Elliptic Operators with Infinite-Dimensional State Spaces

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Abstract. Motivated by applications to problems from physics, we study elliptic operators with operator-valued coefficients acting on Banach-space-valued distributions. After giving a definition of ellipticity, normal ellipticity in particular, generalizing the classical concepts, we show that normally elliptic operators are negative generators of analytic semigroups on $L_p(\mathbb{R}^n, E)$ for $1 \leq p < \infty$, and on $BUC(\mathbb{R}^n, E)$ and $C_0(\mathbb{R}^n, E)$, as well as on all Besov spaces of E-valued distributions on $\mathbb{R}^n$, where E is any Banach space. This is true under minimal regularity assumptions for the coefficients, thanks to a point-wise multiplier theorem for E-valued distributions proven in the appendix.

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Introduction

In this paper we derive resolvent estimates for linear elliptic differential operators

$$\mathcal{A} := \mathcal{A}(x, D) := \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha \tag{0.1}$$

acting on E-valued distributions on $\mathbb{R}^n$, where E is an arbitrary (nontrivial) Banach space E. Thus $a_\alpha(x)$ is, in general, a bounded linear operator on E. Of course, in this generality, a concept of ellipticity has to be defined encompassing the finite-dimensional case, that is, $E = \mathbb{C}^N$.

A particularly simple situation occurs if (Y, μ) is a σ-finite measure space and $E := L_q(Y, \mu)$ for some $q \in [1, \infty]$. Then, given functions $a_\alpha : \mathbb{R}^n \to L_\infty(Y, \mu)$ and setting $a_\alpha(x, y) := a_\alpha(x)(y)$, we can interpret

$$\sum_{|\alpha| \leq m} a_\alpha(x, y) D_x^\alpha \tag{0.2}$$

as a family of scalar differential operators depending on the parameter $y \in Y$. In this case $a_\alpha(x)$ acts as a multiplication operator on E via $\big[a_\alpha(x)v\big](y) = a_\alpha(x,y)v(y)$ for $(x,y) \in \mathbb{R}^n \times Y$ and $v \in L_q(Y,\mu)$. Of course, our concept of ellipticity for (0.1) has to be wide enough to include the case where there exists a constant $\nu > 0$ such that

$$\operatorname{Re} \sum_{|\alpha|=m} a_\alpha(x,y)\xi^\alpha \geq \nu\,|\xi|^m\,, \qquad (x,y) \in \mathbb{R}^n \times Y, \quad \xi \in \mathbb{R}^n\,, \tag{0.3}$$

that is, where the family (0.2) is uniformly strongly elliptic, uniformly with respect to $y \in Y$.

The motivation for studying elliptic differential operators with operator-valued coefficients comes from our previous investigation of coagulation-fragmentation equations describing certain physical situations involving a very large number of particles (cf. [7], [10]). Corresponding mathematical models lead to quasilinear reaction-diffusion systems

$$\partial_t u - \nabla_x \cdot \big(a(x,y,u)\nabla u\big) = f(x,y,u), \qquad x \in \mathbb{R}^n, \quad t > 0, \tag{0.4}$$

the parameter y being a measure for the cluster size, running either through $Y := \mathbb{N}$ in discrete models, or through $\mathbb{R}^+$ in the continuous case. For physical reasons the unknown $u\colon \mathbb{R}^n \times Y \times \mathbb{R}^+ \to \mathbb{R}$, the particle size distribution function, has to be nonnegative and to satisfy

$$\int_Y u(x,y,t)(1+y)\,dy < \infty, \qquad x \in \mathbb{R}^n, \quad t \geq 0.$$

Here dy denotes the counting measure if $Y = \mathbb{N}$. In a relatively simple situation taking only coagulation into account, the right hand side of (0.4) has the form

$$\begin{aligned} f(x,y,v) := \frac{1}{2}\int_0^y &\gamma(x,y-y',y')v(y-y')v(y')\,dy' \\ &- v(y)\int_0^\infty \gamma(x,y,y')v(y')\,dy' \end{aligned}$$

for $v : Y \to \mathbb{R}$ and a given function $\gamma\colon \mathbb{R}^n \times Y \times Y \to \mathbb{R}$.

In [7] the semilinear case, where the diffusion coefficient $a(x,y,u)$ in (0.4) is independent of u, has been studied. The more general quasilinear case has been investigated in [10]. In the latter paper, a is of the form

$$a(x,y,v)w(y) := a_0(x,y,v)w(y) + \int_Y a_1(x,y,y',v)w(y')\,dy'$$

for $w \in L_1\big(Y,(1+y)\,dy\big)$, and a_0 and a_1 being suitable scalar-valued functions.

The principal idea in those papers is to interpret u as a function on $\mathbb{R}^n$ with values in

$$E := L_1\big(Y, (1+y)\,dy\big). \tag{0.5}$$

Then (0.4) is a quasilinear evolution equation

$$\dot{u} + \mathcal{A}(u)u = F(u) \tag{0.6}$$

whose state space is the infinite-dimensional Banach space (0.5). Thus, if for any given sufficiently smooth $v\colon \mathbb{R}^n \to E$, the operator $-\mathcal{A}(v)$ generates a strongly continuous analytic semigroup, the general theory of abstract quasilinear parabolic evolution equations (e.g., [3]) can be invoked to prove the well-posedness of system (0.4).

A natural choice for the solution space of (0.6) would be

$$(L_1 \cap L_p)\big(\mathbb{R}^n, L_1(Y, (1+y)\,dy)\big) \tag{0.7}$$

for some $p > n$. It is a consequence of this paper that such a choice is possible. (In [7] and [10] less natural Besov spaces have been used since a suitable generation theorem allowing the choice (0.7) had not been available.) Indeed, it is one of the main results of this paper (Corollary 5.11) that every normally elliptic differential operator $\mathcal{A}$ with bounded and uniformly Hölder continuous $\mathcal{L}(E)$-valued coefficients is the negative generator of a strongly continuous analytic semigroup on each one of the spaces

$$L_p(\mathbb{R}^n, E), \quad 1 \le p < \infty, \qquad BUC(\mathbb{R}^n, E), \quad C_0(\mathbb{R}^n, E).$$

There is no restriction whatsoever on E. The price which we have to pay is that we cannot give a precise description of $D(\mathcal{A})$. However, we obtain rather precise inclusion results. In addition, we show that all real interpolation spaces between $D(\mathcal{A})$ and the underlying space $\mathfrak{F}$, where either $\mathfrak{F} = L_p(\mathbb{R}^n, E)$ for $1 \le p < \infty$ or $\mathfrak{F} = BUC(\mathbb{R}^n, E)$, are suitable Besov spaces (Proposition 5.1). In particular, these interpolation spaces are independent of $\mathcal{A}$. As a consequence, one can study time-dependent and quasilinear parabolic problems on $\mathfrak{F}$ by using the general results in [4, Section IV.2].

It should be mentioned that our derivation of the resolvent estimates in BUC is rather simple and completely new, even in the finite-dimensional case. It does not, in fact: cannot, make use of L_p-estimates as is the case for all known proofs if $E = \mathbb{C}^N$.

The results described above are proven in Section 5. Indeed, we consider more general elliptic operators and obtain precise resolvent estimates.

For explicit definitions of the realizations of $\mathcal{A}$ in $\mathfrak{F}$ we need to know that $-\mathcal{A}$ is the generator of an analytic semigroup in suitable superspaces of $\mathfrak{F}$, to be precise, in the Besov spaces $B^0_{p,\infty}(\mathbb{R}^n, E)$ if $\mathfrak{F} = L_p$, and in $B^0_{\infty,\infty}(\mathbb{R}^n, E)$ if $\mathfrak{F} = BUC$. That this is true follows from the results of Section 4. There it is shown that

$-\mathcal{A}$ generates an analytic semigroup on each Besov space $B^s_{p,q}(\mathbb{R}^n, E)$ with $s \in \mathbb{R}$ and $p, q \in [1, \infty]$, and that $D(\mathcal{A}) = B^{s+m}_{p,q}(\mathbb{R}^n, E)$, provided the coefficients of $\mathcal{A}$ are bounded and uniformly ρ-Hölder continuous with $\rho > |s|$ and $\mathcal{A}$ is normally elliptic. In particular, the Hölder scale $BUC^s(\mathbb{R}^n, E)$ for $s \in \mathbb{R}\backslash\mathbb{Z}$ is included. Moreover, the same result holds for little Hölder spaces, $buc^s(\mathbb{R}^n, E)$, little Nikol'skii spaces, $n^s_p(\mathbb{R}^n, E)$ for $1 \leq p < \infty$, and the scale $C^s_0(\mathbb{R}^n, E)$ for $s \in \mathbb{R}\backslash\mathbb{Z}$.

Of course, all this relies on a good definition of ellipticity, which is given in Section 3. There it is also shown that it naturally generalizes corresponding finite-dimensional concepts.

In Section 1 we collect some properties of function spaces, in particular: Besov spaces, which we use throughout. Section 2 contains a technical result which is needed in Section 3 to handle bounded and uniformly Hölder continuous coefficients without assuming additional conditions near infinity.

In view of applications to quasilinear problems it is most important to require minimal regularity for the coefficients a_α only. For this we have to extend the well-known point-wise multiplier theorem, guaranteeing that BUC^s is a multiplier space for $B^t_{p,q}$ if $s > |t|$ and $p, q \in [1, \infty]$, from the finite-dimensional to the infinite-dimensional setting. This result, being of independent interest, is proven in Appendix A2. Note that it is by no means trivial to define a point-wise product between a smooth operator-valued function and a vector-valued distribution, since we cannot use duality. For this we have to rely on Schwartz' theory of vector-valued distributions ([21]; also see [5, Chapter VI] for an exposition of this theory in the somewhat simpler case of Banach-space-valued distributions, as well as [6] for a summary).

In Appendix A1 we give a precise definition of Besov spaces and extend to the infinite-dimensional setting an important criterion, due to Yamazaki [32], for a temperate distribution to belong to a Besov space.

It should be mentioned that the whole paper is based on the Fourier multiplier theorems in [6] for operator-valued symbols.

Notations and conventions Throughout this paper all abstract vector spaces are over $\mathbb{C}$. The real case can be included by complexification. We use standard notation. In particular, if X and Y are locally convex spaces then $\mathcal{L}(X, Y)$ is the space of all continuous linear maps from X into Y, and $\mathcal{L}(X) := \mathcal{L}(X, X)$. It is a Banach space with the operator norm if X and Y are Banach spaces. $\mathcal{L}\mathrm{is}(X, Y)$ is the set of all isomorphisms in $\mathcal{L}(X, Y)$, and $\mathcal{L}\mathrm{aut}(X) := \mathcal{L}\mathrm{is}(X, X)$. The identity on X is often denoted by 1_X, or simply by 1.

We write $X \hookrightarrow Y$ if X is continuously injected in Y, that is, X is a vector subspace of Y and the natural injection $x \mapsto x$ is continuous. $X \doteq Y$ means that $X \hookrightarrow Y$ and $Y \hookrightarrow X$, provided X and Y are normed vector spaces. Thus $X \doteq Y$ iff X and Y coincide as vector spaces and carry equivalent norms. If $X \hookrightarrow Y$ and X is dense in Y then we express this by writing $X \overset{d}{\hookrightarrow} Y$.

