \times \mathbb{R}^n \; ,$$

and

$$b_\pi(y,\xi) := \delta(y)\nu^j(y)a_{jk}(y)\xi^k + \big(1-\delta(y)\big) \; , \qquad (y,\xi) \in \partial\Omega \times \mathbb{R}^n \; .$$

Let S^{n-1} be the unit sphere in $\mathbb{R}^n$. Then $\mathcal{A}$ is said to be **normally elliptic** if

$$\sigma\big(a_\pi(x,\xi)\big) \subset [\mathrm{Re}\, z > 0] := \{\, z \in \mathbb{C} \; ; \; \mathrm{Re}\, z > 0 \,\} \; , \qquad (x,\xi) \in \overline{\Omega} \times S^{n-1} \; . \tag{4.6}$$

The boundary operator $\mathcal{B}$ satisfies the **normal complementing condition** with respect to $\mathcal{A}$ (the parameter-dependent Lopatinskii-Shapiro condition) if zero is, for each $(x,\xi) \in T(\partial\Omega)$ and $\lambda \in [\mathrm{Re}\, z \geq 0]$ with $(\xi,\lambda) \neq (0,0)$, the only exponentially decaying solution of the boundary value problem on the half-line:

$$\big[\lambda + a_\pi\big(x, \xi + \nu(x) i\, \partial_t\big)\big]u = 0 \; , \quad t > 0 \; , \qquad b_\pi\big(x, \xi + \nu(x) i\, \partial_t\big)u(0) = 0 \; . \tag{4.7}$$

Of course, $T(\partial\Omega)$ is the tangent bundle of $\partial\Omega$. Lastly, the boundary value problem $(\mathcal{A},\mathcal{B})$ is **normally elliptic** on Ω if $\mathcal{A}$ is normally elliptic and $\mathcal{B}$ satisfies the normal complementing condition with respect to $\mathcal{A}$.

We denote by

$$\mathcal{E}(\Omega) := \mathcal{E}(\Omega, \delta, N)$$

the set of all normally elliptic boundary value problems on Ω. It follows from [Ama90a, Theorem 2] that

$$\mathcal{E}(\Omega) \text{ is open in } \mathbb{E}(\Omega) \; . \tag{4.8}$$

Besides boundary value problems $(\mathcal{A},\mathcal{B})$ with 'continuous coefficients', that is, besides $\mathbb{E}(\Omega)$, we consider boundary value problems $(\mathcal{A},\mathcal{B})$ with 'C^1–coefficients' which we identify by means of (4.5) with

$$\mathbb{E}^1(\Omega) := \big[C^1(\overline{\Omega}, X)\big]^{n^2+n+n} \times L_\infty(\Omega, X) \times C^1(\partial\Omega, X) \; .$$

Moreover,

$$\mathcal{E}^1(\Omega) := \mathcal{E}(\Omega) \cap \mathbb{E}^1(\Omega) \ .$$

Since $\mathbb{E}^1(\Omega) \hookrightarrow \mathbb{E}(\Omega)$, it follows from (4.8) that $\mathcal{E}^1(\Omega)$ is open in $\mathbb{E}^1(\Omega)$.

Given $(\mathcal{A},\mathcal{B}) \in \mathbb{E}^1(\Omega)$,

$$(\mathcal{A},\mathcal{B}) \in \mathcal{L}\big(W_p^2, L_p \times L_p(\partial\Omega, \mathbb{K}^N)\big) \ , \qquad 1 < p < \infty \ ,$$

where $W_p^2 := W_p^2(\Omega,\mathbb{K}^N)$ and $L_p := L_p(\Omega,\mathbb{K}^N)$. Thus

$$W_{p,\mathcal{B}}^2 := \ker \mathcal{B} = \{\, u \in W_p^2 \ ; \ \mathcal{B}u = 0 \,\} \tag{4.9}$$

is a closed linear subspace of W_p^2, whence a Banach space. Hence

$$A := A_p := \mathcal{A} | \ker \mathcal{B} \in \mathcal{L}(W_{p,\mathcal{B}}^2, L_p) \tag{4.10}$$

is well defined, the **L_p-realization of the boundary value problem** $(\mathcal{A},\mathcal{B})$.

The importance of the class $\mathcal{E}^1(\Omega)$ of normally elliptic boundary value problems with C^1–coefficients is seen from the following

4.1 Theorem *If $(\mathcal{A},\mathcal{B}) \in \mathcal{E}^1(\Omega)$ then $A \in \mathcal{H}(W_{p,\mathcal{B}}^2, L_p)$. Conversely, if $(\mathcal{A},\mathcal{B}) \in \mathbb{E}^1(\Omega)$ and $A \in \mathcal{H}(W_{p,\mathcal{B}}^2, L_p)$ then $\mathcal{A}$ is normally elliptic.*

For a proof of this theorem we refer to [Ama90a, Section 3] and [Ama94].

4.2 Remarks

(a) $\mathcal{A}$ is normally elliptic iff $\partial_t + \mathcal{A}$ is parabolic in the sense of Petrowskii (cf. [LadSU68, Section VII.8]).

(b) Theorem 4.1 is true for general linear normally parabolic boundary value problems of order m which are not necessarily in divergence form.

More precisely, suppose that $\mathcal{A} := \sum_{|\alpha| \le m} a_\alpha D^\alpha$, where

$$a_\alpha \in C(\overline{\Omega}, X) \ , \quad |\alpha| = m \ , \qquad a_\alpha \in L_\infty(\Omega, X) \ , \quad |\alpha| < m \ ,$$

and where $D_j := -i\,\partial_j$, $\ j = 1,\ldots,n$. Let

$$a_\pi(x,\xi) := \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \ , \qquad (x,\xi) \in \overline{\Omega} \times \mathbb{R}^n \ ,$$

be the principal symbol of $\mathcal{A}$. Then $\mathcal{A}$ is normally elliptic if condition (4.6) is satisfied. It is easily verified that this implies that m is even.

Let

$$\mathcal{B}_r := \sum_{|\alpha|\le m_r} b_{\alpha,r}\gamma_\partial D^\alpha \ , \qquad r = 1,\dots,mN/2 \ ,$$

be boundary operators of orders $m_r \in \{0,1,\dots,m-1\}$, where m_r may vary from component to component of $\partial\Omega$, such that

$$b_{\alpha,r} \in C^{m-m_r}\big(\partial\Omega, \mathcal{L}(\mathbb{K}^N,\mathbb{K})\big) \ .$$

Define the principal boundary symbol of $\mathcal{B} := (\mathcal{B}_1,\dots,\mathcal{B}_{mN/2})$ by

$$b_\pi(x,\xi) := \Big(\sum_{|\alpha|=m_r} b_{\alpha,r}(x)\xi^\alpha \ ; \ 1 \le r \le mN/2 \Big) \ , \qquad (x,\xi) \in \partial\Omega \times \mathbb{R}^n \ .$$

Then $(\mathcal{A},\mathcal{B})$ is said to be normally elliptic iff $\mathcal{A}$ is normally elliptic and (4.7) is satisfied.

The analogue to Theorem 4.1 says that $A \in \mathcal{H}(W^m_{p,\mathcal{B}}, L_p)$ if $(\mathcal{A},\mathcal{B})$ is normally elliptic. Conversely, if $A \in \mathcal{H}(W^m_{p,\mathcal{B}}, L_p)$ then $\mathcal{A}$ has to be normally elliptic (cf. [Ama94]).

(c) The fact that the normal ellipticity of $\mathcal{A}$ is necessary for $-A$ to be the generator of an analytic semigroup on L_p relies on the fact that we restrict ourselves to generators with domains $W^2_{p,\mathcal{B}}$. If we omit this condition, it is only necessary that $\sigma\big(a(x,\xi)\big) \subset [\mathrm{Re}\, z > 0] \cup \{0\}$, as has been proven in [Sad91], where also an example of a degenerate parabolic equation is given which shows that this case can occur (also cf. [ElF87] for the case where $\mathrm{dom}(A) = W^2_2(\mathbb{R}^n, \mathbb{C}^N)$, and [BoS86]). In general, $A \in \mathcal{H}(W^2_{p,\mathcal{B}}, L_p)$ does not imply that the boundary operator has to have the property that Dirichlet and Neumann type conditions occur on distinct components, that is, mixed boundary conditions are possible (cf. [Tai91], [GrR89]). □

In general, it is not easy to verify that a given boundary value problem $(\mathcal{A},\mathcal{B})$ is normally elliptic. For this reason we present in the following some useful examples and sufficient conditions which are easier to check than the definition above. Proofs are found in [Ama90a, Section 4].

4.3 Examples

(a) The differential operator $\mathcal{A}$ is **uniformly strongly elliptic** (satisfies the uniform Legendre-Hadamard condition) if

$$\mathrm{Re}\big(a_\pi(x,\xi)\eta|\eta\big) > 0 \ , \qquad (x,\xi) \in \overline{\Omega} \times S^{n-1} \ , \quad \eta \in \mathbb{K}^N \setminus \{0\} \ ,$$

that is, iff the symmetric part of $a_\pi(x,\xi) \in \mathcal{L}(\mathbb{K}^N)$ is positive definite, uniformly with respect to $(x,\xi) \in \overline{\Omega} \times S^{n-1}$. It is obvious that

$$\mathcal{A} \text{ is uniformly stronlgy elliptic} \quad \Longrightarrow \quad \mathcal{A} \text{ is normally elliptic.}$$

The converse is true iff $N = 1$.

(b) With $\mathcal{A}$ we associate a quadratic form on $\mathbb{K}^{n\times N}$, a **Legendre form** $a_{\mathcal{L}}$, by

$$a_{\mathcal{L}}(x,\zeta) := \sum_{r,s=1}^{N} a_{jk}^{rs}(x)\zeta_r^j\overline{\zeta}_s^k \ , \qquad (x,\zeta) \in \overline{\Omega} \times \mathbb{K}^{n\times N} \ ,$$

where $a_{jk} = [a_{jk}^{rs}]_{1\le r,s\le N}$. Then $\mathcal{A}$ satisfies a **uniform Legendre condition** on Ω if

$$\operatorname{Re} a_{\mathcal{L}}(x,\zeta) > 0 \ , \qquad (x,\zeta) \in \overline{\Omega} \times \bigl(\mathbb{K}^{n\times N}\setminus\{0\}\bigr) \ .$$

It is not difficult to see that

$$\begin{aligned} &\mathcal{A} \text{ satisfies a uniform Legendre condition} \\ &\qquad \Longrightarrow \quad \mathcal{A} \text{ is uniformly strongly elliptic.} \end{aligned}$$

The converse is true iff $N = 1$.

It should be noted that the Legendre form $a_{\mathcal{L}}$ is uniquely determined by the matrix coefficients of a_{jk} if the principal part $\mathcal{A}_\pi$ of $\mathcal{A}$ is given the representation $-a_{jk}\partial_j\partial_k$. However, there may be other representations of $\mathcal{A}_\pi$ which generate different Legendre forms. Thus, unlike the principal symbol a_π of $\mathcal{A}$, which is invariantly associated with $\mathcal{A}_\pi$, the Legendre form $a_{\mathcal{L}}$ of $\mathcal{A}$ is not an invariant of $\mathcal{A}_\pi$. (The author is grateful to P. Quittner for pointing this out.)

(c) If $\mathcal{A}$ is uniformly strongly elliptic and $\delta = 0$, that is, if $\mathcal{B}$ is the Dirichlet boundary operator γ_∂ then $(\mathcal{A},\mathcal{B})$ is normally elliptic.

(d) If $\mathcal{A}$ satisfies a uniform Legendre condition, $(\mathcal{A},\mathcal{B})$ is normally elliptic.

(e) Suppose that $(\mathcal{A},\mathcal{B}) \in \mathbb{E}(\Omega)$ is of **separated divergence form**, that is, it satisfies

$$a_{jk} = a\alpha_{jk} \ , \qquad 1 \le j,k \le n \ ,$$

where

$$a \in C(\overline{\Omega}, X) \ , \quad \alpha := [\alpha_{jk}]_{1\le j,k\le n} \in C\bigl(\overline{\Omega}, \mathcal{L}(\mathbb{R}^n)\bigr)$$

and α is symmetric and uniformly positive definite. Then $(\mathcal{A},\mathcal{B})$ is normally elliptic provided

$$\sigma\bigl(a(x)\bigr) \subset [\operatorname{Re} z > 0] \ , \qquad x \in \overline{\Omega} \ ,$$

and

$$\bigl(1-\delta(y)\bigr)a(y)\delta(y) = 0 \ , \qquad y \in \partial\Omega \ . \tag{4.11}$$

Observe that (4.11) is automatically satisfied for

$$\delta \in C\bigl(\partial\Omega, \{0, 1_{\mathbb{K}^N}\}\bigr) \ ,$$

that is, if on each component Γ of $\partial\Omega$ the boundary operator $\mathcal{B}|\Gamma$ is a Dirichlet boundary operator, i.e., $\mathcal{B}|\Gamma = \gamma_\partial|\Gamma$, or a Neumann type boundary operator of the form

$$\mathcal{B}|\Gamma = (a\partial_{\nu_\alpha} + \nu^j a_j + c)|\Gamma \ ,$$

where ∂_{ν_α} is the derivative with respect to the conormal $\alpha\vec{\nu}$.

(f) Suppose that $\mathcal{A}$ is normally elliptic and $\delta = 0$, that is, $\mathcal{B} = \gamma_\partial$, the Dirichlet boundary operator. Then $(\mathcal{A}, \mathcal{B})$ is not necessarily normally elliptic (cf. [LadSU68, p. 625]). If $\mathcal{A}$ is uniformly strongly elliptic and $\delta = 1$ (Neumann type boundary conditions), it does not seem to be known if $(\mathcal{A}, \mathcal{B})$ is normally elliptic. □

The last example illustrates the difficulties in verifying that $(\mathcal{A}, \mathcal{B}) \in \mathbb{E}(\Omega)$ belongs to $\mathcal{E}(\Omega)$ in the general case. Note, however, that the reaction-diffusion systems of Section 1 are of separated divergence form with $\alpha_{jk} = \delta_{jk}$ (the Kronecker symbol). Thus, fortunately enough, in this important case these difficulties are drastically reduced. In particular, it follows from Example (e) that a boundary value problem $(\mathcal{A}, \mathcal{B})$ of separated divergence form is normally elliptic iff $\mathcal{A}$ is normally elliptic, that is, $\partial_t + \mathcal{A}$ is Petrowskii parabolic, provided $\delta \in C\big(\partial\Omega, \{0, 1_{\mathbb{K}^N}\}\big)$.

5 Generalized Sobolev Spaces

In the following we shall make essential use of Sobolev spaces of noninteger order. There are several possible extensions of the classical Sobolev spaces W_p^k, $k \in \mathbb{N}$, to intermediate spaces of noninteger order. Among them the Slobodeckii spaces constitute certainly the best-known class — at least in the theory of partial differential equations. This is mainly due to the fact that they occur naturally as trace spaces of the classical Sobolev spaces and to the fact that they can be characterized by a relatively simple intrinsic norm. Namely, putting

$$[u]_{s,p} := \Big(\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, d(x,y)\Big)^{1/p} , \qquad 1 \le p < \infty , \quad 0 < s < 1 ,$$

and denoting by

$$\|\cdot\|_{k,p} := \Big(\sum_{|\alpha|\le k} \|\partial^\alpha u\|_p^p\Big)^{1/p} , \qquad 1 \le p < \infty , \quad k \in \mathbb{N} ,$$

the usual Sobolev norms, let

$$\|u\|_{s,p} := \Big(\|u\|_{[s],p}^p + \sum_{|\alpha|=[s]} [\partial^\alpha u]_{s-[s],p}^p\Big)^{1/p} , \qquad 1 \le p < \infty , \quad s \in \mathbb{R}^+ \setminus \mathbb{N} ,$$

where $[s]$ is the greatest integer smaller than or equal to s. Then the **Slobodeckii spaces** are the Banach spaces defined by

$$W_p^s := W_p^s(\Omega, \mathbb{K}^N) := \big(\{\, u \in W_p^{[s]} \ ;\ [u]_{s,p} < \infty \,\},\ \|\cdot\|_{s,p}\big) , \qquad 1 \le p < \infty , \quad s \in \mathbb{R}^+ \setminus \mathbb{N} .$$

The Slobodeckii spaces are very special cases of the scales of Besov and Triebel-Lizorkin spaces. To introduce the latter spaces we first suppose that $\mathbb{K} = \mathbb{C}$. Then we choose a smooth function φ_0 on $\mathbb{R}^n$ vanishing identically for $|x| > 2$ and being equal to 1 for $|x| \le 1$. We also put

$$\varphi_1(x) := \varphi_0(x/2) - \varphi_0(x) , \quad \varphi_k(x) := \varphi_1(2^{-k}x) , \qquad x \in \mathbb{R}^n , \quad k = 2, 3, \ldots .$$

Then the sequence (φ_j) is a smooth 'dyadic' partition of unity on $\mathbb{R}^n$, that is,

$$\sum_{k=0}^{\infty} \varphi_k(x) = 1 \ , \qquad x \in \mathbb{R}^n \ ,$$

and

$$\operatorname{supp}(\varphi_k) \subset \{\, x \in \mathbb{R}^n \ ; \ 2^{k-1} < |x| < 2^{k+1} \,\} \ , \qquad k \in \dot{\mathbb{N}} := \mathbb{N}\backslash\{0\} \ .$$

We denote by $\mathcal{F}$ the Fourier transform on $\mathcal{S}'(\mathbb{R}^n, \mathbb{C}^N)$, the Schwartz space of tempered $\mathbb{C}^N$–valued distributions, and we put

$$\varphi_k(D) := \mathcal{F}^{-1}\varphi_k\mathcal{F} \ , \qquad k \in \mathbb{N} \ .$$

Given a Banach space E, we use the notations

$$\ell_p(E) := \ell_p(\mathbb{N}, E) \quad \text{and} \quad L_p(E) := L_p(\mathbb{R}^n, E) \ , \qquad 1 \le p \le \infty \ ,$$

in the following definitions.

We put

$$\|u\|_{B^s_{p,q}} := \big\| \big(2^{sk}\varphi_k(D)u\big)_{k\in\mathbb{N}} \big\|_{\ell_q(L_p)} \ , \qquad 1 \le p,q \le \infty \ , \quad s \in \mathbb{R} \ ,$$

and

$$\|u\|_{F^s_{p,q}} := \big\| \big(2^{sk}\varphi_k(D)u\big)_{k\in\mathbb{N}} \big\|_{L_p(\ell_q)} \ , \qquad 1 \le p,q < \infty \ , \quad s \in \mathbb{R} \ .$$

Then the **Besov spaces**, $B^s_{p,q}(\mathbb{R}^n, \mathbb{C}^N)$, are defined by

$$B^s_{p,q}(\mathbb{R}^n, \mathbb{C}^N) := \big(\{\, u \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C}^N) \ ; \ \|u\|_{B^s_{p,q}} < \infty \,\}, \ \|\cdot\|_{B^s_{p,q}}\big)$$

for $1 \le p,q \le \infty$ and $s \in \mathbb{R}$. Similarly, the **Triebel-Lizorkin spaces**, $F^s_{p,q}(\mathbb{R}^n, \mathbb{C}^N)$, are given by

$$F^s_{p,q}(\mathbb{R}^n, \mathbb{C}^N) := \big(\{\, u \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C}^N) \ ; \ \|u\|_{F^s_{p,q}} < \infty \,\}, \ \|\cdot\|_{F^s_{p,q}}\big)$$

for $1 \le p,q < \infty$ and $s \in \mathbb{R}$. Lastly, we put

$$\|u\|_{H^s_p} := \big\| (1-\Delta)^{s/2} u \big\|_p \ , \qquad 1 < p < \infty \ , \quad s \in \mathbb{R} \ ,$$

where $(1-\Delta)^{s/2} := \mathcal{F}^{-1}(1+|\xi|^2)^{s/2}\mathcal{F}$. Then the **Bessel potential spaces**, $H^s_p(\mathbb{R}^n, \mathbb{C}^N)$, are defined by

$$H^s_p(\mathbb{R}^n, \mathbb{C}^N) := \big(\{\, u \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C}^N) \ ; \ \|u\|_{H^s_p} < \infty \,\}, \ \|\cdot\|_{H^s_p}\big)$$

for $1 < p < \infty$ and $s \in \mathbb{R}$.

Observe that the usual definition of $\mathcal{S}'(\mathbb{R}^n, \mathbb{C}^N)$ applies equally well to the real case to give the Schwartz space, $\mathcal{S}'(\mathbb{R}^n, \mathbb{R}^N)$, of $\mathbb{R}^N$-valued distributions on $\mathbb{R}^n$. Moreover, $\mathcal{S}'(\mathbb{R}^n, \mathbb{R}^N)$ is a *real* closed linear subspace of $\mathcal{S}'(\mathbb{R}^n, \mathbb{C}^N)$. Hence the following definitions of the scales of real Besov, Triebel-Lizorkin, and Bessel potential spaces,

$$S(\mathbb{R}^n, \mathbb{R}^N) := S(\mathbb{R}^n, \mathbb{C}^N) \cap \mathcal{S}'(\mathbb{R}^n, \mathbb{R}^N) \ ,$$

where $S := B^s_{p,q}$, or $S := F^s_{p,q}$, or $S := H^s_p$, respectively, are meaningful.

Lastly, we denote by $\mathcal{D} := \mathcal{D}(\Omega, \mathbb{K}^N)$ and $\mathcal{D}' := \mathcal{D}'(\Omega, \mathbb{K}^N)$ the space of $\mathbb{K}^N$-valued test functions and distributions on Ω, respectively, and by

$$r_\Omega \in \mathcal{L}\big(\mathcal{S}'(\mathbb{R}^n, \mathbb{K}^N), \mathcal{D}'\big)$$

the 'restriction operator for Ω', given by

$$\langle r_\Omega u, \varphi\rangle := \langle u, \varphi\rangle \ , \qquad u \in \mathcal{S}'(\mathbb{R}^n, \mathbb{K}^N) \ , \quad \varphi \in \mathcal{D} \ ,$$

where $\langle \cdot, \cdot \rangle$ denote the obvious duality pairings. Then we define the **Besov**, **Triebel-Lizorkin**, and **Bessel potential spaces over** Ω by restriction, that is,

$$S := S(\Omega, \mathbb{K}^N) := r_\Omega S(\mathbb{R}^n, \mathbb{K}^N) \ ,$$

where $S := B^s_{p,q}$, or $S := F^s_{p,q}$, or $S := H^s_p$, respectively, endowed with the obvious quotient topologies.

The spaces introduced above are all Banach spaces (except for $\mathcal{S}'$, $\mathcal{D}$, and $\mathcal{D}'$, of course) and are independent of the function φ_0, except for equivalent norms. Many of the classical function spaces are contained as special cases in the above scales. In particular,

$$B^s_{\infty,\infty} \doteq C^s \ , \qquad s \in \mathbb{R}^+ \setminus \mathbb{N} \ , \tag{5.1}$$

where $C^s := C^s(\overline{\Omega}, \mathbb{K}^N)$,

$$H^k_p \doteq W^k_p \ , \qquad 1 < p < \infty \ , \quad k \in \mathbb{N} \ , \tag{5.2}$$

and

$$B^s_{p,p} \doteq F^s_{p,p} \doteq W^s_p \ , \qquad 1 \le p < \infty \ , \quad s \in \mathbb{R}^+ \setminus \mathbb{N} \ . \tag{5.3}$$

Moreover,

$$F^s_{p,2} \doteq H^s_p \ , \qquad 1 < p < \infty \ , \quad s \in \mathbb{R} \ . \tag{5.4}$$

In the following, we collect some important properties of these spaces which will be needed — implicitly or explicitly — in later sections. For the benefit of the readers more familiar with the theory of partial differential equations than with function spaces, we shall later on restrict our considerations to the Sobolev-Slobodeckii spaces. However, for reasons which will become clear below, for the development of the theory we need both, the scales of Besov and of Bessel potential spaces. Hence we review the needed results for these scales only and leave aside the more general scale of Triebel-Lizorkin spaces.

It is known that, letting $C^k := C^k(\overline{\Omega}, \mathbb{K}^N)$ for $k \in \mathbb{N} \cup \{\infty\}$,

$$C^\infty \overset{d}{\hookrightarrow} B^s_{p,1} \overset{d}{\hookrightarrow} B^s_{p,q} \hookrightarrow B^s_{p,\infty} \overset{d}{\hookrightarrow} B^t_{p,1} \overset{d}{\hookrightarrow} \mathcal{D}' \tag{5.5}$$

for $1 \le p, q < \infty$ and $-\infty < t < s < \infty$, and that

$$B^s_{p,1} \overset{d}{\hookrightarrow} H^s_p \hookrightarrow B^s_{p,\infty} \ , \qquad 1 < p < \infty \ , \quad s \in \mathbb{R} \ . \tag{5.6}$$

If $s_0, s_1 \in \mathbb{R}$ and $p_0, p_1 \in [1, \infty]$ satisfy

$$s_1 - n/p_1 \ge s_0 - n/p_0 \ , \tag{5.7}$$

it follows that

$$B^{s_1}_{p_1,q} \hookrightarrow B^{s_0}_{p_0,q} \ , \qquad 1/p_1 \ge 1/p_0 \ , \quad 1 \le q \le \infty \ , \tag{5.8}$$

and

$$H^{s_1}_{p_1} \hookrightarrow H^{s_0}_{p_0} \ , \qquad 1 > 1/p_1 \ge 1/p_0 > 0 \ . \tag{5.9}$$

Moreover,

$$\partial^\alpha \in \mathcal{L}(B^s_{p,q}, B^{s-|\alpha|}_{p,q}) \ , \qquad 1 \le p,q \le \infty \ , \quad s \in \mathbb{R} \ , \quad \alpha \in \mathbb{N}^n \ , \tag{5.10}$$

and

$$\partial^\alpha \in \mathcal{L}(H^s_p, H^{s-|\alpha|}_p) \ , \qquad 1 < p < \infty \ , \quad s \in \mathbb{R} \ , \quad \alpha \in \mathbb{N}^n \ . \tag{5.11}$$

It is an important fact that

$$\mathcal{D} \overset{d}{\hookrightarrow} B^s_{p,q} \ , \qquad -\infty < s < 1/p \ , \quad 1 \le p,q < \infty \ , \tag{5.12}$$

and

$$\mathcal{D} \overset{d}{\hookrightarrow} H^s_p \ , \qquad -\infty < s < 1/p \ , \quad 1 < p < \infty \ . \tag{5.13}$$

As for duality properties,

$$(B^s_{p,q})' \doteq B^{-s}_{p',q'} \ , \qquad -1 + 1/p < s < 1/p \ , \quad 1 < p,q < \infty \ , \tag{5.14}$$

and

$$(H^s_p)' \doteq H^{-s}_{p'} \ , \qquad -1 + 1/p < s < 1/p \ , \quad 1 < p < \infty \ , \tag{5.15}$$

with respect to the duality pairing naturally induced by $\langle \cdot, \cdot \rangle$, that is, with respect to the standard duality pairing between $\mathcal{D}$ and $\mathcal{D}'$.

By means of local coordinates it is not difficult to define Besov spaces, $B^s_{p,q}(\partial\Omega, \mathbb{K}^N)$, on $\partial\Omega$. It is clear that the imbeddings (5.5) and (5.8) carry over to these spaces provided, of course, n is replaced by $n-1$. (If $n = 1$, the boundary of Ω is a two-point set and all spaces on $\partial\Omega$ are isomorphic to $\mathbb{K}^N \times \mathbb{K}^N$.) As for duality,

$$\left[B^s_{p,q}(\partial\Omega, \mathbb{K}^N)\right]' \doteq B^{-s}_{p',q'}(\partial\Omega, \mathbb{K}^N) \ , \qquad 1 < p,q < \infty \ , \quad s \in \mathbb{R} \ , \tag{5.16}$$

with respect to the duality pairing induced by $\langle \cdot, \cdot \rangle_\partial$.

Let E and F be Banach spaces. Then R is said to be a **retraction** from E onto F if $R \in \mathcal{L}(E,F)$ and there exists a right inverse $R^c \in \mathcal{L}(F,E)$ for R, a **coretraction**.

The trace theorem asserts that the trace operator, γ_∂, is a retraction

$$\gamma_\partial : B^s_{p,q} \to B^{s-1/p}_{p,q}(\partial\Omega, \mathbb{K}^N) \ , \qquad 1 \le p,q < \infty \ , \quad s > 1/p \ , \tag{5.17}$$

and a retraction

$$\gamma_\partial : H^s_p \to B^{s-1/p}_{p,p}(\partial\Omega, \mathbb{K}^N) \ , \qquad 1 < p < \infty \ , \quad s > 1/p \ . \tag{5.18}$$

For proofs of the facts listed above we refer to [Tri78], [Tri83], and [Tri92], where the reader finds many useful additional facts and historical remarks.

In the following, we denote by $[\cdot,\cdot]_\theta$, $0 < \theta < 1$, the complex, and by $(\cdot,\cdot)_{\theta,q}$, $0 < \theta < 1$, $1 \le q \le \infty$, the standard real interpolation functors. We refer to [BeL76] and [Tri78], for example, for the general theory of interpolation spaces and assume that the reader has a working knowledge of that theory.

Let (E_0, E_1) be a real interpolation couple, that is, E_0 and E_1 are real Banach spaces and there exists a locally convex space F such that $E_j \hookrightarrow F$ for $j = 0, 1$. Then we define the 'complex interpolation spaces $[E_0, E_1]_\theta$ in the real setting' by

$$[E_0, E_1]_\theta := \big[(E_0)_{\mathbb{C}}, (E_1)_{\mathbb{C}}\big]_\theta \cap (E_0 + E_1) , \qquad 0 < \theta < 1 ,$$

equipped with the topology induced by $\big[(E_0)_{\mathbb{C}}, (E_1)_{\mathbb{C}}\big]_\theta$. Of course, $(E_j)_{\mathbb{C}}$ denotes the complexification of E_j. Thus $[E_0, E_1]_\theta$ is simply the 'real subspace' of $\big[(E_0)_{\mathbb{C}}, (E_1)_{\mathbb{C}}\big]_\theta$. It is not difficult to verify that $[\cdot,\cdot]_\theta$ is an exact interpolation functor of exponent θ in the category of real interpolation couples. Of course, it inherits the properties of the standard complex interpolation functor, namely the reiteration, commutation, and duality properties. These 'complex interpolation functors' are used throughout if $[\cdot,\cdot]_\theta$ appears in connection with real Banach spaces.

5.1 Remark Suppose that $(F, \|\cdot\|_F)$ is a complex Banach space of $\mathbb{C}^N$–valued distributions on Ω and $(E, \|\cdot\|)$ is the real subspace of $\mathbb{R}^N$-valued distributions in F. Then $E_{\mathbb{C}} \doteq F$, but easy examples show that $\|\cdot\|_{E_{\mathbb{C}}} \neq \|\cdot\|_F$, in general. $\square$

We collect now some important interpolation properties of Besov and Bessel potential spaces. For proofs we refer again to Triebel's books cited above (also cf. [BeL76]).

Suppose that $-\infty < s_0 < s_1 < \infty$ and $0 < \theta < 1$, and put $s_\theta := (1-\theta)s_0 + \theta s_1$. Then

$$[H_p^{s_0}, H_p^{s_1}]_\theta \doteq H_p^{s_\theta} , \qquad 1 < p < \infty , \tag{5.19}$$

and

$$[B_{p,q}^{s_0}, B_{p,q}^{s_1}]_\theta \doteq B_{p,q}^{s_\theta} , \qquad 1 \le p, q < \infty . \tag{5.20}$$

As for the real interpolation functors,

$$(H_p^{s_0}, H_p^{s_1})_{\theta,q} \doteq B_{p,q}^{s_\theta} , \qquad 1 < p < \infty , \quad 1 \le q \le \infty , \tag{5.21}$$

and

$$(B_{p,q_0}^{s_0}, B_{p,q_1}^{s_1})_{\theta,q} \doteq B_{p,q}^{s_\theta} , \qquad 1 \le p, q, q_0, q_1 \le \infty . \tag{5.22}$$

Suppose now that $(\mathcal{A}, \mathcal{B}) \in \mathbb{E}^1(\Omega, \delta, N)$ and $1 < p < \infty$, and let A be the L_p-realization of $(\mathcal{A}, \mathcal{B})$. It is an easy consequence of Theorem 4.1 that

$$D(A) \doteq W_{p,\mathcal{B}}^2$$

if $(\mathcal{A}, \mathcal{B}) \in \mathcal{E}^1(\Omega)$. In the following, we characterize the interpolation spaces

$$\big(L_p, D(A)\big)_\theta \doteq (L_p, W_{p,\mathcal{B}}^2)_\theta$$

for most values of $\theta \in (0,1)$, where $(\cdot,\cdot)_\theta$ denotes either the complex interpolation functor or any one of the real ones. For this we introduce the Banach spaces $H_{p,\mathcal{B}}^s$ and $B_{p,q,\mathcal{B}}^s$ as follows:

Suppose that $(\mathcal{A},\mathcal{B})\in\mathbb{E}(\Omega)$ and $S_p^s\in\{\,H_p^s, B_{p,q}^s\ ;\ 1\le q\le\infty\,\}$. Then

$$S_{p,\mathcal{B}}^s := \begin{cases} \{\,u\in S_p^s\ ;\ \mathcal{B}u=0\,\}\ , & 1+1/p<s\le 2\ ,\\ \{\,u\in S_p^s\ ;\ (1-\delta)\gamma_\partial u=0\,\}\ , & 1/p<s<1+1/p\ ,\\ S_p^s\ , & 0\le s<1/p\ . \end{cases} \tag{5.23}$$

If $\delta=0$ (Dirichlet boundary operator: $\mathcal{B}=\gamma_\partial$), we extend this definition to include the value $s=1+1/p$ by putting

$$S_{p,\mathcal{B}}^s := \{\,u\in S_p^s\ ;\ \gamma_\partial u=0\,\}\ ,\qquad 1/p<s\le 2\ . \tag{5.24}$$

Similarly, if $\delta=1$ (Neumann type boundary operator), we include the value $s=1/p$ by putting

$$S_{p,\mathcal{B}}^s := S_p^s\ ,\qquad 0\le s<1+1/p\ . \tag{5.25}$$

To have a concise notation we define the set Σ_p^+ of 'singular positive values' by

$$\Sigma_p^+ := \begin{cases} \{1/p\} & \text{if } \delta=0\ ,\\ \{1+1/p\} & \text{if } \delta=1\ ,\\ \{1/p,1+1/p\} & \text{otherwise.} \end{cases} \tag{5.26}$$

Using these notations we have the following fundamental

5.2 Theorem *Suppose that $(\mathcal{A},\mathcal{B})\in\mathcal{E}^1(\Omega,\delta,N)$ and $1<p<\infty$. Then*

$$[L_p, W_{p,\mathcal{B}}^2]_\theta \doteq H_{p,\mathcal{B}}^{2\theta}$$

and

$$(L_p, W_{p,\mathcal{B}}^2)_{\theta,q} \doteq B_{p,q,\mathcal{B}}^{2\theta}\ ,\qquad 1\le q\le\infty\ ,$$

for $2\theta\in(0,2)\backslash\Sigma_p^+$.

The proof of this theorem is obtained by obvious modifications of the techniques of Grisvard [Gri69] and Seeley [See72] (cf. [Ama94]). Also cf. [Gui91b] for related results.

5.3 Remarks

(a) The space $S_{p,\mathcal{B}}^s$ is a closed linear subspace of S_p^s for $s\in[0,2]\backslash\Sigma_p^+$. If $0\le s<1+1/p$, it is independent of $(\mathcal{A},\mathcal{B})$. It depends only on δ if $1/p<s<1+1/p$.

(b) Given an interpolation couple (E_0,E_1), we define the continuous interpolation space $(E_0,E_1)_{\theta,\infty}^0$ to be the closure of $E_0\cap E_1$ in $(E_0,E_1)_{\theta,\infty}$ for $0<\theta<1$. By defining 'interpolated morphisms' in an obvious way by restriction, one obtains an exact interpolation functor of exponent θ, the 'continuous interpolation functor' $(\cdot,\cdot)_{\theta,\infty}^0$. It can be characterized by the 'continuous trace method' introduced by Da Prato and Grisvard [DaPG79] as has been shown in [DoF87]. This explains its name.

If $1 \le p < \infty$ and $0 < s < \infty$ then $B^s_{p,\infty} \doteq N^s_p$, where $N^s_p := N^s_p(\Omega, \mathbb{K}^N)$ denote the **Nikolskii spaces** (e.g., [KuJF77]). For this reason, the spaces

$$n^s_p := n^s_p(\Omega, \mathbb{K}^N) := \text{closure of } C^\infty \text{ in } N^s_p \ , \qquad 1 \le p < \infty \ , \quad s > 0 \ ,$$

are the **little Nikolskii spaces**. Similarly, the spaces

$$c^s := c^s(\overline{\Omega}, \mathbb{K}^N) := \text{closure of } C^\infty \text{ in } C^s$$

are the **little Hölder spaces**. It follows from (5.2), (5.5), (5.6), and (5.21) that

$$(L_p, W^k_p)^0_{\theta,\infty} \doteq n^{\theta k}_p \ , \qquad 1 < p < \infty \ , \quad k \in \dot{\mathbb{N}} \ , \quad 0 < \theta < 1 \ .$$

Consequently,

$$(L_p, W^2_{p,\mathcal{B}})^0_{\theta,\infty} \doteq n^{2\theta}_{p,\mathcal{B}} \ , \qquad 1 < p < \infty \ , \quad 2\theta \in (0,2)\backslash\Sigma^+_p \ .$$

(c) If one is interested in generalized Sobolev spaces of order s with $|s| \le 2$ only, as is the case in connection with second order differential equations, everything said above remains valid if it is only assumed that Ω is of class C^2. More generally, if $\overline{\Omega}$ is a compact n-dimensional C^2–submanifold of $\mathbb{R}^n$, all the results (5.1)–(5.22) above remain valid if the orders of all spaces, that is, all the superscripts, are additionally restricted to belong to $[-2,2]$. □

In the study of nonlinear problems it is often crucial to work in the category of real Banach spaces. This is, among other things, mainly due to the fact that — in the complex case — a map is (Fréchet) differentiable iff it is analytic. Thus differentiability is in the complex setting a condition which is by far too restrictive for most applications. This is the main reason for formulating all results in this paper for the general case $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

6 Abstract Interpolation-Extrapolation Scales

Let E_0 and E_1 be Banach spaces with $E_1 \overset{d}{\hookrightarrow} E_0$. Then (E_0, E_1) is said to be a **densely injected Banach couple**. By an **admissible interpolation functor** $(\cdot,\cdot)_\theta$, $0 < \theta < 1$, we mean an interpolation functor of exponent θ for the category of densely injected Banach couples (whose morphisms are the bounded linear operators, of course) such that

$$E_1 \overset{d}{\hookrightarrow} (E_0, E_1)_\theta$$

whenever (E_0, E_1) is such a couple. Note that each one of the following standard interpolation functors is admissible:

$$\begin{aligned} &\text{the complex interpolation functor } [\cdot,\cdot]_\theta \ , \\ &\text{the real interpolation functors } (\cdot,\cdot)_{\theta,q} \ , \qquad 1 \le q < \infty \ , \\ &\text{the continuous interpolation functor } (\cdot,\cdot)^0_{\theta,\infty} \ . \end{aligned} \tag{6.1}$$

Let (E_0, E_1) be a densely injected Banach couple and denote by $\|\cdot\|_j$ the norm in E_j. Also assume that

$$A_0 \in \mathcal{H}(E_1, E_0) \ . \tag{6.2}$$

Then there exists $\omega \geq 0$ such that

$$\sigma(\omega + A_0) \subset [\mathrm{Re}\, z > 0] \ . \tag{6.3}$$

It is a consequence of (6.2) that $E_1 \doteq D(A_0)$. Thus it follows from (6.3) that

$$E_1 \ \doteq \ \big(\mathrm{dom}(A_0), \|(\omega + A_0)\cdot\|_0\big) \ .$$

Let A_1 be the E_1-realization of A_0 and let

$$E_2 := \big(\mathrm{dom}(A_1), \|(\omega + A_1)\cdot\|_1\big) \ .$$

Then (E_1, E_2) is a densely injected Banach couple and

$$A_1 \in \mathcal{H}(E_2, E_1)$$

with $\sigma(A_1) = \sigma(A_0)$.

We define now inductively a discrete scale of Banach spaces and operators $(E_k, A_k)_{k\in\mathbb{N}}$ by

$$E_{k+1} := (E_{k+1}, \|\cdot\|_{k+1}) := \big(\mathrm{dom}(A_k), \|(\omega + A_k)\cdot\|_k\big)$$

and

$$A_{k+1} := E_{k+1}\text{-realization of } A_k$$

for $k \in \dot{\mathbb{N}}$. Then

$$E_k \doteq D(A_0^k) \ , \qquad k \in \mathbb{N} \ ,$$

(E_k, E_{k+1}) is a densely injected Banach couple, and

$$A_k \in \mathcal{H}(E_{k+1}, E_k)$$

with $\sigma(A_k) = \sigma(A_0)$ for $k \in \mathbb{N}$.

On the other hand, we can recover A_0 and E_0 from A_1 and E_1 since $\big(E_1, \|(\omega + A_1)^{-1}\cdot\|_1\big)$ is a dense linear subspace of E_0 whose norm is equivalent to $\|\cdot\|_0$. Thus E_0 is a completion of the normed vector space $\big(E_1, \|(\omega + A_1)^{-1}\cdot\|_1\big)$ and $A_0 \in \mathcal{L}(E_1, E_0)$ is the continuous extension of $A_1 \in \mathcal{L}(E_2, E_1)$, or, considered as a linear operator in E_0, A_0 is the closure of A_1 in E_0. This observation suggests the following definition of a superspace E_{-1} of E_0 and an extension A_{-1} of A_0.

We put

$$E_{-1} := \big(E_0, \|(\omega + A_0)^{-1}\cdot\|\big)^{\sim} \ ,$$

where $^{\sim}$ denotes completion. Then

$$E_0 \overset{d}{\hookrightarrow} E_{-1} \ ,$$

that is, (E_{-1}, E_0) is a densely injected Banach couple. It follows that the closure A_{-1} of A_0 in E_{-1} is well defined and

$$A_{-1} \in \mathcal{H}(E_0, E_{-1})$$

with $\sigma(A_{-1}) = \sigma(A_0)$.

We fix now for each $\theta \in (0,1)$ an admissible interpolation functor $(\cdot,\cdot)_\theta$ arbitrarily and put

$$\mathfrak{F} := \big\{ (\cdot,\cdot)_\theta \ ; \ 0 < \theta < 1 \big\} \ .$$

Then we define

$$E_{k+\theta} := (E_{k+\theta}, \|\cdot\|_{k+\theta}) := (E_k, E_{k+1})_\theta$$

and

$$A_{k+\theta} := E_{k+\theta}\text{-realization of } A_k$$

for $k \in \mathbb{N} \cup \{-1\}$ and call

$$\{ (E_\alpha, A_\alpha) \ ; \ -1 \leq \alpha < \infty \} \tag{6.4}$$

the **interpolation-extrapolation scale (over $[-1,\infty)$) generated by A_0 and $\mathfrak{F}$.**

It follows that $(E_\alpha, E_{\alpha+1})$ is a densely injected Banach couple,

$$A_\alpha \in \mathcal{H}(E_{\alpha+1}, E_\alpha) \ , \quad \sigma(A_\alpha) = \sigma(A_0) \ , \qquad -1 \leq \alpha < \infty \ ,$$

and, given $-1 \leq \beta < \alpha < \infty$, the following diagram is commutative

$$\begin{array}{ccccc} E_{\alpha+1} & \xrightarrow{\ A_\alpha\ } & E_\alpha & \xrightarrow{\ e^{-tA_\alpha}\ } & E_\alpha \\ \Big\downarrow d & & \Big\downarrow d & & \Big\downarrow d \\ E_{\beta+1} & \xrightarrow{\ A_\beta\ } & E_\beta & \xrightarrow{\ e^{-tA_\beta}\ } & E_\beta \end{array} \tag{6.5}$$

for $t \geq 0$. Observe that (6.5) shows that the operators A_α and A_β, as well as the corresponding semigroups $\{ e^{-tA_\alpha} \ ; \ t \geq 0 \}$ and $\{ e^{-tA_\beta} \ ; \ t \geq 0 \}$, respectively, are obtained from each other by natural restrictions and extensions, respectively.

Once the 'extrapolation space' E_{-1} and the extension A_{-1} of A_0 have been constructed, everything above follows easily from the basic facts about analytic semigroups and interpolation theory (cf. [Ama86d] and, in particular, [Ama94]). The above construction of E_{-1} and A_{-1} occurs perhaps for the first time in [Nag83] (also cf. [Wal86]). Without being aware of this it has been reintroduced in [Ama87] (also cf. [Ama86d]) and, again independently, in [Liu89]. In [DaPG84] there has been given a different, rather complicated construction of an extrapolation space $\widetilde{E}_{-1}$ and an extension $\widetilde{A}_{-1}$ of A_0. In [Ama88c, Remark 6.2] it has been shown that the Da Prato-Grisvard extrapolation pair $(\widetilde{E}_{-1}, \widetilde{A}_{-1})$ is equivalent to (E_{-1}, A_{-1}) (see also a corresponding statement in [DaP84]).

In [Nag83] and [Wal86] discrete scales $\{ (E_k, A_k) \ ; \ k \in \mathbb{Z}, \ k \geq -1 \}$ are considered only. In [Liu89] there is studied the 'scale of fractional power spaces' $E_\alpha := D(A_0^\alpha)$ for $\alpha \geq -1$, as is the case in [Ama87]. In addition, in [Liu89] the author also introduces at each point $\alpha \in [-1,\infty)$ inductive limit spaces $\bigcup_{\beta>\alpha} E_\beta$ and projective limit spaces $\bigcap_{\beta<\alpha} E_\beta$. (It should be mentioned that in all these papers the scales are not restricted below; see, however, Remark 6.1(a) below.) Lastly, in [DaPG84] the spaces E_α, $-1 < \alpha < 0$, have been defined by means of the continuous interpolation functors $(\cdot,\cdot)^0_{\theta,\infty}$, $0 < \theta < 1$.

Lastly, it should be noted that recently an extrapolation theory has been developed by B. Jawerth and M. Milman (cf. [JaM91]) which is of an entirely different nature than the one considered here.

6.1 Remarks

(a) Of course, we can replace in the above procedure A_0 by A_{-1} and construct the interpolation-extrapolation scale $\{ (F_\alpha, B_\alpha) \ ; \ -1 \leq \alpha < \infty \}$ over $[-1, \infty)$ generated by A_{-1} and $\mathfrak{F}$. Then $\{ (F_{\alpha+1}, B_{\alpha+1}) \ ; \ \alpha \geq -1 \}$ is naturally isomorphic to $\{ (E_\alpha, A_\alpha) \ ; \ \alpha \geq -1 \}$. This means that we can extend the scale (6.4) to a scale over $[-2, \infty)$,

$$\{ (E_\alpha, A_\alpha) \ ; \ -2 \leq \alpha < \infty \} \ .$$

By repeating this argument, given any $m \in \mathbb{N}$, we can construct the interpolation–extrapolation scale over $[-m, \infty)$ generated by A_0 and $\mathfrak{F}$,

$$\{ (E_\alpha, A_\alpha) \ ; \ -m \leq \alpha < \infty \} \ .$$

Observe, however, that, thanks to the fact that we have to take a completion at each 'extrapolation step', we have to replace that part of the scale which has already been constructed in the previous step by an isomorphic image. Thus, in general, $E_{-\infty} := \bigcup_{k \geq 0} E_{-k}$ is not defined. (I am grateful to P. Koch Medina for pointing out this fact. It has been overlooked in my earlier papers — as well as in [Liu89], [Nag83], and [Wal86].) On the other hand, a 'universal superspace' $E_{-\infty}$ can be constructed by taking the inductive limit of the spaces $\{ E_{-k} \ ; \ k \in \mathbb{N} \}$ with respect to the natural injections $j_k : E_{-k} \to E_{-k-1}$ induced by completions. In all of our applications of the interpolation-extrapolation scales only parameter values in $[-1, 1]$ are relevant so that a universal superspace is not needed. For this reason we restrict ourselves to scales over $[-1, \infty)$.

(b) It is obvious that for the construction of an interpolation-extrapolation scale it is by no means necessary that A_0 be the negative infinitesimal generator of a semigroup. For more general situations we refer to [Ama94]. Moreover, the different choices of ω lead to equivalent norms on E_α. Thus we do not indicate the ω-dependence.

(c) Suppose that $-1 \leq \beta < \alpha < \infty$. It follows from (6.5) that (E_β, E_α) is a densely injected Banach couple. Thus, given any $\theta \in (0,1)$ and any other admissible interpolation functor $(\!(\cdot,\cdot)\!)_\theta$, the interpolation space $(\!(E_\beta, E_\alpha)\!)_\theta$ is well defined. In general, $(\!(E_\beta, E_\alpha)\!)_\theta$ will not coincide with any one of the spaces E_γ, $\beta < \gamma < \alpha$. The following 'sandwich property' shows, however, that $(\!(E_\beta, E_\alpha)\!)_\theta$ can be arbitrarily closely approximated by spaces of the original interpolation-extrapolation scale. Namely,

$$E_\xi \hookrightarrow (\!(E_\beta, E_\alpha)\!)_\theta \hookrightarrow E_\eta \ , \qquad \beta < \eta < (1-\theta)\beta + \theta\alpha < \xi < \alpha \ .$$

This is a consequence of an extension of the 'almost reiteration theorem' [Ama86c, Theorem 8.3] (cf. [Ama94]).

(d) There is an important case in which the scale $\{ (E_\alpha, A_\alpha) \ ; \ -1 \leq \alpha < \infty \}$ is stable under reiteration. Namely, suppose, in addition to (6.2) and (6.3), that $\omega + A_0$ has bounded imaginary powers and $(\cdot,\cdot)_\theta = [\cdot,\cdot]_\theta$ for $0 < \theta < 1$. Then

$$[E_\beta, E_\alpha]_\theta \doteq E_{(1-\theta)\beta + \theta\alpha} \ , \qquad -1 \leq \alpha < \beta < \infty \ .$$

Moreover, in this case $E_\alpha \doteq D(A_0^\alpha)$ for $\alpha \in [-1, \infty)$, where $D(A_0^\alpha)$ denotes the completion of E_0 in the norm $\|(\omega + A_0)^\alpha \cdot\|_0$ if $\alpha < 0$.

This has been proven in [Ama87, Theorem 3.3].

(e) The following are equivalent:

(i) A_0 has a compact resolvent;

(ii) $E_\alpha \overset{cc}{\hookrightarrow} E_\beta$ for $-1 \le \beta < \alpha < \infty$;

(iii) There exist $\alpha, \beta \in [-1,\infty)$ with $\beta < \alpha$ such that $E_\alpha \overset{cc}{\hookrightarrow} E_\beta$.

This is an extension of [Ama86d, Theorem 8] for which we refer to [Ama94]. □

Suppose now that $(\mathcal{A},\mathcal{B}) \in \mathcal{E}^1(\Omega)$, let A_0 be the L_p-realization of $(\mathcal{A},\mathcal{B})$, where p is fixed in $(1,\infty)$, and put $E_0 := L_p$ and $E_1 := W^2_{p,\mathcal{B}}$. Then $A_0 \in \mathcal{H}(E_1,E_0)$ by Theorem 4.1. Hence we can construct the interpolation-extrapolation scale generated by A_0 and $\mathfrak{F}$. Thus if we restrict $(\cdot,\cdot)_\theta$ to any one of the choices (6.1), it follows from Theorem 5.2 that E_α is — except for equivalent norms — explicitly known, provided $2\alpha \in [0,2]\backslash\Sigma_p^+$. For this one has to observe that Theorem 5.2 implies

$$(L_p, W^2_{p,\mathcal{B}})^0_{\theta,\infty} \doteq n^{2\theta}_{p,\mathcal{B}}\ , \qquad 2\theta \in (0,2)\backslash\Sigma_p^+\ . \tag{6.6}$$

On the other hand, the 'negative' spaces $E_\alpha,\ -1 \le \alpha < 0$, are defined in a rather abstract way which does not seem to admit an easy concrete description. For applications of the above theory of interpolation-extrapolation spaces it is, of course, most important to possess a 'computable' description of the negative spaces E_α and the 'negative realizations' A_α of A_0, where $-1 \le \alpha < 0$. In the following, we shall show that this can be achieved by duality, provided we restrict the generality slightly.

We suppose now that

$$E_0 \text{ is a reflexive Banach space} \tag{6.7}$$

and

$$(\cdot,\cdot)_\theta \in \bigl\{\, [\cdot,\cdot]_\theta, (\cdot,\cdot)^0_{\theta,\infty}, (\cdot,\cdot)_{\theta,q}\ ;\ 1 < q < \infty \,\bigr\}\ , \qquad 0 < \theta < 1\ . \tag{6.8}$$

We put

$$(\cdot,\cdot)^\#_\theta := \begin{cases} [\cdot,\cdot]_\theta & \text{if } (\cdot,\cdot)_\theta = [\cdot,\cdot]_\theta\ , \\ (\cdot,\cdot)_{\theta,q'} & \text{if } (\cdot,\cdot)_\theta = (\cdot,\cdot)_{\theta,q}\ , \\ (\cdot,\cdot)_{\theta,1} & \text{if } (\cdot,\cdot)_\theta = (\cdot,\cdot)^0_{\theta,\infty} \end{cases} \tag{6.9}$$

for $0 < \theta < 1$, so that $(\cdot,\cdot)^\#_\theta$ is an admissible interpolation functor. We denote by $E_0^\#$ the dual space $(E_0)'$ of E_0 and by $A_0^\#$ the dual $(A_0)'$ of A_0. Moreover,

$$E_1^\# := D(A_0^\#)\ .$$

Then it is known that

$$A_0^\# \in \mathcal{H}(E_1^\#, E_0^\#) \quad \text{and} \quad e^{-tA_0^\#} = (e^{-tA_0})'\ , \qquad t \ge 0\ .$$

Thus the **dual interpolation-extrapolation scale**

$$\bigl\{\, (E_\alpha^\#, A_\alpha^\#)\ ;\ -1 \le \alpha < \infty \,\bigr\}$$

over $[-1,\infty)$ generated by $A_0^\#$ and $\mathfrak{F}^\# := \bigl\{\, (\cdot,\cdot)^\#_\theta\ ;\ 0 < \theta < 1 \,\bigr\}$ is well defined.

6.2 Theorem *Let conditions* (6.7)–(6.9) *be satisfied. Then*

$$(E_\alpha)' \doteq E^\#_{-\alpha} \quad \textit{and} \quad (A_\alpha)' = A^\#_{-\alpha}\ , \qquad -1 \le \alpha \le 1\ ,$$

with respect to the duality pairing naturally induced by $\langle\cdot,\cdot\rangle_E$. If $(\cdot,\cdot)_\theta \ne (\cdot,\cdot)^0_{\theta,\infty}$ for $0 < \theta < 1$ then E_α is reflexive for $-1 \le \alpha < \infty$.

This theorem has been proven in [Ama86d, Theorem 11], for the case that $(\cdot,\cdot)_\theta \neq (\cdot,\cdot)^0_{\theta,\infty}$ for $0<\theta<1$. The same proof extends to the case where the continuous interpolation functor is admitted too (cf. [Sit92, Theorem 2.3]). An analoguous result was proven earlier in [Ama87, Theorem 1.3] for the scale of fractional power spaces. That result was rediscovered in [Liu89, Theorem I.2.8].

7 Sobolev Interpolation-Extrapolation Scales

We extend now the definition of the Sobolev-Slobodeckii spaces to cover spaces of negative order by putting

$$W_p^{-s} := W_p^{-s}(\Omega,\mathbb{K}^N) := \begin{cases} H_p^{-s} , & s\in\mathbb{N} , \\ B_{p,p}^{-s} , & s\in\mathbb{R}^+\setminus\mathbb{N} , \end{cases}$$

for $1<p<\infty$. Henceforth we shall restrict our considerations to the **Sobolev-Slobodeckii and the Bessel potential scales**

$$W_p^s \quad\text{and}\quad H_p^s , \qquad s\in\mathbb{R} ,$$

respectively, which are scales of Banach spaces of distributions.

We define the **formally dual boundary value problem** $(\mathcal{A}^\#,\mathcal{B}^\#)$ of $(\mathcal{A},\mathcal{B})\in\mathbb{E}(\Omega)$ by

$$\mathcal{A}^\# v := -\partial_j(a_{jk}^\#\partial_k v + a_j^\# v) + b_j^\#\partial_j v + a_0^\# v$$

and

$$\mathcal{B}^\# v := \delta\bigl\{\nu^j\gamma_\partial(a_{jk}^\#\partial_k v + a_j^\# v) + c^\#\gamma_\partial v\bigr\} + (1-\delta)\gamma_\partial v ,$$

respectively, where

$$a_{jk}^\# := {}^t a_{kj} , \quad a_j^\# := {}^t b_j , \quad b_j^\# := {}^t a_j , \quad a_0^\# := {}^t a , \quad c^\# := {}^t c .$$

and ${}^t a\in X := \mathcal{L}(\mathbb{K}^N)$ is the transposed of $a\in X$.

Note that, given $(\mathcal{A},\mathcal{B})\in\mathbb{E}(\Omega)$ and $S\in\{H,W\}$, the set consisting of the two letters H and W,

$$\mathcal{B}\in\mathcal{L}\bigl(S_p^s, L_p(\partial\Omega,\mathbb{K}^N)\bigr) , \qquad 1<p<\infty , \quad 1+1/p<s\le 2 ,$$

and

$$\mathcal{B}^\#\in\mathcal{L}\bigl(S_{p'}^{2-t}, L_{p'}(\partial\Omega,\mathbb{K}^N)\bigr) , \qquad 1<p<\infty , \quad 0\le t<1/p .$$

If $\delta=0$ (Dirichlet boundary operator), we can admit $s\in(1/p,2]$ and $t\in[0,1+1/p)$. Hence the spaces

$$S_{p,\mathcal{B}}^s \quad\text{and}\quad S_{p',\mathcal{B}^\#}^{2-s} , \qquad p\in(1,\infty) , \quad s\in[0,2]\setminus\Sigma_p^+ ,$$

are well defined by (5.23) and closed linear subspaces of S_p^s and $S_{p'}^{2-s}$, respectively. Note that

$$S_{p',\mathcal{B}^\#}^s = S_{p',\mathcal{B}}^s , \qquad s\in[0,1+1/p')\setminus\Sigma_{p'}^+ , \tag{7.1}$$

and that

$$S^s_{p',\mathcal{B}^\#} \overset{d}{\hookrightarrow} L_{p'} \ , \qquad p \in (1,\infty) \ , \qquad s \in [0,2]\backslash \Sigma^+_{p'} \ .$$

Hence

$$L_p \overset{d}{\hookrightarrow} (S^s_{p',\mathcal{B}^\#})' =: S^{-s}_{p,\mathcal{B}} \ , \qquad p \in (1,\infty) \ , \quad -s \in [-2,0]\backslash(-\Sigma^+_{p'}) \ .$$

with respect to the duality pairing naturally induced by $\langle\cdot,\cdot\rangle$ so that

$$\langle v,u\rangle_{S^{-s}_{p,\mathcal{B}}} = \langle v,u\rangle \ , \qquad v \in S^s_{p',\mathcal{B}^\#} \ , \quad u \in L_p \ , \tag{7.2}$$

for $1 < p < \infty$, and $-s \in [-2,0]\backslash(-\Sigma^+_{p'})$. Thus, without fearing confusion, we simply write $\langle\cdot,\cdot\rangle$ for $\langle\cdot,\cdot\rangle_{S^{-s}_{p,\mathcal{B}}}$.

Observe that, thanks to (5.14) and (5.15),

$$S^s_{p,\mathcal{B}} = \begin{cases} \{\, u \in S^s_p \ ; \ \mathcal{B}u = 0 \,\} \ , & 1+1/p < s \le 2 \ , \\ \{\, u \in S^s_p \ ; \ (1-\delta)\gamma_\partial u = 0 \,\} \ , & 1/p < s < 1+1/p \ , \\ S^s_p \ , & -1+1/p < s < 1/p \ , \\ \{\, v \in S^{-s}_{p'} \ ; \ (1-\delta)\gamma_\partial v = 0 \,\}' \ , & -2+1/p < s < -1+1/p \ , \\ \{\, v \in S^{-s}_{p'} \ ; \ \mathcal{B}^\# v = 0 \,\}' \ , & -2 \le s < -2+1/p \ , \end{cases} \tag{7.3}$$

if $\delta \notin \{0,1\}$. If $\delta = 0$ then

$$S^s_{p,\mathcal{B}} = \begin{cases} \{\, u \in S^s_p \ ; \ \gamma_\partial u = 0 \,\} \ , & 1/p < s \le 2 \ , \\ S^s_p \ , & -2+1/p < s < 1/p \ , \quad s \ne -1+1/p \ , \\ \{\, v \in S^{-s}_{p'} \ ; \ \gamma_\partial v = 0 \,\}' \ , & -2 \le s \le -2+1/p \ , \end{cases} \tag{7.4}$$

whereas

$$S^s_{p,\mathcal{B}} = \begin{cases} \{\, u \in S^s_p \ ; \ \mathcal{B}u = 0 \,\} \ , & 1+1/p < s \le 2 \ , \\ S^s_p \ , & -1+1/p < s < 1+1/p \ , \\ (S^{-s}_{p'})' \ , & -2+1/p < s \le -1+1/p \ , \\ \{\, v \in S^{-s}_{p'} \ ; \ \mathcal{B}^\# v = 0 \,\}' \ , & -2 \le s < -2+1/p \ , \end{cases} \tag{7.5}$$

if $\delta = 1$. Moreover,

$$S^s_p = (S^{-s}_{p'})' \ , \qquad -1+1/p < s < 1/p \ . \tag{7.6}$$

It is convenient to put

$$\Sigma_p := \Sigma^+_p \cup (-\Sigma^+_{p'}) = \begin{cases} \{-1+1/p, 1/p\} & \text{if } \delta = 0 \ , \\ \{-2+1/p, 1+1/p\} & \text{if } \delta = 1 \ , \\ (\mathbb{Z}+1/p) \cap (-2,2) & \text{otherwise} \ . \end{cases} \tag{7.7}$$

Note that

$$S^s_{p,\mathcal{B}} \overset{d}{\hookrightarrow} S^t_{p,\mathcal{B}} \ , \qquad -2 \le t < s \le 2 \ , \quad s,t \notin \Sigma_p \ . \tag{7.8}$$

Also note that $S^s_{p,\mathcal{B}}$ is a space of distributions iff $s > -1 + 1/p$ and $\delta \neq 0$, or $s > -2 + 1/p$ and $\delta = 0$, thanks to the fact that the space of test functions is dense in $S^s_{p',\mathcal{B}^\#}$ iff $s < 1/p'$ and $\delta \neq 0$, or $s < 1 + 1/p'$ and $\delta = 0$ (cf. [Tri78, Theorems 4.3.2/1 and 4.7.1]).

In the following,

$$H^s_{p,\mathcal{B}} \quad \text{and} \quad W^s_{p,\mathcal{B}} \ , \qquad p \in (1,\infty) \ , \quad s \in [-2,2] \backslash \Sigma_p \ ,$$

are said to be **Bessel potential** and **Sobolev-Slobodeckii scales induced by** $(\mathcal{A},\mathcal{B})$, respectively. Note that the 'center parts'

$$S^s_{p,\mathcal{B}} \ , \qquad p \in (1,\infty) \ , \quad s \in (-2 + 1/p, 1 + 1/p) \backslash \Sigma_p \ ,$$

are, despite of notations and names, independent of $(\mathcal{A},\mathcal{B})$. They are completely determined by the boundary characterization map δ. This fact will be of utmost importance for the study of quasilinear parabolic problems.

The following theorem shows that the spaces $S^s_{p,\mathcal{B}}$ characterize interpolation-extrapolation spaces generated by normally elliptic boundary value problems.

7.1 Theorem *Suppose that $(\mathcal{A},\mathcal{B}) \in \mathcal{E}^1(\Omega,\delta,N)$ and $1 < p < \infty$. Also suppose that*

$$(\cdot,\cdot)_\alpha \begin{cases} = [\cdot,\cdot]_\alpha & \text{if } \alpha = 1/2 \ , \\ \in \{ [\cdot,\cdot]_\alpha, (\cdot,\cdot)_{\alpha,p} \} & \text{if } \alpha \in (0,1) \backslash \{1/2\} \ , \end{cases} \tag{7.9}$$

and put $\mathfrak{F} := \{ (\cdot,\cdot)_\alpha \ ; \ 0 < \alpha < 1 \}$. Let $\{ (E_\alpha, A_\alpha) \ ; \ -1 \leq \alpha < \infty \}$ be the interpolation-extrapolation scale generated by the L_p-realization of $(\mathcal{A},\mathcal{B})$ and by $\mathfrak{F}$. Then

$$E_\alpha \doteq \begin{cases} H^{2\alpha}_{p,\mathcal{B}} & \text{if } (\cdot,\cdot)_{\theta(\alpha)} = [\cdot,\cdot]_{\theta(\alpha)} \\ W^{2\alpha}_{p,\mathcal{B}} & \text{if } (\cdot,\cdot)_{\theta(\alpha)} = (\cdot,\cdot)_{\theta(\alpha),p} \end{cases}$$

for $2\alpha \in [-2,2] \backslash (\Sigma_p \cup \{0\})$, where

$$\theta(\alpha) := \begin{cases} \alpha & \text{if } \quad 0 < \alpha < 1 \ , \\ \alpha + 1 & \text{if } -1 < \alpha < 0 \ . \end{cases}$$

This follows from Theorems 5.2 and 6.2 and the duality theory of normally elliptic boundary value problems (cf. [Ama90a] and [Ama94]).

Besides of the fact that the 'center parts' of the Bessel potential and Sobolev-Slobodeckii scales induced by $(\mathcal{A},\mathcal{B})$ have the property that they are independent of $(\mathcal{A},\mathcal{B})$, they enjoy the following important interpolation properties.

7.2 Theorem *Suppose that $1 < p < \infty$ and $-2 + 1/p < s_o < s_1 < 1 + 1/p$. Also suppose that $0 < \theta < 1$ and $s_0, s_1, s_\theta := (1-\theta)s_0 + \theta s_1 \notin \Sigma_p$. Then*

$$[H^{s_0}_{p,\mathcal{B}}, H^{s_1}_{p,\mathcal{B}}]_\theta \doteq H^{s_\theta}_{p,\mathcal{B}} \ , \tag{7.10}$$

and

$$(H^{s_0}_{p,\mathcal{B}}, H^{s_1}_{p,\mathcal{B}})_{\theta,p} \doteq W^{s_\theta}_{p,\mathcal{B}}\ , \qquad s_\theta \notin \mathbb{N}\ .$$

Moreover,

$$[W^{s_0}_{p,\mathcal{B}}, W^{s_1}_{p,\mathcal{B}}]_{\theta} \doteq W^{s_\theta}_{p,\mathcal{B}}\ , \qquad s_\theta \notin \mathbb{N}\ ,$$

and

$$(W^{s_0}_{p,\mathcal{B}}, W^{s_1}_{p,\mathcal{B}})_{\theta,p} \doteq W^{s_\theta}_{p,\mathcal{B}}\ , \qquad s_\theta \notin \mathbb{N}\ .$$

The proof of [Ama90a, Proposition 5.5] gives (7.10). The remaining assertions follow by reiteration, duality, and commutation arguments for the real and complex interpolation functors (cf. [Ama91b, Lemma 6.3] and [Ama94]).

7.3 Remark Let $\mathcal{A}_0 := (1-\Delta)1_{\mathbb{K}^N}$ and $\mathcal{B}_0 := \delta\partial_\nu + (1-\delta)\gamma_\partial$. Then the proof of Theorem 7.2 in the case that $s_0 < 0 < s_1$ relies on the fact that A_0, the L_p-realization of $(\mathcal{A}_0, \mathcal{B}_0)$, has bounded imaginary powers, and on Remark 6.1(d). The boundedness of A_0^{it} follows from a theorem of Seeley [See71] which requires a C^∞–domain. This is the only place in the whole paper where $\Omega \in C^\infty$ is essential. For all the other results, which do not employ Theorem 7.2 with $s_0 < 0 < s_1$, it suffices to assume that $\Omega \in C^2$. So far the boundedness of the imaginary powers of A_0, in the case where $\Omega \in C^2$, is only known if $\delta = 0$ and $N = 1$, thanks to [PrS93]. The proof of the latter paper does, however, not extend to Neumann boundary conditions. □

8 Extrapolated Elliptic Boundary Value Problems

Having obtained in Theorem 7.1 concrete descriptions of the interpolation-extrapolation spaces E_α for most values of $\alpha \in [-1,1]$, we turn now to the problem of characterizing the operators $A_{\alpha-1}$ for $0 \le \alpha \le 1$ if A_0 is the L_p-realization of a normally elliptic boundary value problem.

We define $\delta_0 \in \mathbb{N}$ by

$$\delta_0 := \begin{cases} 0 & \text{if } \delta = 0\ , \\ 1 & \text{otherwise}\ . \end{cases}$$

Suppose that $S \in \{H, W\}$. For dealing with quasilinear parabolic problems it will be of importance to know rather weak regularity conditions for the coefficients of $(\mathcal{A}, \mathcal{B})$ guaranteeing that, given $\delta_0 + 1/p < s \le 2$,

$$(\mathcal{A}, \mathcal{B}) \in \mathcal{L}\big(S^s_p, S^{s-2}_p \times L_p(\partial\Omega, \mathbb{K}^N)\big)$$

and, given $0 \le t < 1 - \delta_0 + 1/p$,

$$(\mathcal{A}^\#, \mathcal{B}^\#) \in \mathcal{L}\big(S^{2-t}_{p'}, S^{-t}_{p'} \times L_{p'}(\partial\Omega, \mathbb{K}^N)\big)\ .$$

For this observe that, given $1 < p < \infty$ and $s \in \mathbb{R}$,

$$S^s_p \xrightarrow{\ \partial_k\ } S^{s-1}_p \xrightarrow{\ w \mapsto a_{jk} w\ } S^{s-1}_p \xrightarrow{\ \partial_j\ } S^{s-2}_p \tag{8.1}$$

implies, thanks to (5.10) and (5.11), that

$$\big[u \mapsto \partial_j(a_{jk}\partial_k u)\big] \in \mathcal{L}(S_p^s, S_p^{s-2}) \ ,$$

provided the middle arrow in (8.1) represents a continuous linear map, that is, if a_{jk} is a pointwise multiplier for S_p^{s-1}.

8.1 Remarks

(a) Let E be a Banach space of $\mathbb{K}^N$-valued distributions on Ω, that is, E is a Banach space such that $E \hookrightarrow \mathcal{D}'$. Then $a \in L_\infty(\Omega, X)$ is a (pointwise) **multiplier** for E if

$$[u \mapsto au] \in \mathcal{L}(E) \ . \tag{8.2}$$

We denote by $M(E)$ the set of all multipliers for E. It is a Banach space if the norm of $a \in M(E)$ is defined to be the norm of the linear map (8.2) (cf. [Tri83] or [Boud87] for more details).

(b) It is well known (e.g., [Proposition III.6][Boud87]) and easily seen that

$$M(L_p) = L_\infty(\Omega, X) \ , \qquad 1 \le p \le \infty \ ,$$

Multipliers on Besov and Triebel-Lizorkin spaces have been studied in detail in [Tri83], [Fra86], [Sic86], and [Sic93] (also cf. [MaS85] and [Ama91d]). In general, there is no easy description of $M(S_p^s)$, except for the case where $|s| > n/p$. Indeed,

$$M(S_p^s) = \begin{cases} H_p^{|s|}(\Omega, X) & \text{if } S = H \ , \\ W_p^{|s|}(\Omega, X) & \text{if } S = W \ , \end{cases}$$

for $1 < p < \infty$ and $|s| > n/p$. However,

$$C^\rho(\overline{\Omega}, X) \hookrightarrow M(S_p^s) \ , \qquad 1 < p < \infty \ , \quad s \in \mathbb{R} \ , \tag{8.3}$$

provided

$$\rho \begin{cases} > |s| & \text{if } s \notin \mathbb{Z} \ , \\ = |s| & \text{if } s \in \mathbb{Z} \ . \end{cases} \tag{8.4}$$

Thus Hölder spaces are kind of ‘universal’ multiplier spaces for S_p^s, independently of p. □

We fix now an increasing function $\sigma : [0,1] \to [0,1]$ satisfying

$$\sigma(0) = 0 \ , \qquad \sigma(t) > t \ , \quad 0 < t < 1 \ , \tag{8.5}$$

and put

$$\rho(\alpha) := \sigma(|2\alpha - 1|) \ , \qquad 0 \le \alpha \le 1 \ . \tag{8.6}$$

Moreover, we define $\mathbb{E}^\alpha(\Omega) := \mathbb{E}^\alpha(\Omega, N)$ by

$$\mathbb{E}^\alpha(\Omega) := \big[C^{\rho(\alpha)}(\overline{\Omega}, X)\big]^{n^2+n+n} \times L_\infty(\Omega, X) \times C^{\rho(\alpha)}(\partial\Omega, X) \ , \qquad 0 \le \alpha \le 1 \ .$$

Note that $\mathbb{E}^1(\Omega)$ is the space introduced in Section 4, that

$$\mathbb{E}^1(\Omega) \hookrightarrow \mathbb{E}^\alpha(\Omega) \hookrightarrow \mathbb{E}^\beta(\Omega) \hookrightarrow \mathbb{E}^{1/2}(\Omega) = \mathbb{E}(\Omega) \ , \qquad 1/2 < \beta < \alpha < 1 \ , \tag{8.7}$$

and that

$$\mathbb{E}^\alpha(\Omega) = \mathbb{E}^{1-\alpha}(\Omega) \ , \qquad 0 \le \alpha \le 1 \ . \tag{8.8}$$

Of course, $\mathbb{E}^\alpha(\Omega)$ depends on σ too for $\alpha \in (0,1)\backslash\{1/2\}$.

We choose a boundary characterization map δ and define **scales of boundary spaces**

$$\partial W_p^s \ , \qquad 1 < p < \infty \ , \quad 0 \le s \le 2 \ ,$$

by

$$\partial W_p^s := \partial W_p^s(\Omega, \delta, N) := \bigoplus_{\Gamma \in \mathbf{\Gamma}} \bigoplus_{r=1}^{N} W_p^{s-\delta_r(\Gamma)-1/p}(\Gamma, \mathbb{K}) \ ,$$

where $\mathbf{\Gamma}$ is the set of components of $\partial\Omega$ and $\delta_r(\Gamma) \in \{0,1\} \subset \mathbb{N}$ denotes the value of δ_r on $\Gamma \in \mathbf{\Gamma}$. Note that $\mathbf{\Gamma}$ is finite since $\mathbf{\Gamma}$ is an open covering of $\partial\Omega$ and $\partial\Omega$ is compact. We also put

$$\partial_1 W_p^{s-1} := \left\{ u \in W_p^{s-1-1/p}(\partial\Omega, \mathbb{K}^N) \ ; \ (1-\delta)u = 0 \right\}$$

and

$$\partial_0 W_p^s := \left\{ u \in W_p^{s-1/p}(\partial\Omega, \mathbb{K}^N) \ ; \ \delta u = 0 \right\} \ .$$

Then

$$\partial W_p^s = \partial_1 W_p^{s-1} \oplus \partial_0 W_p^s \ , \qquad 1 < p < \infty \ , \quad 0 \le s \le 2 \ . \tag{8.9}$$

It is an easy consequence of (8.3), (8.4), arguments of the type of (8.1), and the trace theorem that

$$(\mathcal{A}, \mathcal{B}) | S_p^{2\alpha} \in \mathcal{L}(S_p^{2\alpha}, S_p^{2\alpha-2} \times \partial W_p^{2\alpha}) \ , \qquad 1 < p < \infty \ , \quad \delta_0 + 1/p < 2\alpha \le 2 \ ,$$

whenever $(\mathcal{A}, \mathcal{B}) \in \mathbb{E}^\alpha(\Omega)$. In fact, the map

$$\mathbb{E}^\alpha(\Omega) \to \mathcal{L}(S_p^{2\alpha}, S_p^{2\alpha-2} \times \partial W_p^{2\alpha}) \ , \quad (\mathcal{A}, \mathcal{B}) \mapsto (\mathcal{A}, \mathcal{B}) | S_p^{2\alpha} \tag{8.10}$$

is linear and continuous for $1 < p < \infty$ and $\delta_0 + 1/p < 2\alpha \le 2$. Similarly, the map

$$\mathbb{E}^\beta(\Omega) \to \mathcal{L}(S_{p'}^{2-2\beta}, S_{p'}^{-2\beta} \times \partial W_{p'}^{2-2\beta}) \ , \quad (\mathcal{A}, \mathcal{B}) \mapsto (\mathcal{A}^\#, \mathcal{B}^\#) | S_{p'}^{2-2\beta} \tag{8.11}$$

is linear and continuous for $1 < p < \infty$ and $0 \le 2\beta < 1 - \delta_0 + 1/p$.

8.2 Remark The ‘multiplier spaces’ $\mathbb{E}^\alpha(\Omega)$ have been introduced to have relatively simple conditions guaranteeing (8.10) and (8.11). Of course, the corresponding regularity assumptions for the coefficients of $(\mathcal{A}, \mathcal{B})$ are by no means optimal. More general conditions can be obtained by using ‘better’ multiplier spaces, yet at the expense of more cumbersome formulations. □

With $(\mathcal{A},\mathcal{B}) \in \mathbb{E}(\Omega)$ we associate the **Dirichlet form $\boldsymbol{a}$**, defined by

$$\boldsymbol{a}(v,u) := \langle \partial_j v, a_{jk}\partial_k u + a_j u\rangle + \langle v, b_j \partial_j u + a_0 u\rangle + \langle \gamma_\partial v, c\gamma_\partial u\rangle_\partial \ .$$

It follows from Remark 8.1, the duality theorems (5.14) and (5.15), the imbeddings (5.5) and (5.6), and by arguments of the type of (8.1) that

$$\boldsymbol{a} \in \mathcal{L}(S^{2\beta}_{p'}, S^{2\alpha}_p; \mathbb{K}) \ , \tag{8.12}$$

in fact, that the map

$$\mathbb{E}^\alpha(\Omega) \to \mathcal{L}(S^{2\beta}_{p'}, S^{2\alpha}_p; \mathbb{K}) \ , \quad (\mathcal{A},\mathcal{B}) \mapsto \boldsymbol{a} \tag{8.13}$$

is linear and continuous for $1 < p < \infty$ and $1/p < 2\alpha \le 2,\ 1/p' < 2\beta \le 2$ satisfying $\alpha + \beta \ge 1$.

With $(\mathcal{A},\mathcal{B}) \in \mathbb{E}(\Omega)$ we also associate two additional boundary operators defined by

$$\mathcal{C}u := (\delta - 1)\bigl\{\nu^j \gamma_\partial(a_{jk}\partial_k u + a_j u) + c\gamma_\partial u\bigr\} + \delta\gamma_\partial u$$

and

$$\mathcal{C}^\# v := (\delta - 1)\bigl\{\nu^j \gamma_\partial(a^\#_{jk}\partial_k v + a^\#_j v) + c^\#\gamma_\partial v\bigr\} + \delta\gamma_\partial v \ ,$$

respectively.

Suppose that $1 < p < \infty$. If $1 + 1/p < 2\alpha \le 2$ and $1/p' < 2\beta \le 2$ satisfy $\alpha + \beta \ge 1$, and if $(\mathcal{A},\mathcal{B}) \in \mathbb{E}^\alpha(\Omega)$, it is not difficult to verify that **Green's formula**

$$\langle v, \mathcal{A}u\rangle + \langle \delta\gamma_\partial v, \mathcal{B}u\rangle_\partial + \bigl\langle (\delta - 1)\gamma_\partial v, \mathcal{C}u\bigr\rangle_\partial = \boldsymbol{a}(v,u) \ , \qquad (v,u) \in S^{2\beta}_{p'} \times S^{2\alpha}_p \ , \tag{8.14}$$

is valid. Similarly, if $(\mathcal{A},\mathcal{B}) \in \mathbb{E}^\beta(\Omega)$ and if $1/p < 2\alpha \le 2$ and $1 + 1/p' < 2\beta \le 2$ satisfy $\alpha + \beta \ge 1$, the **Green formula**

$$\langle \mathcal{A}^\# v, u\rangle + \langle \mathcal{B}^\# v, \delta\gamma_\partial u\rangle_\partial + \bigl\langle \mathcal{C}^\# v, (\delta - 1)\gamma_\partial u\bigr\rangle_\partial = \boldsymbol{a}(v,u) \ , \qquad (v,u) \in S^{2\beta}_{p'} \times S^{2\alpha}_p \ , \tag{8.15}$$

is true. Lastly, if $(\mathcal{A},\mathcal{B}) \in \mathbb{E}^\alpha(\Omega)$ and if $1 + 1/p < 2\alpha \le 2$ and $1 + 1/p' < 2\beta \le 2$ satisfy $\alpha + \beta \ge 1$, the **second Green formula**

$$\langle v, \mathcal{A}u\rangle + \langle \mathcal{C}^\# v, \mathcal{B}u\rangle_\partial = \langle \mathcal{A}^\# v, u\rangle + \langle \mathcal{B}^\# v, \mathcal{C}u\rangle_\partial \ , \qquad (v,u) \in S^{2\beta}_{p'} \times S^{2\alpha}_p \ , \tag{8.16}$$

is valid.