Suppose that X and Y are Banach spaces. If A is a linear operator in X then $\sigma(A)$ and $\rho(A)$ denote its spectrum and resolvent set, respectively. If $X \hookrightarrow Y$ and $A \in \mathcal{L}(X,Y)$ then it is always understood that $\sigma(A)$ and $\rho(A)$ refer to the linear operator A *in* Y with domain X.

Let $A\colon \mathrm{dom}(A) \subset Y \to Y$ be linear and $X \hookrightarrow Y$. Then the X-realization, A_X, of A is the map in X with domain $\{\, x \in X \cap \mathrm{dom}(A) \; ; \; Ax \in X \,\}$ and $A_X x = Ax$ for $x \in \mathrm{dom}(A_X)$. It is easily verified that A_X is closed if this is true for A.

If A is a linear operator in X then we write $D(A)$ for its domain endowed with the graph norm.

Let $E := (E, |\cdot|)$ be a Banach space. Then, as a rule, we use $|\cdot|$ also for the norm in $\mathcal{L}(E)$. We denote by $\mathcal{D}_{\mathbb{C}}$ the space of scalar test functions on $\mathbb{R}^n$, that is, the locally convex space of all smooth complex functions with compact supports, equipped with its usual inductive limit topology. We write $\mathcal{S}(E)$ for the Schwartz space of all rapidly decreasing smooth E-valued functions on $\mathbb{R}^n$, endowed with the usual family of seminorms (as in the scalar case). If $E = \mathbb{C}$ we denote this space by $\mathcal{S}_{\mathbb{C}}$. Then $\mathcal{S}'(E) := \mathcal{L}(\mathcal{S}_{\mathbb{C}}, E)$ is the space of all temperate E-valued distributions on $\mathbb{R}^n$. It is given the topology of uniform convergence on bounded subsets of $\mathcal{S}_{\mathbb{C}}$.

We use $\mathcal{F}$ to denote the Fourier transform on $\mathcal{S}'(E)$, defined by $\widehat{u}(\varphi) := u(\widehat{\varphi})$ for $u \in \mathcal{S}'(E)$ and $\varphi \in \mathcal{S}_{\mathbb{C}}$, where $\widehat{u} := \mathcal{F}u$. Then $\mathcal{F} \in \mathcal{L}\mathrm{aut}\big(\mathcal{S}(E)\big) \cap \mathcal{L}\mathrm{aut}\big(\mathcal{S}'(E)\big)$. Also recall that the distributional derivative $\partial^\alpha u$ is defined for $\alpha \in \mathbb{N}^n$ and $u \in \mathcal{S}'(E)$ by $(\partial^\alpha u)(\varphi) = (-1)^{|\alpha|} u(\partial^\alpha \varphi)$ for all $\varphi \in \mathcal{D}_{\mathbb{C}} \overset{d}{\hookrightarrow} \mathcal{S}_{\mathbb{C}}$. Of course, we employ standard multiindex notation.

If Z is a nonempty subset of some vector space then $\dot{Z} := Z \backslash \{0\}$. In particular, $\dot{\mathbb{N}}$ stands for all integers ≥ 1.

We denote by c (and $c(\alpha, \beta, \ldots)$) various constants whose values may be different from occurrence to occurrence (and depend on the indicated quantities) but are always independent of all free variables of a given formula.

Throughout the rest of this paper all spaces of distributions are subspaces of $\mathcal{S}'(E)$. Therefore we always drop $\mathbb{R}^n$, the domain of definition, in the notation. If it clear which Banach space E is being considered, or if this choice is unimportant, we simply write $\mathcal{S}'$ etc. For example, L_p always means $L_p(E)$, more precisely, $L_p(\mathbb{R}^n, E)$, etc. Noteworthy exceptions are $\mathcal{D}_{\mathbb{C}}$ and $\mathcal{S}_{\mathbb{C}}$, defined above.

1. Spaces

Let $E := (E, |\cdot|)$ be a Banach space. We use standard notation for function spaces. Thus BUC^s is, for $s \in \mathbb{R}^+$, the Banach space of all $u : \mathbb{R}^n \to E$ whose derivatives of orders at most $[s]$ are bounded and uniformly continuous, and whose derivatives of order $[s]$ are uniformly $(s - [s])$-Hölder continuous, if $s \notin \mathbb{N}$. Here and below, given $t \in \mathbb{R}$, we denote by $[t]$ the largest integer less than or equal to t. The space BUC^s

is given the usual norm which we denote by $\|\cdot\|_{s,\infty}$. Moreover,

$$u \mapsto [u]_{s,\infty} := \sup_{x \neq y} |u(x) - u(y)| / |x - y|^s$$

is the s-Hölder seminorm for $s \in (0,1)$.

Similarly, W_p^s denotes for $s \in \mathbb{R}^+$ and $1 \leq p < \infty$ the Sobolev space of E-valued distributions on $\mathbb{R}^n$ of order s if $s \in \mathbb{N}$, and the corresponding Slobodeckii space if $s \notin \mathbb{N}$. Its norm, the usual one, is denoted by $\|\cdot\|_{s,p}$. In particular, $\|\cdot\|_p$ is the norm in $L_p = W_p^0$.

Suppose that $s < 0$ and $\mathfrak{F} \in \{ BUC, W_p \ ; \ 1 \leq p < \infty \}$. Then $\mathfrak{F}^s$ is defined to consist of all E-valued distributions on $\mathbb{R}^n$ such that there exist $u_\alpha \in \mathfrak{F}^{s-[s]}$ for $|\alpha| \leq -[s]$ satisfying

$$u = \sum_{|\alpha| \leq -[s]} \partial^\alpha u_\alpha. \tag{1.1}$$

It is a Banach space with the norm

$$u \mapsto \|u\|_{s,q} := \inf \sum_{|\alpha| \leq -[s]} \|u_\alpha\|_{s-[s],q},$$

where $q := p$ if $\mathfrak{F} = W_p$, and $q := \infty$ if $\mathfrak{F} = BUC$, the infimum being taken with respect to all representations (1.1) of u.

For $s \in \mathbb{R}$, we write buc^s for the closure of BUC^{s+1} in BUC^s. By mollifying, it is not difficult to see that

$$buc^k = BUC^k, \qquad k \in \mathbb{N}. \tag{1.2}$$

If $s \in \mathbb{R}^+ \setminus \mathbb{N}$ then buc^s is a 'little Hölder space', that is, $u \in buc^s$ iff $u \in BUC^{[s]}$ and

$$\lim_{t \to 0} \sup_{0 < |x-y| < t} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^{s-[s]}} = 0, \qquad |\alpha| = [s].$$

For $s \in \mathbb{R}$ and $p, q \in [1, \infty]$, we denote by $B_{p,q}^s$ the Besov spaces of E-valued distributions on $\mathbb{R}^n$. For a precise definition we refer to Appendix A1. We also denote by $b_{p,q}^s$ the closure of $B_{p,q}^{s+1}$ in $B_{p,q}^s$. These 'little Besov spaces' differ from $B_{p,q}^s$ only if $q = \infty$, that is,

$$b_{p,q}^s = B_{p,q}^s, \qquad 1 \leq p \leq \infty, \quad 1 \leq q < \infty, \quad s \in \mathbb{R}. \tag{1.3}$$

It follows that

$$b_{p,\infty}^s = \mathrm{cl}_{B_{p,\infty}^s}(B_{p,\infty}^t) = \mathrm{cl}_{B_{p,\infty}^s}(BUC^\infty), \qquad s < t, \quad 1 \leq p \leq \infty, \tag{1.4}$$

where $\mathrm{cl}_X(\cdot)$ denotes the closure in X, and $BUC^\infty := \bigcap_s BUC^s$. Furthermore,

$$B_{p,p}^s \doteq W_p^s, \qquad s \in \mathbb{R} \setminus \mathbb{Z}, \quad 1 \leq p < \infty, \tag{1.5}$$

and

$$B^s_{\infty,\infty} \doteq BUC^s, \qquad s \in \mathbb{R}\backslash\mathbb{Z}. \tag{1.6}$$

Consequently,

$$b^s_{\infty,\infty} \doteq buc^s, \qquad s \in \mathbb{R}\backslash\mathbb{Z}. \tag{1.7}$$

It should be noted that $B^s_{p,\infty}$ equals, for $s \in \mathbb{R}^+\backslash\mathbb{N}$ and $1 \leq p < \infty$, the Nikols'kii space of order s and integrability index p, except for equivalent norms. For this reason we put

$$n^s_p := b^s_{p,\infty}, \qquad s \in \mathbb{R}, \quad p \in [1,\infty), \tag{1.8}$$

and call it little Nikols'kii space. It is also convenient to set

$$B^s_p := B^s_{p,p}, \quad b^s_\infty := b^s_{\infty,\infty}, \qquad s \in \mathbb{R}, \quad p \in [1,\infty].$$

Besov spaces enjoy the following important embedding properties:

$$\mathcal{S} \hookrightarrow B^{s_1}_{p,q_1} \overset{d}{\hookrightarrow} B^{s_0}_{p,q_0} \overset{d}{\hookrightarrow} b^{s_0}_{p,\infty} \overset{d}{\hookrightarrow} B^t_{p,1} \overset{d}{\hookrightarrow} \mathcal{S}', \qquad s_1 \geq s_0 > t, \tag{1.9}$$

provided either $s_1 = s_0$ and $1 \leq q_1 \leq q_0 < \infty$, or $s_1 > s_0$ and $q_0, q_1 \in [1,\infty)$. Moreover,

$$B^{s_1}_{p_1,q_1} \hookrightarrow B^{s_0}_{p_0,q_0}, \qquad s_1 > s_0, \quad s_1 - n/p_1 \geq s_0 - n/p_0. \tag{1.10}$$

It is also true that

$$B^k_{p,1} \overset{d}{\hookrightarrow} W^k_p \overset{d}{\hookrightarrow} b^k_{p,\infty}, \qquad k \in \mathbb{Z}, \quad p \in [1,\infty), \tag{1.11}$$

and

$$B^k_{\infty,1} \overset{d}{\hookrightarrow} BUC^k \overset{d}{\hookrightarrow} b^k_\infty, \qquad k \in \mathbb{Z}. \tag{1.12}$$

We write $\mathring{B}^s_{p,q}$ for the closure of $\mathcal{S}$ in $B^s_{p,q}$. Then

$$\mathring{B}^s_{p,q} = \begin{cases} B^s_{p,q}, & p \vee q < \infty, \\ n^s_p, & p < \infty, \quad q = \infty. \end{cases} \tag{1.13}$$

Hence only the spaces $\mathring{B}^s_{\infty,q}$ for $1 \leq q < \infty$ and $\mathring{B}^s_\infty := \mathring{B}^s_{\infty,\infty}$ are different from the ones already introduced.