Suppose that $1 < p < \infty$ and $(\mathcal{A},\mathcal{B}) \in \mathbb{E}^\alpha(\Omega)$. We define a unique

$$A_{\alpha-1} \in \mathcal{L}(S^{2\alpha}_{p,\mathcal{B}}, S^{2\alpha-2}_{p,\mathcal{B}}) \ ,$$

the **$S^{2\alpha-2}_{p,\mathcal{B}}$-realization of** $(\mathcal{A},\mathcal{B})$, as follows:

- If $\delta_0 + 1/p < 2\alpha \le 2$,

$$A_{\alpha-1} := \mathcal{A} | S^{2\alpha}_{p,\mathcal{B}} \ . \tag{8.17}$$

- If $1/p < 2\alpha < 1 + 1/p$ then $A_{\alpha-1}$ is the unique linear operator induced by the Dirichlet form $\boldsymbol{a}$ associated with $(\mathcal{A}, \mathcal{B})$, that is,

$$\langle v, A_{\alpha-1} u \rangle = \boldsymbol{a}(v, u) \ , \qquad (v, u) \in S^{2-2\alpha}_{p', \mathcal{B}^\#} \times S^{2\alpha}_{p, \mathcal{B}} \ . \tag{8.18}$$

  Observe that, thanks to (8.12) and (7.3)–(7.6), the operator $A_{\alpha-1}$ is well defined. Moreover, it follows from (8.13) that

$$\big[(\mathcal{A}, \mathcal{B}) \mapsto A_{\alpha-1}\big] \in \mathcal{L}\big(\mathbb{E}^\alpha(\Omega), \mathcal{L}(S^{2\alpha}_{p,\mathcal{B}}, S^{2\alpha-2}_{p,\mathcal{B}})\big) \ . \tag{8.19}$$

  (Thanks to Green's formula (8.14), the two definitions (8.17) and (8.18) for $A_{\alpha-1}$ agree in the case that $\delta = 0$ and $1/p < 2\alpha < 1 + 1/p$.)

- Lastly, if $0 \le 2\alpha < 1 - \delta_0 + 1/p$,

$$\langle v, A_{\alpha-1} u \rangle = \langle \mathcal{A}^\# v, u \rangle \ , \qquad (v, u) \in S^{2-2\alpha}_{p', \mathcal{B}^\#} \times S^{2\alpha}_{p} \ , \tag{8.20}$$

  that is,

$$A_{\alpha-1} := (\mathcal{A}^\# | S^{2-2\alpha}_{p', \mathcal{B}^\#})' \ .$$

  Note that $A_{\alpha-1}$ is well defined in this case too, thanks to (8.11) and (7.3) (and to the Green formula (8.15) if $\delta = 0$ and $1/p < 2\alpha < 1 + 1/p$).

Of course, the $S^{2\alpha-2}_{p,\mathcal{B}}$-realization $A_{\alpha-1}$ of $(\mathcal{A}, \mathcal{B})$ depends on $S \in \{H, W\}$ too. But we do not indicate this in order not to overburden the notation.

The following theorem justifies the notation $A_{\alpha-1}$ for the $S^{2\alpha-2}_{p,\mathcal{B}}$-realization of $(\mathcal{A}, \mathcal{B})$.

8.3 Theorem *Suppose that $(\mathcal{A}, \mathcal{B}) \in \mathcal{E}^1(\Omega)$, that $1 < p < \infty$, and that $\mathfrak{F}$ is defined by (7.9). Let $\{(E_\alpha, A_\alpha) \ ; \ -1 \le \alpha < \infty\}$ be the interpolation-extrapolation scale generated by the L_p-realization of $(\mathcal{A}, \mathcal{B})$ and by $\mathfrak{F}$. Then, given $2\alpha \in [0, 2] \backslash \Sigma_p$,*

$$A_{\alpha-1} = \begin{cases} H^{2\alpha-2}_{p,\mathcal{B}}\text{-realization of } (\mathcal{A}, \mathcal{B}) & \text{if } (\cdot,\cdot)_\alpha = [\cdot,\cdot]_\alpha \ , \\ W^{2\alpha-2}_{p,\mathcal{B}}\text{-realization of } (\mathcal{A}, \mathcal{B}) & \text{otherwise} \ . \end{cases}$$

For the proof of this theorem it suffices to observe that, thanks to (8.7), the first Green formula (8.14) implies

$$\langle v, A_0 u \rangle = \boldsymbol{a}(v, u) \ , \qquad (v, u) \in S^{2-2\alpha}_{p', \mathcal{B}^\#} \times W^2_{p,\mathcal{B}} \ , \quad 1/p < 2\alpha < 1 + 1/p \ ,$$

and the second Green formula (8.16) gives

$$\langle v, A_0 u \rangle = \langle \mathcal{A}^\# v, u \rangle \ , \qquad (v, u) \in S^{2-2\alpha}_{p', \mathcal{B}^\#} \times W^2_{p,\mathcal{B}} \ , \qquad 0 \le 2\alpha < 1/p \ .$$

Now the assertion follows from Theorem 7.1, formulas (7.2) and (6.5), and an obvious density argument (cf. [Ama94] for more details).

8.4 Remarks

(a) Suppose that $(\mathcal{A}, \mathcal{B}) \in \mathcal{E}^1(\Omega)$. Since $W^2_p \overset{d}{\hookrightarrow}\hookrightarrow L_p$, it follows from Remark 6.1(e) and Theorem 8.3 that the $S^{2\alpha-2}_{p,\mathcal{B}}$-realization of $(\mathcal{A}, \mathcal{B})$ has a compact resolvent for $2\alpha \in [0, 2] \backslash \Sigma_p$.

(b) Suppose that $(\mathcal{A},\mathcal{B}) \in \mathcal{E}^1(\Omega)$. It is a consequence of Theorem 8.3 that
$$A_{\alpha-1} \in \mathcal{H}(S^{2\alpha}_{p,\mathcal{B}}, S^{2\alpha-2}_{p,\mathcal{B}}) \tag{8.21}$$
and that
$$\sigma(A_{\alpha-1}) = \sigma(A_0)$$
for $2\alpha \in [0,2]\backslash\Sigma_p$.

(c) Using (a) and the fact that the 'center parts' of the Bessel potential and Sobolev-Slobodeckii scales induced by $(\mathcal{A},\mathcal{B})$ are independent of $(\mathcal{A},\mathcal{B})$, we deduce from Remark 6.1(e) that
$$S^{2\alpha}_{p,\mathcal{B}} \overset{cc}{\hookrightarrow} S^{2\beta}_{p,\mathcal{B}} \ , \qquad 1 < p < \infty \ , \quad 2\alpha, 2\beta \in (-2+1/p, 1+1/p)\backslash\Sigma_p \ ,$$
provided $2\alpha > 2\beta$. □

Thanks to Remark 8.4(b) Theorem 8.3 implies generation theorems for analytic semigroups on the spaces $S^{2\alpha-2}_{p,\mathcal{B}}$ for the corresponding realizations of $(\mathcal{A},\mathcal{B}) \in \mathcal{E}^1(\Omega)$. As will be shown in detail in Sections 9 and 11 below, these realizations are closely related to weak formulations of elliptic and parabolic boundary value problems of various 'degrees of weakness'. In this connection it is of importance to know that (8.21) is true if $(\mathcal{A},\mathcal{B})$ belongs to
$$\mathcal{E}^\alpha(\Omega) := \mathcal{E}^\alpha(\Omega,\delta,N) := \mathcal{E}(\Omega) \cap \mathbb{E}^\alpha(\Omega) \ , \qquad 0 \le \alpha \le 1 \ ,$$
only. Note that
$$\mathcal{E}^\alpha(\Omega) \text{ is open in } \mathbb{E}^\alpha(\Omega) \ , \qquad 0 \le \alpha \le 1 \ , \tag{8.22}$$
thanks to (4.8) and (8.7).

The following theorem contains the desired 'weak generation' result.

8.5 Theorem *Suppose that $1 < p < \infty$ and $1 \le 2\alpha \le 2$. If $(\mathcal{A},\mathcal{B}) \in \mathcal{E}^\alpha(\Omega)$ then*
$$A_{\beta-1} \in \mathcal{H}(S^{2\beta}_{p,\mathcal{B}}, S^{2\beta-2}_{p,\mathcal{B}}) \ , \qquad 2\beta \in [2-2\alpha, 2\alpha]\backslash\Sigma_p \ , \tag{8.23}$$
and $A_{\beta-1}$ has a compact resolvent. Moreover, the map
$$\mathcal{E}^\alpha(\Omega) \to \mathcal{H}(S^{2\beta}_{p,\mathcal{B}}, S^{2\beta-2}_{p,\mathcal{B}}) \ , \quad (\mathcal{A},\mathcal{B}) \mapsto A_{\beta-1} \tag{8.24}$$
is analytic, provided either $1/p < 2\beta < 1+1/p$, or $\delta = 0$ and $2\beta \in [0,2]\backslash\{1/p\}$.

The generation theorem (8.23) in the case $1/p < 2\alpha < 1+1/p$ has been proven in [Ama89a] under an additional hypothesis (cf. [Ama89a, Erratum]). The proof is based on Theorem 8.3. By a somewhat more sophisticated argument it can be shown that the additional hypothesis is not needed. Similar arguments can be used to prove (8.23) for $1+1/p < 2\alpha < 2$. The case $0 \le 2\alpha < 1/p$ follows then by duality, of course. The compactness of the resolvent is an easy consequence of $S^{2\alpha}_{p,\mathcal{B}} \overset{cc}{\hookrightarrow} L_p \hookrightarrow S^{2\alpha-2}_{p,\mathcal{B}}$ which, in turn, follows from $W^s_p \overset{cc}{\hookrightarrow} L_p$ (also see Remark 8.4(c)). Lastly, (8.13) and (8.22) imply (8.24). (For details we refer to [Ama94].)

8.6 Remarks

(a) The regularity assumptions for the 'lower order coefficients' a_j, b_j, a_0, and c in

Theorem 8.5 can be considerably weakened (cf. [Ama89a] and [Ama94]). A particularly simple situation occurs if $p > n$ and $2\beta = 1$. Indeed, let

$$\mathbb{E}_p(\Omega) := C(\overline{\Omega}, X)^{n^2} \times L_p(\Omega, X)^n \times L_{p' \vee n}(\overline{\Omega}, X)^n \times L_q(\Omega, X) \times L_r(\partial\Omega, X) ,$$

where $1 \wedge pn/(n+p) \leq q < \infty$ and $1 \wedge p(n-1)/(p+n-1) \leq r < \infty$. Then

$$\mathcal{E}_p(\Omega) := \mathbb{E}_p(\Omega) \cap \mathcal{E}(\Omega)$$

is open in $\mathbb{E}(\Omega)$, and it is not difficult to verify that the map

$$\mathcal{E}_p(\Omega) \to \mathcal{H}(W^1_{p,\mathcal{B}}, W^{-1}_{p,\mathcal{B}}) , \quad (\mathcal{A}, \mathcal{B}) \mapsto A_{-1/2}$$

is well defined and analytic.

(b) Analoguous 'weak' generation theorems are valid for normally elliptic systems of higher order (cf. [Ama94]).

(c) It is a consequence of (3.1) and (8.23) that there exist constants $\kappa \geq 1$ and $\omega \geq 0$ such that

$$|\lambda| \, \|u\|_{S^{2\beta-2}_{p,\mathcal{B}}} + \|u\|_{S^{2\beta}_{p,\mathcal{B}}} \leq \kappa \, \|(\lambda + A_{\beta-1})u\|_{S^{2\beta-2}_{p,\mathcal{B}}} , \qquad u \in S^{2\beta}_{p,\mathcal{B}} ,$$

for all $\lambda \in [\mathrm{Re}\, z \geq \omega]$ and $2\beta \in [2-2\alpha, 2\alpha] \backslash \Sigma_p$.

(d) If we restrict our consideration to values of β with $1 + 1/p < 2\beta \leq 2$, Theorem 8.5 remains true for second order normally elliptic boundary value problems of the general form

$$\mathcal{A}u := -a_{jk}\partial_j\partial_k u + a_j \partial_j u + a_0 u$$

and

$$\mathcal{B}u := \delta\{b_j \gamma_\partial \partial_j u + c\gamma_\partial u\} + (1-\delta) du ,$$

provided $a_{jk} \in C^\rho(\overline{\Omega}, X)$ for some $\rho > 2 - 2\beta$, and $a_j, a_0 \in L_\infty(\Omega, X)$, and where $b_j, c \in C^1(\partial\Omega, X)$ and $d \in C^2(\partial\Omega, X)$ (cf. Remark 4.2(b) and observe that every regular boundary operator of order at most one can be written in the above form). For a proof we refer to [Ama94]. □

In the very special situation that $\alpha = 1/2$, it follows from Theorem 8.5 and Remark 8.6(a), in particular, that the Dirichlet form

$$(v, u) \mapsto \langle \partial_j v, a_{jk}\partial_k u + a_j u \rangle + \langle v, b_j \partial_j u + a_0 u \rangle + \langle \gamma_\partial v, c\gamma_\partial u \rangle_\partial$$

with continuous top-order coefficients a_{jk} and lower-order coefficients belonging to L_∞ (for example) defines a negative generator $A_{-1/2}$ of an analytic semigroup on $W^{-1}_{p,\mathcal{B}}$ with domain $W^1_{p,\mathcal{B}}$, provided $(\mathcal{A}, \mathcal{B}) \in \mathcal{E}(\Omega)$. If A is uniformly strongly elliptic and $\mathcal{B} = \gamma_\partial$ is the Dirichlet boundary operator, this has also been proven (for continuous coefficients and systems of order $2m$) in [Ves89] by a different technique. If $N = 1$ and $\mathcal{A}$ is uniformly strongly elliptic with continuous coefficients, the fact that $A_{-1/2} \in \mathcal{H}(W^1_{p,\mathcal{B}}, W^{-1}_{p,\mathcal{B}})$ has again been shown in [LuV91] (also cf. [Ves91]). In the latter paper, spaces of continuous and Hölder continuous functions are considered too (also cf. [Ves88]). The methods of proofs in these papers do not extend to general normally elliptic boundary value problems of the class $\mathcal{E}(\Omega)$ (or to $\alpha \neq 1/2$) since they rely (implicitly or explicitly) on the coercivity of the quadratic form $u \mapsto \boldsymbol{a}(\overline{u}, u)$ on $H^1_{\mathcal{B}} := H^1_{2,\mathcal{B}}$, which is not available in the general case.

Note that the Dirichlet form $\boldsymbol{a}$ is a continuous bilinear form on $W^1_{p',\mathcal{B}^\#} \times W^1_{p,\mathcal{B}}$ if it is only assumed that it has L_∞-coefficients. If $p = 2$ and the associated quadratic form $u \mapsto \boldsymbol{a}(\overline{u}, u)$ is coercive, it is well known that $A_{-1/2} \in \mathcal{H}(H^1_{\mathcal{B}}, H^{-1}_{\mathcal{B}})$, since $A_{-1/2}$ is 'regularly accretive' (e.g., [Tan79, Theorem 3.6.1]). In [GrR89] (also cf. [Grö89], [Grö92]) this result is extended — under weak conditions on $\partial\Omega$ and without requiring that Dirichlet and Neumann boundary parts do not touch — to values of p different from 2 but close to 2.

Given $(\mathcal{A}, \mathcal{B}) \in \mathbb{E}^\alpha(\Omega) = \mathbb{E}^{1-\alpha}(\Omega)$, we define

$$A^\#_{-\alpha} \in \mathcal{L}(S^{2-2\alpha}_{p',\mathcal{B}^\#}, S^{-2\alpha}_{p',\mathcal{B}^\#}) ,$$

the $S^{-2\alpha}_{p',\mathcal{B}^\#}$**-realization** of $(\mathcal{A}^\#, \mathcal{B}^\#)$, as follows:

- If $0 \leq 2\alpha < 1 - \delta_0 + 1/p$,

$$A^\#_{-\alpha} := \mathcal{A}^\# | S^{2-2\alpha}_{p',\mathcal{B}^\#} .$$

- If $1/p < 2\alpha < 1 + 1/p$,

$$\langle A^\#_{-\alpha} v, u \rangle := \boldsymbol{a}(v, u) , \qquad (v, u) \in S^{2-2\alpha}_{p',\mathcal{B}^\#} \times S^{2\alpha}_{p,\mathcal{B}} .$$

- If $\delta_0 + 1/p < 2\alpha \leq 2$,

$$\langle A^\#_{-\alpha} v, u \rangle := \langle v, \mathcal{A} u \rangle , \qquad (v, u) \in S^{2-2\alpha}_{p',\mathcal{B}^\#} \times S^{2\alpha}_{p,\mathcal{B}} .$$

Similarly as above, one verifies the unambiguity of this definition.

Let $(\mathcal{A}, \mathcal{B}) \in \mathcal{E}^\alpha(\Omega)$. Then it follows from the fact that $A_{-\alpha}$ has a nonempty resolvent set, from Theorem 8.5, and by Green's formulas that

$$(A_{\alpha-1})' = A^\#_{1-\alpha} := S^{2-2\alpha}_{p',\mathcal{B}^\#}\text{-realization of } A^\#_{-\alpha} \tag{8.25}$$

for $2\alpha \in [0, 2] \backslash (\mathbb{N} + 1/p)$.

For simplicity of presentation, we have restricted ourselves to the 'classical scales' H^s_p and W^s_p. It should be clear to the reader that all the results presented so far extend to the scales of Besov spaces $B^s_{p,q}$ with $1 < q < \infty$. For this it suffices to admit the interpolation functors $(\cdot, \cdot)_{\theta,q}$ with $1 < q < \infty$. If one considers the continuous interpolation functors $(\cdot, \cdot)^0_{\theta,\infty}$ too, it is possible to extend (suitable modifications of) the above results to include the scales of little Nikolskii spaces, $n^s_{p,\mathcal{B}}$, induced by $(\mathcal{A}, \mathcal{B})$ (cf. [Sit92], [Sit93a]; also see [Gui91b] for a study of elliptic and parabolic boundary value problems in little Nikolskii spaces of positive orders).

Let now $\mathbb{K} = \mathbb{R}$. We always equip $\mathbb{R}^N$ with its natural, that is, componentwise, order, so that $\mathbb{R}^N$ is a Banach lattice with positive cone $(\mathbb{R}^N)^+ := (\mathbb{R}^+)^N$. The natural order of L_p, $1 \le p \le \infty$, is induced by the positive cone

$$L_p^+ := \{\, u \in L_p \ ; \ u \ge 0 \text{ a.e. } \} \ .$$

Then L_p is a Banach lattice. Note that $L_{p'}^+ = (L_p^+)'$ for $1 < p < \infty$.

Suppose that $S \in \{H, W\}$, that $1 < p < \infty$, and that $s \in \dot{\mathbb{R}}^+$. Then $S_p^s \hookrightarrow L_p$ so that S_p^s is an ordered Banach space with respect to the natural order canonically induced by L_p. By mollifying, for example, it is easily seen that $(S_p^s)^+ = S_p^s \cap L_p^+$, the positive cone of S_p^s, is total.

If $1 < p < \infty$ and $s \in \mathbb{R}^+$, the natural order of $W_p^1(\partial\Omega, \mathbb{R}^N)$ is defined similarly. It follows that the positive cone $W_p^s(\partial\Omega, \mathbb{R}^N)^+$ of $W_p^s(\Omega, \mathbb{R}^N)$ is total. Thus $W_p^{-s}(\partial\Omega, \mathbb{R}^N)$ is an ordered Banach space with respect to the canonical dual order and, given $u \in W_p^{-s}(\partial\Omega, \mathbb{R}^N)$, it follows that

$$u \ge 0 \quad \text{iff} \quad \langle v, u \rangle_\partial \ge 0 \ , \quad v \in W_{p'}^s(\partial\Omega, \mathbb{R}^N)^+ \ .$$

Consequently, the spaces

$$\partial W_p^s \ , \qquad 1 < p < \infty \ , \quad 0 \le s \le 2 \ ,$$

are ordered Banach spaces with respect to the obvious natural order.

Suppose now that $N = 1$. If $1 < p < \infty$ and $0 < s < 1 + 1/p$, it follows from [BouM91] and [BouK92] that S_p^s is a sublattice of L_p and that there exists a constant c such that

$$\| \, |u| \, \|_{S_p^s} \le c \|u\|_{S_p^s} \ , \qquad u \in S_p^s \ . \tag{8.26}$$

Let $1/p < s < 1 + 1/p$. By means of local coordinates and the fact that the translation group is strongly continuous on $S_p^s(\mathbb{R}^n, \mathbb{R}^N)$, it is not difficult to see that

$$C^2_{(1-\delta)\gamma_\partial} := \{\, u \in C^2 \ ; \ (1-\delta)\gamma_\partial u = 0 \,\}$$

is dense in $S^s_{p,\mathcal{B}}$ (cf. the proof of [Ada75, Theorem 3.18]). Thus, given $u \in S^s_{p,\mathcal{B}}$, there exists a sequence (u_j) in $C^2_{(1-\delta)\gamma_\partial}$ such that $u_j \to u$ in S_p^s. From (8.26) and the reflexivity of S_p^s we deduce the existence of $v \in S_p^s$ and a subsequence, again denoted by (u_j), such that $|u_j| \rightharpoonup v$ in S_p^s, where $\rightharpoonup$ means 'weak convergence'. Since $|u_j| \to |u|$ in L_p, we see that $v = |u|$, that is, $|u_j| \rightharpoonup |u|$. Note that $(1-\delta)\gamma_\partial$ is continuous, whence weakly continuous, from S_p^s into ∂W_p^s, and that $(1-\delta)\gamma_\partial \, |u_j| = |(1-\delta)\gamma_\partial u_j| = 0$. Consequently $(1-\delta)\gamma_\partial \, |u| = 0$, that is, $|u| \in S^s_{p,\mathcal{B}}$. This shows that

$$S^s_{p,\mathcal{B}} \text{ is a sublattice of } L_p \quad \text{if} \quad 0 < s < 1 + 1/p \text{ and } s \notin \Sigma_p \ .$$

Note, however, that $S^s_{p,\mathcal{B}}$ is not a Banach lattice since the positive cone is not normal. But, of course, the positive cone $(S^s_{p,\mathcal{B}})^+ = (S^s_{p,\mathcal{B}}) \cap L_p^+$ is total in $S^s_{p,\mathcal{B}}$. Hence the dual positive cone

$$(S^{-s}_{p,\mathcal{B}})^+ := \big((S^s_{p',\mathcal{B}\#})^+\big)' \ , \qquad -2 + 1/p < -s < 0 \ , \quad s \notin \Sigma_p \ ,$$

is a proper cone in $S^{-s}_{p,\mathcal{B}}$ inducing the natural order. Thus, given $u \in S^{-s}_{p,\mathcal{B}}$, it follows that

$$u \ge 0 \quad \text{iff} \quad \langle v, u \rangle \ge 0 \ , \quad v \in (S^s_{p',\mathcal{B}\#})^+ \ . \tag{8.27}$$

Suppose now that $2\alpha \in [1,2]\backslash\Sigma_p$ and that $(\mathcal{A},\mathcal{B}) \in \mathcal{E}^\alpha(\Omega)$. Given $\beta \in [2-2\alpha, 2\alpha]\backslash\Sigma_p$, let $A_{\beta-1}$ be the $S^{2\beta-2}_{p,\mathcal{B}}$-realization of $(\mathcal{A},\mathcal{B})$. If $(\mathcal{A},\mathcal{B}) \in \mathbb{E}^1(\Omega)$, it follows from the maximum principle for uniformly elliptic equations that A_0 is resolvent positive (cf. [Ama83]). Hence $A_0^\# = (A_0)'$ is resolvent positive too, thanks to $L^+_{p'} = (L^+_p)'$. Thus, given $2\beta \in (1/p, 2]\backslash\Sigma_p$ and $(u,v) \in (S^{2\beta-2}_{p,\mathcal{B}})^+ \times (S^{2-2\beta}_{p',\mathcal{B}^\#})^+$, it follows from Theorem 6.2 and (8.27) that

$$\big\langle v, (\lambda + A_{\beta-1})^{-1}u\big\rangle = \big\langle (\lambda + A^\#_{1-\beta})^{-1}v, u\big\rangle = \big\langle (\lambda + A^\#_0)^{-1}v, u\big\rangle \geq 0 \tag{8.28}$$

for all sufficiently large $\lambda > 0$. Thus $A_{\beta-1}$ is resolvent positive for $2\beta \in (1/p, 2]\backslash\Sigma_p$, provided $(\mathcal{A},\mathcal{B}) \in \mathcal{E}^1(\Omega)$.

Suppose now that $1 \leq 2\alpha < 2$. Since $\rho(\alpha) > 2\alpha - 1$ for $2\alpha > 1$ and since $C^s \hookrightarrow c^t$ and, similarly, $C^s(\partial\Omega,\mathbb{R}) \hookrightarrow c^t(\partial\Omega,\mathbb{R})$ for $0 < t < s < 1$ and for $s = t = 0$, we can assume, by replacing $\rho(\alpha)$ by a slightly smaller number if $2\alpha > 1$, that

$$(\mathcal{A},\mathcal{B}) \in \big[c^{\rho(\alpha)}(\overline{\Omega},\mathbb{R})\big]^{n^2+n+n} \times L_\infty(\Omega,\mathbb{R}) \times c^{\rho(\alpha)}(\partial\Omega,\mathbb{R}) \ .$$

Note that $\mathbb{E}^1(\Omega)$ is dense in the latter space. Hence, thanks to (8.22), there exists a sequence $\big((\mathcal{A}_j,\mathcal{B}_j)\big) \in \mathcal{E}^1(\Omega)$ such that $(\mathcal{A}_j,\mathcal{B}_j) \to (\mathcal{A},\mathcal{B})$ in $\mathcal{E}^\alpha(\Omega)$. Now it follows from (8.24) and the continuity of the inversion map that, denoting by $(A_j)_{\beta-1}$ the $S^{2\beta-2}_{p,\mathcal{B}}$-realization of $(\mathcal{A}_j,\mathcal{B}_j)$,

$$\big(\lambda + (A_j)_{\beta-1}\big)^{-1} \to (\lambda + A_{\beta-1})^{-1} \qquad \text{in } \mathcal{L}(S^{2\beta-2}_{p,\mathcal{B}})$$

for each sufficiently large $\lambda > 0$, provided $2\beta \in [2-2\alpha, 2\alpha] \cap (1/p, 1+1/p)$. Since

$$(\lambda + A_{\gamma-1})^{-1} \subset (\lambda + A_{\beta-1})^{-1} \qquad \text{for} \quad \beta < \gamma \leq 1 \ ,$$

it is clear that $A_{\beta-1}$ is resolvent positive for each $2\beta \in (1/p, 2\alpha] \cap [2-2\alpha, 2\alpha]\backslash\Sigma_p$.

Since $\mathbb{E}^\alpha(\Omega) = \mathbb{E}^{1-\alpha}(\Omega)$, it follows that

$$A^\#_{-\beta} \text{ is resolvent positive for} \quad 2\beta \in [2-2\alpha, 2\alpha] \cap [2-2\alpha, 1+1/p)\backslash\Sigma_p \ .$$

Note that the positive cone $(S^{-2\beta}_{p',\mathcal{B}^\#})^+$ is total, thanks to the fact that $L_{p'}$ is dense in $S^{-2\beta}_{p',\mathcal{B}^\#}$ and $L_{p'}$ is a lattice so that $L_{p'} = L^+_{p'} - L^+_{p'}$. Choose $\lambda > 0$ such that $(\lambda + A^\#_{-\beta})^{-1}$ is a positive continuous isomorphism from $S^{-2\beta}_{p',\mathcal{B}^\#}$ onto $S^{2-2\beta}_{p',\mathcal{B}^\#}$. Then it follows that $(S^{2-2\beta}_{p',\mathcal{B}^\#})^+ = S^{2-2\beta}_{p',\mathcal{B}^\#} \cap L^+_{p'}$ is total too. Hence the dual positive cone

$$(S^{2\beta-2}_{p,\mathcal{B}})^+ := \big((S^{2-2\beta}_{p',\mathcal{B}^\#})^+\big)'$$

is a proper cone defining the natural order of $S^{2\beta-2}_{p,\mathcal{B}}$ for $2\beta \in [2-2\alpha, 2\alpha] \cap [2-2\alpha, 1/p)$. Now the argument (8.28) applies again for these values of 2β to show that $A_{\beta-1}$ is resolvent positive.

Lastly, if $2\beta \in [2-2\alpha, 2\alpha]\backslash\Sigma_p$, it follows that there exists $\varepsilon_0 > 0$ such that

$$(1 + \varepsilon A_{\beta-1})^{-1}(S^{2\beta-2}_{p,\mathcal{B}})^+ \subset (S^{2\gamma}_{p,\mathcal{B}})^+ \ , \qquad 2\gamma \in [2\beta-2, 2\beta \wedge 2\alpha]\backslash\Sigma_p \ , \quad 0 < \varepsilon < \varepsilon_0 \ .$$

Since $(1 + \varepsilon A_{\beta-1})^{-1}$ converges strongly to the identity in $S^{2\beta-2}_{p,\mathcal{B}}$ as $\varepsilon \to 0$, we see that $(S^{2\gamma}_{p,\mathcal{B}})^+$ is dense in $(S^{2\beta}_{p,\mathcal{B}})^+$ if $2\alpha - 2 \leq 2\beta \leq 2\gamma \leq 2\alpha$ and $2\gamma, 2\beta \notin \Sigma_p$. This proves

8.7 Theorem *Suppose that $\mathbb{K}=\mathbb{R}$ and $N=1$. Also suppose that $1<p<\infty$, that $1\le 2\alpha\le 2$, and that $(\mathcal{A},\mathcal{B})\in\mathcal{E}^\alpha(\Omega)$. Then each one of the Banach spaces*

$$S^{2\beta}_{p,\mathcal{B}}\ , \qquad 2\beta\in[2-2\alpha,2\alpha]\backslash\Sigma_p\ ,$$

is an ordered Banach space with the natural order, which is canonically induced by L_p^+ if $2\beta\ge 0$ and which is the canonical dual order otherwise. If $2\alpha-2\le 2\beta\le 2\gamma\le 2\alpha$ with $2\beta,2\gamma\notin\Sigma_p$, the injection $S^{2\gamma}_{p,\mathcal{B}}\hookrightarrow S^{2\beta}_{p,\mathcal{B}}$ is positive and

$$(S^{2\gamma}_{p,\mathcal{B}})^+ \text{ is dense in } (S^{2\beta}_{p,\mathcal{B}})^+\ .$$

Lastly, each one of the operators

$$A_{\beta-1}\in\mathcal{H}(S^{2\beta}_{p,\mathcal{B}},S^{2\beta-2}_{p,\mathcal{B}})\ , \qquad 2\beta\in[2-2\alpha,2\alpha]\backslash\Sigma_p\ , \tag{8.29}$$

is resolvent positive.

Observe that the above theorem is a generalization of the classical naximum principle for strongly uniformly elliptic equations. It appears here for the first time in print.

8.8 Remark It should be noted that for the validity of the resolvent positivity of (8.29) the regularity assumptions for the 'lower order coefficients' of $A_{\beta-1}$ can be considerably weakened. In particular, if $p>n$, the operators

$$A_{-1/2}\in\mathcal{H}(W^1_{p,\mathcal{B}},W^{-1}_{p,\mathcal{B}})\ ,\quad (\mathcal{A},\mathcal{B})\in\mathcal{E}_p(\Omega)$$

are resolvent positive. This follows from the fact that $\{\,(\mathcal{A},\mathcal{B})\in\mathcal{E}^1(\Omega)\ ;\ a_0\in C(\overline{\Omega},X)\,\}$ is dense in $\mathcal{E}_p(\Omega)$, from Remark 8.6(a), and from the proof of Theorem 8.7. □

9 Nonhomogeneous Elliptic Boundary Value Problems

Suppose that $1<p<\infty$ and $2\alpha\in[1,2]\backslash\Sigma_p$. Given

$$(\mathcal{A},\mathcal{B})\in\mathbb{E}^\alpha(\Omega):=\mathbb{E}^\alpha(\Omega,\delta,N) \tag{9.1}$$

and

$$(\lambda,f,g)\in\mathbb{K}\times S^{2\alpha-2}_{p,\mathcal{B}}\times\partial W^{2\alpha}_p\ , \tag{9.2}$$

we consider the linear boundary value problem

$$(\lambda+\mathcal{A})u=f\quad\text{in }\Omega\ ,\qquad \mathcal{B}u=g\quad\text{on }\partial\Omega\ . \tag{9.3}$$

If $\delta_0+1/p<2\alpha\le 2$ then u is said to be a **strong $S^{2\alpha}_p$-solution** of (9.3) if $u\in S^{2\alpha}_{p,\mathcal{B}}$ and

$(\lambda + \mathcal{A}, \mathcal{B})u = (f, g)$ (in the sense of distributions).

If $1/p < 2\alpha < 1 + 1/p$ then u is a **weak $S_p^{2\alpha}$-solution** if $u \in S_p^{2\alpha}$ and satisfies

$$\begin{aligned}
&(1-\delta)\gamma_\partial u = (1-\delta)g \\
&\text{and} \\
&\boldsymbol{a}_\lambda(v, u) = \langle v, f\rangle + \langle \gamma_\partial v, \delta g\rangle_\partial\ , \qquad v \in S_{p'\mathcal{B}}^{2-2\alpha}\ ,
\end{aligned} \tag{9.4}$$

where

$$\boldsymbol{a}_\lambda := \boldsymbol{a} + \lambda\langle\cdot,\cdot\rangle$$

is the Dirichlet form associated with $(\lambda + \mathcal{A}, \mathcal{B})$.

Lastly, if $0 \le 2\alpha < 1 - \delta_0 + 1/p$ then u is a **very weak $S_p^{2\alpha}$-solution** of (9.3) if $u \in S_p^{2\alpha}$ and

$$\big\langle(\lambda + \mathcal{A}^\#)v, u\big\rangle = \langle v, f\rangle + \langle \mathcal{C}^\# v, g\rangle_\partial\ , \qquad v \in S_{p',\mathcal{B}^\#}^{2-2\alpha}\ . \tag{9.5}$$

In general, $u \in S_p^{2\alpha}$ is said to be an **$S_p^{2\alpha}$-solution** of (9.3) provided it is

- a strong $S_p^{2\alpha}$-solution if $\delta_0 + 1/p < 2\alpha \le 2$,
- a weak $S_p^{2\alpha}$-solution if $1/p < 2\alpha < 1 + 1/p$,
- a very weak $S_p^{2\alpha}$-solution if $0 \le 2\alpha < 1 - \delta_0 + 1/p$.

The following remarks show that the above definitions are natural and consistent.

9.1 Remarks

(a) Let (9.1) and (9.2) be satisfied. If u is an $S_p^{2\alpha}$-solution of (9.3), it is an $S_p^{2\beta}$-solution for each $2\beta \in [2\alpha - 2, 2\alpha]\backslash(\mathbb{N} + 1/p)$.

This is an easy consequence of (5.5), (7.4), (8.7), (8.8), and the Green formulas (8.14)–(8.16).

(b) Suppose that $\delta = 0$ and $1/p < 2\alpha < 1 + 1/p$. In this case it is an easy consequence of Green's formulas that the three concepts: strong, weak, and very weak $S_p^{2\alpha}$-solution all coincide.

(c) The concept of a strong $S_p^{2\alpha}$-solution is, of course, meaningful for general second order boundary value problems which are not necessarily in divergence form. □

Suppose now that

$$(\mathcal{A}, \mathcal{B}) \in \mathcal{E}^\alpha(\Omega)\ . \tag{9.6}$$

If $1 + 1/p < 2\alpha \le 2$, it is a consequence of the normal ellipticity and the trace theorem that

$$\mathcal{B}|S_p^{2\alpha} \in \mathcal{L}(S_p^{2\alpha}, \partial W_p^{2\alpha}) \tag{9.7}$$

is a retraction. Hence we can choose a coretraction

$$\mathcal{T}_\alpha := (\mathcal{B}|S_p^{2\alpha})^c \in \mathcal{L}(\partial W_p^{2\alpha}, S_p^{2\alpha}) \tag{9.8}$$

(cf. [Ama90a, Appendix B], for example). Then

$$\mathcal{S}_{\alpha-1,\lambda} := -(\lambda+\mathcal{A})\mathcal{T}_\alpha \in \mathcal{L}(\partial W_p^{2\alpha}, S_p^{2\alpha-2}) \ , \qquad \lambda \in \mathbb{K} \ . \tag{9.9}$$

Moreover, given (9.2), u is an $S_p^{2\alpha}$-solution of (9.3) iff $u = v + \mathcal{T}_\alpha g$ and v is an $S_p^{2\alpha}$-solution of

$$(\lambda+\mathcal{A})v = f + \mathcal{S}_{\alpha-1,\lambda} g \quad \text{in } \Omega \ , \qquad \mathcal{B}v = 0 \quad \text{on } \partial\Omega \ .$$

Hence we see from (8.17) that (9.3) is equivalent to

$$\begin{aligned}&\text{find } v \in S_{p,\mathcal{B}}^{2\alpha} \text{ satisfying}\\ &(\lambda + A_{\alpha-1})v = f + \mathcal{S}_{\alpha-1,\lambda} g\\ &\text{and put } u := v + \mathcal{T}_\alpha g \ ,\end{aligned} \tag{9.10}$$

provided $1 + 1/p < 2\alpha \le 2$.

Suppose that $1/p < 2\alpha < 1 + 1/p$. Then $(1-\delta)\gamma_\partial \in \mathcal{L}(S_p^{2\alpha}, \partial_0 W_p^{2\alpha})$ is a retraction by the trace theorem. Choose a coretraction

$$\mathcal{T}_\alpha := \left[(1-\delta)\gamma_\partial | S_p^{2\alpha}\right]^c \in \mathcal{L}(\partial_0 W_p^{2\alpha}, S_p^{2\alpha}) \ . \tag{9.11}$$

It follows from (8.12) that there is a unique

$$\mathcal{S}_{\alpha-1,\lambda} \in \mathcal{L}(\partial_0 W_p^{2\alpha}, S_{p,\mathcal{B}}^{2\alpha-2}) \ , \qquad \lambda \in \mathbb{K} \ , \tag{9.12}$$

satisfying

$$\langle v, \mathcal{S}_{\alpha-1,\lambda} w\rangle = -\boldsymbol{a}_\lambda(v, \mathcal{T}_\alpha w) \ , \qquad (v,w) \in S_{p',\mathcal{B}^\#}^{2-2\alpha} \times \partial_0 W_p^{2\alpha} \ . \tag{9.13}$$

Moreover, $\delta\gamma_\partial \in \mathcal{L}(S_{p',\mathcal{B}^\#}^{2-2\alpha}, \partial_1 W_{p'}^{2-2\alpha})$ and, consequently,

$$\mathcal{R}_{\alpha-1} := (\delta\gamma_\partial | S_{p',\mathcal{B}^\#}^{2-2\alpha})' \in \mathcal{L}(\partial_1 W_p^{2\alpha-1}, S_{p,\mathcal{B}}^{2\alpha-2}) \tag{9.14}$$

is well defined, thanks to (7.1) and the fact that $(\partial_1 W_{p'}^{2-2\alpha})' \doteq \partial_1 W_p^{2\alpha-1}$, as follows from (5.15). Now, given (9.2), we deduce that u is an $S_p^{2\alpha}$-solution of (9.3) iff $u = v + \mathcal{T}_\alpha(1-\delta)g$ and v is an $S_p^{2\alpha}$-solution of

$$(\lambda+\mathcal{A})v = f + \mathcal{S}_{\alpha-1,\lambda}(1-\delta)g \quad \text{in } \Omega \ , \qquad \mathcal{B}v = \delta g \quad \text{on } \partial\Omega \ .$$

We see from (9.4) that this is the case iff $v \in S_{p,\mathcal{B}}^{2\alpha}$ and

$$\boldsymbol{a}_\lambda(w,v) = \left\langle w, f + \mathcal{S}_{\alpha-1,\lambda}(1-\delta)g\right\rangle + \langle \gamma_\partial w, \delta g\rangle_\partial = \left\langle w, f + \mathcal{S}_{\alpha-1,\lambda}(1-\delta)g + \mathcal{R}_{\alpha-1}\delta g\right\rangle \ .$$

for $w \in S_{p',\mathcal{B}^\#}^{2-2\alpha}$. Thus, thanks to (8.18), it follows that (9.4) is equivalent to

$$\begin{aligned}&\text{find } v \in S_{p,\mathcal{B}}^{2\alpha} \text{ satisfying}\\ &(\lambda + A_{\alpha-1})v = f + \mathcal{S}_{\alpha-1,\lambda}(1-\delta)g + \mathcal{R}_{\alpha-1}\delta g\\ &\text{and put } u := v + \mathcal{T}_\alpha(1-\delta)g\end{aligned} \tag{9.15}$$

provided $1/p < 2\alpha < 1 + 1/p$.

Lastly, suppose that $0 \le 2\alpha < 1/p$. Then

$$\mathcal{S}_{\alpha-1} := (\mathcal{C}^{\#} | S^{2-2\alpha}_{p',\mathcal{B}^{\#}})' \in \mathcal{L}(\partial W^{2\alpha}_p, S^{2\alpha-2}_{p,\mathcal{B}}) \tag{9.16}$$

and it is clear that (9.5) is equivalent to

$$\begin{aligned} &\text{find } u \in S^{2\alpha}_{p,\mathcal{B}} \text{ satisfying} \\ &(\lambda + A_{\alpha-1})u = f + \mathcal{S}_{\alpha-1} g \end{aligned} \tag{9.17}$$

provided $0 \le 2\alpha < 1/p$.

Thanks to (9.10), (9.15), and (9.17) the problem of the $S^{2\alpha}_p$-solvability of the boundary value problem (9.3) has been reduced to the study of a linear operator equation of the form

$$(\lambda + A_{\alpha-1})u = F \ , \tag{9.18}$$

where $F \in S^{2\alpha-2}_{p,\mathcal{B}}$ depends linearly and continuously upon (f,g). Since $A_{\alpha-1}$ has a compact resolvent by Theorem 8.5, provided $(\mathcal{A},\mathcal{B}) \in \mathcal{E}^{\alpha}(\Omega)$, the Riesz-Schauder theory, (8.25), and Theorem 8.5 imply the following 'Fredholm alternative theorem' (for details see [Ama94]).

9.2 Theorem *Suppose that $1 < p < \infty$ and $2\alpha \in [0,2]\backslash(\mathbb{N} + 1/p)$. Also suppose that $(\mathcal{A},\mathcal{B}) \in \mathcal{E}^{\alpha}(\Omega)$. Then the normally elliptic boundary value problem*

$$(\lambda + \mathcal{A})u = f \quad in\ \Omega \ , \qquad \mathcal{B}u = g \quad on\ \partial\Omega \tag{9.19}$$

has for each

$$(f,g) \in W^{2\alpha-2}_{p,\mathcal{B}} \times \partial W^{2\alpha}_p \tag{9.20}$$

a unique $W^{2\alpha}_p$-solution u, provided $\lambda \notin \sigma(A_{\alpha-1})$. The set $\sigma(A_{\alpha-1})$ is a nonempty countable subset of a sector of the form $\omega + \big([\,|\arg z| \le \varphi] \cup \{0\}\big)$ for some $\omega \ge 0$ and $\varphi := \varphi(\mathcal{A},\mathcal{B}) \in [0,\pi/2)$, without finite limit point. If $(\mathcal{A},\mathcal{B})$ is formally self-dual, that is, if $(\mathcal{A},\mathcal{B}) = (\mathcal{A}^{\#},\mathcal{B}^{\#})$, then $\varphi = 0$.

For $\lambda \in \sigma(A_{\alpha-1})$ the homogeneous problems

$$(\lambda + \mathcal{A})u = 0 \quad in\ \Omega \ , \qquad \mathcal{B}u = 0 \quad on\ \partial\Omega \tag{9.21}$$

and

$$(\lambda + \mathcal{A}^{\#})v = 0 \quad in\ \Omega \ , \qquad \mathcal{B}^{\#}v = 0 \quad on\ \partial\Omega \tag{9.22}$$

possess the same finite positive number of linearly independent $W^{2\alpha}_p$-solutions and $W^{2-2\alpha}_{p'}$-solutions, respectively. Moreover, the inhomogeneous problem (9.19) is solvable iff the following 'orthogonality conditions' are satisfied:

- *if $1 + 1/p < 2\alpha \le 2$,*

$$\langle w, f\rangle - \big\langle w, (\lambda + \mathcal{A})\mathcal{T}_{\alpha} g\big\rangle = 0 \tag{9.23}$$

- *if $1/p < 2\alpha < 1 + 1/p$,*

$$\boldsymbol{a}_{\lambda}\big(w, \mathcal{T}_{\alpha}(1-\delta)g\big) = \langle w, f\rangle + \langle \gamma w, \delta g\rangle_{\partial} \tag{9.24}$$

- *if* $0 \le 2\alpha < 1/p$,

$$\langle w, f\rangle + \langle \mathcal{C}^{\#} w, g\rangle_{\partial} = 0 \tag{9.25}$$

for each $W_{p'}^{2-2\alpha}$*-solution* w *of the homogeneous boundary value problem* (9.22).

Lastly, suppose that $2\beta \in [2-2\alpha, 2\alpha]\backslash(\mathbb{N}+1/p)$. *Then*

$$\sigma(A_{\beta-1}) = \sigma(A_{\alpha-1})$$

and, given (9.20), u *is a* $W_p^{2\beta}$*-solution of* (9.19) *iff it is a* $W_p^{2\alpha}$*-solution.*

9.3 Remarks

(a) Suppose that $(\mathcal{A}, \mathcal{B}) \in \mathcal{E}^1(\Omega)$. Then $\sigma(A_{\alpha-1}) = \sigma(A_0)$ for $2\alpha \in [0,2]\backslash(\mathbb{N}+1/p)$. Moreover, thanks to the regularity assertions contained in the last part of Theorem 9.2, it follows that we can restrict w in (9.23)–(9.25) to belong to $\ker(\lambda + A_0^{\#})$. Then each one of the orthogonality conditions (9.23)–(9.25) takes the form

$$\langle w, f\rangle + \langle \mathcal{C}^{\#} w, g\rangle_{\partial} = 0$$

for $w \in W_{p'}^2$ satisfying $(\lambda + \mathcal{A}^{\#})w = 0$ in Ω and $\mathcal{B}^{\#} w = 0$ on $\partial\Omega$, as follows from (9.8), (9.11), Green's formulas and a density argument.

(b) If $\delta = 0$ [resp. $\delta = 1$], the exceptional value $2\alpha = 1 + 1/p$ [resp. $2\alpha = 1/p$] can be admitted too (cf. (5.24) [resp. (5.25)]).

(c) Using the fact that

$$W_{p,\mathcal{B}}^{2\alpha} \hookrightarrow W_{q,\mathcal{B}}^{2\alpha} \quad \text{and} \quad \partial W_p^{2\alpha} \hookrightarrow \partial W_q^{2\alpha}\ , \qquad 1 < q < p < \infty\ ,$$

it is obvious that, given (9.20), each $W_q^{2\alpha}$-solution, where $1 < q < p < \infty$, is a $W_p^{2\alpha}$-solution.

(d) It is an easy consequence of Remark 8.6(c) and (9.9)–(9.17) that there exists a constant c such that the 'a priori estimate'

$$\|u\|_{W_p^{2\alpha}} \le c\big(\|f\|_{W_{p,\mathcal{B}}^{2\alpha-2}} + \|g\|_{\partial W_p^{2\alpha}} + \|u\|_{W_{p,\mathcal{B}}^{2\alpha-2}}\big)$$

is true.

(e) Theorem (9.2) and (a) remain true if the Sobolev-Slobodeckii scale W_p^s is replaced by the Bessel potential scale H_p^s.

(f) If $1 + 1/p < 2\alpha \le 2$, the first part of Theorem 9.2 is valid for general second order normally elliptic boundary value problems (cf. Remark 8.6(d)). □

It may be worthwhile to elucidate the general results by means of the following simple

9.4 Example Suppose that $a_{jk} = a_{kj} \in C^1(\overline{\Omega}, \mathbb{R})$ and that $a := [a_{jk}] \in \mathbb{R}^{n \times n}$ is uniformly positive definite. Also suppose that $a_0 \in L_\infty(\Omega, \mathbb{R})$ and put

$$\mathcal{A}u = -\partial_j(a_{jk}\partial_k u) + a_0 u$$

so that $\mathcal{A}$ is a uniformly strongly elliptic differential operator. Let ∂_{ν_a} be the conormal derivative with respect to a and suppose that $\delta \in \{0, 1\}$. Put

$$\mathcal{B}u := \delta\partial_{\nu_a} u + (1-\delta)\gamma_\partial u \ .$$

Then $(\mathcal{A}, \mathcal{B}) \in \mathcal{E}^1(\Omega)$ with $N = 1$ (scalar case), and $(\mathcal{A}, \mathcal{B})$ is formally self-dual.

Suppose that $1 < p < \infty$ and $s \in [0, 2] \backslash \{\delta + 1/p\}$. Also suppose that

$$(f, g) \in W_{p,\mathcal{B}}^{s-2} \times W_p^{s-\delta-1/p}(\partial\Omega) \ .$$

Then the scalar linear elliptic boundary value problem

$$(\lambda + \mathcal{A})u = f \quad \text{in } \Omega \ , \qquad \mathcal{B}u = g \quad \text{on } \partial\Omega \tag{9.26}$$

has for each $\lambda \notin \sigma(A_0)$ a unique W_p^s-solution. The spectrum $\sigma(A_0)$ of the $L_p(\Omega)$-realization of $(\mathcal{A}, \mathcal{B})$ consists of a countably infinite sequence of real eigenvalues of finite multiplicity possessing ∞ as its only limit point. If $\lambda \in \sigma(A_0)$ then (9.26) is solvable iff

$$\langle v, f\rangle + \langle \gamma_\partial v, \delta g\rangle_\partial - \big\langle \partial_{\nu_a} v, (1-\delta)g\big\rangle_\partial = 0$$

for each $v \in W_{p'}^2$ satisfying

$$(\lambda + \mathcal{A})v = 0 \quad \text{in } \Omega \ , \qquad \mathcal{B}v = 0 \quad \text{on } \partial\Omega \ .$$

Suppose that u is a W_p^s-solution of (9.26). Then

- $u \in W_p^s$ and $(\lambda + \mathcal{A})u = f$ in Ω in the sense of distributions and $\mathcal{B}u = g$ on $\partial\Omega$ in the sense of traces, provided $\delta + 1/p < s \leq 2$.

- $u \in W_p^s$ and
$$(1-\delta)\gamma_\partial u = (1-\delta)g$$
and
$$\langle \partial_j v, a_{jk}\partial_k u\rangle + \big\langle v, (\lambda + a_0)u\big\rangle = \langle v, f\rangle + \langle \gamma_\partial v, \delta g\rangle_\partial \tag{9.27}$$
for $v \in W_{p',\mathcal{B}^\#}^{2-s}$, provided $1/p < s < 1 + 1/p$.

- $u \in W_p^s$ and
$$\big\langle -\partial_j(a_{jk}\partial_k v) + (\lambda + a_0)v, u\big\rangle = \langle v, f\rangle + \langle \gamma_\partial v, \delta g\rangle_\partial - \big\langle \partial_{\nu_a} v, (1-\delta)g\big\rangle_\partial$$
for $v \in W_{p',\mathcal{B}^\#}^{2-s}$, provided $0 \leq s < \delta + 1/p$.

Observe that

$$\langle v,f\rangle + \langle \gamma_\partial v,\delta g\rangle_\partial - \langle \partial_{\nu_a} v,(1-\delta)g\rangle_\partial = \int_\Omega v f\,dx + \delta\int_{\partial\Omega} vg\,d\sigma - (1-\delta)\int_{\partial\Omega} \partial_{\nu_a} vg\,d\sigma$$

for $v\in W_{p'}^2$, if $(f,g)\in L_p\times L_p(\partial\Omega)$. On the other hand, if $2-s>n/p'$, it follows from $W_{p'}^{2-s}\hookrightarrow C$ that $\langle v,f\rangle$ is well defined for $v\in W_{p'}^{2-s}$ if f is a Radon measure on $\overline{\Omega}$. Similarly, $\langle \gamma_\partial v,g\rangle_\partial$ and $\langle \partial_{\nu_a} v,g\rangle_\partial$ are well defined for $v\in W_{p'}^{2-s}$ if g is a Radon measure on $\partial\Omega$ provided $2-s>n/p'$ and $1-s>n/p'$, respectively. Thus (9.26) can be solved for rather singular data, provided we choose p sufficiently close to 1. □

9.5 Remarks

(a) Presuppose the hypotheses of Example 9.4 and consider the Neumann problem

$$-\partial_j(a_{jk}\partial_k u) = f \quad \text{in } \Omega\ , \qquad \partial_{\nu_a} u = g \quad \text{on } \partial\Omega\ . \tag{9.28}$$

Suppose that $1<p<\infty$ and $s\in[0,2]\backslash\{1+1/p\}$. Given

$$(f,g)\in W_{p,\mathcal{B}}^{s-2}\times W_p^{s-1-1/p}(\partial\Omega)\ ,$$

it follows from Example 9.4 that (9.28) has a W_p^s-solution iff

$$\langle \mathbf{1},f\rangle + \langle \mathbf{1},g\rangle_\partial = 0\ , \tag{9.29}$$

where $\mathbf{1}(x):=1$ for $x\in\overline{\Omega}$, thanks to the fact that $\ker(A_{\alpha-1})=\ker(A_0)=\mathbb{K}\mathbf{1}$, where $\alpha:=s/2$. Since 0 is a simple eigenvalue of A_0, whence of $A_{\beta-1}$, as follows from the maximum principle, for example, the spectral theory for operators with compact resolvents implies the existence of a subspace F_β of $W_{p,\mathcal{B}}^{2\beta}$ such that

$$W_{p,\mathcal{B}}^{2\beta} = \mathbb{K}\mathbf{1}\oplus F_\beta\ , \qquad 2\beta\in[-2,2]\backslash\{-2+1/p,1+1/p\}\ ,$$

and that this decomposition reduces the linear operator $A_{\beta-1}$. Hence we deduce from Theorem 8.3 that

$$B_{\alpha-1} := A_{\alpha-1}|F_{\alpha-1}\in\mathcal{L}\mathrm{is}(F_\alpha,F_{\alpha-1})\ . \tag{9.30}$$

Let

$$\mathcal{R}_{\beta-1} := (\gamma_\partial|W_{p',\mathcal{B}\#}^{2-2\beta})'\in\mathcal{L}\big(W_p^{2\beta-1-1/p}(\partial\Omega),W_{p,\mathcal{B}}^{2\beta-2}\big) \tag{9.31}$$

for $0\le 2\beta<1+1/p$ and note that $\mathcal{S}_{\alpha-1}$ as defined in (9.16) equals $\mathcal{R}_{\alpha-1}$ if $0\le 2\alpha<1/p$ and $\delta=1$. Thus, if u is a $W_p^{2\alpha}$-solution of (9.28), we deduce from (9.10)–(9.17) that

$$u=\xi\mathbf{1}+\begin{cases} B_{\alpha-1}^{-1}(f+\mathcal{S}_{\alpha-1,0}\,g)+\mathcal{T}_\alpha g & \text{if } 1+1/p<2\alpha\le 2\ ,\\ B_{\alpha-1}^{-1}(f+\mathcal{R}_{\alpha-1}g) & \text{if } \quad 0\le 2\alpha<1+1/p\ ,\end{cases}$$

where $\xi\in\mathbb{K}$, provided (9.29) is satisfied. Consequently, it follows from (9.8), (9.30), and (9.31) that there exists a constant c such that

$$\|u-\langle \mathbf{1},u\rangle\mathbf{1}\|_{W_p^s}\le c\big(\|f\|_{W_{p,\mathcal{B}}^{s-2}}+\|g\|_{W_p^{s-1-1/p}(\partial\Omega)}\big)\ , \tag{9.32}$$

whenever (9.29) is satisfied and u is a solution of (9.28).

In [vW90] von Wahl proved (among other things) by estimating directly integral operators on $\partial\Omega$ that every solution of the three-dimensional Neumann problem (i.e., $n=3$)

$$-\Delta u = 0 \quad \text{in } \Omega\ , \qquad \partial_\nu u = g \quad \text{on } \partial\Omega$$

satisfies the a priori estimate

$$\|\nabla u\|_{L_p(\Omega)} \le c\,\|g\|_{W_p^{-1/p}(\partial\Omega)}\ ,$$

which is a very special case of (9.32).

(b) Let the assumptions of Example 9.4 be satisfied and suppose, for example, that

$$f \in L_{p,\mathrm{loc}}\ .$$

Then, given any W_p^s-solution u of (9.26),

$$u \in W_{p,\mathrm{loc}}^2\ .$$

Indeed, given $\varphi \in \mathcal{D}$, it follows that φu is a W_p^s-solution of

$$(\lambda + \mathcal{A})v = f_\varphi \quad \text{in } \Omega\ , \qquad \mathcal{B}v = 0 \quad \text{on } \partial\Omega\ , \tag{9.33}$$

where (using formal notation)

$$f_\varphi := \varphi f - \partial_j(a_{jk}u\partial_k\varphi) - \partial_j\varphi a_{jk}\partial_k u \in W_{p,\mathcal{B}}^{(s-1)\wedge 0}\ .$$

From this we deduce that φu is a W_p^t-solution of (9.33), where $t := 2\wedge(s+1)$. By repeating this 'boot strapping argument', if necessary, the assertion follows.

(c) Suppose that $s=1$, $p=2$, and $\delta=0$ in (9.27). Then (9.27) is the usual weak formulation of the Dirichlet problem

$$(\lambda + \mathcal{A})u = f \quad \text{in } \Omega\ , \qquad u = 0 \quad \text{on } \partial\Omega$$

in $\mathring{H}^1 := W_{2,\mathcal{B}}^1$. Since $\mathcal{A}$ is a scalar uniformly strongly elliptic operator, Gårding's inequality implies the coerciveness of the quadratic form $u \mapsto \boldsymbol{a}(\overline{u},u)$ on $\mathring{H}^1$. Hence the assertions are in this particular case well known consequences of the Lax-Milgram lemma and the compact imbedding $\mathring{H}^1 \overset{c}{\hookrightarrow} L_2$ (e.g., [DauL90, Chapter VII]). More generally, the case $p=2$ and $2\alpha=1$ of Theorem 9.2 follows from the Lax-Milgram lemma and from $W_{2,\mathcal{B}}^1 \overset{c}{\hookrightarrow} L_2$ whenever the quadratic form $u \mapsto \boldsymbol{a}_\lambda(\overline{u},u)$ is coercive on $W_{2,\mathcal{B}}^1$ for some $\lambda \ge 0$. This is, however, in general not the case for $(\mathcal{A},\mathcal{B}) \in \mathcal{E}(\Omega)$ if $N>1$. (We refer to [deF63], [LiM72], and [Neč67], for example, for discussions of the coercivity of Dirichlet forms.) Thus, even in the 'classical' case $p=2$ and $2\alpha=1$, the weak solvability of (9.19) is not accessible by 'variational techniques' if $N>1$. □

In the literature there is only one result concerning the weak $S_p^{2\alpha}$-solvability of problem (9.13), namely [Sim72]. Simader studies the weak W_p^1-solvability of the Dirichlet problem (9.14), (9.15) for Dirichlet forms with continuous coefficients and arbitrary $p \in (1,\infty)$, but in the case of a single equation ($N = 1$), by a completely different technique. (As mentioned earlier, we restrict ourselves here to the case of second order problems. However, our techniques extend straightforwardly to normally elliptic boundary value problems of order $2m$ (cf. [Ama94]). Hence our results also cover the higher order case treated by Simader.)

Nonhomogeneous elliptic boundary value problems of arbitrary order, but for smooth coefficients and in the case $N = 1$, have been studied in the L_p-setting in detail by Lions and Magenes in [LiM60]. An extension of the Lions-Magenes theory to systems is given in [Gey65]. The simpler L_2-case is treated in [LiM72]. There is, however, an important difference between the Lions-Magenes theory and the theory exposed in this section. Namely, in the Lions-Magenes theory the inhomogeneity f in (9.19) is restricted to belong to a space of distributions (which cannot be 'optimally' described), whereas we admit inhomogeneities f from the space $W_{p,\mathcal{B}}^{2\alpha-2}$ which is, in general, not a space of distributions. However, $W_{p,\mathcal{B}}^{2\alpha-2}$ has a natural description as the dual of $W_{p',\mathcal{B}^\#}^{2-2\alpha}$ which makes it a very reasonable and optimal choice. Thus the above theory can be considered to be an extension and simplification of the Lions-Magenes theory, even in the case of smooth coefficients, for the class of systems considered here.

The results of this section occur here for the first time in print. However, special cases have already been presented by the author in lecture series at TH Delft in the fall of 1989 and at Tokyo University in the spring of 1990.

10 Abstract Linear Parabolic Equations

Let E be a Banach space and let J be a perfect subinterval of $\mathbb{R}^+$ containing 0. For each $t \in J$ let $A(t)$ be a linear operator in E, and let $f : J \to E$. By a **solution** of the **linear evolution equation**

$$\dot{u} + A(t)u = f(t) \ , \qquad t \in \dot{J} := J\backslash\{0\} \ , \tag{10.1}$$

where $\dot{u} := \partial_t u$, we mean a function $u \in C(J,E) \cap C^1(\dot{J},E)$ such that $u(t) \in \operatorname{dom}\big(A(t)\big)$ for $t \in \dot{J}$ and u satisfies (10.1) pointwise. If $u(0) = x$ then u is a solution of the **linear Cauchy problem** (in E)

$$\dot{u} + A(t)u = f(t) \ , \quad t \in \dot{J} \ , \qquad u(0) = x \ . \tag{10.2}$$

Lastly, u is a **strict solution** of (10.2) if $u \in C^1(J,E)$ and $u(0) \in \operatorname{dom}\big(A(0)\big)$.

The linear evolution equation (10.1) and the linear Cauchy problem (10.2) are said to be **parabolic** if $-A(t)$ is the infinitesimal generator of a strongly continuous analytic semigroup on E for each $t \in J$.

In the following, $(\cdot,\cdot)_\theta$ is for each $\theta \in (0,1)$ an admissible interpolation functor and $E_\theta := (E_0,E_1)_\theta$, if (E_0,E_1) is a densely injected Banach couple.

As for the solvability of parabolic linear Cauchy problems we have the following fundamental

10.1 Theorem *Suppose that* (E_0, E_1) *is a densely injected Banach couple and*

$$\big(x, (A, f)\big) \in E_0 \times C^\rho\big(J, \mathcal{H}(E_1, E_0) \times E_0\big)$$

for some $\rho \in (0,1)$*. Then the linear parabolic Cauchy problem* (10.2) *possesses a unique solution* $u := u(\cdot, x, A, f)$*, and*

$$u \in C^\rho(\dot{J}, E_1) \cap C^{1+\rho}(\dot{J}, E_0) \ . \tag{10.3}$$

If $0 \le \beta \le \alpha \le 1$ *and* $x \in E_\alpha$*,*

$$u \in C^{\alpha-\beta}(J, E_\beta) \ .$$

If $x \in E_1$ *then* u *is a strict solution.*

The existence of a unique solution, which is strict if $x \in E_1$, has been proven by Sobolevskii [Sob66] and, independently, by Tanabe [Tan60]. A different proof has more recently been given by Acquistapace and Terreni [AcT85]. The regularity result (10.3) is contained in [Sob66] and [AcT85]. The continuity assertion involving interpolation spaces is due to the author (cf. [Ama86b] and [Ama94], also cf. [DaK92, Section 5]).

It is an important property of parabolic evolution equations that they 'regularize'. In fact, (10.3) shows that the solution is more regular than its initial value, in general. The following 'automatic regularity theorem' implies that the solution of (10.2) has even better regularity properties if f is more regular.

10.2 Theorem *Let* (E_0, E_1) *be a densely injected Banach couple. Suppose that* $0 < \gamma < \rho < 1$ *and* $0 < \varepsilon \le 1 - \gamma$*, and that*

$$(x, A) \in E_0 \times C^\rho\big(J, \mathcal{H}(E_1, E_0)\big)$$

and

$$f \in C^\varepsilon(J, E_\gamma) + C(J, E_{\gamma+\varepsilon}) \ .$$

Then the Cauchy problem (10.2) *has a unique solution* u*, and*

$$u \in C(J, E_0) \cap C(\dot{J}, E_1) \cap C^1(\dot{J}, E_\gamma) \ .$$

It is also a solution of the parabolic equation

$$\dot{v} + A_\gamma(t)v = f(t) \ , \qquad t \in \dot{J} \ , \tag{10.4}$$

in E_γ*, where* A_γ *is the* E_γ*-realization of* A*. If* $x \in \operatorname{dom}\big(A_\gamma(0)\big)$ *then* u *is a strict solution of* (10.4).

This theorem has been proven in [Ama88c, Theorem 8.2].

It should be noted that the domains, $\mathrm{dom}\big(A_\gamma(t)\big)$, of the linear operators $A_\gamma(t)$ depend on $t \in J$, in general, whereas the operators $A(t),\ t \in J$, have 'constant domains'. Also observe that the assertion that u is a solution of (10.4) implies $u(t) \in \mathrm{dom}\big(A_\gamma(t)\big)$ for $t \in \dot{J}$. This means that $u(t)$ has, in general, better 'spacial' regularity than follows from $u(t) \in E_1$.

Theorem 10.2 guarantees the unique solvability of the abstract Cauchy problem for operators 'with variable domains but a constant interpolation space'. Related recent results can be found in [AcT86] and [Yag88], [Yag90].

Suppose that $\mu \in \mathbb{R}^+$ and $\rho \in (0,1)$. Put

$$\|u\|_{C^\rho_{\rho,\mu}} := \sup_{t\in\dot{J}} t^\mu \,\|u(t)\| + \sup_{2\varepsilon\in\dot{J}} \varepsilon^{\rho+\mu}[u]_{\rho,[\varepsilon,2\varepsilon]}$$

and denote by

$$BUC^\rho_{\rho,\mu}(J,E)$$

the subspace of $C^\rho(\dot{J},E)$ consisting of all u satisfying $\|u\|_{C^\rho_{\rho,\mu}} < \infty$ and

$$\lim_{\varepsilon\to 0} \varepsilon^{\rho+\mu}[u]_{\rho,[\varepsilon,2\varepsilon]} = 0 \ , \tag{10.5}$$

equipped with the norm $\|\cdot\|_{C^\rho_{\rho,\mu}}$. It is not difficult to see that $BUC^\rho_{\rho,\mu}(J,E)$ is a Banach space, a 'singular Hölder space'. Moreover,

$$BUC^\rho(J,E) \hookrightarrow BUC^\rho_{\rho,\mu}(J,E) \hookrightarrow BUC^\sigma_{\sigma,\nu}(J,E) \hookrightarrow C^\sigma(\dot{J},E)$$

for $0 < \sigma \le \rho < 1$ and $0 \le \mu \le \nu < \infty$. Also put

$$BUC^{1+\rho}_{\rho,\mu}(J,E) := \big(\big\{\, u \in BUC^\rho_{\rho,\mu}(J,E) \ ;\ \partial u \in BUC^\rho_{\rho,\mu}(J,E) \,\big\},\ \|\cdot\|_{C^{1+\rho}_{\rho,\mu}}\big) \ ,$$

where

$$\|u\|_{C^{1+\rho}_{\rho,\mu}} := \|u\|_{C^\rho_{\rho,\mu}} + \|\partial u\|_{C^\rho_{\rho,\mu}} \ .$$

Lastly, $BUC^\rho_\rho := BUC^\rho_{\rho,0}$. Note that the elements of $BUC^\rho_{\rho,\mu}(J,E)$ are allowed to 'become singular' as $t \to 0$ at a controlled rate.

10.3 Theorem *Let (E_0,E_1) be a densely injected Banach couple, let J be bounded, and let $\rho \in (0,1)$ and $\mu \in [0,1)$. Put*

$$(\mathbb{E}_0,\mathbb{E}_1) := \big(BUC^\rho_{\rho,\mu}(J,E_0), BUC^\rho_{\rho,\mu}(J,E_1) \cap BUC^{1+\rho}_{\rho,\mu}(J,E_0)\big)$$

and

$$E_{1-\mu} := (E_0,E_1)^0_{1-\mu,\infty} \ .$$

Denote by γ_0 the 'trace operator at $t=0$', that is, $\gamma_0 u = u(0)$. Then

$$(\partial + A, \gamma_0) \in \mathcal{L}\mathrm{is}(\mathbb{E}_1, \mathbb{E}_0 \times E_{1-\mu}) \ ,$$

provided $A \in C^\rho_\rho\big(J, \mathcal{H}(E_1,E_0)\big)$.

Theorem 10.3 is a slight extension of some results contained in [AcT87] and [Lun87]. The spaces $BUC^{\rho}_{\rho,\mu}$ — but without the requirement (10.5) — are the spaces $Z_{\mu,\rho}$ in [AcT87]. Related spaces also occur in [Sob64]. On the other hand, the spaces BUC^{ρ}_{ρ} correspond to the spaces z^{ρ} in [Lun87].

Theorem 10.3 is a 'maximal regularity' result. There are other maximal regularity theorems: in spaces of continuous functions, due to Da Prato and Grisvard [DaPG79], and in Sobolev spaces, due to Dore and Venni [DoV87]. All these maximal regularity results require additional restrictions either for the class of admissible initial values (as in Theorem 10.3), or for the class of admissible Banach spaces E_0, E_1, and the space of initial values (as in the Da Prato-Grisvard case), or — in addition — for the class of admissible operators (as in the case of the Dore-Venni result). In contrast, there are no restrictions of this type in Theorems 10.1 and 10.2. But, of course, the above theorems do not give maximal regularity. Since maximal regularity plays a minor rôle in this paper, we refer the interested reader to the literature (e.g., [DaPG79], [CH*87], [Lun87], [DoV87], or [Ama94]).

11 Nonhomogeneous Linear Parabolic Boundary Value Problems

Let J be a perfect subinterval of $\mathbb{R}^+$ containing 0, and let $1 < p < \infty$ and $1 \leq 2\alpha \leq 2$. Also suppose that $\rho \in (0,1)$ and

$$(\mathcal{A},\mathcal{B}) \in C^{\rho}\big(J,\mathcal{E}^{\alpha}(\Omega)\big) \ . \tag{11.1}$$

Given $S \in \{H,W\}$, we write

$$S^s_{p,\mathcal{B}} = S^s_{p,\mathcal{B}(t)}$$

to mean that the spaces $S^s_{p,\mathcal{B}(t)}$ are independent of $t \in J$. Note that this is the case, for example, if one of the following conditions is satisfied:

- $\mathcal{B}(t)$ is independent of $t \in J$ and $s \in [-2,2]\backslash\Sigma_p$;
- $s \in (-2+1/p, 1+1/p)\backslash\Sigma_p$.

Recall that

$$S^s_{p,\mathcal{B}} = S^s_p \ , \qquad -1+1/p < s < 1/p \ .$$

Also assume that 2β satisfies

$$2\alpha - 2 \leq 2\beta \leq 2\alpha \tag{11.2}$$

and

$$2\beta \in \begin{cases} [0,2]\backslash\Sigma_p & \text{if } S^{2\beta}_{p,\mathcal{B}} = S^{2\beta}_{p,\mathcal{B}(t)} \ , \\ (1/p,2]\backslash\Sigma_p & \text{otherwise} \ . \end{cases} \tag{11.3}$$

Lastly, assume that $\sigma \in [0,1)$ and

$$\big(u^0,(f,g)\big) \in S^{2\beta-2}_{p,\mathcal{B}} \times C^{\sigma}(J, S^{2\beta-2}_{p,\mathcal{B}} \times \partial W^{2\beta}_p) \ . \tag{11.4}$$

Then we consider the nonhomogeneous linear **parabolic boundary value problem**

$$\begin{aligned} \partial_t u + \mathcal{A}(t)u &= f(t) && \text{in} \quad \Omega \times \dot{J} \ , \\ \mathcal{B}(t)u &= g(t) && \text{on } \partial\Omega \times \dot{J} \ , \\ u(\cdot,0) &= u^0 && \text{on} \quad \Omega \ . \end{aligned} \tag{11.5}$$

Similarly as in the case of elliptic boundary value problems, we introduce several types of generalized solutions of (11.5). Namely, u is said to be

- a **strong $S_p^{2\beta}$-solution** if $1+1/p<2\beta\le 2$ and

$$u\in C(J,S_p^{2\beta-2})\cap C(\dot J,S_p^{2\beta})\cap C^1(\dot J,S_p^{2\beta-2}) \ ,$$

  and if it satisfies (11.5) pointwise (in the obvious sense);

- a **weak $S_p^{2\beta}$-solution** if $1/p<2\beta<1+1/p$ and

$$u\in C(J,S_{p,\mathcal{B}}^{2\beta-2})\cap C(\dot J,S_p^{2\beta})\cap C^1(\dot J,S_{p,\mathcal{B}}^{2\beta-2}) \ ,$$

  and if it satisfies $u(0)=u^0$,

$$(1-\delta)\gamma_\partial u(t)=(1-\delta)g(t) \ , \qquad t\in\dot J \ ,$$

  and

$$\langle v,\dot u(t)\rangle+\boldsymbol{a}(t)\bigl(v,u(t)\bigr)=\langle v,f(t)\rangle+\langle\gamma_\partial v,\delta g(t)\rangle_\partial$$

  for $t\in\dot J$ and $v\in S_{p',\mathcal{B}^\#}^{2-2\beta}$;

- a **very weak $S_p^{2\beta}$-solution** if $0\le 2\beta<1/p$ and $S_{p,\mathcal{B}}^{2\beta}$ and $S_{p',\mathcal{B}^\#}^{2-2\beta}$ are independent of t, and if

$$u\in C(J,S_{p,\mathcal{B}}^{2\beta})$$

  and u satisfies

$$\int_J\langle-\dot\varphi+\mathcal{A}^\#\varphi,u\rangle\,dt=\int_J\bigl\{\langle\varphi,f\rangle+\langle\mathcal{C}^\#\varphi,g\rangle_\partial\bigr\}\,dt+\langle\varphi(0),u^0\rangle$$

  for all

$$\varphi\in C(J,S_{p',\mathcal{B}^\#}^{2-2\beta})\cap C^1(J,S_{p'}^{-2\beta})$$

  having compact support in $J\backslash\{\sup J\}$.

In each one of the above cases u is simply said to be an **$S_p^{2\beta}$-solution** of (11.5).

11.1 Remarks

(a) Every strong $S_p^{2\beta}$-solution is a weak $S_p^{2\gamma}$-solution if $1/p<2\gamma<1+1/p$. Similarly, if u is a weak $S_p^{2\gamma}$-solution for some $2\gamma\in(1/p,1+1/p)$ and if $0\le 2\beta<1/p$ and $S_{p,\mathcal{B}}^{2\beta}$ and $S_{p',\mathcal{B}^\#}^{2-2\beta}$ are independent of t, it is a very weak $S_p^{2\beta}$-solution.

This is an easy consequence of Green's formulas and obvious integrations.

(b) Of course, it suffices to require

$$(\mathcal{A},\mathcal{B})\in C\bigl(J,\mathcal{E}(\Omega)\bigr)$$

for the definition of $S_p^{2\beta}$-solutions to be meaningful.

(c) In the definition of a very weak solution we require u to be continuous on J into $S^{2\beta}_{p,\mathcal{B}}$ and not on $\dot{J}$ as may be expected. Thus is done to guarantee the existence of the occurring integrals. $\square$

Consider now first the case $1/p < 2\beta < 1 + 1/p$. Suppose that

$$\begin{aligned}&\text{there exists}\\ &u_g \in C^\sigma(J, S^{2\beta}_p) \cap C^{1+\sigma}(J, S^{2\beta-2}_{p,\mathcal{B}})\\ &\text{such that } (1-\delta)\gamma_\partial u_g = (1-\delta)g\ .\end{aligned} \tag{11.6}$$

Define $v_g(t) \in S^{2\beta-2}_{p,\mathcal{B}}$ by

$$\langle v, v_g(t)\rangle := \boldsymbol{a}(t)\big(v, u_g(t)\big)\ , \qquad t \in J\ , \quad v \in S^{2-2\beta}_{p',\mathcal{B}^\#}\ ,$$

and note that (11.6), (8.7), and (8.13) imply

$$v_g \in C^{\rho\wedge\sigma}(J, S^{2\beta-2}_{p,\mathcal{B}})\ .$$

Consequently,

$$F_{\beta-1} := f + \mathcal{R}_{\beta-1}\delta g - v_g - \dot{u}_g \in C^{\rho\wedge\sigma}(J, S^{2\beta-2}_{p,\mathcal{B}})\ . \tag{11.7}$$

Also note that, thanks to (11.1) and (8.24),

$$A_{\beta-1} \in C^\rho\big(J, \mathcal{H}(S^{2\beta}_{p,\mathcal{B}}, S^{2\beta-2}_{p,\mathcal{B}})\big)\ . \tag*{(11.8)$_\beta$}$$

It is now obvious that

$$\begin{aligned}&u \text{ is an } S^{2\beta}_{p,\mathcal{B}}\text{-solution of (11.5) iff}\\ &u = v + u_g \text{ and } v \text{ is a solution of the Cauchy problem}\\ &\dot{v} + A_{\beta-1}(t)v = F_{\beta-1}(t)\ , \quad t \in \dot{J}\ , \qquad v(0) = u^0 - u_g(0)\\ &\text{in } S^{2\beta-2}_{p,\mathcal{B}}\ ,\end{aligned} \tag{11.9}$$

provided $1/p < 2\beta < 1 + 1/p$ and (11.6) is satisfied.

Clearly, Theorem 10.1 guarantees the unique solvability of the Cauchy problem in (11.9) provided $\sigma > 0$. On the other hand, suppose that

$$\sigma = 0 \quad \text{and} \quad 2 - 2\alpha < 2\beta < 2\alpha\ . \tag{11.10}$$

Fix $\gamma \in (1/p, \beta)$ and put $E_0 := S^{2\gamma-2}_{p,\mathcal{B}}$ and $E_1 := S^{2\gamma}_{p,\mathcal{B}}$. Note that

$$A_{\gamma-1} \in C^\rho\big(J, \mathcal{H}(E_1, E_0)\big)\ .$$

Choose $\theta \in (0, \beta - \gamma)$, select an admissible interpolation functor $(\cdot,\cdot)_\theta$, put $E_\theta := (E_1, E_0)_\theta$, and observe that

$$S^{2\gamma-2}_{p,\mathcal{B}} \hookrightarrow E_\theta \hookrightarrow E_0\ .$$

Hence it follows from (11.7) that

$$F_{\beta-1} \in C(J, E_\theta)\ .$$

Consequently, we can apply Theorem 10.2 to deduce that the Cauchy problem in (11.9) has a unique solution, considered as a Cauchy problem in E_0. These arguments prove the following

11.2 Theorem *Let* (11.1), (11.2), (11.4), *and* (11.6) *be satisfied with* $S := W$, *and suppose that* $1/p < 2\beta < 1 + 1/p$. *Then the parabolic boundary value problem* (11.5) *has a unique weak* $W_p^{2\beta}$*-solution, provided* $\sigma > 0$. *If* $\sigma = 0$ *and* $2 - 2\alpha < 2\beta < 2\alpha$, *it has u unique weak* $S_p^{2\gamma}$*-solution for any* γ *satisfying* $1/p < 2\gamma < 2\beta$.

Next we consider the case

$$1 + 1/p < 2\beta \le 2\alpha \le 2 \;, \tag{11.11}$$

and assume that $\sigma \in [0, 1)$ and

$$\begin{aligned}&\text{there exists}\\ &u_g \in C^\sigma(J, S_p^{2\beta}) \cap C^{1+\sigma}(J, S_p^{2\beta-2})\\ &\text{such that } \mathcal{B}u_g = g \;.\end{aligned} \tag{11.12}$$

Then

$$F_{\beta-1} := f - \mathcal{A}u_g - \dot{u}_g \in C^{\rho\wedge\sigma}(J, S_p^{2\beta-2}) \;,$$

thanks to (11.1) and (8.10). Note that Theorem 8.5 implies

$$A_{\beta-1}(t) \in \mathcal{H}(S_{p,\mathcal{B}(t)}^{2\beta}, S_p^{2\beta-2}) \;, \qquad t \in J \;.$$

It is obvious that (11.9) is true in this case too.

Suppose that $2\rho > 2\beta - 1 - 1/p$ and fix γ such that $1/p < 2\gamma < 1 + 1/p$ and $\beta - \gamma < \rho$. Let $E_0 := H_{p,\mathcal{B}}^{2\gamma-2}$ and $E_1 := H_{p,\mathcal{B}}^{2\gamma}$ and note that Theorem 7.2 implies $S_p^{2\beta-2} \doteq (E_0, E_1)_{\beta-\gamma}$, provided $(\cdot,\cdot)_{\beta-\gamma} := [\cdot,\cdot]_{\beta-\gamma}$ if $S = H$, and $(\cdot,\cdot)_{\beta-\gamma} := (\cdot,\cdot)_{\beta-\gamma,p}$ if $S = W$ and $2\beta \ne 1$. Also note that $(11.8)_\gamma$ is true. Now, as an easy consequence of Theorem 10.2, we obtain the following result.