Similarly, we denote by C^s_0 the closure of $\mathcal{S}$ in BUC^s for $s \in \mathbb{R}$. Then $C_0 := C^0_0$ is the space of E-valued continuous functions on $\mathbb{R}^n$ vanishing at infinity. Moreover, (1.6) implies

$$C^s_0 \doteq \mathring{B}^s_\infty \hookrightarrow buc^s, \qquad s \in \mathbb{R}\backslash\mathbb{N}. \tag{1.14}$$

Besides of these embedding properties we need the following characterization of Besov spaces: If $m \in \dot{\mathbb{N}}$ then

$$u \in B^s_{p,q} \Longleftrightarrow \partial^\alpha u \in B^{s-m}_{p,q}, \qquad |\alpha| \leq m, \tag{1.15}$$

and

$$u \mapsto \sum_{|\alpha|\le m} \|\partial^\alpha u\|_{B^{s-m}_{p,q}} \tag{1.16}$$

is an equivalent norm for $B^s_{p,q}$, for $s \in \mathbb{R}$ and $p,q \in [1,\infty]$. This implies, in particular, that $\partial^\alpha \in \mathcal{L}(\mathcal{B}^{s+|\alpha|}_{p,q}, \mathcal{B}^s_{p,q})$ for $\mathcal{B} \in \{B, b\}$ and $\alpha \in \mathbb{N}^n$.

Similarly, if $m \in \dot{\mathbb{N}}$, $s \in \mathbb{R}$, and $p,q \in [1,\infty]$ then $u \in B^{s-m}_{p,q}$ iff there exist $u_\alpha \in B^s_{p,q}$ for $|\alpha| \le m$ such that

$$u = \sum_{|\alpha|\le m} \partial^\alpha u_\alpha. \tag{1.17}$$

Moreover,

$$u \mapsto \inf \sum_{|\alpha|\le m} \|u_\alpha\|_{B^s_{p,q}} \tag{1.18}$$

is an equivalent norm for $B^{s-m}_{p,q}$, where the infimum is taken over all representations (1.17).

We denote by $(\cdot,\cdot)_{\theta,q}$ the real interpolation functor of exponent $\theta \in (0,1)$ and index $q \in [1,\infty]$, and by $(\cdot,\cdot)^0_{\theta,\infty}$ the continuous one. (We refer to [4, Section I.2] for a summary of and to [11] and [27] for more details on interpolation theory.) It follows that, given $\theta \in (0,1)$ and $p, q_0, q_1, q \in [1,\infty]$ as well as $s_0, s_1 \in \mathbb{R}$ with $s_0 \ne s_1$,

$$(B^{s_0}_{p,q_0}, B^{s_1}_{p,q_1})_{\theta,q} \doteq B^{(1-\theta)s_0+\theta s_1}_{p,q} \tag{1.19}$$

and

$$(B^{s_0}_{p,q_0}, B^{s_1}_{p,q_1})^0_{\theta,\infty} \doteq (b^{s_0}_{p,q_0}, b^{s_1}_{p,q_1})^0_{\theta,\infty} \doteq b^{(1-\theta)s_0+\theta s_1}_{p,\infty}. \tag{1.20}$$

For more details and proofs of the above results we refer to [6] and [5]. The proof of characterization (1.17), (1.18) of [8, Theorem 2.1] carries over to the infinite-dimensional case without any change, thanks to [6, Theorem 6.1].

It should be noted that our definition of W^s_p and BUC^s for $s < 0$ is consistent with (1.5), (1.6), and characterization (1.17), (1.18) of $B^{s-m}_{p,q}$. If E is infinite-dimensional then we cannot describe negative spaces by duality, as in the finite-dimensional situation, unless we impose restrictive conditions on E (cf. [6, (5.22)]).

2. Retractions

Fix $\delta > 0$ and a sequence (U_j) of nonempty open subsets of $\mathbb{R}^n$ with $\operatorname{diam}(U_j) \le \delta$, covering $\mathbb{R}^n$, and being of finite multiplicity, that is, there exists $k \in \mathbb{N}$ such that any intersection of more than k distinct sets U_j is empty.

Let (φ_j) and (ψ_j) be sequences in $\mathcal{D}_{\mathbb{C}}$ satisfying $0 \le \varphi_j \le \psi_j \le 1$ with $\operatorname{supp}(\psi_j)$ contained in U_j and $\psi_j \,|\, \operatorname{supp}(\varphi_j) = 1$. Also suppose that

$$\sup_j \|\varphi_j\|_{k,\infty} + \sup_j \|\psi_j\|_{k,\infty} \le \nu_k < \infty, \qquad k \in \mathbb{N}.$$

Set

$$\Phi\big((u_j)\big) := \sum_{j=0}^{\infty} \varphi_j u_j, \quad \Phi^c(u) := (\varphi_j u), \qquad u, u_j : \mathbb{R}^n \to E.$$

Also set

$$\mathfrak{F}_p^s := \begin{cases} W_p^s, & 1 \le p < \infty, \\ BUC^s, & p = \infty, \end{cases} \tag{2.1}$$

and let $\ell_p(\mathfrak{F}_p^s) := \ell_p(\mathbb{N}, \mathfrak{F}_p^s)$ for $s \in \mathbb{R}$.

Lemma 2.1. *Suppose that $s \in \mathbb{R}$ and $p \in [1, \infty]$. Then*

$$\Phi \in \mathcal{L}\big(\ell_p(\mathfrak{F}_p^s), \mathfrak{F}_p^s\big), \quad \Phi^c \in \mathcal{L}\big(\mathfrak{F}_p^s, \ell_p(\mathfrak{F}_p^s)\big).$$

Proof (i) If $1 \le p < \infty$ then the assertion follows by the arguments of the proofs of [9, Lemma 9.1 and Corollary 9.2] if $0 \le s \le 1$, and from [10, Remark 3.15] if $-1 \le s < 0$. Thus suppose that $p = \infty$.

(ii) If $s \in \{0, 1\}$ then the lemma is an immediate consequence of the finite multiplicity of (U_j).

(iii) Suppose that $0 < s < 1$. Then, given $v \in BUC^s$,

$$\begin{aligned} \varphi_j v(x) - \varphi_j v(y) = \big(\varphi_j(x) - \varphi_j(y)\big)\psi_j v(x) + \varphi_j(y)\big(\psi_j(x) - \psi_j(y)\big)v(x) \\ + \varphi_j(y)\psi_j(y)\big(v(x) - v(y)\big) \end{aligned}$$

implies, thanks to $\operatorname{diam}\big(\operatorname{supp}(\lambda_j)\big) \le \delta$ for $\lambda_j \in \{\varphi_j, \psi_j\}$,

$$|\varphi_j v(x) - \varphi_j v(y)| \le 2\nu_1^2 \, \|v\|_{s,\infty} \, \big(\chi_j(x) + \chi_j(y)\big) \, |x-y|^s, \qquad x, y \in \mathbb{R}^n,$$

where χ_j is the characteristic function of U_j. Now, by the finite multiplicity of (U_j), the assertion is obvious.

(iv) Assume that $-1 \le s < 0$. For $v \in BUC^s$, there are $v_i \in BUC^{s+1}$ such that

$$v = v_0 + \textstyle\sum_{k=1}^n \partial_k v_k. \tag{2.2}$$

Hence

$$\varphi_j v = \varphi_j v_0 + \sum_{k=1}^n \big[\varphi_{j,k} v_k + \partial_k(\varphi_j v_k)\big], \tag{2.3}$$

where $\varphi_{j,k} := -\partial_k \varphi_j$ belong to $\mathcal{D}_{\mathbb{C}}$ with $\operatorname{supp}(\varphi_{j,k}) \subset \operatorname{supp}(\varphi_j)$ for $1 \le k \le n$ and $\sup_{j,k} \|\varphi_{j,k}\|_{1,\infty} \le \nu_2$. Define Φ_k and Φ_k^c for $1 \le k \le n$ by replacing φ_j in the definition of Φ and Φ^c, respectively, by $\varphi_{j,k}$. Given $(u_j) \in \ell_\infty(BUC^s)$, choose $u_{j,i} \in BUC^{s+1}$ for $0 \le i \le n$ satisfying

$$u_j = u_{j,0} + \textstyle\sum_{k=1}^n \partial_k u_{j,k}, \qquad j \in \mathbb{N}. \tag{2.4}$$

Then it follows from (2.3) that

$$\Phi\bigl((u_j)\bigr) = \Phi\bigl((u_{j,0})\bigr) + \sum_{k=1}^{n} \Phi_k\bigl((u_{j,k})\bigr) + \sum_{k=1}^{n} \partial_k \Phi\bigl((u_{j,k})\bigr)$$

and

$$\Phi^c(v) = \Phi^c(v_0) + \sum_{k=1}^{n} \Phi_k^c(v_k) + \sum_{k=1}^{n} \partial_k \Phi_k^c(v).$$

From this and from (ii) and (iii) we infer that $\Phi\bigl((u_j)\bigr) \in BUC^s$, that $\Phi^c(u)$ belongs to $\ell_\infty(BUC^s)$, and that

$$\|\Phi\bigl((u_j)\bigr)\|_{s,\infty} \le c \sup_{j\in\mathbb{N}} \sum_{i=0}^{n} \|u_{j,i}\|_{s+1,\infty}, \quad \|\Phi^c(v)\|_{\ell_\infty(BUC^s)} \le c \sum_{i=0}^{n} \|v_i\|_{s+1,\infty},$$

where c is independent of (u_j), $(u_{j,i})$, v, and v_i for $0 \le i \le n$. This being true for every representation (2.4) and (2.2), respectively, it follows that

$$\bigl\|\Phi\bigl((u_j)\bigr)\bigr\|_{s,\infty} \le c\,\|(u_j)\|_{\ell_\infty(BUC^s)}\,, \quad \|\Phi^c(v))\|_{\ell_\infty(BUC^s)} \le c\,\|v\|_{s,\infty}\,,$$

where c is independent of (u_j) and v.

(v) The assertion for $|s| > 1$ is now deduced by an obvious induction argument. □

Henceforth we fix an enumeration (x_j) of $\mathbb{Z}^n$ satisfying $|x_j|_\infty \ge |x_k|_\infty$ if $j \ge k$. We denote by Q the open cube $\{\, x \in \mathbb{R}^n \ ;\ |x|_\infty < 1 \,\}$ and set $U_{\varepsilon,j} := \varepsilon(x_j + Q)$ for $\varepsilon > 0$. Then $(U_{\varepsilon,j})_{j\in\mathbb{N}}$ covers $\mathbb{R}^n$, has finite multiplicity being independent of ε, and satisfies $\dim(U_{\varepsilon,j}) \le 2\varepsilon\sqrt{n}$.

For $j \in \mathbb{N}$, we write $\lambda_{\varepsilon,j}$ for the smooth diffeomorphism

$$U_{\varepsilon,j} \to Q, \quad x \mapsto -x_j + x/\varepsilon.$$

We also fix $\pi, \psi \in \mathcal{D}_{\mathbb{C}}$ satisfying $0 \le \pi \le \psi \le 1$ and $\operatorname{supp}(\psi) \subset Q$ as well as $\pi(x) = 1$ for $|x|_\infty \le 1/2$ and $\psi(x) = 1$ for $x \in \operatorname{supp}(\pi)$, and set

$$\pi_{\varepsilon,j} := \frac{\pi \circ \lambda_{\varepsilon,j}}{\bigl[\sum_{j=0}^{\infty} (\pi \circ \lambda_{\varepsilon,j})^2\bigr]^{1/2}}, \quad \psi_{\varepsilon,j} := \psi \circ \lambda_{\varepsilon,j}, \qquad j \in \mathbb{N}.$$

Then $\pi_{\varepsilon,j}, \psi_{\varepsilon,j}$ belong to $\mathcal{D}_{\mathbb{C}}$ with $0 \le \pi_{\varepsilon,j} \le \psi_{\varepsilon,j} \le 1$, and $\psi_{\varepsilon,j} \,|\, \operatorname{supp}(\pi_{\varepsilon,j}) = 1$ with $\operatorname{supp}(\psi_{\varepsilon,j}) \subset U_{\varepsilon,j}$,

$$\sum_{j=0}^{\infty} \pi_{\varepsilon,j}^2(x) = 1, \qquad x \in \mathbb{R}^n, \tag{2.5}$$

and

$$\sup_j \|\pi_{\varepsilon,j}\|_{k,\infty} + \sup_j \|\psi_{\varepsilon,j}\|_{k,\infty} \le c(k)\varepsilon^{-k}, \qquad k \in \mathbb{N}.$$

Given Banach spaces X and Y, a linear map $r : X \to Y$ is said to be a retraction in $\mathcal{L}(X,Y)$ if $r \in \mathcal{L}(X,Y)$ and there exists $r^c \in \mathcal{L}(Y,X)$, a coretraction for r, satisfying $r \circ r^c = 1_Y$.

Proposition 2.2. *Suppose that $s \in \mathbb{R}$ and $1 \le p \le \infty$. Put*

$$R_\varepsilon\big((u_j)\big) := \sum_{j=0}^{\infty} \pi_{\varepsilon,j} u_j, \quad R_\varepsilon^c(u) := (\pi_{\varepsilon,j} u)$$

for $(u_j) \in \ell_p(\mathfrak{F}_p^s)$ and $u \in \mathfrak{F}_p^s$. Then R_ε is a retraction in $\mathcal{L}\big(\ell_p(\mathfrak{F}_p^s), \mathfrak{F}_p^s\big)$ and R_ε^c is a coretraction for R_ε.

Proof Thanks to Lemma 2.1 and the preceding observations it suffices to verify that $R_\varepsilon \circ R_\varepsilon^c = \mathrm{id}_{\mathfrak{F}_p^s}$. But this is obvious by (2.5). □

Similar results are given in [19, pp. 148ff] in the finite-dimensional setting. However, Peetre uses duality arguments to cover spaces of negative order. Hence in our case his method is not applicable without additional restrictions on E.

3. Elliptic Operators

Suppose that $m \in \dot{\mathbb{N}}$ and $a_\alpha : \mathbb{R}^n \to \mathcal{L}(E)$ for $|\alpha| \le m$. Then, setting $D_j := -i\partial_j$,

$$\mathcal{A} := \mathcal{A}(x,D) := \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha$$

is a (formal) linear differential operator on $\mathbb{R}^n$ with $\mathcal{L}(E)$-valued coefficients. We denote by $\sigma\mathcal{A}$ its principal symbol, that is,

$$\sigma\mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{L}(E), \quad (x,\xi) \mapsto \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha .$$

We also set $\Sigma_\vartheta := \{\, z \in \mathbb{C} \ ; \ |\arg z| \le \vartheta \,\} \cup \{0\}$ for $\vartheta \in [0,\pi]$.

Given $\kappa \ge 1$ and $\vartheta \in [0,\pi)$, the operator $\mathcal{A}$ is (uniformly) (κ,ϑ)**-elliptic** if

$$\rho\big(-\sigma\mathcal{A}(x,\xi)\big) \supset \Sigma_\vartheta \tag{3.1}$$

and

$$(1+|\lambda|)\,\big|\big(\lambda + \sigma\mathcal{A}(x,\xi)\big)^{-1}\big| \le \kappa, \qquad \lambda \in \Sigma_\vartheta, \tag{3.2}$$

for $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ with $|\xi| = 1$. It is called ϑ**-elliptic** if it is (κ,ϑ)-elliptic for some $\kappa \ge 1$, and $\mathcal{A}$ is said to be **normally elliptic** if it is $\pi/2$-elliptic.

Remarks 3.1. **(a)** Condition (3.2) is equivalent to

$$\left|\left(\lambda+\sigma\mathcal{A}(x,\xi)\right)^{-1}\right|\le\kappa(|\xi|^m+|\lambda|)^{-1},\qquad (x,\xi)\in\mathbb{R}^n\times(\mathbb{R}^n)^\bullet,\quad \lambda\in\Sigma_\vartheta.$$

Proof The m-homogeneity of $\sigma\mathcal{A}(x,\cdot)$ implies

$$\lambda+\sigma\mathcal{A}(\cdot,\xi)=|\xi|^m\left(\lambda\,|\xi|^{-m}+\sigma\mathcal{A}(\cdot,\xi/|\xi|)\right),\qquad \xi\in(\mathbb{R}^n)^\bullet,\quad \lambda\in\mathbb{C}.$$

Now the assertion is obvious. □

(b) If $\mathcal{A}$ is normally elliptic then m is even.

Proof If m were odd then (3.1) and (3.2) would imply, upon replacing ξ by $-\xi$, that $\rho\bigl(a(x,\xi)\bigr)=\mathbb{C}$, which is impossible. □

(c) Suppose that $M\in\mathbb{R}$ and

$$\sum_{|\alpha|=m}\|a_\alpha\|_\infty\le M. \tag{3.3}$$

If $\mathcal{A}$ is (κ,ϑ)-elliptic, then there exist $0<r<R$ such that the spectrum of $\sigma\mathcal{A}(x,\xi)$ is contained in

$$\{\,z\in\mathbb{C}\ ;\ r\le|z|\le R\,\}\cap\Sigma_{\pi-\vartheta} \tag{3.4}$$

for $(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n$ with $|\xi|=1$. In particular, if $\mathcal{A}$ is normally elliptic, then these spectra are contained in

$$\{\,z\in\mathbb{C}\ ;\ \operatorname{Re}z\ge r\,\}\cap\{\,z\in\mathbb{C}\ ;\ |z|\le R\,\}. \tag{3.5}$$

If E is finite-dimensional and (3.4) or (3.5) is satisfied then $\mathcal{A}$ is (κ,ϑ)-elliptic or normally elliptic, respectively.

Proof The necessity of (3.4) and (3.5) is obvious from (3.1), (3.2), and (3.3). Their sufficiency is an easy consequence of [9, Lemma 4.1] (see [5, Section VII.2.3] for details). □

Remark 3.1(c) shows that our definitions of ellipticity are the correct extensions of these concepts from the finite- to the infinite-dimensional setting. It should also be remarked that, in the finite-dimensional case, $\mathcal{A}$ is normally elliptic iff $\partial_t+\mathcal{A}$ is (Petrowskii) parabolic. This property is also called parameter-ellipticity on the rays $\lambda=re^{i\vartheta}$ for $|\vartheta|\le\pi/2$ (e.g., [13]).

The next example is related to the applications mentioned in the introduction.

Example 3.2. Let (Y,μ) be a σ-finite measure space and $E:=L_q(Y,\mu;\mathbb{C}^N)$ for some $q\in[1,\infty]$ and $N\in\dot{\mathbb{N}}$. Identify $a_\alpha\in BUC\bigl(\mathbb{R}^n,L_\infty(Y,\mu;\mathbb{C}^{N\times N})\bigr)$ for $|\alpha|\le m$ with the multiplication operator induced by a_α.

Then $\mathcal{A} = \sum_{|\alpha|\le m} a_\alpha D^\alpha$ is ϑ-elliptic if there exist $\theta \in [0, \pi - \vartheta)$ and $\varepsilon > 0$ such that

$$\sigma\Bigl(\sum_{|\alpha|=m} a_\alpha(x,y)\xi^\alpha\Bigr) \in \Sigma_\theta \cap \{\, z \in \mathbb{C} \,;\ |z| \ge \varepsilon \,\}$$

for $x \in \mathbb{R}^n$, μ-a.a. $y \in Y$, and $|\xi| = 1$. If, in particular, $N = 1$ then $\mathcal{A}$ is normally elliptic if there exists $\varepsilon > 0$ such that

$$\operatorname{Re} \sum_{|\alpha|=m} a_\alpha(x,y)\xi^\alpha \ge \varepsilon\, |\xi|^m\,, \qquad x, \xi \in \mathbb{R}^n, \quad \mu\text{-a.a. } y \in Y. \tag{3.6}$$

Proof Remark 3.1(c) implies the existence of $\kappa_0 \ge 1$ such that

$$\rho\bigl(-\sigma\mathcal{A}(x,y,\xi)\bigr) \supset \Sigma_\vartheta, \quad (1+|\lambda|)\, \bigl|\bigl(\lambda + \sigma\mathcal{A}(x,y,\xi)\bigr)^{-1}\bigr|_{\mathbb{C}^{N\times N}} \le \kappa_0$$

for $x \in \mathbb{R}^n$, $|\xi| = 1$, μ-a.a. $y \in Y$, and $\lambda \in \Sigma_\vartheta$. Now the assertion follows from well-known properties of multiplication operators on L_q-spaces. □

For easy reference we include the following simple perturbation lemma which will be of repeated use for us.

Lemma 3.3. *Let E_0, E_1, and E_2 be Banach spaces satisfying $E_1 \hookrightarrow E_2 \hookrightarrow E_0$. If $A \in \mathcal{L}\mathrm{is}(E_1, E_0)$ and $B \in \mathcal{L}(E_2, E_0)$ satisfy $\|BA^{-1}\|_{\mathcal{L}(E_0)} \le 1/2$, then $A + B$ belongs to $\mathcal{L}\mathrm{is}(E_1, E_0)$ with $\|(A+B)^{-1}\| \le 2\,\|A^{-1}\|$.*

Proof This follows from $A + B = (1 + BA^{-1})A$ and the Neumann series. □

As a first application we prove a perturbation result for ϑ-elliptic operators.

Proposition 3.4. *Let $\mathcal{A}$ be (κ, ϑ)-elliptic and $\mathcal{B} := \sum_{|\alpha|\le m} b_\alpha D^\alpha$ satisfy*

$$|\sigma\mathcal{B}(x,\xi)| \le 1/2\kappa, \qquad (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n, \quad |\xi| = 1.$$

Then $\mathcal{A} + \mathcal{B}$ is $(2\kappa, \vartheta)$-elliptic.

Proof Since

$$\bigl|\bigl(\lambda + \sigma\mathcal{A}(x,\xi)\bigr)^{-1}\bigr| \le \kappa(1+|\lambda|)^{-1} \le \kappa, \qquad (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n, \quad |\xi| = 1, \quad \lambda \in \Sigma_\vartheta,$$

the assertion is a consequence of Lemma 3.3. □

If $a_\alpha \in BUC^\infty\bigl(\mathcal{L}(E)\bigr)$ for $|\alpha| \le m$ then the theory of vector-valued distributions guarantees that

$$\mathcal{A} := \sum_{|\alpha|\le m} a_\alpha D^\alpha \in \mathcal{L}(\mathcal{S}) \cap \mathcal{L}(\mathcal{S}') \tag{3.7}$$

(see [6, Theorem 2.1] and observe that $BUC^\infty\big(\mathcal{L}(E)\big) \hookrightarrow \mathcal{O}_M\big(\mathcal{L}(E)\big)$). However, we are interested in differential operators $\mathcal{A}$ whose coefficients possess limited smoothness. In this case $\mathcal{A}$ will be defined on suitable subspaces of $\mathcal{S}'$ only. To formulate our basic continuity result we suppose that $\rho > 0$ and set $m(n) := \sum_{|\alpha|\le m} 1$. Then, using multiplication $\mathcal{L}(E) \times E \to E$, $(a,e) \mapsto ae$, it follows from Theorem A.2.5 and the continuity properties of ∂^α that

$$\Big[(a_\alpha)_{|\alpha|\le m} \mapsto \mathcal{A} := \sum_{|\alpha|\le m} a_\alpha D^\alpha\Big] \in \mathcal{L}\Big(\big[BUC^\rho\big(\mathcal{L}(E)\big)\big]^{m(n)}, \mathcal{L}(\mathcal{B}^{s+m}_{p,q}, \mathcal{B}^s_{p,})\Big) \tag{3.8}$$

for $\mathcal{B} \in \{B, b, \mathring{B}\}$ and $-\rho < s < \rho$.