11.3 Theorem *Let assumptions* (11.1), (11.4), (11.11), *and* (11.12) *be satisfied with* $S := W$, *and suppose that* $2\rho > 2\beta - 1 - 1/p$. *If* $\sigma > 0$, *the parabolic boundary value problem* (11.5) *has a unique strong* $W_p^{2\beta}$*-solution. If* $\sigma = 0$ *and* $\beta < \alpha$, *it has a unique strong* $W_p^{2\gamma}$*-solution for each* γ *satisfying* $1 + 1/p < 2\gamma < 2\beta$.

Lastly, we consider the case of very weak solutions. Thus we assume that $0 \le 2\beta < 1/p$ and $S_{p,\mathcal{B}}^{2\beta}$ and $S_{p',\mathcal{B}^\#}^{2-2\beta}$ are independent of t. We also presuppose (11.2).

Note that

$$F_{\beta-1} := f + \mathcal{S}_{\beta-1}g \in C^{\rho\wedge\sigma}(J, S_{p,\mathcal{B}}^{2\beta-2}) \;,$$

thanks to (8.7), (8.8), (8.10), (8.11), (11.1), and the definition of $\mathcal{S}_{\beta-1}$ in (9.16). Also note that $(11.8)_\beta$ is satisfied, as follows from Theorem 8.5, the definition of $A_{\beta-1}(t)$ in (8.20), and (8.11) and (11.1).

We associate with (11.5) the Cauchy problem

$$\dot{u} + A_{\beta-1}(t)u = F_{\beta-1}(t) \;, \quad t \in \dot{J} \;, \qquad u(0) = u^0 \tag{11.13}$$

in $S^{2\beta-2}_{p,\mathcal{B}}$. Suppose that u is a strict $S^{2\beta}_p$-solution of (11.13). Then an obvious integration argument implies that u is a very weak $S^{2\beta}_p$-solution of (11.5). Conversely, let u be a very weak $S^{2\beta}_p$-solution of (11.5) satisfying

$$u \in C(J, S^{2\beta}_p) \cap C^1(J, S^{2\beta-2}_{p,\mathcal{B}}) \ .$$

Then it follows from (8.25) and the arguments of the proof of [Ama88b, Theorem 10.2] that u is a strict solution of (11.13).

Using these facts and similar arguments as above one obtains the following

11.4 Theorem *Let* (11.1) *and* (11.2) *be satisfied, where* $0 \le 2\beta < 1/p$, *and let* $S^{2\beta}_{p,\mathcal{B}}$ *and* $S^{2-2\beta}_{p',\mathcal{B}^\#}$ *be independent of* t. *Also suppose that* $u^0 \in W^{2\beta}_p$ *and*

$$(f, g) \in C^\sigma(J, W^{2\beta-2}_{p,\mathcal{B}} \times \partial W^{2\beta}_p)$$

for some $\sigma \in [0,1)$. *If* $\sigma > 0$, *the parabolic boundary value problem* (11.5) *has a unique very weak* $S^{2\beta}_p$*-solution* u *satisfying*

$$u \in C(J, W^{2\beta}_p) \cap C^1(J, W^{2\beta-2}_{p,\mathcal{B}}) \ . \tag{11.14$_\beta$}$$

If $\sigma = 0$ *and* $2 - 2\alpha < 2\beta < 2\alpha$ *then, given any* $\gamma \in [0, \beta)$, *problem* (11.5) *has a unique very weak* $S^{2\gamma}_p$*-solution satisfying* $(11.14)_\gamma$.

11.5 Remarks

(a) The hypotheses (11.6) and (11.12) imply additional restrictions for g, in general. They are trivially satisfied if (11.4) is true and g is independent of t. Note that (11.6) and (11.12) are necessary conditions, provided $\sigma = 0$ and we are interested in strict $S^{2\beta}_p$-solutions of (11.5), that is, in $S^{2\beta}_p$-solutions satisfying

$$u \in C(J, S^{2\beta}_p) \cap C^1(J, S^{2\beta-2}_{p,\mathcal{B}}) \ .$$

In this case u^0 has to belong to $S^{2\beta}_p$ too, of course.

We do not discuss conditions (11.6) and (11.12) here in more detail, but refer to [Ama94]. Of course, it is easy to give (nonoptimal) sufficient conditions for their validity (along the lines of [Ama90a, Appendix B], for example).

(b) It is obvious that Theorems 11.2–11.4 remain valid if W is replaced by H.

(c) The arguments preceding Theorem 11.3 show that the spaces $H^s_{p,\mathcal{B}}$ have to be considered too, even if one is interested in the Sobolev-Slobodeckii spaces $W^s_{p,\mathcal{B}}$ only. This stems from the fact that L_p and $W^1_{p,\mathcal{B}}$ cannot be obtained from Besov spaces by real or complex interpolation if $p \neq 2$.

(d) Suppose that $1/p < 2\beta < 1 + 1/p$. Then u is a strict $S_p^{2\beta}$-solution of (11.5) in the sense of (a) iff

$$u \in C(J, S_p^{2\beta})$$

and u satisfies

$$(1-\delta)\gamma_\partial u = (1-\delta)g$$

and

$$\int_J \{-\langle\dot\varphi, u\rangle + \boldsymbol{a}(\varphi, u)\}\, dt = \int_J \{\langle\varphi, f\rangle + \langle\gamma_\partial\varphi, \delta g\rangle_\partial\}\, dt + \langle\varphi(0), u^0\rangle$$

for all $\varphi \in C(J, S_{p',\mathcal{B}\#}^{2-2\beta}) \cap C^1(J, S_{p',\mathcal{B}\#}^{-2\beta})$ having compact support in $J\backslash\{\sup J\}$.

This follows by the arguments of the proof of [Ama88b, Theorem 10.3].

(e) The above considerations show that the parabolic boundary value problem (11.5) is equivalent to a Cauchy problem of the form

$$\dot v + A_{\beta-1}(t)v = F_{\beta-1}(t)\ , \quad t \in \dot J\ , \qquad v(0) = v^0 \tag{11.15}$$

in $S_{p,\mathcal{B}}^{2\beta-2}$, provided assumption (11.6) or (11.12) is satisfied if 2β belongs to the corresponding interval. Hence it is easy to deduce from Theorem 10.1 additional smoothness properties of the solutions provided u^0 is sufficiently regular. Moreover, by applying abstract maximal regularity theorems to (11.15) (e.g., Theorem 10.3), it is possible to obtain maximal regularity results for (11.5) (cf. [Ama94]).

(f) Of course, Theorems 11.2–11.4 allow bootstrapping arguments (in the obvious way) to improve the (local) regularity of the solutions if the data enjoy better (local) regularity properties (cf. Remark 9.5(b) above). Moreover, if the data are smooth, it can be shown (by employing classical results about linear parabolic boundary value problems in Hölder spaces; see [LadSU68], for example) that the solutions constructed above are smooth classical solutions of (11.5). For details we refer to [Ama94].

(g) It is obvious that Theorem 11.3 remains valid for general normally elliptic boundary value problems $(\mathcal{A}, \mathcal{B})$ which are not necessarily in divergence form (cf. Remark 8.6(d)).

(h) Nonhomogeneous (nonlocal) boundary data possessing little regularity occur naturally in control problems for parabolic equations (cf. [Ama89b] and [Koc92], for example, as well as the references given there). In this connection it is of particular interest to admit Radon measures as boundary data. Observe that this is possible in the weak settings studied in Theorems 11.2 and 11.4, provided p is sufficiently close to 1 (cf. Remark 9.5(b)).

(i) Suppose, for simplicity, that $(\mathcal{A}, \mathcal{B}) \in \mathcal{E}^1(\Omega)$, that $\sigma(A_0) \subset [\operatorname{Re} z > 0]$, and that $\delta = 1$. Consider the 'semi-autonomous' Neumann type parabolic boundary value problem

$$\begin{aligned} \partial_t u + \mathcal{A}u &= f(t) &&\text{in} \quad \Omega \times \dot{J} \ , \\ \mathcal{B}u &= g(t) &&\text{on } \partial\Omega \times \dot{J} \ , \\ u(\cdot, 0) &= u^0 &&\text{on} \quad \Omega \ . \end{aligned} \tag{11.16}$$

Let $W^2_{p,\mathcal{A}} := \ker(\mathcal{A})$. It is not difficult to see that

$$W^2_p = W^2_{p,\mathcal{B}} \oplus W^2_{p,\mathcal{A}} \tag{11.17}$$

and that

$$\mathcal{B}^c := (\mathcal{B} | W^2_{p,\mathcal{A}})^{-1} \in \mathcal{L}\mathrm{is}\big(W^{1-1/p}_p(\partial\Omega, \mathbb{R}^N), W^2_{p,\mathcal{A}}\big) \ .$$

By decomposing a solution u of (11.16) according to (11.17) in the form $u = v + w$, it is easily verified that $w = \mathcal{B}^c g$ and v satisfies

$$\dot{v} + A_0 v = f(t) - (\mathcal{B}^c g)^{\cdot} \ , \quad t \in \dot{J} \ , \qquad v(0) = u_0 - w(0) \ . \tag{11.18}$$

Since $A_0 \in \mathcal{H}(W^2_{p,\mathcal{B}}, L_p)$, the solution of (11.18) can be represented by the variation-of-constants formula. Using the fact that $(e^{-tA_0})^{\cdot} = -A_0 e^{-tA_0}$ for $t > 0$, a formal partial integration then leads to the representation

$$u(t) = u^0 + \int_0^t e^{-(t-\tau)A_0} \big\{ f(\tau) + A_0 \mathcal{B}^c g(\tau) \big\} \, d\tau \ , \qquad t \in J \ , \tag{11.19}$$

where we also used the fact that a semigroup commutes with its generator. Unfortunately, formula (11.19) is not meaningful since, in general, $\mathcal{B}^c g(\tau)$ will not be contained in $\mathrm{dom}(A_0)$. However, if we replace A_0 in (11.19) by $A_{\alpha-1}$ for $2\alpha \in (1/p, 1 + 1/p)$, we obtain the **generalized varation-of-constants formula**

$$u(t) = u^0 + \int_0^t e^{-(t-\tau)A_{\alpha-1}} \big\{ f(\tau) + A_{\alpha-1} \mathcal{B}^c g(\tau) \big\} \, d\tau \ , \qquad \tau \in J \ , \tag{11.20}$$

for solutions of (11.16), which is meaningful, thanks to

$$\mathrm{im}(\mathcal{B}^c) = W^2_{p,\mathcal{A}} \hookrightarrow W^2_p \hookrightarrow W^{2\alpha}_p = W^{2\alpha}_{p,\mathcal{B}} \ .$$

Moreover, it follows from Theorem 9.2 that $\mathcal{B}^c$ has a unique continuous extension to a map

$$\mathcal{B}^c_\alpha \in \mathcal{L}(\partial W^{2\alpha}_p, W^{2\alpha}_p) \ .$$

Hence (11.20) is well defined for $(f, g) \in C(J, W^{2\alpha-2}_{p,\mathcal{B}} \times \partial W^{2\alpha}_p)$.

Note that u of (11.20) is a 'mild' solution of the Cauchy problem

$$\dot{u} + A_{\alpha-1} u = f + A_{\alpha-1} \mathcal{B}^c_\alpha g \quad \text{in } \dot{J} \ , \qquad u(0) = u^0$$

in $W^{2\alpha-2}_{p,\mathcal{B}}$. Using Green's formula (8.14) and a density argument, it is not difficult to see that

$$A_{\alpha-1} \mathcal{B}^c_\alpha = \mathcal{R}_{\alpha-1}$$

where $\mathcal{R}_{\alpha-1}$ is defined in (9.14).

The above approach to the nonhomogeneous linear parabolic boundary value problem (11.6) has been proposed by the author in [Ama86a] and extended to nonautonomous (linear and semilinear) problems in [Ama88b] (also cf. [Ama86d] for a heuristic explanation of this method). Later we discovered that the representation formula (11.19) had already been introduced by Balakrishnan [Bal76] and exploited by Lasiecka and others (e.g., [Las80], [Trig80]) for problems in control theory. These authors work in a Hilbert space setting which is crucial for their approach.

It should be mentioned that our investigations around the generalized variation-of-constants formula (11.20) eventually led us to introduce the interpolation-extrapolation scales described in Section 6 above and to our approach to quasilinear parabolic systems to be described in the following sections. □

Suppose now that $\mathbb{K}=\mathbb{R}$ and $N=1$, and endow all spaces with their respective natural orders as in Theorem 8.7. Also suppose that

$$g=0 \qquad \text{if} \quad 1+1/p<2\beta\le 2\alpha\le 2 \tag{11.21}$$

and

$$(1-\delta)g=0 \qquad \text{if} \quad 2-2\alpha\le 2\beta<1+1/p\;. \tag{11.22}$$

Then it follows from (11.9), (11.12), and (11.13), respectively, that the unique $S_p^{2\beta}$-solution u of (11.5) is the solution of the Cauchy problem (11.13), where $A_{\beta-1}$ satisfies $(11.8)_\beta$ and $F_{\beta-1}=f+\mathcal{R}_{\beta-1}\delta g$. Thus u is given by

$$u(t)=U_{\beta-1}(t,0)u^0+\int_0^t U_{\beta-1}(t,\tau)F_{\beta-1}(\tau)\,d\tau\;, \qquad t\in J\;, \tag{11.23}$$

where $U_{\beta-1}$ is the parabolic evolution operator for $A_{\beta-1}$ (cf. [DaK92] or [Ama94], for example). Note that $\mathcal{R}_{\beta-1}\ge 0$, thanks to (9.14) and the positivity of the trace operator. Hence $F_{\beta-1}(t)\ge 0$ if $\big(f(t),g(t)\big)\ge 0$. By Theorem 8.7 we know that $A_{\beta-1}(t)$ is resolvent positive for each $t\in J$. Hence [DaK92, Corollary 10.10] implies that $U_{\beta-1}(t,s)\ge 0$ for $s,t\in J$ with $s\le t$. Now the following theorem is an obvious consequence of (11.23).

11.6 Theorem *Suppose that $\mathbb{K}=\mathbb{R}$ and $N=1$, and let conditions* (11.21) *and* (11.22), *respectively, be satisfied. If $u^0\ge 0$ and $\big(f(t),g(t)\big)\ge 0$ for $t\in J$ then $u(t)\ge 0$ for $t\in J$, where u is the unique $S_p^{2\beta}$-solution of Theorems* 11.2–11.4, *respectively.*

As always, the regularity assumptions for the 'lower order coefficients' of $(\mathcal{A},\mathcal{B})$ can be weakened. For later purposes we exhibit the following simple special case.

11.7 Remark Suppose that $p>n$ and $\rho\in(0,1)$. Also suppose that

$$(\mathcal{A},\mathcal{B})\in C^\rho\big(J,\mathcal{E}_p(\Omega)\big)$$

and

$$\big(u^0,(f,g)\big)\in W_{p,\mathcal{B}}^{-1}\times C^\rho(J,W_{p,\mathcal{B}}^{-1}\times\partial W_p^1)$$

with $(1-\delta)g = 0$. Then problem (11.5) has a unique weak W_p^1-solution u. If $\mathbb{K} = \mathbb{R}$ and $N = 1$ then $u^0 \geq 0$ and $\big(f(t), g(t)\big) \geq 0$ for $t \in J$ imply $u(t) \geq 0$ for $t \in J$.

This follows from Remarks 8.6 and 8.8 and the above considerations. □

The results of the present section occur in the given generality in this paper for the first time in print. The basic observation of Remark 11.5(e) is due to the author. It has been used — in the case $1/p < 2\beta < 1 + 1/p$ and under the assumption that $(1-\delta)g = 0$ — implicitly or explicitly in many of the papers of the author cited in the references, but almost all the times under more restrictive assumptions (see Remark 11.5(h)).

Nonhomogeneous linear parabolic boundary value problems in W_p^2 (in fact, in W_p^{2m} for systems of order $2m$, which we do not consider here) have been studied by Solonnikov in great generality (cf. [Sol65] and [LadSU68]), where optimal regularity results are given. The case $p = 2$ is also considered in [AgV64]. A different approach to W_p^2-solutions has been proposed by Terreni [Ter89], who obtained optimal regularity results in spaces of functions which are Hölder continuous in t with values in W_p^2. Terreni's technique is inspired by the 'operator theoretic' approach of Grisvard [Gri69], based on Dunford integral representations. For further results in this direction using spaces of continuous and Hölder continuous functions and Nikolskii spaces we refer to [LuSW92], [Lun89], and [Gui90], respectively, as well as to the references in those papers (also see [Gui91a]).

Whereas the papers cited above all deal with strong W_p^2-solutions (or even more regular ones), there are very few results for nonhomogeneous parabolic problems with nonregular data. Lions and Magenes have extended their 'elliptic' theory, referred to in Section 9, to nonhomogeneous linear parabolic problems in [LiM72, Chapter 4]. However, due to the considerable 'technical' difficulties they encounter, an L_2-theory has been developed only (see the relevant remarks in [LiM72, vol. II pp. 85–88]). In addition, the Lions-Magenes theory requires rather smooth coefficients and differs from our theory (in the case $p = 2$, of course) in that we do not necessarily require that f be a distribution, whereas f has to belong to a suitable (nonoptimal) space of distributions on Ω in their theory.

As already mentioned in Remark 11.5(h), Lasiecka [Las80] and Triggiani [Trig80] used a semigroup approach to autonomous problems with irregular data, based on Balakrishnan's representation formula (11.19). However, their theory is a Hilbert space theory which uses fractional powers in an essential way. For these reasons it is neither applicable to problems with nonautonomous principal parts nor to W_p^s-spaces with $p \neq 2$.

Lastly, in [LuV91] the authors prove Hölder regularity results for weak W_p^1-solutions in the case of a single equation ($N = 1$).

12 Abstract Quasilinear Parabolic Problems

Let J be a perfect subinterval of $\mathbb{R}^+$ containing 0, let E be a Banach space, and let V be a nonempty subset of E. Suppose that $A(t, x)$ is for each $(t, x) \in J \times V$ a linear operator in E and that $f : J \times V \to E$.

By a **solution** u on $\dot{J}_u$ of the **quasilinear evolution equation**

$$\dot{u} + A(t, u)u = f(t, u) , \qquad t \in J , \tag{12.1}$$

we mean a function

$$u \in C(J_u, V) \cap C^1(\dot{J}_u, E) \ ,$$

where J_u is a perfect subinterval of J containing 0, such that $u(t) \in \operatorname{dom} A\bigl(t, u(t)\bigr)$ and

$$\dot{u}(t) + A\bigl(t, u(t)\bigr)u(t) = f\bigl(t, u(t)\bigr) \ , \qquad t \in \dot{J}_u \ .$$

If, in addition, $u(0) = x$ then u is a solution on J_u of the **quasilinear Cauchy problem**

$$\dot{u} + A(t,u)u = f(t,u) \quad \text{in } \dot{J}_u \ , \qquad u(0) = x \ . \tag{12.2}$$

The evolution equation (12.1) and the Cauchy problem (12.2) are said to be **parabolic** if $A(t,x)$ is for each $(t,x) \in J \times V$ a negative infinitesimal generator of an analytic semigroup on E.

Given $u \colon J_u \to V$, put

$$(A_u, f_u)(t) := \bigl(A(t, u(t)), f(t, u(t))\bigr) \ , \qquad t \in J \ .$$

Then u is a solution of (12.2) iff u is a solution of the linear Cauchy problem

$$\dot{v} + A_u(t)v = f_u(t) \ , \quad t \in \dot{J}_u \ , \qquad v(0) = x \ . \tag{12.3}$$

It is a **strict** solution of the Cauchy problem (12.2) on J_u if u is a strict solution of (12.3). Lastly, a solution u of (12.1) is **maximal** if there does not exist a solution of (12.1) which is a proper extension of u.

Let X be a metric space and let $t^+ : X \to (0, \infty]$. Put

$$\mathfrak{X} := \bigcup_{x \in X} \bigl[0, t^+(x)\bigr) \times \{x\}$$

and, given $\varphi \colon \mathfrak{X} \to X$, let $\varphi^t(x) := \varphi(t,x)$. Then φ is a (local) **semiflow** on X if

(i) $\mathfrak{X}$ is open in $\mathbb{R}^+ \times X$;

(ii) $\varphi \in C(\mathfrak{X}, X)$;

(iii) $\varphi^0 = \mathrm{id}_X$;

(iv) if $x \in X$ and $0 \le s < t^+(x)$, and if $0 \le t < t^+\bigl(\varphi^s(x)\bigr)$ then $s + t < t^+(x)$ and $\varphi^{s+t}(x) = \varphi^t \circ \varphi^s(x)$.

If $t^+(x) = \infty$ for each $x \in X$ then φ is a **global** semiflow on X. The set

$$\gamma^+(x) := \bigl\{\, \varphi^t(x) \ ; \ 0 \le t < t^+(x) \,\bigr\}$$

is the (positive semi-) **orbit** through x, and $t^+(x)$ is the (positive) **escape time** of x.

Throughout the remainder of this section we assume that

$$(E_0, E_1) \text{ is a densely injected Banach couple} \ ,$$

and $(\cdot,\cdot)_\theta$ is for each $\theta \in (0,1)$ an admissible interpolation functor. Moreover,

$$E_\theta := (E_0, E_1)_\theta \ , \qquad 0 < \theta < 1 \ ,$$

and, given $X \subset E_\theta$ for some $\theta \in (0,1)$, we put $X_\eta := X \cap E_\eta$ for $\eta \in [\theta, 1]$, equipped with the topology induced by E_η, as explained in Section 3.

After these preparations we can formulate the basic existence, uniqueness, and continuity theorem for quasilinear parabolic Cauchy problems.

12.1 Theorem *Suppose that $0 < \gamma \le \beta < \alpha < 1$, that X_β is open in E_β, and that*

$$(A, f) \in C^{1\text{-}}\bigl(X_\beta, \mathcal{H}(E_1, E_0) \times E_\gamma\bigr) \ .$$

Then the autonomous quasilinear parabolic Cauchy problem

$$\dot u + A(u)u = f(u) \ , \quad 0 < t < \infty \ , \qquad u(0) = x \tag{12.4}$$

possesses for each $x \in X_\alpha$ a unique maximal solution

$$u(\cdot, x) \in C\bigl([0, t^+(x)), X_\alpha\bigr) \cap C^\alpha\bigl([0, t^+(x)), E_0\bigr) \ .$$

The map $(t,x) \mapsto u(t,x)$ is a semiflow on X_α.

This theorem is due to the author. A proof is contained implicitly in [Ama86c]. A detailed derivation has also been given in the lecture series at TH Delft in the fall of 1989, referred to already in Section 9 (cf. [Li92, Chapter IV, Appendix A], where this proof has been written up for the particular case $f = 0$). A complete proof will be given in [Ama94]. It relies on a careful study of linear parabolic evolution equations (cf. [Ama86c], [Ama88a]) and on Banach's fixed point theorem. (In [Ama88a] a more complicated version (containing some unnecessary restrictions) has been proven which was needed to obtain the results of [Ama90a] since at that time the generation Theorem 8.5, which simplifies the study of quasilinear parabolic boundary value problems considerably, was not yet available.)

The first existence theorem for the quasilinear Cauchy problem (12.4) is due to Sobolevskii [Sob66]. Roughly speaking, Sobolevskii uses fractional power spaces $D\bigl(A^\alpha(0)\bigr)$ and $D\bigl(A^\beta(0)\bigr)$ instead of the interpolation spaces E_α and E_β, respectively. However, fractional power spaces are not particularly well suited for quasilinear problems and their use causes considerable technical problems. In particular, Sobolevskii did not prove that

$$u(\cdot, x) \in C\bigl([0, t^+(x)), D(A^\alpha(0))\bigr) \ , \qquad x \in D\bigl(A^\alpha(0)\bigr) \ .$$

The continuity of the solution is only shown with respect to a topology which is weaker than the topology of the space of initial values. Hence the results of [Sob66] do not guarantee that the solutions generate a semiflow.

The use of interpolation spaces simplifies the theory considerably (cf. [DaK92] for an exposition of this fact in the semilinear case). Moreover, it has the advantage that in concrete applications the spaces E_θ often can be identified with well known function spaces as is shown in this article. In contrast to this, the spaces $D\bigl(A^\alpha(0)\bigr)$, $0 < \alpha < 1$, are not explicitly known, in general, or only if restrictive regularity assumptions are satisfied, which is of little value for quasilinear parabolic boundary value problems.

More recently, Angenent [Ang90] has studied autonomous quasilinear parabolic Cauchy problems in 'spaces of maximal regularity' in the sense of Da Prato and Grisvard [DaPG79] (also cf. [Sit92], [Sit93a], [Sit93b] for some extensions and complements). The maximal regularity setting requires, however, serious restrictions for the Banach couple (E_0, E_1). In particular, E_0 (whence E_1) cannot be reflexive if $E_0 \neq E_1$, as has been shown in [Bai80].

It should be noted that Theorem 12.1 has the advantage of being of great generality and flexibility. Namely, there are no restrictions for the Banach spaces, except for $E_1 \overset{d}{\hookrightarrow} E_0$, and there are practically no restrictions for the interpolation functors, except for the density requirements $E_1 \overset{d}{\hookrightarrow} (E_0, E_1)_\theta$.

12.2 Remarks

(a) It is easy to deduce from the 'autonomous' Theorem 12.1 an existence, uniqueness, and continuity theorem for parameter-dependent nonautonomous Cauchy problems. For this we suppose that Λ is an open subset of some Banach space F and put

$$\boldsymbol{E}_\theta := \mathbb{R} \times E_\theta \times F , \qquad 0 \leq \theta \leq 1 .$$

Note that $(\boldsymbol{E}_0, \boldsymbol{E}_1)$ is a densely injected Banach couple and $(\boldsymbol{E}_0, \boldsymbol{E}_1)_\theta \doteq \boldsymbol{E}_\theta$ for $0 < \theta < 1$. We also assume that $0 < \gamma \leq \beta < \alpha < 1$ and that

$$(A, f) \in C^{1\text{-}}\big(J \times X_\beta \times \Lambda, \mathcal{H}(E_1, E_0) \times E_\gamma\big) . \tag{12.5}$$

Then we consider the parameter-dependent nonautonomous quasilinear parabolic Cauchy problem

$$\dot{u} + A(t, u, \lambda)u = f(t, u, \lambda) , \quad t \in \dot{J} , \qquad u(0) = x^0 . \tag{12.6}$$

Put $\boldsymbol{X}_\beta := \mathbb{R} \times X_\beta \times \Lambda$, fix any $T \in \dot{J}$, and extend $(A, f)|\big([0,T] \times X_\beta \times \Lambda\big)$ over all of $\boldsymbol{X}_\beta$ in the class $C^{1\text{-}}$. Denoting this extension again by (A, f), define

$$(\boldsymbol{A}, \boldsymbol{f}) \in C^{1\text{-}}\big(\boldsymbol{X}_\beta, \mathcal{H}(\boldsymbol{E}_1, \boldsymbol{E}_0) \times \boldsymbol{E}_\gamma\big)$$

by

$$(\boldsymbol{A}, \boldsymbol{f}) := \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 0 \end{bmatrix} , \begin{bmatrix} 1 \\ f \\ 0 \end{bmatrix} \right) .$$

Then (12.6) is equivalent to the autonomous parabolic Cauchy problem

$$\dot{\boldsymbol{u}} + \boldsymbol{A}(\boldsymbol{u})\boldsymbol{u} = \boldsymbol{f}(\boldsymbol{u}) , \quad 0 < t < \infty , \qquad \boldsymbol{u}(0) = \boldsymbol{x} , \tag{12.7}$$

where $\boldsymbol{u} := (t, u, \lambda)$ and $\boldsymbol{x} := (0, x, \lambda)$. Of course, if $\boldsymbol{u} := (t, u, \lambda)$ is a solution to (12.7) on $[0, t^+)$ then $u|[0,T] \cap [0, t^+)$ is a solution to (12.6) on this interval. Since $T \in \dot{J}$ was arbitray, we obtain a maximal solution to (12.5) by uniqueness.

However, by extending the proof of Theorem 12.1 to deal with (12.6) directly, it is possible to weaken the continuity requirements for A and f with respect to t and λ. This is important for some applications. For a proof we refer to [Ama94] (also cf. [Ama88a]).

(b) Of course, the above results also apply to the **semilinear Cauchy problem**

$$\dot{u} + A(t)u = f(t,u) \ , \quad t \in \dot{J} \ , \qquad u(0) = x \tag{12.8}$$

(and to the parameter-dependent analogue), where

$$A \in C^{\rho}\big(J, \mathcal{H}(E_1, E_0)\big) \tag{12.9}$$

for some $\rho \in (0,1)$. In this case standard arguments based upon the integral equation deduced from the variation-of-constants formula guarantee the solvability of (12.8) also in E_1 if f maps into E_γ for some $\gamma \in (0,1)$. More precisely, suppose that

$$X_\alpha \text{ is an open subset of } E_\alpha \text{ for some } \alpha \in [0,1]$$

and

$$f \in C^{1\text{-}}(J \times X_\alpha, E_\gamma) \qquad \text{for some } \gamma \in [0,\alpha] \quad \text{with } \alpha - \gamma < 1 \ .$$

Then, given $x \in X_\alpha$, there exists a unique maximal solution $u(\cdot,x)$ of (12.8) satisfying

$$u(\cdot,x) \in C\big([0,t^+(x)), X_\alpha\big) \cap C^{\alpha}\big([0,t^+(x)), X_0\big) \ .$$

Moreover, $u(\cdot,x)$ depends continuously upon $x \in X_\alpha$ in the topology of X_α. In particular, if (12.8) is autonomous, $(t,x) \mapsto u(t,x)$ is a semiflow on X_α. Lastly, $u(\cdot,x)$ is the unique solution of the Volterra integral equation

$$u(t) = U(t,0)x + \int_0^t U(t,\tau) f\big(\tau, u(\tau)\big)\, d\tau \ , \qquad t \in J \ ,$$

in E_α, where U is the unique parabolic fundamental solution for (12.9) (cf. [Ama84] and [Ama94]). Recall that $U(t,\tau) = e^{-(t-\tau)A}$ if $A(t) := A \in \mathcal{H}(E_1, E_0)$ for $t \in J$. □

For simplicity we restrict our considerations in the following to the autonomous quasilinear Cauchy problem (12.4) and we leave it to the reader to extend the results below to the nonautonomous case (along the lines of Remark 12.2(a), for example; cf. [Ama94]). We also presuppose the hypotheses of Theorem 12.1 for what follows.

The fact that $(t,x) \mapsto u(t,x)$ is a semiflow on X_α has important consequences for the long time behavior of solutions to (12.4). For example, that the ω-limit set $\omega(x)$ of $x \in X_\alpha$, defined by

$$\omega(x) := \{\, y \in X_\alpha \ ; \text{ there exists a sequence } t_j \to \infty \text{ such that } u(t_j,x) \to y \,\} \ ,$$

is a nonempty compact connected subset of X_α, provided the orbit $\gamma^+(x)$ is relatively compact. In particular, $t^+(x) = \infty$, that is, the solution to problem (12.4) exists globally, if $\gamma^+(x)$ is relatively compact. (We refer to [Ama90b] for the elementary theory of semiflows.)

It is an important consequence of Theorem 12.1 that the semiflow $(t,x) \mapsto u(t,x)$ is smoothing. In fact, $u(\cdot,x) \in C\big((0,t^+(x)),E_1\big)$ for $x \in X_\alpha$. Thus, if we are interested in the long time behavior of $u(\cdot,x)$, it is no restriction to assume that $x \in X_1$, which we will do in the following (since it will simplify the presentation somewhat).

Now suppose that $E_1 \overset{cc}{\hookrightarrow} E_0$ and that $\gamma^+(x)$ is bounded in E_δ for some $\delta \in (\alpha,1)$. Then it follows from $E_\delta \overset{cc}{\hookrightarrow} E_\alpha$ (cf. Remark 6.1(e)) that $\gamma^+(x)$ is relatively compact in X_α, provided it is bounded away from ∂X_α. Thus $t^+(x)=\infty$ in this case. In fact, one can do better if one is interested in global existence only. Namely, by analyzing the proof of Theorem 12.1 carefully and using the smoothing properties of the semiflow one proves the following

12.3 Theorem *Suppose that $E_1 \overset{cc}{\hookrightarrow} E_0$ and there exists $\delta \in (\beta,\alpha)$ such that*

$$u(\cdot,x)[0,T] := u(\cdot,x)\big|\big[0,t^+(x)\big) \cap [0,T]$$

is bounded in E_δ and bounded away from ∂X_β for each $T>0$. Then $t^+(x)=\infty$.

12.4 Remark Theorem 12.3 guarantees the global existence of $u(\cdot,x)$ for a given x, provided it is possible to establish **a priori bounds** for $u(\cdot,x)$ in the norm of E_δ for some $\delta>\beta$. This shows that one should try to choose β as close to zero as possible since then one has to establish a priori bounds in relatively 'weak norms' only. □

Theorem 12.3 is closely related to the following 'blow up' result for solutions having a finite 'lifetime' only.

12.5 Theorem *Suppose that $E_1 \overset{cc}{\hookrightarrow} E_0$ and $t^+(x)<\infty$. Then either*

$$u(t,x) \to \partial X_\beta \quad \textit{or} \quad \|u(t,x)\|_\delta \to \infty \qquad \textit{as} \quad t \to t^+(x)$$

for each $\delta \in (\beta,1]$. If $u(\cdot,x)$ is bounded in E_δ for some $\delta \in (\beta,1]$ and uniformly continuous from $\big[0,t^+(x)\big)$ to E_β then $u(\cdot,x)$ converges towards a single point of ∂X_β.

Lastly, we give a sufficient condition for a bounded orbit to be relatively compact.

12.6 Theorem *Suppose that $E_1 \overset{cc}{\hookrightarrow} E_0$ and that $\gamma^+(x)$ is bounded away from ∂X_β for some $x \in X_\alpha$. Also suppose that*

$$u \in BUC^\rho\big([0,t^+(x)),E_\delta\big)$$

for some $\delta \in (\beta,\alpha]$. Then $\gamma^+(x)$ is relatively compact in X_α, whence $t^+(x)=\infty$. Moreover, $\big\{\,u(t,x)\ ;\ t \ge \tau\,\big\}$ is bounded in E_ε for $\varepsilon \in [0,1)$ and $\tau>0$.

Proofs of the above results will be found in [Ama94], where the case that E_1 is not compactly injected in E_0 will be considered too.

Suppose now that

$$(A,f) \in C^k\big(X_\beta, \mathcal{H}(E_1,E_0) \times E_\gamma\big)$$

for some $k \in \dot{\mathbb{N}} \cup \{\infty, \omega\}$, where C^ω means 'analytic'. Then it can be shown that the semiflow $(t,x) \mapsto u(t,x)$ belongs to the class C^k too for $t > 0$. Moreover, considering $\partial_2 u(\cdot,x)$, for example, it follows that, given any $h \in E_\alpha$, the function $\partial_2 u(\cdot,x)h$ is the unique solution of the linearized Cauchy problem

$$\begin{gathered}\dot{v} + A\big(u(t,x)\big)v + \partial A\big(u(t,x)\big)\big[v,u(t,x)\big] = \partial f\big(u(t,x)\big)v\ , \qquad 0 < t < t^+(x)\ , \\ v(0) = h\ .\end{gathered}$$

Thus, roughly speaking, the results known from the theory of ordinary differential equations hold in the case of abstract quasilinear parabolic equations too. In fact, this analogy extends further: the principle of linearized stability is true; given suitable hypotheses, there are stable, unstable, and center manifolds possessing properties similar to the ones for ordinary differential equations; Hopf bifurcation theorems are valid; etc.

A qualitative theory analoguous to the one for ordinary differential equations has first been established by D. Henry in [Hen81] for the case of semilinear parabolic evolution equations. In the last years such a theory has also been developed for 'fully nonlinear parabolic problems' by Da Prato and Lunardi (jointly, and separately by Lunardi; cf. [Lun91] for a survey). By using maximal regularity too, Angenent [Ang90] and Simonett [Sit92], [Sit93a], [Sit93b] developed aspects of a qualitative theory for quasilinear parabolic evolution equations. The use of maximal regularity results greatly simplifies this task. In fact, given maximal regularity, the proofs follow pretty closely the corresponding ones of the theory of ordinary differential equations (modulo 'technical' difficulties, of course). As explained already, maximal regularity requires restrictive assumptions which we try to avoid. Without maximal regularity proofs are somewhat harder and technically more involved since one has to handle a 'loss of regularity'. This can be seen by comparing, for example, the 'differentiability proof' in the presence of maximal regularity assumptions in [Ang90] or [Sit93b] with the corresponding proof without maximal regularity hypotheses in [Ama88a]. In fact, a center-manifold theory has not yet been established without using maximal regularity.

We shall not go into further detail here but refer the interested reader to [Ama94]. However, it should be mentioned that the applicability of the qualitative theory of quasilinear parabolic evolution equations to quasilinear parabolic systems, to reaction-diffusion systems, for example — though of great importance — is somewhat limited due to the great complexity of those problems. This should be clear to any one knowing a bit about the great complexity of a single semilinear parabolic equation (cf. [Ama91c] and [AmR93] for explicit formulations of simple looking open problems which have to be solved for obtaining deeper qualitative information).

'Fully nonlinear parabolic problems' are of the form

$$\dot{u} + \varphi(u) = 0\ , \quad t > 0\ , \qquad \varphi(0) = x\ , \tag{12.10}$$

where $\varphi \in C^2(X_1, E_0)$ is such that

$$\partial\varphi \in C^1\big(X_1, \mathcal{H}(E_1,E_0)\big)\ . \tag{12.11}$$

The investigations of this class of problems are all based on 'maximal regularity properties' either in the sense of [DaPG79] (namely in [DaPG79]) and [Ang90], for example) or in the sense of singular Hölder spaces (e.g., [Lun87]). In fact, the following theorem is a consequence of [Lun87, Theorem 2].

12.7 Theorem *Let condition* (12.11) *be satisfied. Then the Cauchy problem* (12.10) *possesses for each* $x \in X_1$ *a unique maximal solution*

$$u(\cdot, x) \in C\big([0, t^+(x)), X_1\big) \cap C^1\big([0, t^+(x)), E_0\big) \ .$$

The map $(t, x) \mapsto u(t, x)$ *is a semiflow on* X_1.

This theorem can, of course, also be applied to quasilinear Cauchy problems. In general, it is, however, not advisable to use this approach since it requires the initial values to belong to E_1 so that the solutions do not exhibit any smoothing effect. Of course, there are situations where no smoothing does occur and one has to have recourse to this approach (cf. Section 17 below).

13 General Quasilinear Parabolic Boundary Value Problems

We show now how the abstract results of Section 12 can be combined with the theory developed above to prove that quasilinear parabolic boundary value problems generate semiflows on suitable phase spaces. In this section we consider the general case where the nonlinearities are not supposed to be local. Thus we study problems of the form

$$\begin{aligned} \partial_t u + \mathcal{A}(u)u &= F(u) && \text{in} \quad \Omega \times (0, \infty) \ , \\ \mathcal{B}(u)u &= G(u) && \text{on} \ \partial\Omega \times (0, \infty) \ , \\ u(\cdot, 0) &= u^0 && \text{on} \quad \Omega \ . \end{aligned} \tag{13.1}$$

We start by considering weak solutions. For this we suppose that

$$1 \le \sigma < 1 + 1/p < 2 \quad \text{and} \quad \sigma - 2 < \tau \le r < s < \sigma \tag{13.2}$$

such that $\tau, r, s \notin \Sigma_p$. We also suppose that

$$\sigma \le 2\widehat{\alpha} \le 2 \ ,$$

that

$$V^r_{p,\mathcal{B}} \text{ is a nonempty open subset of } W^r_{p,\mathcal{B}}$$

and that

$$(\mathcal{A}, \mathcal{B}) \in C^{1\text{-}}\big(V^r_{p,\mathcal{B}}, \mathcal{E}^{\widehat{\alpha}}(\Omega)\big) \tag{13.3}$$

and

$$(F, G) \in C^{1\text{-}}(V^r_{p,\mathcal{B}}, W^\tau_{p,\mathcal{B}} \times \partial W^{\tau+2}_p) \ . \tag{13.4}$$

Lastly, we assume that

$$(1 - \delta)G = 0 \ . \tag{13.5}$$

Let $S \in \{H, W\}$ and suppose that $\rho \in (r, 1+1/p) \backslash \Sigma_p$ satisfies $2 - 2\widehat{\alpha} \le \rho \le 2\widehat{\alpha}$. Then u is said to be a **weak S_p^ρ-solution** on J of (13.1) if J is a perfect subinterval of $\mathbb{R}^+$ containing 0, if

$$u \in C(J, V_{p,\mathcal{B}}^r) \ , \tag{13.6}$$

and u is a weak S_p^ρ-solution of the linear parabolic boundary value problem

$$\begin{aligned} \partial_t v + \mathcal{A}\big(u(t)\big) v &= F\big(u(t)\big) && \text{in} \quad \Omega \times \dot{J} \ , \\ \mathcal{B}\big(u(t)\big) v &= G\big(u(t)\big) && \text{on } \partial\Omega \times \dot{J} \ , \\ u(\cdot, 0) &= u^0 && \text{on} \quad \Omega \ . \end{aligned} \tag{13.7}$$

Note that (13.3)–(13.6) imply that the concept of weak S_p^ρ-solutions of (13.7) is well defined.