4. Elliptic Operators in Besov Spaces

In this section we prove the basic resolvent estimates for elliptic operators in the Besov space setting.

Throughout the following, we fix a Banach space E, a number $\rho > 0$, and constants $\kappa_0, M \ge 1$ and $\vartheta \in [0,\pi)$. We also suppose that $\mathcal{B} \in \{B, b\}$.

Given $s \in (-\rho,\rho)$ and $p,q \in [1,\infty]$, we denote by $\mathcal{A}_{\mathcal{B}^s_{p,q}}$ the $\mathcal{B}^s_{p,q}$-realization of $\mathcal{A}$,

$$\mathcal{A}_{\mathcal{B}^s_{p,q}} : \mathcal{B}^{s+m}_{p,q} \to \mathcal{B}^s_{p,q}, \quad u \mapsto \mathcal{A}u,$$

which is well-defined by (3.8).

Theorem 4.1. *Suppose that $-\rho < s < \rho$ and $1 \le p,q \le \infty$. Then there exist $\kappa \ge 1$ and $\omega > 0$ such that $\omega + \Sigma_\vartheta \subset \rho(-\mathcal{A}_{\mathcal{B}^s_{p,q}})$ and*

$$|\lambda|^{1-j}\, \|(\lambda + \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{B}^s_{p,q}, \mathcal{B}^{s+jm}_{p,q})} \le \kappa, \qquad \lambda \in \omega + \Sigma_\vartheta, \quad j = 0,1, \tag{4.1$_j$}$$

for every (κ_0,ϑ)-elliptic operator $\mathcal{A} = \sum_{|\alpha|\le m} a_\alpha D^\alpha$ satisfying

$$\sum_{|\alpha|\le m} \|a_\alpha\|_{\rho,\infty} \le M. \tag{4.2}$$

Proof First we show that it suffices to prove some simpler versions of the assertion. Then we consider the case of constant coefficients, and finally, by perturbation arguments and by using the retractions of Section 2, we obtain the desired result for general coefficients.

(i) We can assume that $\mathcal{B} = B$, $p = q$, and $s \notin \mathbb{Z}$.

Indeed, setting $(\cdot,\cdot)^0_{\theta,q} := (\cdot,\cdot)_{\theta,q}$ for $1 \le q < \infty$, it follows from (1.3), (1.19), and (1.20) that

$$\mathcal{B}^t_{p,q} \doteq (\mathcal{B}^{t-\varepsilon}_p, \mathcal{B}^{t+\varepsilon}_p)^0_{1/2,q}, \qquad t \in \mathbb{R}, \quad \varepsilon > 0.$$

Now we obtain the claim by interpolation.

(ii) It suffices to prove the assertion for $s \in (-1,0) \cup (0,1)$.

To see this, first suppose that $s = \sigma + 1$ for some $\sigma \in (0,1)$ and let the assertion be true for σ (and $\mathcal{B} = B$ and $p = q$).

Suppose that $f \in B_p^s$ and $\lambda \in \omega + \Sigma_\vartheta$. Since $B_p^s \hookrightarrow B_p^\sigma$, there exists $u \in B_p^{\sigma+m}$ satisfying $(\lambda + \mathcal{A})u = f$. By differentiating we find

$$(\lambda + \mathcal{A})\partial_k u = \partial_k f - \mathcal{A}_k u, \qquad 1 \leq k \leq n,$$

with $\mathcal{A}_k := \sum_{|\alpha| \leq m} \partial_k a_\alpha D^\alpha$. Note that $\partial_k a_\alpha \in BUC^{\rho-1}\big(\mathcal{L}(E)\big)$ and $\rho - 1 > \sigma$. Thus (3.8) guarantees that $\partial_k f - \mathcal{A}_k u \in B_p^\sigma$. Consequently,

$$\partial_k u = (\lambda + \mathcal{A})^{-1}(\partial_k f - \mathcal{A}_k u) \in B_p^{\sigma+m}, \qquad 1 \leq k \leq n,$$

and, thanks to $(4.1)_1$ (for $s = \sigma$),

$$\begin{aligned} |\lambda|^{1-j} \|\partial_k u\|_{B_p^{\sigma+jm}} &\leq c\big(\|\partial_k f\|_{B_p^\sigma} + \|u\|_{B_p^{\sigma+m}}\big) \leq c\big(\|\partial_k f\|_{B_p^\sigma} + \|f\|_{B_p^\sigma}\big) \\ &\leq c\,\|f\|_{B_p^s}\,, \end{aligned}$$

where the last estimate follows from (1.15) and (1.16). Thus, by employing these facts once more, we see that $u = (\lambda + \mathcal{A})^{-1} f \in B_p^{s+m}$ and that (4.1) is satisfied.

Next suppose that $s = \sigma - 1$ for some $\sigma \in (-1,0)$ and $f \in B_p^s$. Also suppose that the assertion holds for σ (and $\mathcal{B} = B$ and $p = q$). Thus, by (1.17) and (1.18), there exist $f_k \in B_p^\sigma$ such that

$$f = f_0 + \sum_{k=1}^n \partial_k f_k. \tag{4.3}$$

Note that $\partial_k a_\alpha \in BUC^{\rho-1}\big(\mathcal{L}(E)\big)$, where $\rho - 1 > |\sigma|$, so that (3.8) implies that $\mathcal{A}_k \in \mathcal{L}(B_p^{\sigma+m}, B_p^\sigma)$. Set

$$\begin{aligned} u &:= (\lambda + \mathcal{A})^{-1}\Big(f_0 - \sum_{k=1}^n \mathcal{A}_k(\lambda + \mathcal{A})^{-1} f_k\Big) + \sum_{k=1}^n \partial_k(\lambda + \mathcal{A})^{-1} f_k \\ &=: u_0 + \sum_{k=1}^n \partial_k u_k. \end{aligned}$$

Then $u_i \in B_p^{\sigma+m}$ for $0 \leq i \leq n$ so that $u \in B_p^{s+m}$, thanks to (1.17). Moreover, we obtain from (4.1)

$$|\lambda|^{1-j} \|u\|_{B_p^{s+jm}} \leq |\lambda|^{1-j} \sum_{i=0}^m \|u_i\|_{B_p^{\sigma+jm}} \leq c \sum_{i=0}^m \|f_i\|_{B_p^\sigma} \tag{4.4}$$

for $j=0,1$. Since $\mathcal{A}_k = \partial_k \circ (\lambda+\mathcal{A}) - (\lambda+\mathcal{A})\circ \partial_k$ we see that

$$\begin{aligned}(\lambda+\mathcal{A})u &= f_0 - \sum_{k=1}^n \mathcal{A}_k(\lambda+\mathcal{A})^{-1}f_k + \sum_{k=1}^n (\lambda+\mathcal{A})\partial_k(\lambda+\mathcal{A})^{-1}f_k \\ &= f_0 + \sum_{k=1}^n \partial_k f_k = f.\end{aligned}$$

Note that the constant in (4.4) is independent of $f_0,\dots,f_n$ and that these considerations hold for every representation (4.3). Hence it follows that

$$|\lambda|^{1-j}\,\|u\|_{B_p^{s+jm}} \le c\,\|f\|_{B_p^s}\,, \qquad j=0,1.$$

Now the validity of the assertion in this case is clear.

If $s\in\mathbb{R}\backslash\mathbb{Z}$ satisfies $|s|>2$ then an obvious induction argument applies. Thanks to (i), this proves (ii).

(iii) Suppose that $\mathcal{A}$ has constant coefficients, that is, $a_\alpha \in \mathcal{L}(E)$ for $|\alpha|\le m$. Set

$$\mathcal{A}(\xi) := \sum_{|\alpha|\le m} a_\alpha \xi^\alpha$$

and $a := \sigma\mathcal{A}$. Also put $b(\xi) := \mathcal{A}(\xi) - a(\xi)$ and note that $|b(\xi)| \le cM(1\vee|\xi|^{m-1})$ for $\xi\in\mathbb{R}^n$. Then we infer from Remark 3.1(a) that

$$\bigl|b(\xi)\bigl(\lambda+a(\xi)\bigr)^{-1}\bigr| \le cM\kappa_0(1\vee|\xi|^{m-1})\,(|\xi|^m+|\lambda|)^{-1} \le 1/2$$

for $\xi\in(\mathbb{R}^n)^\bullet$ and $\lambda\in\omega+\Sigma_\vartheta$, where $\omega:=\omega(\kappa_0,M)\ge 1$ is sufficiently large. Hence Lemma 3.3 and Remark 3.1(a) show that $\rho\bigl(-\mathcal{A}(\xi)\bigr)\supset\omega+\Sigma_\vartheta$ and

$$\bigl|\bigl(\lambda+\mathcal{A}(\xi)\bigr)^{-1}\bigr| \le 2\kappa_0\bigl(|\xi|^m+|\lambda|\bigr)^{-1}, \qquad \lambda\in\omega+\Sigma_\vartheta, \quad \xi\in(\mathbb{R}^n)^\bullet.$$

Now the assertion follows in this case from [6, Theorem 7.1].

(iv) Let the assertions of the theorem be true for $\mathcal{A}$ and suppose that

$$\mathcal{C}\in\mathcal{L}(B_p^{s+m},B_p^s), \qquad \|\mathcal{C}\|\le 1/2\kappa.$$

Then it holds for $\mathcal{A}+\mathcal{C}$, as follows from Lemma 3.3.

(v) For $\varepsilon>0$ denote by

$$r_\varepsilon(x) := \begin{cases} x, & |x|_\infty \le \varepsilon, \\ \varepsilon x/|x|_\infty, & |x|_\infty > \varepsilon, \end{cases}$$

the radial retraction from $\mathbb{R}^n$ onto $\varepsilon\overline{Q}$. Then r_ε is uniformly Lipschitz continuous with constant 2 (e.g., [1, Lemma 19.8]). Recall that $U_{\varepsilon,j} = \varepsilon(x_j+Q)$ with $x_j\in\mathbb{Z}^n$.

Given $a_\alpha \in BUC^\rho\big(\mathcal{L}(E)\big)$ with $0 < \rho < 1$, set

$$a_{\alpha,\varepsilon,j}(x) := a_\alpha\big(\varepsilon x_j + r_\varepsilon(x - \varepsilon x_j)\big), \qquad x \in \mathbb{R}^n.$$

Then $a_{\alpha,\varepsilon,j} \in BUC^\rho\big(\mathcal{L}(E)\big)$ and

$$a_{\alpha,\varepsilon,j} \,|\, U_{\varepsilon,j} = a_\alpha \,|\, U_{\varepsilon,j}, \qquad j \in \mathbb{N}. \tag{4.5}$$

Furthermore,

$$|a_{\alpha,\varepsilon,j}(x) - a_\alpha(\varepsilon x_j)| \le \sup_{|y-z|_\infty \le \varepsilon} |a_\alpha(y) - a_\alpha(z)| \le [a_\alpha]_{\rho,\infty}\varepsilon^\rho \tag{4.6}$$

and

$$\big[a_{\alpha,\varepsilon,j} - a_\alpha(\varepsilon x_j)\big]_{\rho,\infty} \le 2^\rho [a_\alpha]_{\rho,\infty} \tag{4.7}$$

for $j \in \mathbb{N}$ and $x \in \mathbb{R}^n$. Fix $\rho' \in (|s|, \rho)$ and observe that

$$[\cdot]_{\rho',\infty} \le 2^{1-\rho'/\rho}\, [\cdot]_{\rho,\infty}^{\rho'/\rho}\, \|\cdot\|_\infty^{1-\rho'/\rho}$$

implies, together with (4.6) and (4.7), that

$$\|a_{\alpha,\varepsilon,j} - a_\alpha(\varepsilon x_j)\|_{\rho',\infty} \le 8\, [a_\alpha]_{\rho,\infty}\, \varepsilon^{\rho-\rho'}, \qquad 0 < \varepsilon \le 1, \quad j \in \mathbb{N}.$$

Thus it follows from (3.8), (4.2), and (iii) and (iv) that we can fix $\varepsilon_0 \in (0,1)$ such that the assertions (with $\mathcal{B} = B$ and $p = q$) hold for

$$\mathcal{A}_j := \sum_{|\alpha| \le m} a_{\alpha,\varepsilon_0,j} D^\alpha = \mathcal{A}(\varepsilon_0 x_j, D) + \big(\mathcal{A}_j - \mathcal{A}(\varepsilon_0 x_j, D)\big),$$

uniformly with respect to $j \in \mathbb{N}$.

(vi) Set $U_j := U_{\varepsilon_0,j}$ and $\pi_j := \pi_{\varepsilon_0,j}$. Then, by Leibniz' rule (and by identifying π_j with the multiplication operator $u \mapsto \pi_j u$),

$$\mathcal{B}_j := \mathcal{A} \circ \pi_j - \pi_j \circ \mathcal{A} =: \sum_{|\alpha| \le m-1} b_{\alpha,j} D^j, \tag{4.8}$$

where $b_{\alpha,j} \in BUC^\rho\big(\mathcal{L}(E)\big)$ satisfy

$$\operatorname{supp}(b_{\alpha,j}) \subset \operatorname{supp}(\pi_j) \tag{4.9}$$

and

$$\sum_{|\alpha| \le m-1} \|b_{\alpha,j}\|_{\rho,\infty} \le c(M), \qquad j \in \mathbb{N}.$$

Thus (3.8) implies

$$\mathcal{B}_j \in \mathcal{L}(B_p^{s+m-1}, B_p^s), \quad \|\mathcal{B}_j\| \le c(M), \qquad j \in \mathbb{N}. \tag{4.10}$$

(vii) Now we construct a left inverse for $\lambda + \mathcal{A}$. Set

$$C_{j,k}(\lambda) := \mathcal{B}_j \circ \pi_k \circ (\lambda + \mathcal{A}_k)^{-1} \in \mathcal{L}(B_p^s), \qquad j, k \in \mathbb{N}.$$

The finite multiplicity of the covering (U_j) and (4.9) guarantee the existence of $\ell \in \dot{\mathbb{N}}$ such that $C_{j,k} = 0$ for $|j-k| > \ell$. Since, by (1.19),

$$B_p^{s+m-1} \doteq (B_p^s, B_p^{s+m})_{1-1/m,p}, \tag{4.11}$$

we infer from the corresponding interpolation inequality (e.g., Proposition I.2.2.1 in [4]) and (4.1) that, thanks to (v),

$$\|(\lambda + \mathcal{A}_j)^{-1}\|_{\mathcal{L}(B_p^s, B_p^{s+m-1})} \le c\,|\lambda|^{-1/m}\,, \qquad \lambda \in \omega + \Sigma_\vartheta, \quad j \in \mathbb{N}.$$

From this, (2.5), and (4.10) it follows that there exists $\omega_1 \ge \omega$ such that

$$\|C_{j,k}(\lambda)\|_{\mathcal{L}(B_p^s)} \le 1/(4\ell+1), \qquad \lambda \in \omega_1 + \Sigma_\vartheta, \quad j,k \in \mathbb{N}.$$

Consequently, setting

$$\big(C(\lambda)u\big)_j := \sum_{k=0}^{\infty} C_{j,k}(\lambda) u_k, \qquad u = (u_j) \in \ell_p(B_p^s), \quad j \in \mathbb{N},$$

we see that

$$C(\lambda) \in \mathcal{L}\big(\ell_p(B_p^s)\big), \quad \|C(\lambda)\| \le 1/2$$

for $\lambda \in \omega_1 + \Sigma_\vartheta$. Hence $1 - C(\lambda) \in \mathcal{L}\mathrm{aut}\big(\ell_p(B_p^s)\big)$ and

$$\big\|\big[1 - C(\lambda)\big]^{-1}\big\|_{\mathcal{L}(\ell_p(B_p^s))} \le 2, \qquad \lambda \in \omega_1 + \Sigma_\vartheta. \tag{4.12}$$

Define $A \in \mathcal{L}\big(\ell_p(B_p^{s+m}), \ell_p(B_p^s)\big)$ by $Au := (\mathcal{A}_j u_j)$ for $u = (u_j)$. Then (4.1) implies $\omega_1 + \Sigma_\vartheta \subset \rho(-A)$ and

$$|\lambda|^{1-j}\,\|(\lambda + A)^{-1}\|_{\mathcal{L}(\ell_p(B_p^s), \ell_p(B_p^{s+jm}))} \le \kappa, \qquad \lambda \in \omega_1 + \Sigma_\vartheta, \quad j = 0, 1. \tag{4.13}$$

Put $Bv := (\mathcal{B}_j v)$ for $v \in B_p^{s+m-1}$ and note that, by (4.10),

$$B \in \mathcal{L}\big(B_p^{s+m-1}, \ell_p(B_p^s)\big), \quad \|B\| \le c(M).$$

Also note that, setting $R := R_{\varepsilon_0}$,

$$(BRu)_j = \mathcal{B}_j \sum_{k=0}^{\infty} \pi_k u_k, \qquad u = (u_j) \in \ell_p(B_p^{s+m})$$

implies $\lambda + A - BR = \big(1 - C(\lambda)\big)(\lambda + A)$ for $\lambda \in \omega_1 + \Sigma_\vartheta$. Hence we infer from (4.12) and (4.13) that

$$(\lambda + A - BR)^{-1} = (\lambda + A)^{-1}\big(1 - C(\lambda)\big)^{-1}$$

and

$$|\lambda|^{1-j}\,\|(\lambda + A - BR)^{-1}\|_{\mathcal{L}(\ell_p(B_p^s),\ell_p(B_p^{s+jm}))} \le c, \qquad j = 0, 1, \tag{4.14}$$

for $\lambda \in \omega_1 + \Sigma_\vartheta$.

Set $R^c := R^c_{\varepsilon_0}$ and $L(\lambda) := R(\lambda + A - BR)^{-1}R^c$. Then Proposition 2.2 and (4.14) guarantee that $L(\lambda) \in \mathcal{L}(B_p^s, B_p^{s+m})$ and

$$|\lambda|^{1-j}\,\|L(\lambda)\|_{\mathcal{L}(B_p^s,B_p^{s+jm})} \le c, \qquad \lambda \in \omega_1 + \Sigma_\vartheta, \quad j = 0, 1. \tag{4.15}$$

From (4.8) it follows that $R^c(\lambda + \mathcal{A}) = (\lambda + A)R^c - B$. Consequently,

$$L(\lambda)(\lambda + \mathcal{A}) = R(\lambda + A - BR)^{-1}R^c(\lambda + \mathcal{A}) = RR^c = 1_{B_p^{s+m}}$$

for $\lambda \in \omega_1 + \Sigma_\vartheta$.

(viii) Lastly, we construct a right inverse for $\lambda + \mathcal{A}$. For this we set

$$Du := \sum_{j=0}^{\infty} \mathcal{B}_j u_j, \qquad u = (u_j) \in \ell_p(B_p^{s+m-1}).$$

Since $\mathcal{B}_j v(x) = 0$ for $x \notin \operatorname{supp}(\pi_j)$, the finite multiplicity of (U_j) and (4.10) guarantee that $D \in \mathcal{L}\big(\ell_p(B_p^{s+m-1}), B_p^s\big)$ is well-defined. Thus, by Proposition 2.2,

$$R^c D \in \mathcal{L}\big(\ell_p(B_p^{s+m-1}), \ell_p(B_p^s)\big).$$

Lemma 3.3 and (4.11) imply that $(\lambda + A + R^c D)^{-1}$ exists for $\lambda \in \omega_2 + \Sigma_\vartheta$ and is uniformly bounded, provided $\omega_2 \ge \omega_1$ is suitably chosen. Hence

$$\mathsf{R}(\lambda) := R(\lambda + A + R^c D)^{-1}R^c \in \mathcal{L}(B_p^s, B_p^{s+m})$$

is well-defined for $\lambda \in \omega_2 + \Sigma_\vartheta$. Since (4.8) implies $(\lambda + \mathcal{A})R = R(\lambda + A) + D$, it follows that

$$(\lambda + \mathcal{A})\mathsf{R}(\lambda) = (\lambda + \mathcal{A})R(\lambda + A + R^c D)^{-1}R^c = RR^c = 1_{B_p^s}$$

for $\lambda \in \omega_2 + \Sigma_\vartheta$.

From this and (vii) we deduce that $(\lambda + \mathcal{A})^{-1} = L(\lambda) = \mathsf{R}(\lambda)$ for $\lambda \in \omega_2 + \Sigma_\vartheta$, and the assertion follows from (4.15). $\square$

The basic ingredient of this proof is the Fourier multiplier theorem of [6] for operator-value symbols. It gives the fundamental estimates in the case of constant coefficients. Steps (vii) and (viii) are similar to the arguments used in the proof of [9, Theorem 9.4], where resolvent estimates in the finite-dimensional case and in the L_p-setting for $1 < p < \infty$ have been derived. Since the interpolation argument used in [9] is not available if $p = \infty$, we have given here a more direct proof.

Suppose that $\mathcal{A}$ is normally elliptic. Then, by Theorem 4.1,

$$\|(\lambda + \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{B}^s_{p,q})} \le \kappa/|\lambda|, \qquad \operatorname{Re}\lambda \ge \omega.$$

This is well-known to imply that $-\mathcal{A}_{\mathcal{B}^s_{p,q}}$ generates an analytic semigroup on $\mathcal{B}^s_{p,q}$. It is not strongly continuous if $\mathcal{B}^{s+m}_{p,q}$ is not dense in $\mathcal{B}^s_{p,q}$ (cf. [18, Chapter 2]). Note that $\mathcal{B}^{s+m}_{p,\infty}$ is never dense in $\mathcal{B}^s_{p,\infty}$.