Choose $\sigma_1 \in [1, 1+1/p)$ such that

$$\sigma_1 = 1 \quad \text{if } 2\widehat{\alpha} = 1 \qquad \text{and} \qquad \sigma - 1 < \sigma_1 - 1 < \rho(\widehat{\alpha}) \wedge (\tau + 1) \quad \text{otherwise} \ ,$$

where $\rho(\widehat{\alpha})$ is the Hölder exponent (8.6). Put $E_0 := H_{p,\mathcal{B}}^{\sigma_1 - 2}$ and $E_1 := H_{p,\mathcal{B}}^{\sigma_1}$ and define $\mathfrak{F} := \{\, (\cdot,\cdot)_\theta \ ; \ 0 < \theta < 1 \,\}$ by

$$(\cdot,\cdot)_\theta := \begin{cases} [\cdot,\cdot]_\theta & \text{if } \theta \in \big\{(3-\sigma_1)/2, (2-\sigma_1)/2\big\} \ , \\ (\cdot,\cdot)_\theta & \text{otherwise} \ . \end{cases} \tag{13.8}$$

Then (E_0, E_1) is a densely injected Banach couple and Theorem 7.2 implies

$$E_\theta := (E_0, E_1)_\theta \doteq W_{p,\mathcal{B}}^{\sigma_1 - 2 + 2\theta} \ , \qquad 0 < \theta < 1 \ , \quad \sigma_1 - 2 + 2\theta \notin \Sigma_p \ . \tag{13.9}$$

Let

$$\alpha := 1 - (\sigma_1 - s)/2 \ , \quad \beta := 1 - (\sigma_1 - r)/2 \ , \quad \gamma := 1 - (\sigma_1 - \tau)/2 \ .$$

Then $0 < \gamma \le \beta < \alpha < 1$ and

$$E_\alpha \doteq W_{p,\mathcal{B}}^s \ , \quad E_\beta \doteq W_{p,\mathcal{B}}^r \ , \quad E_\gamma \doteq W_{p,\mathcal{B}}^\tau \ .$$

Hence $X_\beta := V_{p,\mathcal{B}}^r$ is a nonempty open subset of E_β. Let $A := A_{(\sigma_1/2)-1}$ and note that (13.3) and Theorem 8.5 imply

$$A \in C^{1\text{-}}\big(X_\beta, \mathcal{H}(E_1, E_0)\big) \ .$$

Also note that

$$f := F + \mathcal{R}_{\tau/2} G \in C^{1\text{-}}(X_\beta, E_\gamma) \ ,$$

thanks to (13.4), where $\mathcal{R}_{\tau/2}$ has been defined in (9.14).

It is an obvious consequence of (11.9) that u is a weak $H_p^{\sigma_1}$-solution of (13.1) on J iff u is a solution of the quasilinear Cauchy problem

$$\dot{u} + A(u)u = f(u) \ , \quad 0 < t < \infty \ , \qquad u(0) = u^0 \tag{13.10}$$

in E_0. Thanks to Theorem 12.1 the Cauchy problem (13.10) generates a local semiflow on

$$X_\alpha := V_{p,\mathcal{B}}^s := V_{p,\mathcal{B}}^r \cap W_p^s$$

consisting of weak $H_p^{\sigma_1}$-solutions of (13.1). It is obvious that each weak $H_p^{\sigma_1}$-solution of (13.1) is a weak W_p^{σ}-solution too.

Suppose now that $2-2\widehat{\alpha} \le s \le 2\widehat{\alpha}$ and let $\gamma^+(u^0)$ be bounded in X_α and bounded away from ∂X_β for some $u^0 \in X_\alpha$. Since $E_\alpha \overset{cc}{\hookrightarrow} E_\beta$, it follows that $\gamma^+(u^0)$ is relatively compact in X_β. Hence $(A,f)\bigl(\gamma^+(u^0)\bigr)$ is relatively compact in $\mathcal{L}(W_{p,\mathcal{B}}^s, W_{p,\mathcal{B}}^{s-2}) \times W_{p,\mathcal{B}}^{s-2}$, whence bounded. Thus we deduce from (13.10) that

$$\|\dot{u}(t,u^0)\|_{W_{p,\mathcal{B}}^{s-2}} \le c\ , \qquad t \in \bigl[0, t^+(u^0)\bigr)\ .$$

Consequently,

$$u(\cdot,u^0) \in B\bigl([0,t^+(u^0)), W_{p,\mathcal{B}}^s\bigr) \cap BUC^{1\text{-}}\bigl([0,t^+(u^0)), W_{p,\mathcal{B}}^{s-2}\bigr)\ . \tag{13.11}$$

Fix $s_\delta \in (r,s)\backslash\mathbb{N}$ and note that Theorem 7.2 implies

$$E_\delta \doteq W_{p,\mathcal{B}}^{s_\delta} = (W_{p,\mathcal{B}}^{s-2}, W_{p,\mathcal{B}}^s)_{\theta,p}\ , \qquad \theta := 1-(s-s_\delta)/2\ .$$

From this and from (13.11) we easily deduce that

$$u(\cdot,u^0) \in BUC^{1-\theta}\bigl([0,t^+(u^0)), E_\delta\bigr)\ .$$

Since $\theta \in (0,1)$ and $\delta \in (\beta,\alpha)$, we can now invoke Theorem 12.6. This shows that the following fundamental existence, uniqueness, and continuity theorem is true, where the last part of the assertion follows from Theorem 12.3.

13.1 Theorem *The quasilinear parabolic boundary value problem* (13.1) *possesses for each $u^0 \in V_{p,\mathcal{B}}^s$ a unique maximal weak W_p^{σ}-solution $u(\cdot,u^0)$, and $(t,u^0) \mapsto u(t,u^0)$ is a semiflow on $V_{p,\mathcal{B}}^s$. If $2-2\widehat{\alpha} \le s \le 2\widehat{\alpha}$, bounded orbits which are bounded away from $\partial V_{p,\mathcal{B}}^r$ are relatively compact in $V_{p,\mathcal{B}}^s$. If there exists $\rho \in (r,s)$ such that $u(\cdot,u^0)|[0,T]$ is bounded in W_p^{ρ} and bounded away from $\partial V_{p,\mathcal{B}}^r$ for each $T>0$ then $t^+(u^0)=\infty$.*

13.2 Remarks

(a) The assumption (13.5) that the Dirichlet part of the boundary conditions has to be homogeneous can easily be substituted by the condition

$$(1-\delta)G = (1-\delta)g^0$$

for some fixed $g^0 \in \partial W_p^{\sigma}$. Then Theorem 13.1 remains valid provided $V_{p,\mathcal{B}}^r$ is replaced by a nonempty open subset of the closed affine subspace

$$\bigl\{\, u \in W_p^r\ ;\ (1-\delta)\gamma_\partial u = (1-\delta)g^0 \,\bigr\}$$

of W_p^r.

(b) From (13.9) it should be clear why we have to consider both scales, the Sobolev-Slobodeckii scale and the Bessel potential scale, even if we are interested in the Sobolev-Slobodeckii scale only. Namely, if we would restrict ourselves to the latter scale only, we would have to put $E_1 := W_{p,\mathcal{B}}^{\sigma_1}$ and $E_0 := W_{p,\mathcal{B}}^{\sigma_1-2}$. But then we would not obtain the — in many cases most natural — space $W_{p,\mathcal{B}}^1$ by interpolating between E_0 and E_1. $\square$

By imposing additional restrictions it is easy to get better regularity properties for a given weak solution $u(\cdot, u^0)$ of (13.1). Indeed, the following theorem is an easy consequence of Theorem 11.3 and the fact that $u(\cdot, u^0)$ is a weak W_p^σ-solution of the linear problem (13.7).

13.3 Theorem *Let $u := u(\cdot, u^0)$ and $t^+ := t^+(u^0)$ for some $u^0 \in V_{p,\mathcal{B}}^s$. Suppose that $G = 0$ and $1 + 1/p < 2\alpha \leq 2$ and that*

$$\big(\mathcal{A}(u(\cdot)), \mathcal{B}(u(\cdot))\big) \in C^\rho\big((0,t^+), \mathbb{E}^\alpha(\Omega)\big)$$

and

$$F\big(u(\cdot)\big) \in C^\mu\big((0,t^+), W_p^{2\alpha-2}\big) \ ,$$

where $2\rho > 2\alpha - 1 - 1/p$ and $\mu > 0$. Then

$$u \in C\big((0,t^+), W_p^{2\alpha}\big) \cap C^1\big((0,t^+), W_p^{2\alpha-2}\big)$$

and u satisfies

$$\begin{aligned} \partial_t u + \mathcal{A}(u)u &= F(u) && \text{in} \quad \Omega \times (0,t^+) \ , \\ \mathcal{B}(u)u &= 0 && \text{on } \partial\Omega \times (0,t^+) \end{aligned}$$

in the strong sense.

We turn now to the particular situation that

$$\begin{aligned} &\text{there exists } (\mathcal{A}_0, \mathcal{B}_0) \in \mathcal{E}^1(\Omega) \text{ such that} \\ &S^\rho_{p,\mathcal{B}_0} = S^\rho_{p,\mathcal{B}(v)} \ , \qquad 1 + 1/p < \rho \leq 2 \ , \quad v \in \operatorname{dom}\big(\mathcal{A}(\cdot), \mathcal{B}(\cdot)\big) \ , \quad S \in \{H, W\} \ , \end{aligned} \tag{13.12}$$

that is, to the case where the Bessel potential and the Sobolev-Slobodeckii scales induced by $\big(\mathcal{A}(v), \mathcal{B}(v)\big)$ are independent of v for $\rho \in [0,2] \backslash \Sigma_p$. (By $\operatorname{dom}\big(\mathcal{A}(\cdot), \mathcal{B}(\cdot)\big)$ we mean the domain of the map $v \mapsto \big(\mathcal{A}(v), \mathcal{B}(v)\big)$.)

13.4 Examples

(a) $\delta = 0$, that is, $\mathcal{B} = \gamma_\partial$, the Dirichlet boundary operator. Then we can choose for $(\mathcal{A}_0, \mathcal{B}_0)$ the (diagonal) normally elliptic boundary value problem $(-\Delta, \gamma_\partial)$.

(b) $\delta \in \{0,1\}$ and $(\mathcal{A}, \mathcal{B})$ is of separated divergence form with $a_j = 0, \ \ j = 1, \ldots, n$, and $c = 0$. In this case we can put $(\mathcal{A}_0, \mathcal{B}_0) := \big(-\Delta, \delta\partial_{\nu_\alpha} + (1-\delta)\gamma_\partial\big)$, where we use the notations of Example 4.3(e).

(c) $\big(\mathcal{A}(\cdot), \mathcal{B}(\cdot)\big) = (\mathcal{A}_0, \mathcal{B}_0)$ is independent of v, and $(\mathcal{A}_0, \mathcal{B}_0) \in \mathcal{E}^1(\Omega)$. □

Given condition (13.12), we replace assumption (13.2) by

$$1 + 1/p < \sigma \leq 2 \quad \text{and} \quad \sigma - 2 < \tau \leq r < s < \sigma \tag{13.13}$$

with $\tau, r, s \notin \Sigma_p$. In this case we fix $\sigma_1 \in (1 + 1/p, 2]$ such that

$$\sigma_1 = 2 \quad \text{if} \quad \sigma = 2 \qquad \text{and} \qquad \sigma - 1 < \sigma_1 - 1 < \rho(\widehat{\alpha}) \wedge (\tau + 1) \quad \text{otherwise} \ ,$$

and we define $\mathfrak{F}$ again by (13.8).

Suppose that $S \in \{H, W\}$ and that $\rho \in (r, 2] \backslash \Sigma_p$ satisfies $2 - 2\widehat{\alpha} \leq \rho \leq 2\widehat{\alpha}$. Then a **strong S_p^ρ-solution** of the quasilinear parabolic boundary value problem

$$\begin{aligned} \partial_t u + \mathcal{A}(u)u &= F(u) && \text{in} \quad \Omega \times (0, \infty) \ , \\ \mathcal{B}(u)u &= 0 && \text{on } \partial\Omega \times (0, \infty) \ , \\ u(\cdot, 0) &= u^0 && \text{on} \quad \Omega \end{aligned} \tag{13.14}$$

on J is a function u satisfying (13.6) such that u is a strong S_p^ρ-solution of the linear parabolic boundary value problem (13.7) (with $G = 0$, of course).

It follows that u is a strong $H_p^{\sigma_1}$-solution of (13.14) on J iff it is a solution on J of the quasilinear Cauchy problem

$$\dot{u} + A(u)u = F(u) \ , \quad 0 < t < \infty \ , \qquad u(0) = u^0$$

in E_0. An obvious modification of the arguments leading to Theorem 13.1, based on the 'almost reiteration theorem' of Remark 6.1(c), proves the following

13.5 Theorem *Let* (13.3) *and* (13.4) *be satisfied. Then the assertions of Theorem* 13.1 *remain valid for strong solutions, except for the following modification: the map* $(t, u^0) \mapsto u(t, u^0)$ *is a semiflow on*

$$V_\alpha := (H_p^{\sigma_1 - 2}, H_{p,\mathcal{B}}^{\sigma_1})_{\alpha,p} \cap V_{p,\mathcal{B}}^r \ .$$

13.6 Remarks

(a) It follows from Remark 6.1(c) and from Theorem 7.1 that

$$V_{p,\mathcal{B}}^{s_1} \hookrightarrow V_\alpha \hookrightarrow V_{p,\mathcal{B}}^{s_2} \ ,$$

provided $r \leq s_2 < s < s_1 \leq 2$ and $s_1, s_2 \notin \Sigma_p$.

(b) It is obvious that Theorem 13.5 remains valid if $\big(\mathcal{A}(v), \mathcal{B}(v)\big)$ is for each $v \in V_{p,\mathcal{B}}^r$ a second order normally elliptic boundary value problem, which is not necessarily of divergence form, provided suitable regularity conditions are satisfied. This is a consequence of Remark 8.6(d). □

Lastly, we turn to the study of very weak solutions. For this we impose the assumption that

$$\begin{gathered} \text{there exists } (\mathcal{A}_0, \mathcal{B}_0) \in \mathcal{E}^1(\Omega) \text{ such that} \\ S_{p',\mathcal{B}_0^\#}^{2-\rho} = S_{p',\mathcal{B}^\#(v)}^{2-\rho} \ , \qquad 0 \leq \rho < 1/p \ , \quad v \in \operatorname{dom}\big(\mathcal{A}(\cdot), \mathcal{B}(\cdot)\big) \ , \quad S \in \{H, W\} \ . \end{gathered} \tag{13.15}$$

Then we replace assumption (13.2) by

$$0 \leq \sigma < 1/p \quad \text{and} \quad \sigma - 2 < \tau \leq r < s < \sigma \tag{13.16}$$

such that $\tau, r, s \notin \Sigma_p$.

Suppose that $\rho \in (r, 1/p)$ satisfies $2 - 2\widehat{\alpha} \leq \rho \leq 2\widehat{\alpha}$. Then a **very weak S_p^ρ-solution** of the quasilinear parabolic boundary value problem (13.1) on J is a function u satisfying (13.6) such that u is a very weak S_p^ρ-solution of the linear problem (13.7).

Assume that $\sigma \geq 2 - 2\widehat{\alpha}$, fix $\sigma_1 \in \big(\sigma, (\tau+2) \wedge 1/p\big)$, and define $\mathfrak{F}$ by

$$(\cdot,\cdot)_\theta := \begin{cases} [\cdot,\cdot]_\theta & \text{if } \theta \in \big\{(2-\sigma_1)/2, (1-\sigma_1)/2\big\}\ , \\ (\cdot,\cdot)_{\theta,p} & \text{otherwise}\ . \end{cases}$$

Note that (13.3) implies

$$S_\gamma := (\mathcal{C}^\# | W_{p',\mathcal{B}_0^\#}^{-\tau})' \in C^{1\text{-}}\big(V_{p,\mathcal{B}}^r, \mathcal{L}(\partial W_p^{\tau+2}, W_p^\tau)\big)$$

and, consequently and thanks to (13.4),

$$f := F + S_\gamma G \in C^{1\text{-}}(V_{p,\mathcal{B}}^r, E_\gamma)\ . \tag{13.17}$$

Then u is a very weak $H_p^{\sigma_1}$-solution of (13.1) on J iff u is a solution of (13.10) on J (where now f is given by (13.17), of course). Hence, similarly as above, we obtain

13.7 Theorem *Let* (13.15) *and* (13.16) *be satisfied. Then the assertions of Theorem* 13.1 *are true for very weak solutions, except for the following modification*: *the map* $(t, u^0) \mapsto u(t, u^0)$ *is a semiflow on*

$$V_\alpha := (H_{p,\mathcal{B}}^{\sigma_1 - 2}, H_p^{\sigma_1})_{\alpha,p} \cap V_{p,\mathcal{B}}^r\ .$$

Of course, the above theorems apply, in particular, to the **semilinear parabolic boundary value problem**

$$\begin{aligned} \partial_t u + \mathcal{A}u &= F(u) && \text{in } \ \Omega \times (0,\infty)\ , \\ \mathcal{B}u &= G(u) && \text{on } \partial\Omega \times (0,\infty)\ , \\ u(\cdot,0) &= u^0 && \text{on } \ \Omega\ , \end{aligned} \tag{13.18}$$

provided $(\mathcal{A},\mathcal{B}) \in \mathcal{E}^1(\Omega)$. However, in this case we can also apply Remark 12.2(b) to get the following useful complement to the above results.

13.8 Theorem *Let* $(\mathcal{A},\mathcal{B}) \in \mathcal{E}^1(\Omega)$ *and suppose that* $s, \tau \in [-2,2] \backslash \Sigma_p$ *satisfy* $0 \leq s \leq 2$ *and* $0 \leq s - \tau < 2$. *Let* $V_{p,\mathcal{B}}^s$ *be a nonempty open subset of* $W_{p,\mathcal{B}}^s$ *and suppose that*

$$(F, G) \in C^{1\text{-}}(V_{p,\mathcal{B}}^s, W_{p,\mathcal{B}}^\tau \times \partial W_p^{\tau+2})\ ,$$

where $G = 0$ *if* $1 + 1/p < s \leq 2$, *and* $(1-\delta)G = 0$ *if* $1/p < s < 1 + 1/p$.

Then the semilinear parabolic boundary value problem (13.18) *has for each* $u^0 \in V_{p,\mathcal{B}}^s$ *a unique maximal* W_p^s*-solution* $u(\cdot, u^0)$, *and* $(t, u^0) \mapsto u(t, u^0)$ *is a semiflow on* $V_{p,\mathcal{B}}^s$.

13.9 Remark On the basis of Remark 12.2(a) it is easy to extend the results of this section to nonautonomous parameter-dependent quasilinear parabolic boundary value problems. □

In the given generality, the above results appear here for the first time in print. The case of weak H_p^s-solutions has been treated in [Ama90a], and preliminary results about strong solutions are contained in [Ama86b] and [Ama86c].

By imposing more smoothness assumptions for the maps $(\mathcal{A}, \mathcal{B})$ and (F, G), it is possible to prove differentiability properties of the semiflow $(t, u^0) \mapsto u(t, u^0)$. For this and for the qualitative theory, i.e., stability, bifurcation, etc., we refer to [Ama94].

14 Reaction-Diffusion Equations: Existence and Regularity

We specialize now the results of the preceding section to the case that $(\mathcal{A}, \mathcal{B})$ and (F, G) are local operators. For simplicity we impose more restrictive regularity hypotheses than really needed and leave it to the reader to find weaker assumptions.

We assume that

$$D_0 \text{ is a nonempty open subset of } \mathbb{R}^N \ ,$$

that

$$a_{jk}, a_j, b_j, a_0 \in C^{2\text{-}}\big(\overline{\Omega} \times D_0, \mathcal{L}(\mathbb{R}^N)\big) \ , \qquad 1 \le j, k \le n \ , \tag{14.1}$$

and that

$$c \in C^{2\text{-}}\big(\partial\Omega \times D_0, \mathcal{L}(\mathbb{R}^N)\big) \ . \tag{14.2}$$

We put

$$\mathcal{A}(\eta)u := -\partial_j\big(a_{jk}(\cdot,\eta)\partial_j u + a_j(\cdot,\eta)u\big) + b_j(\cdot,\eta)\partial_j u + a_0(\cdot,\eta)$$

and

$$\mathcal{B}(\eta)u := \delta\big\{\nu^j\gamma_\partial\big(a_{jk}(\cdot,\eta)\partial_k u + a_j(\cdot,\eta)u\big) + c(\cdot,\eta)\gamma_\partial u\big\} + (1-\delta)\gamma_\partial u$$

for $\eta \in D_0$. (To be precise: the term $c(\cdot,\eta)\gamma_\partial u$ should be interpreted in the following as $\gamma_\partial\big(\widetilde{c}(\cdot,\eta)u\big)$, where $\widetilde{c} \in C^{2\text{-}}\big(\overline{\Omega} \times D_0, \mathcal{L}(\mathbb{R}^N)\big)$ is an arbitrary extension of c.) Then

$$\big(\mathcal{A}(\cdot), \mathcal{B}(\cdot)\big) \in C^{2\text{-}}\big(D_0, \mathbb{E}^1(\Omega)\big) \ , \tag{14.3}$$

and we assume that $\big(\mathcal{A}(\eta), \mathcal{B}(\eta)\big) \in \mathcal{E}(\Omega)$ for at least one $\eta \in \mathbb{R}^N$. Hence

$$D := \big\{\, \eta \in D_0 \ ; \ \big(\mathcal{A}(\eta), \mathcal{B}(\eta)\big) \in \mathcal{E}(\Omega) \,\big\}$$

is a well defined nonempty open subset of D_0, whence of $\mathbb{R}^N$, thanks to (4.8).

Let X be a compact metric space, let E and F be Banach spaces, and let U be a nonempty open subset of E. Then we write

$$\varphi \in C^{1\text{-},2\text{-}}(X \times U, F)$$

provided each point $y_0 \in U$ has a neighborhood $V(y_0)$ in U such that $\varphi(\cdot, y) \in C^{1\text{-}}(X, F)$, uniformly with respect to $y \in V(y_0)$, and such that $\partial_2 \varphi(x, \cdot) \in C^{1\text{-}}\big(U, \mathcal{L}(E, F)\big)$, uniformly with respect to $x \in X$. If X is a differentiable manifold,

$$\varphi \in C^{1,2}(X \times Y, F)$$

if $\varphi \in C^1(X \times Y, F)$ and $\partial_2^2 \in C\big(X \times Y, \mathcal{L}^2(E, F)\big)$, where $\mathcal{L}^2(E, F)$ is the Banach space of all continuous bilinear maps from $E \times E$ to F.

We suppose that

$$\begin{aligned} &f \in C^{1\text{-},2\text{-}}\big((\overline{\Omega} \times D) \times \mathbb{R}^{N \times n}, \mathbb{R}^N\big) \\ &\text{and for each compact subset } K \text{ of } D \text{ there exists a constant } c_K \text{ such that} \\ &|\partial_3 f(x, \eta, \zeta)| \le c_K(1 + |\zeta|) , \qquad (x, \eta, \zeta) \in \overline{\Omega} \times K \times \mathbb{R}^{N \times n} . \end{aligned} \tag{14.4}$$

Note that, thanks to the mean value theorem, (14.4) implies that

$$|f(x, \eta, \zeta)| \le c_K'(1 + |\zeta|^2) , \qquad (x, \eta, \zeta) \in \overline{\Omega} \times K \times \mathbb{R}^{N \times n} , \tag{14.5}$$

that is, f 'grows at most quadratically in the gradient'.

Lastly, we assume that

$$g \in C^{1\text{-}}(\partial\Omega \times D, \mathbb{R}^N) \quad \text{and} \quad (1 - \delta)g = 0 . \tag{14.6}$$

Then we consider the 'local' quasilinear parabolic boundary value problem

$$\begin{aligned} \partial_t u + \mathcal{A}(u)u &= f(\cdot, u, \partial u) && \text{in } \ \Omega \times (0, \infty) , \\ \mathcal{B}(u)u &= g(\cdot, u) && \text{on } \partial\Omega \times (0, \infty) , \\ u(\cdot, 0) &= u^0 && \text{on } \ \Omega . \end{aligned} \tag{14.7}$$

14.1 Remarks

(a) Of course, by redefining f and g appropriately, we can always assume that $a_0 = 0$ and $c = 0$. This shows, in particular, that it suffices to assume that

$$a \in C^{1\text{-}}\big(\overline{\Omega} \times D, \mathcal{L}(\mathbb{R}^N)\big) \quad \text{and} \quad c \in C^{1\text{-}}\big(\partial\Omega \times D, \mathcal{L}(\mathbb{R}^N)\big) .$$

If we allow f to depend upon ∂u, we can also assume that $b_j = 0$ for $1 \le j \le n$. However, the following results will be slightly different if f depends upon ∂u or if it does not. This may already be seen from the regularity assumptions for f which are weaker if f is independent of ∂u than otherwise.

(b) Suppose that there exists $\widetilde{g} \in C^{2\text{-}}\big(\partial\Omega \times D, \mathcal{L}(\mathbb{R}^N)\big)$ such that

$$g(\cdot, \eta) = \widetilde{g}(\cdot, \eta)\eta , \qquad \eta \in D . \tag{14.8}$$

Then, by replacing c by $c - \widetilde{g}$, we can assume that $g = 0$. This observation will be important below for deriving regularity properties. Note that (14.8) is satisfied, for example, if $g \in C^{2\text{-},3\text{-}}(\partial\Omega \times D_0, \mathbb{R}^N)$ and D_0 is star-shaped with respect to

$0 \in D_0$, and if $g(\cdot,0)=0$. Then

$$\widetilde{g}(\cdot,\eta) := \int_0^1 \partial_2 g(\cdot, t\eta)\, dt\ , \qquad \eta \in D\ ,$$

by the mean value theorem. On the other hand, (14.8) is true if $N=1$ and $0 \notin D$. Then we can put $\widetilde{g}(\cdot,\eta) := g(\cdot,\eta)/\eta$. If $N>1$, similar constructions can be used in many cases. □

In order to apply the results of Section 13 we have to verify that $(\mathcal{A},\mathcal{B})$ and the map (F,G) induced by (f,g) possess suitable continuity properties. For this we need some preparation.

Given $\varphi\colon X \times U \to F$, we define $\Phi : U^X \to F^X$, the **Nemytskii operator** induced by φ, by

$$\Phi(u)(x) := \varphi\bigl(x, u(x)\bigr)\ , \qquad u \in U^X\ , \quad x \in X\ .$$

In the next lemma we collect some of the mapping properties of Φ which we shall use in the following.

14.2 Lemma

(i) *$C(X,U)$ is open in $C(X,E)$.*
If $\varphi \in C^{1\text{-}}(X \times U, F)$ then $\Phi \in C^{1\text{-}}\bigl(C(X,U), C(X,F)\bigr)$.

(ii) *If $\varphi \in C^{1\text{-},2\text{-}}(X \times U, F)$ then*

$$\Phi \in C^{1\text{-}}\bigl(C^\rho(X,U), C^\rho(X,F)\bigr)\ , \qquad \rho \in [0,1) \cup \{1\text{-}\}\ .$$

(iii) *Let μ be a positive Radon measure on X and suppose that $\varphi \in C(X \times E, F)$. Also suppose that $1 \le q \le p < \infty$ and there exists a constant c such that*

$$|\varphi(x,\xi)| \le c(1+|\xi|^{p/q})\ , \qquad (x,\xi) \in X \times E\ .$$

Then

$$\Phi \in C\bigl(L_p(X,\mu,E), L_q(X,\mu,F)\bigr)\ .$$

If, in addition, $\partial_2\varphi \in C\bigl(X \times E, \mathcal{L}(E,F)\bigr)$ and

$$|\partial_2\varphi(x,\xi)| \le c(1+|\xi|^{(p/q)-1})\ , \qquad (x,\xi) \in X \times E\ ,$$

then

$$\Phi \in C^{1\text{-}}\bigl(L_p(X,\mu,E), L_q(X,\mu,F)\bigr)\ .$$

In fact, Φ is continuously differentiable provided $p>q$.

Parts (i) and (ii) are easy exercises, thanks to the fact that locally Lipschitz continuous maps are uniformly Lipschitz continuous on compact sets. Part (iii) is a trivial extension of (special cases of) well known results of Krasnosel'skii [Kra64] and Vainberg [Vai64] (cf. [Ama94] for details; also cf. [Goe92], [GoKT92]).

14.3 Remark If $\varphi \in C^{1\text{-}}(X \times U, F)$, it is easily seen that $\Phi\bigl(C^\rho(X,U)\bigr) \subset C^\rho(X,F)$ for $\rho \in (0,1) \cup \{1\text{-}\}$. However, Φ is not continuous, in general, as easy examples show. □

Suppose that $n < p < \infty$ and note that (5.1)–(5.9) and the compactness of $\overline{\Omega}$ imply

$$W_p^r \hookrightarrow C^{r-n/p} \hookrightarrow C \hookrightarrow L_p \tag{14.9}$$

and

$$W_p^{r-1/p}(\partial\Omega) \hookrightarrow C^{r-n/p}(\partial\Omega) \hookrightarrow C(\partial\Omega) \hookrightarrow L_p(\partial\Omega)\ , \tag{14.10}$$

where $W_p^s(\partial\Omega) := W_p^s(\partial\Omega, \mathbb{R}^N)$ etc., provided

$$r > n/p \quad \text{and} \quad r \notin \mathbb{N} + n/p\ . \tag{14.11}$$

Hence Lemma 14.2(i) implies, in particular, that

$$W_p^r(\Omega, D) := \big\{\, u \in W_p^r\ ;\ u(\overline{\Omega}) \subset D \,\big\} \text{ is open in } W_p^r \text{ for } n/p < r \leq 2\ . \tag{14.12}$$

Moreover, we deduce from (14.1), (14.2), and Lemma 14.2(ii) that

$$\begin{aligned}&(\mathcal{A}, \mathcal{B}) \in C^{1\text{-}}\big(W_p^r(\Omega, D), \mathcal{E}^{\widehat{\alpha}(r)}(\Omega)\big)\\ &\text{for } n/p < r \leq 2, \text{ where } 2\widehat{\alpha}(r) \in [1, 1 + r - n/p) \text{ with } \widehat{\alpha}(r) \leq 1\ ,\end{aligned} \tag{14.13}$$

provided we choose σ in (8.5) in such a way that $\rho\big(\widehat{\alpha}(r)\big) := \sigma(|2\widehat{\alpha}(r) - 1|) \leq r - n/p$, which we can do.

Let now F be the Nemytskii operator induced by f. If f is independent of the gradient, the factorization

$$W_p^r(\Omega, D) \hookrightarrow C(\overline{\Omega}, D) \xrightarrow{\ F\ } C(\overline{\Omega}, \mathbb{R}^N) \hookrightarrow L_p$$

and Lemma 14.2(i) show that

$$F \in C^{1\text{-}}\big(W_p^r(\Omega, D), L_p\big)\ , \qquad n/p < r \leq 2\ . \tag{14.14}$$

On the other hand, we deduce from (5.1)–(5.11) that

$$W_p^r(\Omega, D) \xrightarrow{\ (u, \partial u)\ } C(\overline{\Omega}, D) \times (W_p^{r-1})^n \hookrightarrow C(\overline{\Omega}, D) \times (L_q)^n$$

provided

$$\frac{1}{p} \geq \frac{1}{q} \geq \frac{1}{p} - \frac{r-1}{n} > 0\ .$$

Hence it is a consequence of Lemma 14.2(i) and (iii) that

$$F \in C^{1\text{-}}\big(W_p^r(\Omega, D), L_s\big)$$

provided $2 \leq q/s$. Suppose that $0 \geq \tau > -2 + 1/p$ with $\tau \notin \Sigma_p$ and note that

$$W_{p', \mathcal{B}^\#}^{-\tau} \hookrightarrow L_{s'} \qquad \text{if} \quad \frac{1}{p'} \geq \frac{1}{s'} \geq \frac{1}{p'} + \frac{\tau}{n} > 0\ .$$

Hence

$$L_s \hookrightarrow W_{p,\mathcal{B}}^{\tau} \qquad \text{if} \quad \frac{1}{p} \leq \frac{1}{s} \leq \frac{1}{p} - \frac{\tau}{n} < 1\ .$$

Consequently, choosing q and s maximal, we see that

$$F \in C^{1-}\big(W_p^r(\Omega, D), W_{p,\mathcal{B}}^\tau\big) \tag{14.15}$$

provided (14.5) is satisfied and

$$1 \vee \big(1 + (\tau + n/p)/2\big) \le r < 1 + n/p \quad \text{and} \quad 0 \ge \tau > -2 + 1/p \ , \tag{14.16}$$

where $\tau \notin \Sigma_p$.

Lastly, we denote by $\widehat{g}$ the Nemytskii operator induced by g and put

$$G := \widehat{g} \circ \gamma_\partial \ .$$

Then it is a consequence of (14.8), (14.10), Lemma 14.2(i), and (5.17), (5.18) that

$$G \in C^{1-}\big(W_p^r(\Omega, D), L_p(\partial\Omega)\big) \ , \qquad n/p < r \le 2 \ .$$

Thus, thanks to $G = \delta G$, we see that

$$G \in C^{1-}\big(W_p^r(\Omega, D), \partial W_p^{\tau+2}\big) \ , \qquad n/p < r \le 2 \ , \quad \tau < -1 + 1/p \ . \tag{14.17}$$

After these preparations it is easy to apply the theorems of Section 13 to problem (14.7). First assume that

$$0 \in D \qquad \text{if} \quad \delta \ne 1 \ . \tag{14.18}$$

Then it follows from (14.12) that

$$W_{p,\mathcal{B}}^r(\Omega, D) := \big\{\, u \in W_{p,\mathcal{B}}^r \ ; \ u(\overline{\Omega}) \subset D \,\big\}$$

is a nonempty open subset of $W_{p,\mathcal{B}}^r$ if $n/p < r \le 2$ and either $r < 1 + 1/p$ or $W_{p,\mathcal{B}(v)}^r$ is independent of $v \in \mathrm{dom}\big(\mathcal{A}(\cdot), \mathcal{B}(\cdot)\big)$.

We fix now τ, r, s, and σ such that

$$n/p < r < s < \sigma < \tau + 2 < (1 + 1/p) \wedge (1 + r - n/p) \ , \qquad \sigma > 1 \ , \tag{14.19}$$

if f is independent of the gradient, and such that

$$1 = r < s < \sigma < \tau + 2 < (1 + 1/p) \wedge (2 - n/p) \tag{14.20}$$

otherwise. Then it follows from (14.13) that

$$(\mathcal{A}, \mathcal{B}) \in C^{1-}\big(W_{p,\mathcal{B}}^r(\Omega, D), \mathcal{E}^{\widehat{\alpha}}(\Omega)\big) \ , \tag{14.21}$$

where $\sigma \le 2\widehat{\alpha} < 2$, and from (14.14)–(14.17) that

$$(F, G) \in C^{1-}\big(W_{p,\mathcal{B}}^r(\Omega, D), W_{p,\mathcal{B}}^\tau \times \partial W_p^{\tau+2}\big) \ . \tag{14.22}$$

Also note that $2 - 2\widehat{\alpha} \le \sigma \le 2\widehat{\alpha}$ and that, of course, problem (14.7) has the form (13.1). Hence Theorem 13.1 easily implies

14.4 Theorem *Suppose that $n/p < s < (1+1/p) \wedge (2-n/p)$ and that either*

$$f \text{ is independent of the gradient} \tag{14.23}$$

or

$$s > 1\ . \tag{14.24}$$

Then the boundary value problem (14.7) has for each $u^0 \in W^s_{p,\mathcal{B}}(\Omega, D)$ a unique maximal weak W^s_p-solution $u(\cdot, u^0)$. The map $(t, u^0) \mapsto u(t, u^0)$ is a semiflow on $W^s_{p,\mathcal{B}}(\Omega, D)$, and bounded orbits which are bounded away from ∂D are relatively compact. If there exists $\rho \in (n/p, s)$, where $\rho > 1$ if (14.24) is satisfied, such that $u(\cdot, u^0)|[0,T]$ is bounded in W^ρ_p and bounded away from ∂D for each $T > 0$ then $t^+(u^0) = \infty$.

To see that, indeed, ρ can be chosen arbitrarily close to n/p if (14.23) is satisfied, it suffices to choose $s \leq 1$, so that (14.19) is satisfied with $r < \rho < s$.

Throughout the remainder of this section

> we presuppose the hypotheses of Theorem 14.4, fix $u^0 \in W^s_{p,\mathcal{B}}(\Omega, D)$, and put $u := u(\cdot, u^0)$ and $J := [0, t^+)$ with $t^+ := t^+(u^0)$.

We shall now show that u has better regularity properties on $\dot{J}$ than stated above. First note that

$$u \in C\big(\dot{J}, W^\sigma_{p,\mathcal{B}}(\Omega, D)\big) \cap C^1(\dot{J}, W^{\sigma-2}_{p,\mathcal{B}}) \tag{14.25}$$

and that u (whence J) is independent of the particular choice of σ. Thus, by replacing u^0 by $u(t_0)$ for any $t_0 \in \dot{J}$ and t by $t - t_0$, respectively, we can assume that $u^0 \in W^\sigma_{p,\mathcal{B}}(\Omega, D)$. Observe that we can choose r arbitrarily close to 1. Consequently, we can place σ arbitrarily close to $(1+1/p) \wedge (2-n/p)$. Thus we deduce from (14.25) that

$$u \in C(\dot{J}, W^{s_1}_{p_1})\ , \qquad \frac{1}{p} \geq \frac{1}{p_1} > \frac{1}{p} - \frac{(1-n/p) \wedge 1/p}{n} > 0\ , \tag{14.26}$$

provided s_1 is chosen sufficiently close to 1. Similarly as above, we can now assume that $u^0 \in W^{s_1}_{p_1}$ and apply the preceding argument with p replaced by $p_1 > p$. Then we obtain (14.26) with s_1 replaced by some $s_2 > 1$ and p [resp. p_1] replaced by p_1 [resp. p_2]. By repeating this argument a finite number of times we arrive at a $p_k \geq n+1$. Hence, being interested in $u|(0, t^+)$ only, we can assume that $p \geq n+1$ so that $(1-1/p) \wedge 1/p = 1/p$. Then we obtain from (14.26) that $u \in C(\dot{J}, W^{s_1}_{p_1})$, provided $1/p \geq 1/p_1 > (1-1/n)/p$. Thus, given any $k \in \dot{\mathbb{N}}$, we find iteratively that $u \in C(\dot{J}, W^{s_k}_{p_k})$ for some $s_k \in (1, 1+1/p_k)$, provided $1/p \geq 1/p_k > (1-1/n)^k/p$. From this we infer that $u \in C(\dot{J}, W^1_q)$ for each $q \in (1, \infty)$, which implies the following

14.5 Corollary *The function u is a weak W^1_q-solution of (14.7) for each $q \in (1, \infty)$.*

Suppose now that $g=0$. Then we shall show that u is a classical solution of (14.7). Since this concerns the open interval $\dot{J}$ only, we can assume, as above, that $u^0 \in W^{\sigma}_{p,\mathcal{B}}(\Omega, D)$. Thus

$$u \in C(J, W^{\sigma}_{p,\mathcal{B}}) \cap C^1(J, W^{\sigma-2}_{p,\mathcal{B}}) \ .$$

Hence it follows from Theorem 7.2 that

$$u \in C^{\rho}(J, W^{\sigma-2\rho}_{p,\mathcal{B}}) \ , \qquad 0 \le 2\rho \le 2 \ . \tag{14.27}$$

Since we can choose σ arbitrarily close to $1+1/p$ (recall that we can assume $p > n+1$), it follows from (14.13) that

$$\big(\mathcal{A}(u(\cdot)), \mathcal{B}(u(\cdot))\big) \in C^{\rho}\big(J, \mathcal{E}^{\alpha}(\Omega)\big) \ , \qquad 1 \le 2\alpha < 2 - 2\rho - (n-1)/p \ . \tag{14.28}$$

Choose $\alpha, \rho \in (0,1)$ such that

$$2 - 2\alpha - (n-1)/p > 2\rho > 2\alpha - 1 - 1/p > 0 \ , \tag{14.29}$$

which is easily seen to be possible.