In the next theorem we concentrate on strongly continuous semigroups. For this we use the following notation: If E_0 and E_1 are Banach spaces satisfying $E_1 \overset{d}{\hookrightarrow} E_0$, then we write $A \in \mathcal{H}(E_1, E_0)$ iff $A \in \mathcal{L}(E_1, E_0)$ and $-A$, considered as a linear operator in E_0 with domain E_1, generates a strongly continuous analytic semigroup on E_0, that is, in $\mathcal{L}(E_0)$. It is known that $\mathcal{H}(E_1, E_0)$ is open in $\mathcal{L}(E_1, E_0)$ (e.g., [4, Theorem I.1.3.1]).

Theorem 4.2. *Suppose that $\mathcal{A} = \sum_{|\alpha|\le m} a_\alpha D^\alpha$ is normally elliptic and has coefficients in $BUC^\rho\big(\mathcal{L}(E)\big)$. Then*

$$\mathcal{A} \in \mathcal{H}(b^{s+m}_{p,q}, b^s_{p,q}), \qquad |s| < \rho, \quad p, q \in [1, \infty].$$

In particular, if $s \in (-\rho, \rho)\backslash\mathbb{Z}$, then

$$\mathcal{A} \in \mathcal{H}(W^{s+m}_p, W^s_p) \cap \mathcal{H}(n^{s+m}_p, n^s_p) \cap \mathcal{H}(buc^{s+m}, buc^s)$$

for $1 \le p < \infty$.

Proof This follows from Theorem 4.1, Yosida's well-known characterization of generators of analytic semigroups, and from (1.3)–(1.8), thanks to $b^{s+m}_{p,q} \overset{d}{\hookrightarrow} b^s_{p,q}$. □

In the following remarks we collect some useful consequences.

Remarks 4.3. Let $\mathcal{A}$ be (κ_0, ϑ)-elliptic with coefficients in $BUC^\rho\big(\mathcal{L}(E)\big)$.

(a) Suppose that $-\rho < s < t < \rho$. Also suppose that $u \in \mathcal{B}^{s+m}_{p,q}$ and $v \in \mathcal{B}^t_{p,q}$ satisfy $(\lambda + \mathcal{A})u = v$ for some $\lambda \in \mathbb{C}$. Then $u \in \mathcal{B}^{t+m}_{p,q}$.

Proof Suppose that $s + m < t$ and set $\omega_1 := \omega(s + m, p, q)$. Then $\mathcal{B}^t_{p,q} \hookrightarrow \mathcal{B}^{s+m}_{p,q}$ implies

$$(\omega_1 + \mathcal{A})u = v + (\omega_1 - \lambda)u \in \mathcal{B}^{s+m}_{p,q}.$$

Hence $u \in \mathcal{B}^{s+2m}_{p,q}$ by Theorem 4.1. Define $k \in \mathbb{N}$ by $(k-1)m < t - s \le km$. By repeating this bootstrapping argument $(k-1)$-times we obtain $u \in \mathcal{B}^{s+km}_{p,q} \hookrightarrow \mathcal{B}^t_{p,q}$. The assertion follows by invoking Theorem 4.1 once more. □

(b) If $-\rho < s < t < \rho$ then $\rho(\mathcal{A}_{\mathcal{B}^s_{p,q}}) \supset \rho(\mathcal{A}_{\mathcal{B}^t_{p,q}})$.

Proof This is an immediate consequence of (a). $\square$

(c) Suppose that $\mathcal{A}$ has coefficients in $BUC^\infty(\mathcal{L}(E))$. Also suppose that $s \in \mathbb{R}$ and that $u \in B^s_\infty$ and $v \in BUC^\infty$ satisfy $(\lambda + \mathcal{A})u = v$ for some $\lambda \in \mathbb{C}$. Then $u \in BUC^\infty$. If $v \in \mathcal{S}$ and $\lambda \in \rho(-\mathcal{A}_{B^s_p})$ for some $p \in [1, \infty]$ then

$$u \in \bigcap \{\, B^t_{r,q} \ ;\ t \in \mathbb{R},\ r \in [p, \infty],\ q \in [1, \infty] \,\}. \tag{4.16}$$

Proof The first assertion is a consequence of (a). Thus suppose that $v \in \mathcal{S}$. Then (a) implies that $u \in \bigcap \{\, B^t_p \ \ t \in \mathbb{R} \,\}$. Now (4.16) follows from (1.9) and (1.10). $\square$

Finally, we can show that Theorem 4.1 holds for $\mathring{B}^s_{\infty,q}$ as well.

Theorem 4.4. *Suppose that* $-\rho < s < \rho$ *and* $1 \le q \le \infty$*. Then there exist* $\kappa \ge 1$ *and* $\omega > 0$ *such that* $\omega + \Sigma_\vartheta \subset \rho(-\mathcal{A}_{\mathring{B}^s_{\infty,q}})$ *and*

$$|\lambda|^{1-j}\, \|(\lambda + \mathcal{A})^{-1}\|_{\mathcal{L}(\mathring{B}^s_{\infty,q}, \mathring{B}^{s+jm}_{\infty,q})} \le \kappa, \qquad \lambda \in \omega + \Sigma_\vartheta, \quad j = 0, 1, \tag{4.17}$$

for every (κ_0, ϑ)*-elliptic operator* $\mathcal{A}$ *satisfying* (4.2).

Proof (i) From Theorem 4.1 we obtain κ and ω such that $\omega + \Sigma_\vartheta \subset \rho(-\mathcal{A}_{B^s_{\infty,q}})$ and (4.17) holds with $\mathring{B}$ replaced by B. Since $\mathcal{A} \in \mathcal{L}(\mathring{B}^{s+m}_{\infty,q}, \mathring{B}^s_{\infty,q})$ by (3.8), it remains to show that there exists $\omega_\infty \ge \omega$ with

$$(\lambda + \mathcal{A})^{-1}(\mathring{B}^s_{\infty,q}) \subset \mathring{B}^{s+m}_{\infty,q}, \qquad \lambda \in \omega_\infty + \Sigma_\vartheta. \tag{4.18}$$

(ii) Fix $v \in \mathcal{S}$ and suppose that the coefficients of $\mathcal{A}$ belong to $BUC^\infty(\mathcal{L}(E))$. Set $\omega_\infty := \omega \vee \omega(s, 1, 1)$ and let $\lambda \in \omega_\infty + \Sigma_\vartheta$ be given. Then

$$u := (\lambda + \mathcal{A})^{-1} v \in B^{s+m+n}_{1,1}$$

by Remark 4.3(c). Thus we deduce from (1.10) and the density of $\mathcal{S}$ in $B^{s+m+n}_{1,1}$ that $u \in \mathring{B}^{s+m}_{\infty,q}$. Since $(\lambda + \mathcal{A})^{-1} \in \mathcal{L}(B^s_{\infty,q}, B^{s+m}_{\infty,q})$ by Theorem 4.1, the assertion follows in this case by a density argument.

(iii) Suppose that $v \in \mathring{B}^s_{\infty,q}$. Then there exists a sequence (v_j) in $\mathcal{S}$ converging in $B^s_{\infty,q}$ towards v. Thanks to (1.4), (3.8), and Proposition 3.4 we can find a sequence $(\mathcal{A}_j)$ of $(2\kappa_0, \vartheta)$-elliptic operators having coefficients in $BUC^\infty(\mathcal{L}(E))$ and converging in $\mathcal{L}(B^{s+m}_{\infty,q}, B^s_{\infty,q})$ towards $\mathcal{A}$. Since, given Banach spaces X and Y, the set $\mathcal{L}\mathrm{is}(X, Y)$ is open in $\mathcal{L}(X, Y)$ and inversion $A \mapsto A^{-1} : \mathcal{L}\mathrm{is}(X, Y) \to \mathcal{L}(Y, X)$ is smooth, it follows that $(\lambda + \mathcal{A}_j)^{-1} \to (\lambda + \mathcal{A})^{-1}$ in $\mathcal{L}(B^s_{\infty,q}, B^{s+m}_{\infty,q})$ whenever $\lambda \in \omega_\infty + \Sigma_\vartheta$. Hence $(\lambda + \mathcal{A}_j)^{-1} v_j \to (\lambda + \mathcal{A}) v$ in $B^{s+m}_{\infty,q}$. Now (ii) implies (4.18). $\square$

Corollary 4.5. *Suppose that $-\rho < s < \rho$ and $1 \leq q \leq \infty$. Then any normally elliptic operator with coefficients in $BUC^{\rho}\big(\mathcal{L}(E)\big)$ generates a strongly continuous analytic semigroup on $\mathring{B}^s_{\infty,q}$.*

Remark 4.6. If $p = q = \infty$ and $s \in \mathbb{R}^+ \setminus \mathbb{N}$, so that $\mathcal{B}^s_{p,q}$ is one of the Hölder spaces BUC^s, buc^s, or C_0^s, then $\rho = s$ is admissible, provided $BUC^{\rho}\big(\mathcal{L}(E)\big)$ is replaced by $buc^{\rho}\big(\mathcal{L}(E)\big)$ for $\mathcal{B} \in \{b, \mathring{B}\}$.

Proof This follows from Remark A2.6. □

The only paper (apart from [10]) known to the author, in which resolvent estimates for elliptic operators in Besov spaces have been proven, is [14]. There estimate $(4.1)_j$ is derived for scalar (κ_0, ϑ)-elliptic operators with smooth coefficients (see [14, Lemma 2.4]). Thus, even in the finite-dimensional case, Theorem 4.1 is new since our regularity assumptions are optimal. Guidetti's proof relies on Fourier multiplier theorems as well. However, he uses duality so that his arguments cannot be generalized to the infinite-dimensional setting.

In the finite-dimensional case, elliptic operators have been studied in Hölder space settings by several authors. In particular, Lunardi [18, Theorem 3.1.14 and Corollary 3.1.16] shows that if $m = 2$ and $\mathcal{A}$ is normally elliptic with coefficients in BUC^s, where $s \in (0,2)\backslash\{1\}$, then $-\mathcal{A}$ generates an analytic semigroup on BUC^s with $D(\mathcal{A}) = BUC^{s+2}$. On page 117 of [18] it is mentioned that her proof carries over to higher order operators. The arguments in [18] rely on Agmon-Douglis-Nirenberg L_p- and C^s-estimates and cannot be generalized to infinite dimensions.

Other approaches for second order operators in Hölder spaces are due to Campanato [12] and Vespri [30] (who consider boundary value problems as well). Their methods are also restricted to the finite-dimensional setting.

It is clear that for operators of divergence form (if $m = 2$, for example) the continuity requirements on the coefficients can be further relaxed for $-1 < s < 0$ (see [10]). Note, however, that, in [10], non-divergence form operators could not be handled in Besov spaces of negative orders since Theorem A2.5 had not been available.

5. Elliptic Operators in L_p, BUC, and C_0

We again fix a Banach space E and numbers $m \in \dot{\mathbb{N}}$ and $\rho \in (0,1)$ as well as $\kappa_0, M \geq 1$ and $\vartheta \in [0,\pi)$. We also suppose that $\mathcal{A} = \sum_{|\alpha|\leq m} a_\alpha D^\alpha$ is (κ_0,ϑ)-elliptic and that (4.2) is satisfied. Moreover, throughout this section $1 \leq p < \infty$.