From (14.14)–(14.16) and (14.27) we deduce that

$$F\big(u(\cdot)\big) \in C^{\rho}(J, W^{\tau}_{p,\mathcal{B}}) \ , \qquad \tau \in (-2+1/p, 0]\backslash \Sigma_p \ , \quad \tau < (2-n)/p - 4\rho \ .$$

Since, thanks to (14.29), $(2-n)/p - 4\rho > 2\alpha - 2 + 1/p$, it follows that

$$F\big(u(\cdot)\big) \in C^{\rho}(J, W^{2\alpha-2}_{p,\mathcal{B}}) \ . \tag{14.30}$$

Now we deduce from (14.28)–(14.30) and Theorem 11.3 that the linear parabolic boundary value problem

$$\begin{aligned} \partial_t v + \mathcal{A}\big(u(t)\big)v &= F\big(u(t)\big) && \text{in} \quad \Omega \times \dot{J} \ , \\ \mathcal{B}\big(u(t)\big)v &= 0 && \text{on } \partial\Omega \times \dot{J} \ , \\ v(\cdot, 0) &= u^0 && \text{on} \quad \Omega \end{aligned} \tag{14.31}$$

has a unique strong $W^{2\alpha}_p$-solution v. Since a strong $W^{2\alpha}_p$-solution of (14.31) is also a weak W^s_p-solution, since u is a weak W^s_p-solution of (14.31), and since there exists exactly one weak W^s_p-solution of (14.31), it follows that

$$u = v \in C(\dot{J}, W^{2\alpha}_{p,\mathcal{B}}) \cap C^1(\dot{J}, W^{2\alpha-2}_p) \ . \tag{14.32}$$

Similarly as above, we can assume that (14.32) is true with $\dot{J}$ replaced by J. Thus, using (5.22), we see that

$$u \in C^{\rho}(J, W^{2\alpha-2\rho}_p) \ , \qquad 0 \le 2\rho \le 2 \ .$$

Choose $\alpha_1 \in (0,1]$ and $\rho_1 \in (0,1)$ such that

$$1 + 2\alpha - 2\alpha_1 - n/p > 2\rho_1 > 2\alpha_1 - 1 - 1/p > 2\alpha - 1 - 1/p \ ,$$

which is possible. In fact, we can put

$$2\alpha := (1+1/p) + (1-n/p)/4 \ , \quad 2\alpha_1 := (1+1/p) + (1-n/p)/2 \ .$$

Then (14.13) and (14.14)–(14.16) imply

$$\big(\mathcal{A}(u(\cdot)),\mathcal{B}(u(\cdot))\big)\in C^{\rho_1}\big(J,\mathcal{E}^{\alpha_1}(\Omega)\big)$$

and

$$F\big(u(\cdot)\big)\in C^{\rho_1}\big(J,W_p^{2\alpha_1-2}\big)\ ,$$

provided $2\rho_1$ is chosen sufficiently close to $2\alpha_1-1-1/p=(1-n/p)/2$. Hence Theorem 13.3 guarantees — similarly as above — that

$$u\in(\dot{J},W_p^{2\alpha_1})\cap C^1(\dot{J},W_p^{2\alpha_1-2})\ ,$$

where $2\alpha_1-2\alpha=(1-n/p)/2$. By iterating this argument we find after finitely many steps that u is a strong W_p^2-solution of (14.31). Consequently,

$$u\in C^{\rho}(\dot{J},C^{1+\mu})\ ,\qquad 0\le 2\rho+\mu\le 1-n/p\ .$$

From this and from Remark 14.3 we deduce that u is a strong W_p^2-solution of a linear parabolic system having Hölder continuous coefficients and a Hölder continuous right hand side. Hence we can invoke the results of Solonnikov [Sol65] to show that u is a classical solution. This proves the following theorem and its corollary.

14.6 Theorem *Suppose that $g=0$. Then u is a classical solution of* (14.7)*, that is,*

$$u\in C(\overline{\Omega}\times J,D)\cap C^{2,1}(\overline{\Omega}\times\dot{J},\mathbb{R}^N)\ ,$$

and u satisfies (14.7) *pointwise.*

14.7 Corollary *If $g=0$, and f and all coefficients of $(\mathcal{A},\mathcal{B})$ are C^∞–smooth,*

$$u\in C^\infty(\overline{\Omega}\times\dot{J},\mathbb{R}^N)\ .$$

For simplicity we imposed in Theorem 14.6 and its corollary the assumption $g=0$, which, in view of Remark 14.1(b), is rather mild. For the general case as well as for better regularity conditions of u at $t=0$, provided u^0 has more regularity, we refer to [Ama94].

The above results are due to the author [Ama90a]. Earlier results concerning the solvability of quasilinear parabolic systems under nonlinear boundary conditions are contained in [Ama86c]. In the latter paper we have, however, neither been able to prove the uniqeness of the solution in general, nor to show that it generates a semiflow on the natural state space $W_{p,\mathcal{B}}^s(\Omega,D)$. In addition, we had to assume that $s>1+n/p$ even if f was affine in ∂u. More recently, Yagi [Yag91] has proven a local existence theorem for problem (14.7) with $g=0$. He works directly in the pair (L_p,W_p^2), but also uses the fact that the spaces $W_{p,\mathcal{B}(u)}^s$ are independent of u for $0\le s<1+1/p$. His result is comparable to the one of [Ama86c] and has the same shortcomings.

Problem (14.7), under homogeneous Dirichlet conditions and with f independent of ∂u, has also been studied in [Ves91]. This author uses extrapolation techniques too, but works with spaces of continuous and Hölder continuous functions (so that even in the case of Dirichlet boundary conditions the domains of the generators $A(u)$ are not constant). In both papers [Yag91],[Ves91] the case of a single equation is considered.

Lastly, there is a local existence result for quasilinear parabolic systems in [GiM87] which is based upon classical Schauder type a priori techniques. In that paper it is assumed that $\mathcal{A}$ satisfies a uniform Legendre condition and the given proofs heavily depend on this assumption (although the authors claim that they can handle strongly uniformly elliptic operators too). Anyhow, the techniques of [GiM87] are pure p.d.e. techniques and cannot be used to handle more general 'abstractly parabolic' problems.

The theory developed in this paper is an L_p-theory. It has the advantage that the basic function spaces, namely the spaces $W^s_{p,\mathcal{B}}$, on which the theory is based, are — from the viewpoint of functional analysis — simple spaces: they are reflexive and separable, and their duals are explicitly known spaces of the same sort. This can be useful in dealing with questions concerning the qualitative behavior of the semiflows generated by our problems, questions which we do not consider here and which involve more sophisticated techniques from nonlinear functional analysis. It also provides us with a simple duality theory in the linear case, of which we have already made extensive use in connection with very weak solutions. Moreover, the concept of a weak W^s_p-solution is a very natural one, which is the straightforward extension of the notion of weak solutions in the 'classical' L_2-case. Lastly, as shown above, there is a good regularity theory, which immediately implies that weak $W^1_{p,\mathcal{B}}$-solutions are in fact classical solutions. Most importantly, this regularity theory applies to general normally elliptic operators $(\mathcal{A}, \mathcal{B})$ without any additional structural conditions.

Of course, there is no unique choice for the functional setting in which a given problem has to be put. There are many other possibilities, like the use of spaces of continuous functions, Hölder spaces, general Besov spaces, etc. However, as explained above, the L_p-setting seems to be a natural one and, perhaps, the simplest one.

14.8 Remark The regularity assumptions (14.1) and (14.2) seem to be overly restrictive. In fact, it is not difficult to see that the ones for the 'lower order coefficients' can be weakened. As for the 'top order coefficients', we have to know that

$$\left[u \mapsto a_{jk}(u)\right] \in C^{1\text{-}}\left(X_\beta, H^{s_1}_p(\Omega, \mathcal{L}(\mathbb{R}^N))\right) ,$$

where $X_1 \subset W^{s_\beta}_p$. In order to guarantee this we have reduced this continuity problem to one between Hölder spaces and then applied Lemma 14.2(ii), which — thanks to Remark 14.3 — is optimal. Thus, in order to weaken the regularity assumptions for a_{jk}, one has to find rather mild conditions guaranteeing that Nemytskii operators are locally Lipschitz continuous between generalized Sobolev spaces. Although mapping properties of nonlinear operators in generalized Sobolev spaces have been studied intensively during the last few years (cf. [Run86], [Sic89], [Sic93], [Sic92], [Boud91], [BouK92], [BouM91], [Osw92]), the problem of the Lipschitz continuity (or even continuity) of these maps under optimal assumptions is still open. □

15 Positivity and Global Existence

Throughout this section we impose conditions (14.1)–(14.6) and consider the quasilinear parabolic boundary value problem (14.7). In addition,

> we presuppose the hypotheses of Theorem 14.4, fix $u^0 \in W^s_{p,\mathcal{B}}(\Omega, D)$, and put $u := u(\cdot, u^0)$ and $J := [0, t^+)$ with $t^+ := t^+(u^0)$.

In many applications — for example, in some of the model problems of Section 1 — it is of importance to know that the solution (or some components of it) of problem (14.7) stays positive for $t \in \dot{J}$, provided the (corresponding component of the) initial value is positive. Although there is no maximum principle for general strongly coupled systems, it is easy to give sufficient conditions for this 'positive invariance of the positive cone'.

Let $r \in \{1, \ldots, N\}$ be fixed and put

$$\widehat{\eta}_r := (\eta_1, \ldots, \eta_{r-1}, \eta_{r+1}, \ldots, \eta_N) \ , \quad (\widehat{\eta}_r, \eta_r) := \eta = (\eta_1, \ldots, \eta_N) \ .$$

Suppose that

$$\eta \in D \quad \Longrightarrow \quad (\widehat{\eta}_r, t\eta_r) \in D_0 \ , \quad 0 \le t \le 1 \ , \tag{15.1}$$

and that

$$\begin{aligned} &a^{rs}_{jk},\ a^{rs}_j,\ b^{rs}_j,\ a^{rs}_0,\ c^{rs}_r \text{ vanish on} \\ &D_r := \left\{ (\widehat{\eta}_r, 0) \in D_0 \ ; \ \exists\ \tau > 0 : (\widehat{\eta}_r, \tau) \in D \right\} \\ &\text{for } 1 \le j,k \le n \text{ and } 1 \le s \le N \text{ with } s \ne r. \end{aligned} \tag{15.2}$$

Lastly, suppose that

$$\begin{aligned} &f_r \in C^1(\overline{\Omega} \times D_0, \mathbb{R}),\ \ g \in C^1(\partial\Omega \times D_0, \mathbb{R}) \text{ with } (1-\delta)g = 0, \\ &\text{and that} \\ &f_r\big(\cdot, (\widehat{\eta}_r, 0)\big) \ge 0,\ \ g_r\big(\cdot, (\widehat{\eta}_r, 0)\big) \ge 0,\ \ (\widehat{\eta}_r, 0) \in D_r. \end{aligned} \tag{15.3}$$

Then u_r is a weak W^1_p-solution of the linear parabolic evolution system $(N = 1)$

$$\begin{aligned} \partial_t v + \mathcal{A}_r(t) v &= f^0_r(t) && \text{in } \ \Omega \times \dot{J} \ , \\ \mathcal{B}_r(t) v &= g^0_r(t) && \text{on } \partial\Omega \times \dot{J} \ , \\ v(\cdot, 0) &= u^0_r && \text{on } \ \Omega \ , \end{aligned}$$

where

$$\big((\mathcal{A}_r, \mathcal{B}_r), f^0_r, g^0_r\big) \in C^\rho\big(J, \mathcal{E}_p(\Omega) \times W^{-1}_{p,\mathcal{B}} \times \partial W^1_p\big)$$

for a suitable $\rho \in (0,1)$, provided $u^0 \in W^\sigma_p$. In fact, denoting the coefficients of $(\mathcal{A}, \mathcal{B})$ by $(\alpha_{jk}, \alpha_j, \beta_j, \alpha_0, \gamma)$, it follows — using an obvious simplified notation — that

$$\alpha_{jk}(t) := a^{rr}_{jk}\big(u(t)\big) \ , \quad \beta_j(t) := b^{rr}_j\big(u(t)\big) \ ,$$

and

$$\alpha_j(t) := a^{rr}_j\big(u(t)\big) + \sum_{s \ne r} \big[\widetilde{a}^{rs}_{jk}\big(u(t)\big)\partial_k u_s(t) + \widetilde{a}^{rs}_j\big(u(t)\big) u_s(t)\big] \ ,$$

and that

$$\alpha_0(t) := a_0^{rr}\big(u(t)\big) + \sum_{s\neq r}\big[\widetilde{b}_j^{rs}\big(u(t)\big)\partial_j u_s(t) + \widetilde{a}_0^{rs}\big(u(t)\big)u_s(t)\big] - \widetilde{f}_r\big(u(t)\big)$$

and

$$\gamma(t) := c^{rr}\big(u(t)\big) + \sum_{s\neq r}\widetilde{c}^{rs}\big(\gamma_\partial u(t)\big)\gamma_\partial u_s(t) - \widetilde{g}_r\big(\gamma_\partial u(t)\big) \ ,$$

with

$$\widetilde{d}(\eta) := \int_0^1 \partial_{\eta_r} d\big(\cdot,(\widehat{\eta}_r, t\eta_r)\big)\,dt \ , \qquad \eta \in D \ ,$$

for $d \in \{\, a_{jk}^{rs}, a_j^{rs}, b_j^{rs}, a_0^{rs}, c^{rs}, f_r, g_r \ ; \ s \neq r \,\}$, and where

$$f_r^0(t) := f_r\big((\widehat{u}_r(t),0)\big) \ , \quad g_r^0(t) := g_r\big(\gamma_\partial(\widehat{u}_r(t),0)\big)$$

for $t \in J$. Hence we deduce from (15.3) and Remark 11.7 that $u_r^0 \geq 0$ implies $u_r(t) \geq 0$ for $t \in J$, provided $u^0 \in W_p^\sigma$. Since, by Theorem 8.7, the positive cone of $W_{p,\mathcal{B}_r}^\sigma$ is dense in the positive cone of $W_{p,\mathcal{B}_r}^s$, given $u^0 \in W_{p,\mathcal{B}}^s$ with $u_r^0 \geq 0$, we can find a sequence $\big(u_{(j)}^0\big)$ in $W_{p,\mathcal{B}}^\sigma$ satisfying $u_{(j),r}^0 \geq 0$ such that $u_{(j)}^0 \to u^0$ in $W_{p,\mathcal{B}}^s(\Omega, D)$ as $j \to \infty$. Consequently, given any $t \in J$, we see that $u\big(t, u_{(j)}^0\big) \to u(t,u^0)$ in $W_{p,\mathcal{B}}^s(\Omega,D)$ as $j \to \infty$, thanks to the fact that $(t,u^0) \mapsto u(t,u^0)$ is a semiflow on $W_{p,\mathcal{B}}^s(\Omega,D)$. Now the following theorem is obvious.

15.1 Theorem *Let conditions* (15.1)–(15.3) *be satisfied for some* $r \in \{1,\ldots,N\}$ *and let* $u_r^0 \geq 0$. *Then* $u_r(t) \geq 0$ *for* $t \in J$.

15.2 Remarks

(a) For simplicity, we have assumed in the above theorem that f is independent of the gradient. It is easy to extend it to the gradient dependent case too.

(b) Of course, the assumption that f_r be C^1 can be replaced by: there exists $\widetilde{f}_r \in C^{1\text{-}}(\overline{\Omega} \times D, \mathbb{R})$ such that

$$f_r(\cdot,\eta) = \widetilde{f}_r(\cdot,\eta)\eta_r + f_r\big(\cdot,(\widehat{\eta}_r,0)\big) \ , \qquad \eta \in D \ .$$

A similar remark applies to g_r. □

The above theorem occurs here for the first time in print although the basic idea for its proof is already contained in [Ama90a] and [Ama89b].

We are now interested in the global existence of u. Since this problem concerns the behavior of u near t^+ only, we can assume — thanks to Corollary 14.5 — that p is arbitrarily large. Using this fact and $C^\varepsilon \hookrightarrow W_p^\rho$ for $\varepsilon > \rho$, the following theorem is an immediate consequence of Theorem 14.4.

Here and in the following, we use c to denote constants, which may differ from occurrence to occurrence. Similar use is made of functions $c(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$.

15.3 Theorem *Suppose that f is independent of the gradient. Also suppose that $u|[0,T]$ is bounded away from ∂D for each $T>0$. If there exists $\varepsilon>0$ such that*

$$\|u(t)\|_{C^\varepsilon} \le c(T)\ , \qquad 0 \le t \le T < \infty\ , \quad t < t^+\ , \tag{15.4}$$

then $t^+ = \infty$.

Note that (15.4) means that

$$u \in B\big([0,T] \cap J, C^\varepsilon(\overline{\Omega}, \mathbb{R}^N)\big)\ , \qquad T > 0\ , \tag{15.5}$$

for some $\varepsilon > 0$. We shall now show that this condition can be replaced by

$$u \in BUC^\varepsilon\big([0,T] \cap J, C^\rho(\overline{\Omega}, \mathbb{R}^N)\big)\ , \qquad T > 0\ , \tag{15.6}$$

even if f is gradient dependent, where $\rho = 0$ if f is independent of ∂u, and $\rho > n/(n+1)$ otherwise. For this we suppose first that f is independent of ∂u and that u is bounded away from ∂D on any finite interval. Then, if (15.6) is satisfied and $T>0$ is fixed, there exists a unique

$$(\overline{u}, \overline{f}, \overline{g}) \in C^\varepsilon\big([0, T \wedge t^+], C(\overline{\Omega}, \mathbb{R}^N) \times C(\overline{\Omega}, \mathbb{R}^N) \times C(\partial\Omega, \mathbb{R}^N)\big)$$

such that

$$(\overline{u}, \overline{f}, \overline{g})(t) = \big(u(t), f(u(t)), g(\gamma_\partial u(t))\big)\ , \qquad t \in [0,T] \cap J\ .$$

Consequently,

$$(\overline{\mathcal{A}}, \overline{\mathcal{B}}) := \big(\mathcal{A}(\overline{u}(\cdot)), \mathcal{B}(\overline{u}(\cdot))\big) \in C^\varepsilon\big([0, T \wedge t^+], \mathcal{E}^{1/2}(\Omega)\big)\ ,$$

and Theorem 11.2 implies that the linear parabolic boundary value problem

$$\begin{aligned} \partial_t v + \overline{\mathcal{A}}(t) v &= \overline{f}(t) && \text{in} \quad \Omega \times (0, T \wedge t^+]\ , \\ \overline{\mathcal{B}}(t) v &= \overline{g}(t) && \text{on } \partial\Omega \times (0, T \wedge t^+]\ , \\ v(\cdot, 0) &= u^0 && \text{on} \quad \Omega \end{aligned} \tag{15.7}$$

has a unique weak W_p^1-solution $\overline{u}$. Since u is such a solution of (15.7) on $[0,T] \cap J$, it follows that $\overline{u} \supset u$. Hence

$$u \in BUC\big([0,T] \cap J, W^1_{p,\mathcal{B}}\big)\ ,$$

which shows that $u|[0,T]$ is bounded in W_p^1 for each $T>0$. Hence $t^+ = \infty$ by Theorem 14.4. A similar, slightly more sophisticated argument applies if f is gradient dependent. More precisely, we have the following global existence

15.4 Theorem *Suppose that there exists $\varepsilon > 0$ such that (15.6) is true, where*

$$\rho \begin{cases} = 0 & \textit{if } f \textit{ is independent of } \partial u\ , \\ > 0 & \textit{if (13.12) is satisfied and } g = 0\ , \\ > n/(n+1) & \textit{otherwise}\ . \end{cases}$$

Then $t^+ = \infty$, provided $u|[0,T]$ is bounded away from ∂D for each $T > 0$. If, in addition,

$$u \in BUC^\varepsilon\big(J, C^\rho(\overline{\Omega}, \mathbb{R}^N)\big) \tag{15.8}$$

and if the orbit $\gamma^+(u^0)$ is relatively compact in $C(\overline{\Omega}, D)$ then $\gamma^+(u^0)$ is relatively compact in $W^s_{p,\mathcal{B}}(\Omega, D)$.

These global existence theorems are due to the author. Theorem 15.3 is given here explicitly for the first time, and Theorem 15.4 is contained in [Ama89a] (where, in fact, it is shown that condition (15.8) implies the boundedness of u in 'higher norms').

The above theorems reduce the problem of global existence to the problem of establishing suitable a priori bounds. Note that (15.6) is satisfied, if

$$\|u\|_{C^{\rho,\varepsilon}(\overline{\Omega}\times[0,t],\mathbb{R}^N)} \leq c(T) , \qquad 0 \leq t \leq T < \infty , \quad t < t^+ . \tag{15.9}$$

Thus (15.9) and — of course — (15.4) are trivially satisfied, if there exists $\varepsilon \in (0,1)$ such that

$$\|u\|_{C^{\varepsilon,\varepsilon/2}(\overline{\Omega}\times[0,t],\mathbb{R}^N)} \leq c(T) , \qquad 0 \leq t \leq T < \infty , \quad t < t^+ , \tag{15.10}$$

where $\varepsilon > n/(n+1)$ if f is gradient dependent. Recall that $C^{\varepsilon,\varepsilon/2}(Q_T)$ is the space of ε-Hölder continuous functions on $Q_T := \overline{\Omega} \times [0,T]$, provided Q_T is given the 'parabolic metric'

$$\big((x,s),(y,t)\big) \mapsto \sqrt{|x-y|^2 + |s-t|} .$$

These spaces are usually employed in connection with parabolic equations (e.g., [LadSU68]).

It is an important open question whether the Hölder norms occurring in (15.4) and (15.8), respectively, can be replaced by a weaker norm, in particular, by the supremum norm. More precisely, we can formulate the following **a priori bound problem**:

> Suppose that
> $\|u(t)\|_\infty \leq c_0(T) , \qquad 0 \leq t \leq T , \quad t < t^+ .$
> Do there exist $\varepsilon, \rho \in (0,1)$ such that (15.4) or (15.9) are true,
> where $\rho > n/(n+1)$ if f is gradient dependent?

Of course, a positive answer to this problem guarantees that $t^+ = \infty$, provided $u|[0,T]$ is bounded away from ∂D for each $T > 0$.

So far there are only a few cases for which a positive answer to the a priori bound problem has been found. First, if $N = 1$, this is well known to be the case (e.g., [LadSU68]). For systems the situation is much more complicated. The only result known to the author, which applies to strongly coupled systems with 'fully occupied diffusion matrices', is due to Wiegner [Wie92]. He has to assume that $\mathcal{A}$ satisfies a uniform Legendre condition in addition to further structural hypotheses. Then he proves that (15.9) is true for any $\varepsilon \in (0,1)$.

The situation is much better if we restrict ourselves to triangular systems. To be precise, $(\mathcal{A},\mathcal{B})$ is said to be a (lower) **triangular system** if each one of the matrices a_{jk}, a_j, b_j, a_0, and c is lower triangular, that is, all entries above the diagonal are zero. An obvious modification of the proof of [Ama89a, Theorems 5.2 and 6.2] gives a positive answer to the a priori bound problem, provided assumptions (15.11) or (15.12) below are satisfied. Thus we obtain the following theorem which, in fact, is true for more general 'block triangular systems' (cf. [Ama94]).

15.5 Theorem *Suppose that $(\mathcal{A},\mathcal{B})$ is lower triangular and that either*

$$\begin{aligned} &\textit{each } a_{jk} \textit{ is a diagonal matrix and} \\ &|f_r(u,\partial u)| \leq c(|u|)(1 + |\nabla u_r|^\lambda) , \qquad 1 \leq r \leq N , \end{aligned} \tag{15.11}$$

or

$$\begin{aligned}&\text{the } r\text{-th row of each } a_{jk} \text{ is independent of } u_s \text{ for } s>r, \text{ and}\\ &|f_r(u,\partial u)| \le c(|u|)\big(1+|\nabla u_r|^\lambda + \textstyle\sum_{s<r}|\nabla u_s|^2\big)\ , \qquad 1\le r\le N\ ,\end{aligned} \tag{15.12}$$

where $\lambda = 2$ if condition (13.12) is satisfied and $g = 0$, and $\lambda = 1$ otherwise. Also suppose that

$$\|u(t)\|_\infty \le c(T)\ , \qquad 0\le t\le T<\infty\ , \quad t<t^+\ . \tag{15.13}$$

Then $t^+ = \infty$ if $u|[0,T]$ is bounded away from ∂D for each $T>0$. If c in (15.13) can be chosen to be independent of $T \in \dot{\mathbb{R}}^+$, and if the orbit $\gamma^+(u^0)$ is bounded away from ∂D then $\gamma^+(u^0)$ is relatively compact in $W^s_{p,\mathcal{B}}(\Omega, D)$.

The above theorem has been proven in [Ama89a]. It generalizes, to a certain extent, considerably earlier results of Ladyženskaja and Ural'ceva [LadU65], [LadU66], [LadU67] (also cf. [LadSU68, Theorem VII.7.1] and of Giaquinta and Struwe [GiS81]. These authors considered Dirichlet boundary conditions ($\delta = 0$) and diagonal systems with one and the same strongly uniformly elliptic operator in each row, that is, they assume that $a^{rs}_{jk} = 0$ for $s \neq r$ and $a^{rr}_{jk} = a^{11}_{jk}$ for $1 \le r \le N$. They impose a growth restriction of the form

$$|f(u,\partial u)| \le \alpha\,|\partial u|^2 + \beta\ ,$$

involving the gradient of the full vector u, and not only of u_r as we do, but they have to assume that α is sufficiently small. In fact, the size of α is not specified in the work of Ladyženskaja and Ural'ceva, whereas in [GiS81] it is assumed that

$$\alpha \sup_{Q_T} |u| < \underline{\alpha}\ ,$$

where $\underline{\alpha}$ is the constant of ellipticity. Given these conditions, those authors then show that bounded weak W^1_2-solutions are uniformly Hölder continuous (in x and t).

Of course, if $(\mathcal{A}, \mathcal{B})$ is independent of u, that is, in the semilinear case, a positive answer to the a priori bound problem can easily be obtained from the variation of constants formula. In this case the L_∞-norm can often be replaced by weaker L_q-norms (cf. [Ama85], [Red87], [Red89]. We also refer to [Rhe84] as a protoype for the vast literature dealing with 'weakly coupled' semilinear systems, that is, semilinear systems with diagonal operators $(\mathcal{A}, \mathcal{B})$).

It should be observed that the a priori bound problem is *not* a regularity problem. In fact, it follows from (14.27) that

$$u \in C(\dot{J}, C^\rho) \cap C^{\rho/2}(\dot{J}, C)$$

with $\rho := 1 - n/p$. Since we can choose p arbitrarily large, we see that

$$u \in C^{\rho,\rho/2}(\overline{\Omega} \times \dot{J}, \mathbb{R}^N)\ , \qquad 0<\rho<1\ . \tag{15.14}$$

This regularity result shows the difference between our method and the 'classical' approach to quasilinear parabolic systems. The classical method proceeds by establishing — usually by a Galerkin method or by means of the theory of monotone operators — a weak W^1_2-solution.

Then one tries to prove 'regularity results' to show that theses weak solutions are Hölder continuous and — furthermore — classical . However, it is known that weak W_2^1-solutions of *systems* are, in general, not Hölder continuous (e.g., [Gia83] and note that weak W_2^1-solutions to elliptic systems are stationary weak W_2^1-solutions to the corresponding parabolic systems). Our approach avoids this difficulty since we work from the very beginning with weak W_p^1-solutions with $p > n$, which implies the Hölder regularity result (15.14), as we have seen.

As already mentioned earlier, the Galerkin approach cannot work in our case since — even in the W_2^1-case — the Dirichlet form associated with $(\mathcal{A}, \mathcal{B})$ is not coercive, in general. Thus Theorem 14.4 cannot be proven by the 'classical' method, except rather restrictive hypotheses are imposed. The theory of monotone operators also applies to a rather limited class of problems only.

There are examples of parabolic systems possessing weak W_2^1-solutions which are smooth for small t and develop a discontinuity for some $\tau \in (0, t^+)$ (cf. [Str84], [StJM86]). However, in these examples the operators and the nonlinearities do not satisfy our regularity assumptions. Hence they do not provide negative answers to the a priori bound problem.

Lastly, it should be mentioned that it is by no means trivial to find L_∞-a priori bounds for a given solution in the case of systems. This is mainly due to the fact that there is no good maximum principle for systems. The only result in this direction, which applies to a whole (though rather restricted) class of strongly coupled quasilinear parabolic problems with 'fully occupied diffusion matrices', seems to be [Wie92, Theorem 2]. For triangular systems it is in certain cases possible to get L_∞-a priori bounds, provided suitable growth restrictions are satisfied. We refrain from giving details and refer to [Ama89c] and [Ama94].

16 Strong and Very Weak Solutions

We turn now briefly to the case of 'constant boundary operators' and strong solutions. In this case we do not assume that $(\mathcal{A}, \mathcal{B})$ is of divergence form. More precisely, we assume that

$$D \text{ is open in } \mathbb{R}^N \times \mathbb{R}^{N\times n} \tag{16.1}$$

and

$$a_{jk} \in C^{2-}\big(\overline{\Omega} \times D, \mathcal{L}(\mathbb{R}^N)\big) , \qquad 1 \le j,k \le n . \tag{16.2}$$

We put

$$\mathcal{A}(\eta)u := -a_{jk}(\cdot, \eta)\partial_j\partial_k u , \qquad \eta \in D .$$

We also assume that

$$b_j \in C^1\big(\partial\Omega, \mathcal{L}(\mathbb{R}^N)\big) , \quad b_0 \in C^2\big(\partial\Omega, \mathcal{L}(\mathbb{R}^N)\big) \tag{16.3}$$

and put

$$\mathcal{B}u := b_j\gamma_\partial\partial_j u + b_0\gamma_\partial u . \tag{16.4}$$

Then we suppose that

$$\big(\mathcal{A}(\eta), \mathcal{B}\big) \text{ is normally elliptic for } \eta \in D . \tag{16.5}$$

Note that, thanks to Example 4.3(c), this assumption is satisfied, for example, if $\mathcal{A}(\eta)$ is uniformly strongly elliptic and $\mathcal{B} = \gamma_\partial$, the Dirichlet boundary operator.

Lastly, we suppose that

$$f \in C^{2\text{-}}(\overline{\Omega} \times D, \mathbb{R}^N) \tag{16.6}$$

and consider the quasilinear parabolic boundary value problem

$$\begin{aligned} \partial_t u + \mathcal{A}(u, \partial u)u &= f(\cdot, u, \partial u) && \text{in } \ \Omega \times (0,\infty)\ , \\ \mathcal{B}u &= 0 && \text{on } \partial\Omega \times (0,\infty)\ , \\ u(\cdot,0) &= u^0 && \text{on } \ \Omega\ , \end{aligned} \tag{16.7}$$

where we use an obvious notation.

We fix $p \in (n, \infty)$ and numbers

$$0 < \tau < 1/p < 1 + n/p < r < s < 1\ , \quad 0 < \tau < r - n/p\ ,$$

and put

$$\alpha := s/2\ , \quad \beta := r/2\ , \quad \gamma := \tau/2\ ,$$

and

$$E_\xi := W_{p,\mathcal{B}}^{2\xi}\ , \qquad \xi \in \{1, \alpha, \beta, \gamma, 0\}\ .$$

We also put

$$X_\beta := W_{p,\mathcal{B}}^r(\Omega, D) := \big\{\, u \in W_{p,\mathcal{B}}^r\ ;\ (u, \partial u)(\overline{\Omega}) \subset D \,\big\}\ .$$

Then X_β is an open subset of E_β, which we assume to be nonempty. Given $u \in X_\beta$, we denote by $A(u)$ the L_p-realization of $\big(\mathcal{A}(u,\partial u), \mathcal{B}\big)$ and put $F(u) := f(\cdot, u, \partial u)$. It is not difficult to see that

$$(A, F) \in C^{1\text{-}}\big(X_\beta, \mathcal{H}(E_1, E_0) \times E_\gamma\big)$$

and that the Cauchy problem

$$\dot{u} + A(u)u = F(u)\ , \quad 0 < t < \infty\ , \qquad u(0) = u^0$$

is the strong W_p^2-formulation of problem (16.7).

The following theorem is now an easy consequence of Theorems 7.1, 12.1, and 12.3 combined with the fact that we can apply Solonnikov's 'classical theory' [Sol65] to the linear boundary value problem which is obtained from (16.7) by substituting the strong solution $u(\cdot, u^0)$ into $\mathcal{A}(\cdot)$ and $F(\cdot)$.

16.1 Theorem *Let assumptions* (16.1)–(16.6) *be satisfied and suppose that*

$$1 + n/p < r < s < 2\ .$$

Then, given $u^0 \in W_{p,\mathcal{B}}^s(\Omega, D)$, the quasilinear parabolic boundary value problem (16.7) *has a unique maximal classical solution*

$$u(\cdot, u^0) \in C\big(\overline{\Omega} \times [0, t^+(u^0)), D\big) \cap C^{2,1}\big(\overline{\Omega} \times (0, t^+(u^0)), \mathbb{R}^N\big)\ .$$

The map $(t, u^0) \mapsto u(t, u^0)$ is a semiflow on $W_{p,\mathcal{B}}^s(\Omega, D)$. Bounded orbits, which are bounded away from ∂D, are relatively compact. If there exists $\rho \in (r, s)$ such that $u(\cdot, u^0)|[0,T]$ is bounded in W_p^ρ and bounded away from ∂D for each $T > 0$ then $u(\cdot, u^0)$ is a global solution.

16.2 Remarks

(a) Suppose that a_{jk} is independent of ∂u and f is affine in ∂u, that is, f is of the form

$$f(\cdot,u,\partial u)=f^0(\cdot,u)+a_j(\cdot,u)\partial_j u\ ,$$

where now D is open in $\mathbb{R}^N$ and

$$f^0\in C^{1\text{-}}(\overline{\Omega}\times D,\mathbb{R}^N)\ ,\quad a_j\in C^{2\text{-}}\big(\overline{\Omega}\times D,\mathcal{L}(\mathbb{R}^N)\big)\ .$$

Then, by replacing $\mathcal{A}$ in the obvious way, it is clear that it suffices to assume that $r>n/p$ with $r,s\neq 1+1/p$.

(b) It is clear how to apply the abstract results of Section 12 to the above case in order to deduce further information.

(c) Given suitable growth restrictions for f, the global existence of $u(\cdot,u^0)$ can be guaranteed provided an a priori bound in some weaker norm than the one of W_p^ρ is known. This follows by arguments which are similar to the ones of Section 15. □

Theorem 16.1 generalizes and simplifies the main result of [Ama86b], except that we restrict ourselves here to the second order case. It is, however, clear from Remark 4.2(b) that Theorem 16.1 extends to the higher order case too.

Lastly, we consider the semilinear parabolic boundary value problem (13.18) with local nonlinearities. More precisely, we restrict ourselves to the simplest case, namely to the problem

$$\begin{aligned}\partial_t u+\mathcal{A}u&=f(\cdot,u) &&\text{in}\quad \Omega\times(0,\infty)\ ,\\ \mathcal{B}u&=0 &&\text{on}\ \partial\Omega\times(0,\infty)\ ,\\ u(\cdot,0)&=u^0 &&\text{on}\quad \Omega\ ,\end{aligned}\tag{16.8}$$

where we assume that $(\mathcal{A},\mathcal{B})\in\mathcal{E}^1(\Omega)$. This problem has been studied by many authors in great detail under various aspects (e.g., [Hen81], [DaK92], [Rhe84], [vW85a] to mention just a few general references which, however, almost exclusively consider the case of a single equation (i.e., $N=1$) or of weakly coupled systems. For the case of strongly coupled systems we refer to [Ama84], [Ama85]). For this reason we discuss here one point only, namely the solvability of (16.8) on L_p. Of course, it is well known that (16.8) generates a semiflow on L_p provided F, the Nemytskii operator induced by f, maps L_p locally Lipschitz continuously into itself. It follows from Lemma 14.2 that this is the case if $\partial_2 f$ is uniformly bounded on $\overline{\Omega}\times\mathbb{R}^N$, which implies that f is linearly bounded. In fact, it is known (e.g., [Kra64], [Vai64]) that the linear boundedness of f is necessary for F to map L_p into itself. However, we can get a much better result by studying very weak solutions of (16.8).

Suppose that

$$f\in C^1(\overline{\Omega}\times\mathbb{R}^N,\mathbb{R}^N)\ .\tag{16.9}$$

Also suppose that

$$1<p<\infty\quad\text{and}\quad 1<\nu<1+2p/n\ ,\tag{16.10}$$

and there exists a constsnt c such that

$$|\partial_2 f(x,\xi)|\le c(1+|\xi|^{\nu-1})\ ,\qquad (x,\xi)\in\overline{\Omega}\times\mathbb{R}^N\ .\tag{16.11}$$

Fix $\tau \in (0,2)$ such that $\nu \le 1 + \tau p/n$ and note that $W^{\tau}_{p',\mathcal{B}^\#} \hookrightarrow L_{q'}$ so that $L_q \hookrightarrow W^{-\tau}_{p',\mathcal{B}^\#}$, provided $1/q = (1/p) + (\tau/n)$. Thus $\nu - 1 \le \tau p/n = p/q$ and, thanks to Lemma 14.2(ii),

$$F \in C^{1\text{-}}(L_p, L_q) \hookrightarrow C^{1\text{-}}(L_p, W^{-\tau}_{p',\mathcal{B}^\#}) \ .$$

Now we deduce from Theorem 13.8 and a standard boot strapping argument based upon the results of Section 11 and of [Sol65] the following

16.3 Theorem *Suppose that* $(\mathcal{A},\mathcal{B}) \in \mathcal{E}^1(\Omega)$ *and let conditions* (16.9)–(16.11) *be satisfied. Then the semilinear parabolic boundary value problem* (16.8) *possesses for each* $u^0 \in L_p$ *a unique maximal solution*

$$u(\cdot,u^0) \in C\big([0,t^+),L_p\big) \cap C^{2,1}\big(\overline{\Omega} \times (0,t^+),\mathbb{R}^N\big) \ .$$

The map $(t,u^0) \mapsto u(t,u^0)$ *is a semiflow on* L_p*. If* $u|[0,T]$ *is bounded in* L_p *for each* $T > 0$ *then* $t^+ = \infty$*.*

The last part of the assertion follows easily from [Ama84, Corollary 3.2], for example.

Theorem 16.3 is a vast generalization of earlier results of Weissler [Wei80], who studies the case $N = 1$ with $(\mathcal{A},\mathcal{B}) = (-\Delta, \gamma_\partial)$ and with $f(x,\xi) := |\xi|^{\nu-1}\,\xi$. However, that author admits the case $\nu = 1 + 2p/n$ too and also shows that this exponent is optimal.

Theorem 16.3 is included here mainly to give another illustration of the usefulness and simplicity of our extrapolation technique. Of course, it applies to a great many other situations too.

16.4 Remarks

(a) It is clear that the solution of Theorems 16.1 and 16.3 are smooth, that is, C^∞, on $\overline{\Omega} \times (0,t^+)$ if all data are smooth.

(b) By employing Remark 12.2(a) it is not difficult to see that all results of Sections 13–16 are valid — modulo obvious modifications — for nonautonomous quasilinear parabolic systems. Moreover, the solutions depend continuously (or smoothly, if all data are smooth) in the W^s_p-topology on parameters (e.g., the coefficients etc.). □

17 The Model Problems Revisited

As already mentioned in Section 1, we indicate now briefly how the general theory described in this paper applies to our model problems. Unless explicitly stated otherwise, all data are supposed to be smooth.

17.1 The Shigesada-Kawasaki-Teramoto Model In order to guarantee the applicability of our theory to problem (1.11), and since we are interested in positive solutions only, we have to verify that there exists a neigborhood D of $(\mathbb{R}^2)^+$ in $\mathbb{R}^2$ such that

$$\sigma\big(a(\eta)\big) \subset [\operatorname{Re} z > 0] \ , \qquad \eta \in D \ , \tag{17.1}$$

where a is given by (1.12) (cf. Example 4.3(e)). Note that (17.1) is equivalent to

$$\operatorname{trace}\bigl(a(\eta)\bigr) > 0 \quad \text{and} \quad \det\bigl(a(\eta)\bigr) > 0 \ , \qquad \eta \in D \ .$$

Since $\alpha_r > 0$ and $\beta_{rs} \geq 0$, it is obvious that $\operatorname{trace}\bigl(a(\eta)\bigr) > 0$ for η in some neighborhood D of $(\mathbb{R}^2)^+$. It is easily verified that

$$\det\bigl(a(\eta)\bigr) = \alpha_1\alpha_2 + p_2(\eta) \ , \qquad \eta \in \mathbb{R}^2 \ ,$$

where p_2 is a polynomial of degree 2 in $\eta = (\eta_1, \eta_2)$ with nonnegative coefficients. Hence we can find an open neighborhood D of $(\mathbb{R}^2)^+$ such that (17.1) is true and such that ∂D is bounded away from $(\mathbb{R}^2)^+$.

Now the following theorem is an immediate consequence of Theorems 14.4, 14.6, 15.1, 15.3, and 15.5, where Theorem 15.1 guarantees the positive invariance of the positive cone $(W_p^1)^+$ of W_p^1.

17.1 Theorem *Let $p > n$ $(= 2)$ be fixed. The population dynamical model* (1.11) *has for each $u^0 \in (W_p^1)^+$ a unique maximal classical solution $u(\cdot, u^0)$, which is smooth on $\overline{\Omega} \times \bigl(0, t^+(u^0)\bigr)$. The map $(t, u^0) \mapsto u(t, u^0)$ is a semiflow on $(W_p^1)^+$ such that bounded orbits are relatively compact. If there exist $\varepsilon > 0$ and $c \colon \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\|u(t, u^0)\|_{C^\varepsilon} \leq c(T) \ , \qquad 0 \leq t \leq T < \infty \ , \quad t < t^+(u^0) \ , \tag{17.2}$$

or if $\beta_{12}\beta_{21} = 0$ and (17.2) *is true with $\varepsilon = 0$, then $t^+(u^0) = \infty$.*

Note that (1.11) generates for each s with $n/p < s < (1 + 1/p) \wedge (2 - n/p)$ a semiflow on $W_p^s(\Omega, D)$, and that Theorem 15.1 implies that $(W_p^s)^+$ is positively invariant. Moreover, Theorem 17.1 remains true if the coefficients of $\vec{\jmath}_r$ depend smoothly on $x \in \overline{\Omega}$, provided the positivity assumptions are preserved.

17.2 Remark Problem (1.11) can be used to illustrate the importance of the fact that we can handle normally elliptic boundary value problems $(\mathcal{A}, \mathcal{B})$. Indeed, suppose that we were able to treat strongly uniformly elliptic systems only. Then we would have to define the set D by

$$D := \bigl\{\, \eta \in \mathbb{R}^2 \ ; \ a(\eta) + {}^t a(\eta) \text{ is positive definite} \,\bigr\} \ .$$

It is easily seen that in this case $D \not\supset (\mathbb{R}^2)^+$ if either $\beta_{11} = 0$ and $\beta_{12} \neq 0$, or $\beta_{22} = 0$ and $\beta_{21} \neq 0$. Thus in this case we could not solve (1.11) for arbitrary positive initial values $u^0 \in W_p^1$, which would be unreasonable from the point of view of the model. $\square$

The classical solvability of problem (1.11) and the invariance of the positive cone $(W_p^1)^+$ in the above generality has first been indicated by the author in [Ama90a] as an application of the general results proven there. Recently, Yagi [Yag92] has proven a weaker result. Moreover, he assumes that

$$g_r(u) := \rho_r - \sigma_{r1}u_1 - \sigma_{r2}u_2 \ , \qquad \rho_r, \sigma_{rs} \geq 0 \ , \quad 1 \leq r, s \leq 2 \ . \tag{17.3}$$

The question of global existence of positive solutions to (1.11) has already attracted a fair amount of interest. The first global existence theorem seems to be due to Kim [Kim84] for the very special case $\alpha_1 = \alpha_2$, $\beta_{11} = \beta_{22} = \gamma_1 = \gamma_2 = 0$, and $n = 1$, and with g satisfying (17.3) with positive ρ_r and σ_{rs}. Later Deuring [Deu87] proved global existence under the same assumptions as Kim, but for arbitrary n and allowing $\alpha_1 \neq \alpha_2$, provided the coefficients of $(\mathcal{A}, \mathcal{B})$ are sufficiently small, their size depending on u^0. In [PoT90] there is a global existence result under the assumption that $\beta_{11} = \beta_{21} = \beta_{22} = \gamma_1 = \gamma_2 = 0$ and that g is sufficiently nonlinear, but for Dirichlet boundary conditions. In [Yag92] global existence is proven under the assumption (17.3), provided either $0 < \beta_{21} < 8\beta_{11}$ and $0 < \beta_{12} < 8\beta_{22}$, or $\beta_{11} > 0$, $\beta_{21} = \beta_{22} = 0$, and $\sigma_{11} > 0$. Lastly, Redlinger [Red92] considers the case $\beta_{11} = \beta_{21} = \beta_{22} = 0$ and $\gamma_1 = \gamma_2 = 0$ and assumes that g is of the form (17.3) with positive constants, except that the term $\sigma_{11} u_1$ is replaced by $h(u_1)$, where $\liminf_{t\to\infty} h(t)t^{-\nu} > 0$ for some $\nu > 0$ (with $\nu > 1$ if $n \geq 2$) and h is polynominally bounded. Then he proves the existence of a global attractor, that is, of a compact invariant set for the semiflow on $(W_p^1)^+$, which attracts each bounded set.

Note that $u = 0$ is a stationary point of the semiflow $(t, u^0) \mapsto u(t, u^0)$ and that the linearization at $u = 0$ of the stationary part of (1.11) is given by

$$\begin{aligned} -\partial_j\big(a(0)\partial_j v + a_j v\big) &= g(0)v \qquad &&\text{in } \Omega \ , \\ a(0)\partial_\nu v + a_j \nu^j v &= 0 &&\text{on } \partial\Omega \ , \end{aligned} \tag{17.4}$$

where $a(0) = \operatorname{diag}[\alpha_1, \alpha_2]$, the a_j are given by (1.13), and $g(0) := \operatorname{diag}\big[g_1(0), g_2(0)\big]$. Hence, if we consider, for simplicity, the special case $\gamma_1 = \gamma_2 = 0$, we see that all eigenvalues of the linear boundary value problem (17.4) are strictly positive, provided $g_r(0) > 0$ for $r = 1, 2$. This is true, in particular, if g_r is given by (17.3) with $\sigma_r > 0$ for $r = 1, 2$. Then the principle of linearized stability implies that 0 is an asymptotically stable critical point of the semiflow on $(W_p^1)^+$ (cf. [Dra88], [Dra89], where it has even been shown that 0 is asymptotically stable with respect to the W_p^2-topology). This gives a certain generalization of the result in [Deu87].

17.2 The Penrose-Fife Model In order to study the well-posedness of (1.14) under the general boundary conditions (1.23) we put $D := \mathbb{R} \times (0, \infty)$. Since

$$\sigma\big(a(\eta)\big) = \big\{\kappa(\eta_2), M(\eta)/\mu(\eta_1)\eta_2^2\big\} \subset \dot{\mathbb{R}}^+ \ , \qquad \eta \in D \ ,$$

we can directly apply Theorem 14.4. By taking into consideration Remark 14.1(b) too, we obtain the following

17.3 Theorem *Suppose that $p > n$ and $1 < s < (1 + 1/p) \wedge (2 - n/p)$. Then the phase field model* (1.14) *under the general boundary condition* (1.23) *generates a semiflow on $W_p^s(\Omega, \dot{\mathbb{R}}^+)$ such that bounded orbits, satisfying*

$$T(t) \geq \varepsilon > 0 \qquad \text{for some} \quad \varepsilon > 0 \quad \text{and all} \quad t \in [0, t^+) \ , \tag{17.5}$$

are relatively compact. If $g_1 = 0$,

$$(\varphi, T) := \big(\varphi(\cdot, \varphi^0, T^0), T(\cdot, \varphi^0, T^0)\big) \tag{17.6}$$

is a classical solution which is smooth on $\overline{\Omega} \times (0, t^+)$.

As for global existence, we can, of course, apply Theorm 15.4. However, if $g_1 = 0$, it follows from (1.23) and Example 13.4(b) that (13.12) is satisfied. Hence we deduce from (1.15)–(1.18) and from Theorem 15.5 the following global existence

17.4 Theorem *Suppose that $g_1 = 0$. Then each solution (17.6) satisfying (17.5) exists globally, provided*

$$\|\varphi(t)\|_\infty + \|T(t)\|_\infty \leq c(\tau) \ , \qquad 0 \leq t \leq \tau < \infty \ , \quad t < t^+ \ . \tag{17.7}$$

If c is bounded, the orbit through (φ^0, T^0) is relatively compact in $W_p^s(\Omega, \mathbb{R} \times \dot{\mathbb{R}}^+)$.

Of course, the above theorem remains valid if all functions depend smoothly upon $x \in \overline{\Omega}$ (and on t).

The above results are new. The well-posedness of the phase field model (1.14), (1.23) has — so far — only been proven in special cases. In [SpZ92] the authors derive an existence theorem guaranteeing a unique classical global solution, provided assumptions (1.19)–(1.21) hold and, in addition, the function s (the double well potential) satisfies suitable boundedness restrictions. Much of the work in [SpZ92] consists of the derivation of the a priori bounds (17.5) and (17.7) which enable the authors to apply our Theorem 15.5 to get global solutions.

In [Zhe92] a global existence theorem for problem (1.14), (1.19), (1.20) is obtained in space dimension one and for homogeneous Neumann boundary conditions (that is, conditions (1.21) with $\alpha = 0$). Additional results in even simpler situations are contained in [BaZ90] and [ElZ89].

It should be of interest to derive a priori bounds of the form (17.5) and (17.7) for getting global smooth solutions of (1.14), (1.23) in more general cases.

17.3 The Groundwater Flow Problem In order to treat the parabolic-elliptic problem (1.24)–(1.29) we use the fact that in (1.31) it has been reformulated as a nonlocal parabolic problem of the form (13.1) with $F = 0$ and $G = 0$. Thus it remains to verify (13.3) for appropriate choices of the parameters (13.2).

It is obvious that

$$M \in C^{1-}\big(\mathbb{R}^2, \mathcal{L}(\mathbb{R}^n)\big) \tag{17.8}$$

with $n = 2$, and that M is symmetric and uniformly positive definite. Note that Theorem 9.2, Remark 9.3(a), and the commutativity of the diagram

$$\begin{tikzcd} W_p^s \arrow[r, "\partial_1"] \arrow[d, "Q_j"'] & W_{p,\gamma_\partial}^{s-1} \arrow[d, "(-\Delta_D)^{-1}"] \\ W_{p,\gamma_\partial}^s & W_{p,\gamma_\partial}^{s+1} \arrow[l, "(-1)^{j-1}\partial_j"'] \end{tikzcd} \tag{17.9}$$

where $W_p^s := W_p^s(\Omega, \mathbb{R})$ etc., imply

$$Q_j \in \mathcal{L}(W_p^s, W_{p,\gamma_\partial}^s) \ , \qquad s \in [0, 1 + 1/p) \backslash \{1/p\} \ , \quad j = 1, 2 \ . \tag{17.10}$$

Fix $p > n$ and $n/p < r < s < \sigma := 1$. Then we deduce from (17.8), (17.10), (1.30), the imbedding $W_p^r \hookrightarrow C$, and from Lemma 14.2(i) that

$$(\mathcal{A}, \mathcal{B}) \in C^{1\text{-}}\big(W_p^r, \mathcal{E}^{1/2}(\Omega)\big) \ .$$

Consequently, Theorem 13.1 implies the following well-posedness

17.5 Theorem *Let $0 < 2/p < s < 1$. Then the porous media problem* (1.24)–(1.29) *generates a semiflow of weak W_p^1-solutions on W_p^s. The positive cone $(W_p^s)^+$ is positively invariant, and bounded orbits are relatively compact.*

This theorem is essentially contained in [CDL92]. In that paper the authors also consider the much simpler semilinear problem with $a = b = 0$. Then (1.31) takes the form

$$\begin{aligned} \partial_t u - m\Delta u &= -\partial_j\big(a_j(u)u\big) && \text{in} \quad \Omega \times (0,\infty) \ , \\ \partial_\nu u &= 0 && \text{on } \partial\Omega \times (0,\infty) \ . \end{aligned} \tag{17.11}$$

Since $W_{p'}^{n/p} \hookrightarrow L_{(p/2)'}$ for $p > n$, we deduce from (17.9), Hölder's inequality, and the factorization

$$L_p \xrightarrow{\ u \mapsto a_j(u)u\ } L_{p/2} \hookrightarrow W_{p,\mathcal{B}}^{-n/p} \ , \tag{17.12}$$

where $\mathcal{B} := \partial_\nu$, that

$$\big[u \mapsto \partial_j\big(a_j(u)u\big)\big] \in C^{1\text{-}}(L_p, W_{p,\mathcal{B}}^{-\tau}) \tag{17.13}$$

with $\tau := 1 + n/p < 2$. Thus Theorem 13.8 implies that problem (17.11) generates a semiflow of very weak L_p-solutions on L_p. Then an obvious boot strapping argument based upon the results of Section 11 guarantees that, given any $u^0 \in L_p$, the unique maximal very weak L_p-solution is, in fact, a classical solution on $\big(0, t^+(u^0)\big)$ of (17.11).

Put $\varphi(\xi) := |\xi|^p$ for $\xi \in \mathbb{R}$ and note that φ is twice continuously differentiable and convex. By multiplying the first equation in (17.11) by $\varphi'(u)$, where $u := u(\cdot, u^0)$, and integrating over Ω we obtain

$$\big\langle \varphi'(u), \dot u \big\rangle + m\big\langle \partial_j\varphi'(u), \partial_j u \big\rangle = -\big\langle \varphi'(u), \partial_j\big(a_j(u)u\big)\big\rangle \ . \tag{17.14}$$

Observe that

$$m\big\langle \partial_j\varphi'(u), \partial_j u\big\rangle = m\big\langle \varphi''(u)\partial_j u, \partial_j u\big\rangle \geq 0 \ . \tag{17.15}$$

Moreover, since $\partial_j a_j(u) = 0$ and $\gamma_\partial a_j(u)\nu^j = 0$,

$$\begin{aligned} \big\langle \varphi'(u), \partial_j\big(a_j(u)u\big)\big\rangle &= \big\langle \varphi'(u), a_j(u)\partial_j u\big\rangle = \langle \partial_j\varphi(u), a_j(u)\rangle \\ &= \big\langle \partial_j\big(\varphi(u)a_j(u)\big), \mathbf{1}\big\rangle = \big\langle \gamma_\partial\varphi(u), \gamma_\partial a_j(u)\nu^j\big\rangle_\partial = 0 \ . \end{aligned} \tag{17.16}$$

Thus we deduce from (17.13)–(17.16) that

$$\frac{d}{dt}\,\|\varphi(u)\|_1 = \big\langle \varphi'(u), \dot u\big\rangle \leq 0 \qquad \text{on } \dot J \ , \tag{17.17}$$

which implies

$$\|u(t)\|_p \leq \|u^0\|_p \ , \qquad t \in J \ . \tag{17.18}$$

From this estimate and from (17.9) and (17.11) we infer that

$$\sup_{t\in J}\left\|\partial_j\big(a_j(u(t))u(t)\big)\right\|_{W^{-\tau}_{p,\mathcal{B}}} < \infty$$

which, by an easy Gronwall inequality type argument, implies $t^+(u^0) = \infty$ (e.g., [Ama84, Corollary 3.2]). This proves

17.6 Theorem *Suppose that $a = b = 0$ and $p > 2$. Then the porous media problem* (1.24)–(1.29) *generates a global semiflow on L_p. The positive cone L_p^+ is positively invariant and $u(\cdot, u^0)$ is a classical solution of* (17.11) *for $t > 0$.*

In [CDL92] the above theorem has been proven by a completely different method which is based on an abstract theorem for semilinear parabolic equations due to v. Wahl [vW85b]. Our simple proof gives another illustration for the usefulness of very weak solutions even in the case of nonlocal operators.

17.4 The Dissolution-Growth Model We discuss now the well-posedness of problem (1.32). For this we fix numbers

$$0 < 1/p < r < s < \sigma_1 < \tau + 2 < 1 + 1/p < 2 \tag{17.19}$$

and put

$$E_j := H^{\sigma_1 - 2(1-j)}_{p,\mathcal{B}_{11}}\ , \qquad j = 0, 1\ , \tag{17.20}$$

where $H_p^\sigma := H_p^\sigma(\Omega, \mathbb{R})$ etc. We also put

$$E := \mathbb{R}\ , \quad \boldsymbol{E}_j := E_j \times E\ , \qquad j = 0, 1\ ,$$

and define $\mathfrak{F}$ by (13.8) and $0 < \gamma < \beta < \alpha < 1$ by the line subsequent to (13.9). Then it follows that

$$\boldsymbol{E}_\theta := (\boldsymbol{E}_0, \boldsymbol{E}_1)_\theta \doteq E_\theta \times \mathbb{R}\ , \quad E_\theta := (E_0, E_1)_\theta \doteq W^{\sigma_1 - 2 + 2\theta}_{p,\mathcal{B}_{11}}\ , \qquad 0 < \theta < 1\ ,$$

(cf. [DaK92, Proposition 3.4]). Note that

$$X_\beta := W^r_{p,\mathcal{B}_{11}} \times (0, 1)$$

is an open subset of $\boldsymbol{E}_\beta$. It is easily verified that

$$(\mathcal{A}_{11}, \mathcal{B}_{11}) \in C^{1\text{-}}\big(X_\beta, \mathcal{E}^1(\Omega)\big)\ .$$

Thus, denoting by A_{11} the E_0-realization of $(\mathcal{A}_{11}, \mathcal{B}_{11})$,

$$A_{11} \in C^{1\text{-}}\big(X_\beta, \mathcal{H}(E_1, E_0)\big)\ ,$$

thanks to Theorem 8.5. From this it is obvious that

$$A := \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} \in C^{1\text{-}}\big(\mathbb{R}^+ \times X_\beta, \mathcal{H}(\boldsymbol{E}_1, \boldsymbol{E}_0)\big)\ . \tag{17.21}$$

Observe that $\partial W_p^\sigma \doteq \mathbb{R} \oplus \mathbb{R}$ for $0 \leq \sigma \leq 2$. From this we easily deduce that

$$F := \begin{bmatrix} f_1 + \mathcal{R} g_1 \\ f_2 \end{bmatrix} \in C^{1\text{-}}(\mathbb{R}^+ \times X_\beta, \boldsymbol{E}_\gamma) \ ,$$

where $\mathcal{R} \in \mathcal{L}(\mathbb{R}, E_\gamma)$ is the dual of the trace operator $\gamma_1 \in \mathcal{L}(W_{p',\mathcal{B}_{11}}^{-\tau}, \mathbb{R})$ (on $x = 1$). Then the quasilinear evolution equation

$$\dot{u} + A(t,u)u = F(t,u) \ , \qquad t > 0 \ ,$$

is equivalent to the weak formulation — in the obvious sense — of problem (1.33)–(1.40). Thus, by applying Theorem 12.1, Remark 12.2(a), and an obvious boot strapping argument, we obtain

17.7 Theorem *The moving boundary value problem* (1.32) *has for each* $R^0 \in (0,1)$ *and* $c^0 \in W_p^s\bigl((R^0,1), \mathbb{R}\bigr)$ *with* $c^0(1) = \varphi(0)$ *a unique maximal classical solution* (c, R) *depending continuously in the* W_p^s*-topology on* (c^0, R^0).

Of course, the precise formulation of the continuous dependence on the initial data is given via the transformed quantities (u_1, u_2) of the 'fixed boundary value problem'.

It is an easy consequence of Theorem 12.5 that $t^+ := t^+(c^0, R^0) < \infty$ implies $R(t) \to 0$ or $R(t) \to 1$ as $t \to t^+$, provided

$$\sup_{0<t<t^+} \|c(\cdot, t)\|_\infty < \infty \ . \tag{17.22}$$

This follows from the fact that, by standard easy a priori estimation techniques (use u_1 as a test function in the weak W_2^1-formulation of the first two equations in (1.33)), the estimate (17.22) implies a bound for the W_2^1-norm of $u_1(\cdot, t)$, uniformly with respect to $t \in (t_0, t^+)$, where $t_0 \in (0, t^+)$ is arbitrary. Details are left to the reader.

Theorem 17.7 generalizes the results in [CoHS89] and [CoHS90], respectively, where by a rather lengthy argument the well-posedness of problem (1.32) in the framework of weak solutions has been shown. In contrast to this, our proof is nothing but an almost trivial application of our general theorems. It serves as another illustration for the power of the techniques presented in this paper.

17.5 Highly Degenerate Problems In order to prove the well-posedness of (1.41) we study its equivalent form (1.43). For this we fix p, r, s, σ_1, and τ as in (17.19) and define E_j, $j = 0, 1$, by (17.20), where now $\mathcal{B}_{11} = \gamma_\partial$, of course. We also put

$$E := H_p^{\sigma_1} \ , \quad \boldsymbol{E}_j := E_j \times E \ , \qquad j = 0, 1 \ ,$$

and define $\mathfrak{F}$ and $0 < \gamma < \beta < \alpha < 1$ as in Subsection 17.4. Then, similarly as above,

$$\boldsymbol{E}_\theta := (\boldsymbol{E}_0, \boldsymbol{E}_1)_\theta \doteq E_\theta \times E \ , \quad E_\theta := (E_0, E_1)_\theta \doteq W_{p,\mathcal{B}_{11}}^{\sigma_1 - 2 + 2\theta} \ , \qquad 0 < \theta < 1 \ .$$

We denote by $A_{11}(u)$ the E_0-realization of $\bigl(\mathcal{A}_{11}(u), \gamma_\partial\bigr)$ and define $A_{12}(u)$ by

$$\bigl\langle w_1, A_{12}(u) v_2 \bigr\rangle := \bigl\langle \partial_x w_1, H(u_1 + \Psi) \partial_x v_2 \bigr\rangle_\Omega \ , \qquad w_1 \in W_{p',\gamma_\partial}^2 \ , \quad v_2 \in E \ .$$

Thanks to (1.44) it is not difficult to verify that

$$(A_{11}, A_{12}) \in C^2\big(X_\beta, \mathcal{H}(E_1, E_0) \times \mathcal{L}(E_1, E_0)\big) \ , \tag{17.23}$$

where $X_\beta := \boldsymbol{E}_\beta$. (For simplicity, we assume that Ψ is independent of t. The more general case can easily be handled by means of Remark 12.2.) This implies

$$A := \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} \in C^2\big(\mathbb{R}^+ \times X_\beta, \mathcal{H}(\boldsymbol{E}_1, \boldsymbol{E}_0)\big) \tag{17.24}$$

thanks to [Ama91b, Theorem 2.1], [Ang90, Lemma 2.6], or [Ama94, Corollary 1.6.3]. On the other hand, it is easily seen that

$$F := \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in C^2(X_1, \boldsymbol{E}_\gamma) \ , \tag{17.25}$$

where $X_1 := \boldsymbol{E}_1$. Of course, the evolution equation

$$\dot{u} + A(u)u = f(u) \ , \qquad t > 0 \ , \tag{17.26}$$

is the weak $H_p^{\sigma_1}$-formulation — in the obvious sense — of problem (1.43).

Since f_2 is a (smooth) local function depending upon the component u_1 of u, in particular, it is clear that its domain $X_1 = \boldsymbol{E}_1$ in (17.25) cannot be replaced by X_β, no matter how $\beta \in (0,1)$ has been chosen, if one wants to be sure that the image of F is contained in $\boldsymbol{E}_\gamma$. Thus Theorem 12.1 is not applicable to equation (17.26). Therefore we put

$$\varphi(u) := A(u)u - F(u)$$

and note that $\varphi \in C^2(X_1, \boldsymbol{E}_0)$ with

$$\partial\varphi(u) = A(u) + \partial A(u)[\,\cdot\,, u] - \partial F(u) =: A(u) + B(u) \in \mathcal{L}(\boldsymbol{E}_1, \boldsymbol{E}_0) \ , \qquad x \in X_1 \ .$$

Observe that

$$\partial\varphi(u) = \begin{bmatrix} A_{11}(u) + B_{11}(u) & A_{12}(u) + B_{12}(u) \\ B_{21}(u) & B_{22}(u) \end{bmatrix}$$

where

$$B_{11}(u) = \partial_1 A_{11}(u)[\,\cdot\,, u_1] - \partial_1 f_1(u) \ .$$

Thus, thanks to (1.45) and (17.23), it follows that

$$B_{11}(u) \in \mathcal{L}(E_\beta, E_0) \ , \qquad u \in X_1 \ .$$

Hence (17.23) and a well-known perturbation theorem (e.g., [Ama91b, Proposition 1.1(iii)]) imply

$$A_{11}(u) + B_{11}(u) \in \mathcal{H}(E_1, E_0) \ .$$

Now [Ama91b, Theorem 2.1] guarantees that

$$\partial\varphi(u) \in \mathcal{H}(\boldsymbol{E}_1, \boldsymbol{E}_0) \ , \qquad u \in X_1 \ ,$$

and we can apply Theorem 12.7 to deduce the following well-posedness result in which we specify σ_1 to be 1.

17.8 Theorem *The highly degenerate problem* (1.43) *generates a semiflow of weak W_p^1-solutions on $W_{p,\gamma_\partial}^1 \times W_p^1$.*

It is clear that this result guarantees the well-posedness of the original system (1.41) in the sense that the first equation of (1.41) is satisfied in the weak W_p^1-sense and the second one pointwise. If $H(0)=0$, we can apply Theorem 11.6 in the obvious way to the first equation of (1.41) to deduce that $c(t,c^0,\sigma^0)\geq 0$ for $0<t<t^+(c^0,\sigma^0)$, provided $c^0\geq 0$ and $\psi\geq 0$.

Theorem 17.8 is a special case of the results in [Ama91b], where more general situations have been studied too. In the particular semilinear situation where $D=H=\text{const}$, φ and M vanish identically, and $\rho=1$, it has been shown in [Ama91a] that the corresponding system generates a global semiflow, provided β is uniformly bounded.

Although the operator matrices (17.21) and (17.24) have, formally, the same structure, in as much as their second rows consist of zeros, there is a big difference between the two evolution equations of Subsections 17.4 and 17.5, respectively. This is due to the fact that the second row, that is, the second factor E of the products $\boldsymbol{E}_j=E_j\times E$, is finite dimensional in the first case, but infinite dimensional in the second one. For this reason systems of the latter form are said to be 'highly degenerate'. Although the corresponding evolution equations are 'abstractly parabolic' in the sense that $A(u)\in\mathcal{H}(\boldsymbol{E}_1,\boldsymbol{E}_0)$, highly degenerate problems lack the regularizing property of 'standard' quasilinear parabolic systems which can be put in a framework to which Theorem 12.1 applies.

17.6 The Electrolysis Problem By using the notations introduced in Subsection 1.6 we can write the system under consideration in the form

$$\begin{aligned} \partial_t u-\partial_j\big(a(u)\partial_j u+a_j(u,\varphi)u\big)&=0 &&\text{in}\quad \Omega\times(0,\infty)\ ,\\ a(u)\partial_\nu u+a_j(u,\varphi)\nu^j u&=g(u,\varphi) &&\text{on}\ \partial\Omega\times(0,\infty)\ . \end{aligned}\tag{17.27}$$

It has to be complemented by the initial condition

$$u(\cdot,0)=u^0\qquad\text{on }\Omega\tag{17.28}$$

and the electroneutrality condition

$$z(u):=\textstyle\sum_r z_r u_r=0\qquad\text{on }\Omega\times[0,\infty)\ .\tag{17.29}$$

Note that (17.29) is an invariance condition for the semiflow hopefully generated by (17.27). Thus, if (u,φ) is a solution to (17.27)–(17.29), by applying $z(\cdot)$ to (17.27) it is easily seen that (u,φ) necessarily satisfies the stationary problem

$$\begin{aligned} -\partial_j z\big(a(u)\partial_j u\big)-\nabla\cdot\big(\kappa(u)\nabla\varphi\big)&=0 &&\text{in }\Omega\ ,\\ z\big(a(u)\partial_\nu u\big)+\kappa(u)\partial_\nu\varphi&=z\big(g(u,\varphi)\big) &&\text{on }\partial\Omega\ , \end{aligned}\tag{17.30}$$

where

$$\kappa(u):=\textstyle\sum_r z_r m_r u_r\ .$$

In fact, it can be shown that (17.30) is necessary too. More precisely, (u,φ) is a solution to (17.27)–(17.29) iff it is a solution to (17.27), (17.28) with $z(u^0)=0$, and (17.30).

Observe that, given u, the system (17.30) is an elliptic boundary value problem with nonlinear boundary conditions for φ, provided

$$\kappa(u)(x)>0\ ,\qquad x\in\overline{\Omega}\ .\tag{17.31}$$

Thanks to (1.49), this is certainly the case if u is in the interior of the positive cone of $C(\overline{\Omega}, \mathbb{R}^N)$ and if $z_r m_r > 0$ for at least one $r \in \{1, \ldots, N\}$.

Suppose that, given u^0 satisfying (17.31), there exists φ^0 such that (u^0, φ^0) is a solution of (17.30). Also assume that in a suitable neighborhood U of u^0 every solution of (17.30) near φ^0 can be represented in the form $\bigl(u, \varphi(u)\bigr)$. By substituting $\varphi(u)$ in (17.27) we obtain a system of the form

$$\begin{aligned} \partial_t u_r + \mathcal{A}(u)u &= F(u) \qquad \text{in } \Omega \ , \\ \mathcal{B}(u)u &= G(u) \qquad \text{on } \partial\Omega \ , \end{aligned} \tag{17.32}$$

where F and G are suitable nonlocal operators and

$$\mathcal{A}(u)v := -\nabla\cdot\bigl(\textstyle\sum_s a_{rs}(u)\nabla v_s\bigr) \ , \quad \mathcal{B}(u)v := \textstyle\sum_s a_{rs}(u)\partial_\nu v_s \tag{17.33}$$

with

$$a_{rs}(u) := a_r(\cdot, u)\delta_{rs} - \frac{m_r u_r}{\kappa(u)} z_s a_s(\cdot, u) \ , \qquad 1 \le r, s \le N \ .$$

Note that (17.32) is a strongly coupled quasilinear system even if the diffusion coefficients a_r of the original system are constant.

Observe that $\sum_r z_r a_{rs}(u) = 0$. Hence it follows that the matrix $\bigl[a_{rs}(u)\bigr]_{1\le r,s\le N}$ defines at each $x \in \overline{\Omega}$ a linear operator $\alpha(u)(x) \in \mathcal{L}(Z)$, where

$$Z := \bigl\{\, \eta \in \mathbb{R}^N \ ; \ \textstyle\sum_r z_r \eta_r = 0 \,\bigr\} \ .$$

It can be shown that $\sigma\bigl(\alpha(u)(x)\bigr) \subset [\mathrm{Re}\, z > 0]$ for $x \in \overline{\Omega}$ and each u satisfying (17.31). Consequently, it follows from Example 4.3(e) that (17.33) defines a normally elliptic boundary value problem in spaces of Z-valued functions. Thus we can apply the results of Section 13 to (17.32), provided the latter system is well defined.

By means of the above arguments it can be shown that the reaction-diffusion problem of electrolysis is well-posed, provided suitable hypotheses for the nonlinearity g are met. Details can be found in [AmR93]. Note that this problem is rather more complex than the other model cases since it involves a solvability theory for the nonlinear elliptic boundary value problem (17.30).

17.7 Stefan Problems In order to handle problem (1.50)–(1.53) we study the system (1.57)–(1.59). It is an important observation that the boundary condition $\mathcal{B}_{11}(u,s)u = 0$ is equivalent to the 'constant boundary condition'

$$\mathcal{B}u := \delta\partial_\nu u + (1-\delta)u = 0 \ ,$$

where $\delta(1) = 1_2$. Thus we fix numbers p, r, s, and τ such that

$$0 < \tau < 1/p < 1 + 1/p < r < s < 2 \ .$$

Let

$$E_j := W_{p,\mathcal{B}}^{2j} \ , \qquad j = 0, 1 \ ,$$

where now $N = 2$, and define $\mathfrak{F}$, $0 < \gamma < \beta < \alpha < 1$, and the spaces E and $\boldsymbol{E}_\theta$, $0 \le \theta \le 1$, as in Subsection 17.4 (but with $N = 2$). Also put $X_\beta := W_{p,\mathcal{B}}^r \times (0,1)$ and note that it is an open subset of $\boldsymbol{E}_\beta$. Let $\boldsymbol{u} := (u, s)$ and $\bigl(\mathcal{A}_{11}(\boldsymbol{u}), \mathcal{B}_{11}\bigr) := \bigl(\mathcal{A}(u,s), \mathcal{B}\bigr)$. Then

$$(\mathcal{A}_{11}, \mathcal{B}_{11}) \in C^{1\text{-}}\bigl(X_\beta, \mathcal{E}^1(\Omega)\bigr) \ .$$

Thus, denoting by A_{11} the E_0-realization of $(\mathcal{A}_{11}, \mathcal{B}_{11})$, it follows that

$$A := \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} \in C^{1\text{-}}\bigl(X_\beta, \mathcal{H}(\boldsymbol{E}_1, \boldsymbol{E}_0)\bigr) \ .$$

It is also not difficult to see that the map

$$\boldsymbol{u} \mapsto F(\boldsymbol{u}) := \begin{bmatrix} f(u) \\ h(s,u) \end{bmatrix}$$

belongs to $C^{1\text{-}}(X_\beta, \boldsymbol{E}_\gamma)$. Of course, the evolution equation

$$\dot{\boldsymbol{u}} + A(\boldsymbol{u})\boldsymbol{u} = F(\boldsymbol{u}) \ , \qquad t > 0 \ ,$$

is a reformulation of (1.59). Hence we can apply Theorems 12.1 and 12.5, an obvious boot strapping argument based upon the classical regularity theory for linear parabolic equations in one space dimension, and standard a priori techniques to deduce

17.9 Theorem *The Stefan problem possesses for each $s^0 \in (0,1)$ and $T^0 \in C\bigl([0,1], \mathbb{R}\bigr)$, satisfying $T^0 | \Omega_i(0) \in W_p^s\bigl(\Omega_i(0), \mathbb{R}\bigr)$ and the boundary conditions* (1.51)*, a unique maximal classical solution $t \mapsto \bigl(T(t), s(t)\bigr)$ such that $\bigl(T(0), s(0)\bigr) = (T^0, s^0)$. If t^+, the maximal existence time of $t \mapsto \bigl(T(t), s(t)\bigr)$, is finite and if*

$$\sup_{0<t<t^+} \|T(t)\|_\infty < \infty$$

then $s(t) \to 0$ or $s(t) \to 1$ as $t \to t^+$.

There is an enormous literature on Stefan and related free boundary value problems for parabolic equations. We simply refer to [Tar88] for an exhaustive bibliography covering the years up to 1987. Nevertheless, our approach seems to be new. Of course, (1.50)–(1.53) is just a simple model problem, chosen to demonstrate our techniques. It is clear that much more general situations can be treated this way. But even in the simple case of our model problem, Theorem 17.9 does not seem to be in the literature. Closely related, though different, results are contained in [FasP77], [FasPK77], [FasP83].

17.8 Dynamic Boundary Conditions In this subsection we briefly indicate how quasilinear parabolic systems with dynamic boundary conditions of the type (1.60) can be handled within the framework of abstract quasilinear parabolic evolution equations.

First we assume that $(\mathcal{A}, \mathcal{B}) \in \mathcal{E}^1(\Omega)$ with $\delta = 1_N$, and consider the linear system

$$\begin{aligned} \partial_t u + \mathcal{A} u &= f(t) && \text{in} \quad \Omega \times (0,\infty) \ , \\ \partial_t u + \mathcal{B} u &= g(t) && \text{on} \ \partial\Omega \times (0,\infty) \ , \\ u(\cdot, 0) &= u^0 && \text{on} \quad \Omega \ , \end{aligned} \tag{17.34}$$

where, of course, $\partial_t u$ in the second equation means $\gamma_\partial(\partial_t u) = \partial_t(\gamma_\partial u)$. Suppose that $(\mathcal{A}, \gamma_\partial)$ is normally elliptic, fix $p \in (1,\infty)$, and assume, for simplicity, that the spectrum of the L_p-realization of $(\mathcal{A}, \gamma_\partial)$ is contained in $[\mathrm{Re}\, z > 0]$. Then, cf. (11.17),

$$W_p^2 = W_{p,\gamma_\partial}^2 \oplus W_{p,\mathcal{A}}^2$$

and

$$\gamma_\partial^c := (\gamma_\partial | W^2_{p,\mathcal{A}})^{-1} \in \mathcal{L}\mathrm{is}(\partial_D W^2_p, W^2_{p,\mathcal{A}}) \hookrightarrow \mathcal{L}(\partial_D W^2_p, W^2_p) ,$$

where $\partial_D W^s_p := W^{s-1/p}_p(\partial\Omega, \mathbb{R}^N)$ for $s \in \mathbb{R}$. It follows that (17.34) is equivalent to the system

$$\begin{aligned} \partial_t u + \mathcal{A}(u - \gamma_\partial^c z) &= f(t) \qquad \text{in} \quad \Omega \times (0,\infty) , \\ \partial_t z + \mathcal{B}u &= g(t) \qquad \text{on } \partial\Omega \times (0,\infty) , \end{aligned}$$

where $z = \gamma_\partial u$. This suggests that we introduce the spaces

$$\mathbb{E}_0 := L_p \times \partial W^1_p , \quad \mathbb{E}_1 := \{ (u,z) \in W^2_p \times \partial_D W^2_p \ ; \ z = \gamma_\partial u \} = \mathrm{graph}(\gamma_\partial | W^2_p)$$

and the operator

$$\mathbb{A} : \mathbb{E}_1 \to \mathbb{E}_0 , \quad (u,z) \mapsto \big(\mathcal{A}(u - \gamma_\partial^c z), \mathcal{B}u \big) .$$

It has been shown in [Hin88] (also see [Hin89]) that

$$\mathbb{A} \in \mathcal{H}(\mathbb{E}_1, \mathbb{E}_0) .$$

Let

$$E_0 := \{ (u,z) \in W^1_{p'} \times \partial_D W^1_{p'} \ ; \ \gamma_\partial u = z \}'$$

with respect to the duality pairing naturally induced by the $L_p \times L_p(\partial\Omega, \mathbb{R}^N)$ duality pairing $\langle \cdot,\cdot \rangle + \langle \cdot,\cdot \rangle_\partial$, and put

$$E_1 := \{ (u,z) \in W^1_p \times \partial_D W^1_p \ ; \ \gamma_\partial u = z \} .$$

Then (E_0, E_1) is a densely injected Banach couple, and Escher [Esc91] (also cf. [Esc93b]) has proven that there exists a unique $A \supset \mathbb{A}$ such that

$$A \in \mathcal{H}(E_1, E_0) .$$

Moreover, the abstract evolution equation

$$\dot v + Av = F(t) , \quad 0 < t < \infty , \qquad v(0) = v^0 := (u^0, \gamma_\partial u^0)$$

in E_0, where $F := (f,g)$, is equivalent to the weak formulation

$$\begin{aligned} &\int_0^T \big\{ -\big(\langle \dot\varphi, u \rangle + \langle \dot\psi, \gamma_\partial u \rangle_\partial \big) + \boldsymbol{a}(\varphi, u) \big\} \, dt \\ &\qquad = \int_0^T \big(\langle \varphi, f \rangle + \langle \psi, g \rangle_\partial \big) \, dt + \langle \varphi(0), u^0 \rangle + \langle \psi(0), \gamma_\partial u^0 \rangle_\partial \end{aligned} \tag{17.35}$$

for all $T > 0$, all $(\varphi, \psi) \in C^1\big([0,T], E_1'\big) \cap C\big([0,T], E_0'\big)$ vanishing at $t = T$. Observe that (17.35) is precisely the relation one formally obtains from (17.34) by 'multiplying with a test function' $\varphi \in C^1\big([0,T], W^1_{p'}\big)$ and integrating by parts.

Thanks to these results it is now not surprising that one can handle quasilinear parabolic systems of the form (17.34) by reducing them to quasilinear parabolic evolution equations in the interpolation-extrapolation scales generated by A and suitable interpolation functors $\mathfrak{F}$. For details, generalizations, regularity theory, related results, and qualitative studies we refer to [Esc92], [Esc93a], [Esc93c].

It should be mentioned that in certain cases the theory of monotone operators can be applied to problems with dynamic boundary conditions (cf. [Kač80], [Lio69], [Sho74]). In [Gro87] the concept of a 'closed pair of operators' has been applied to problems of this type. All these techniques require coerciveness assumptions, which is not the case for the semigroup technique sketched above.

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Universität Zürich
Rämistr. 74
CH–8001 Zürich
Switzerland