We denote by $\mathcal{A}_p$ the L_p-realization of $\mathcal{A}_{n_p^0}$ and by $\mathcal{A}_\infty$ the BUC-realization of $\mathcal{A}_{b_\infty^0}$. Since, by (1.9) and (1.11),

$$n_p^m \overset{d}{\hookrightarrow} B_{p,1}^0 \overset{d}{\hookrightarrow} L_p \overset{d}{\hookrightarrow} n_p^0 \tag{5.1}$$

and, by (1.9) and (1.12),

$$b_\infty^m \overset{d}{\hookrightarrow} B_{\infty,1}^0 \overset{d}{\hookrightarrow} BUC \overset{d}{\hookrightarrow} b_\infty^0, \tag{5.2}$$

these realizations are well-defined closed and densely defined linear operators in L_p and BUC, respectively. Furthermore, Theorem 4.1 guarantees the existence of $\kappa_q \geq 1$ and $\omega_q > 0$ for $1 \leq q \leq \infty$ such that $\omega_q + \Sigma_\vartheta \subset \rho(-\mathcal{A}_q)$ and

$$(\lambda + \mathcal{A}_q)^{-1} = (\lambda + \mathcal{A}_{b_{q,\infty}^0})^{-1} \,|\, \mathfrak{F}_q, \qquad \lambda \in \omega_q + \Sigma_\vartheta, \quad 1 \leq q \leq \infty, \tag{5.3}$$

where $\mathfrak{F}_q := \mathfrak{F}_q^0$ (see (2.1)). From (5.2) and (5.3) we easily deduce that

$$B_{q,1}^m \overset{d}{\hookrightarrow} \mathfrak{F}_q^m \overset{d}{\hookrightarrow} D(\mathcal{A}_q) \overset{d}{\hookrightarrow} b_{q,\infty}^m, \qquad 1 \leq q \leq \infty. \tag{5.4}$$

In contrast to the case where $\mathcal{A}$ is looked at as a map between Besov spaces, $D(\mathcal{A}_q)$ depends on $\mathcal{A}$, in general. In fact, in the scalar case it has been shown by Guidetti [15, Proposition 1.12 and Remark 1.13] that, if $\widetilde{\mathcal{A}}$ and $\overline{\mathcal{A}}$ are strongly elliptic with constant coefficients, then neither $D(\widetilde{\mathcal{A}}_1)$ is contained in $D(\overline{\mathcal{A}}_1)$ nor conversely, unless the principal symbols of these operators are proportional. Proposition 1.11 of [15] also shows that $D(\widetilde{\mathcal{A}}_1) \neq W_1^m$ if $n \geq 2$. The same proofs apply to BUC (see [5, Theorem VII.2.4.5] and also [23] for a related observation).

In the light of these negative results the following proposition is of considerable interest since it guarantees that all real interpolation spaces between $\mathfrak{F}_q$ and $D(\mathcal{A}_q)$ are independent of $\mathcal{A}$ and explicitly known.

Proposition 5.1. *Suppose that $0 < \theta < 1$. Then*

$$\big(\mathfrak{F}_q, D(\mathcal{A}_q)\big)_{\theta,r} \doteq B_{q,r}^{\theta m}, \quad \big(\mathfrak{F}_q, D(\mathcal{A}_q)\big)_{\theta,\infty}^0 \doteq b_{q,\infty}^{\theta m}, \qquad q, r \in [1,\infty].$$

Proof This follows from (5.2) and (5.4), thanks to (1.19) and (1.20). $\square$

Corollary 5.2. *Suppose that $\theta m \notin \mathbb{N}$. Then*

$$\big(L_p, D(\mathcal{A}_p)\big)_{\theta,p} \doteq W_p^{\theta m}, \quad \big(BUC, D(\mathcal{A}_\infty)\big)_{\theta,\infty} \doteq BUC^{\theta m},$$

and

$$\big(BUC, D(\mathcal{A}_\infty)\big)_{\theta,\infty}^0 \doteq buc^{\theta m}.$$

By interpolating the resolvent estimates of Theorem 4.1 for $\mathcal{A}_{b^0_{q,r}}$ with $(\cdot,\cdot)^0_{s/m,\infty}$ we infer from (1.20) that, given $q,r\in[1,\infty]$,

$$\|(\lambda+\mathcal{A}_{b^0_{q,r}})^{-1}\|_{\mathcal{L}(b^0_{q,r},b^s_{q,r})}\le c(s)\,|\lambda|^{-1+s/m}\,,\qquad \lambda\in\omega_q+\Sigma_\vartheta,\quad 0<s<m.$$

Hence it follows from (5.3) that, given $\varepsilon\in(0,1)$ and $q\in[1,\infty]$, there exists $\kappa\ge1$ such that

$$\|(\lambda+\mathcal{A}_q)^{-1}\|_{\mathcal{L}(\mathfrak{F}_q)}\le\kappa\,|\lambda|^{-1+\varepsilon}\,,\qquad \lambda\in\omega_q+\Sigma_\vartheta.$$

It is the purpose of the following considerations to show that we can set $\varepsilon=0$ in these estimates. This implies that $-\mathcal{A}_q$ generates a strongly continuous analytic semigroup on $\mathfrak{F}_q$ if $\mathcal{A}$ is normally elliptic.

Given a Banach space X, we set

$$\mathcal{F}L_1(X):=\big\{\,u\in\mathcal{S}'(X)\ ;\ \mathcal{F}^{-1}u\in L_1(X)\,\big\}.$$

It is a Banach space with the norm $u\mapsto\|u\|_{\mathcal{F}L_1}:=\|\mathcal{F}^{-1}u\|_1$.

For $r\in\mathbb{R}$ we write S_r for the vector space of all $a\in C^{n+1}\big((\mathbb{R}^n)^\bullet,\mathcal{L}(E)\big)$ for which there exists a constant c satisfying

$$|\partial^\alpha a(\xi)|\le c(1+|\xi|)^{r-|\alpha|},\qquad |\alpha|\le n+1,\quad \xi\in(\mathbb{R}^n)^\bullet. \tag{5.5}$$

It is a Banach space, its norm, $\|\cdot\|_{\mathsf{S}_r}$, being the infimum of all c satisfying (5.5). Note that

$$\mathsf{S}_r\hookrightarrow\mathsf{S}_t,\qquad r<t. \tag{5.6}$$

The following lemma is the basic tool for proving the desired result.

Lemma 5.3. *If $r>0$ then $\mathsf{S}_{-r}\hookrightarrow\mathcal{F}L_1\big(\mathcal{L}(E)\big)$.*

Proof This follows from [6, Corollary 4.4]. □

Our next result, which is taken from [15, Proposition 1.1], shows that $\mathcal{F}^{-1}a$ is integrable near infinity if $a\in\mathsf{S}_0$. For completeness we include its proof. Henceforth all integrals are over $\mathbb{R}^n$.

Lemma 5.4. *If $\varepsilon\in(0,1)$ then there exists $c:=c(\varepsilon)>0$ such that*

$$|\mathcal{F}^{-1}a(x)|\le c\,\|a\|_{\mathsf{S}_0}\,|x|^{-n-\varepsilon},\qquad x\in(\mathbb{R}^n)^\bullet,\quad a\in\mathsf{S}_0.$$

Proof Suppose that $a\in\mathsf{S}_0$. If $|\alpha|=n$ then we see from (5.5) that $\partial^\alpha a$ vanishes at infinity and that $\partial_j\partial^\alpha a\in L_1\big(\mathcal{L}(E)\big)$ for $1\le j\le n$. Consequently,

$$\int\partial_j\partial^\alpha a\,d\xi=0,\qquad 1\le j\le n.$$

From this we deduce that, given $\beta \in \mathbb{N}^n$ with $|\beta| = 1$,

$$x^{\alpha+\beta}\mathcal{F}^{-1}a(x) = (2\pi)^{-n} \int (e^{i\,x\cdot\xi} - 1)\, |x\cdot\xi|^{\varepsilon-1}\, |x\cdot\xi|^{1-\varepsilon}\, D^{\alpha+\beta}a(\xi)\, d\xi$$

for $x \in (\mathbb{R}^n)^\bullet$. Thus, $|e^{i\,x\cdot\xi} - 1|\, |x\cdot\xi|^{\varepsilon-1}$ being uniformly bounded, it follows from (5.5) that

$$|x^{\alpha+\beta}|\, |\mathcal{F}^{-1}a(x)| \le c\, \|a\|_{\mathsf{S}_0}\, |x|^{1-\varepsilon} \int |\xi|^{1-\varepsilon}(1+|\xi|)^{-n-1}\, d\xi \le c\, \|a\|_{\mathsf{S}_0}\, |x|^{1-\varepsilon}$$

for $x \in (\mathbb{R}^n)^\bullet$ and $|\beta| = 1$. Hence

$$|x^\alpha|\, |\mathcal{F}^{-1}a(x)| \le c\, \|a\|_{\mathsf{S}_0}\, |x|^{-\varepsilon}, \qquad x \in (\mathbb{R}^n)^\bullet.$$

Since $|x|^n = \big[(x_1^2 + \cdots + x_n^2)^n\big]^{1/2} \le c \sum_{|\alpha|=n} |x^\alpha|$ by the multinomial theorem, the assertion follows. $\square$

In the following, we set $\mathsf{H} := \Sigma_{\vartheta/m}$ and

$$\Lambda(\xi,\eta) := \sqrt{|\xi|^2 + |\eta|^2}, \quad (\xi^*,\eta^*) := (\xi,\eta)/\Lambda(\xi,\eta), \qquad (\xi,\eta) \in (\mathbb{R}^n \times \mathsf{H})^\bullet.$$

Given $r \in \mathbb{R}$, we denote by H_r the vector space of all $a\colon (\mathbb{R}^n)^\bullet \times \mathsf{H} \to \mathcal{L}(E)$ being positively r-homogeneous (that is, positively homogeneous of degree r) with

$$a_\eta := a(\cdot,\eta) \in C^{n+1}\big((\mathbb{R}^n)^\bullet, \mathcal{L}(E)\big), \qquad \eta \in \mathsf{H},$$

and

$$|\partial_\xi^\alpha a(\xi^*,\eta^*)| \le c, \qquad |\alpha| \le n+1, \quad (\xi,\eta) \in (\mathbb{R}^n \times \mathsf{H})^\bullet. \tag{5.7}$$

It is a normed vector space, its norm being the infimum of all c in (5.7).

Lemma 5.5. *Suppose that $r \ge 0$. Then*

$$\|a_\eta\|_{\mathsf{S}_{-r}} \le c\, \|a\|_{H_{-r}}, \qquad \eta \in \mathsf{H}, \quad |\eta| \ge 1, \quad a \in H_{-r}. \tag{5.8}$$

If $r > 0$ then $a_\eta \in \mathcal{F}L_1\big(\mathcal{L}(E)\big)$ and

$$\|a_\eta\|_{\mathcal{F}L_1} \le c\, \|a\|_{H_{-r}}\, |\eta|^{-r}, \qquad \eta \in \dot{\mathsf{H}}, \quad a \in H_{-r}. \tag{5.9}$$

Proof By differentiating the identity $a(t\xi, t\xi) = t^{-r}a(\xi,\eta)$ we find

$$(\partial_\xi^\alpha a)(t\xi, t\xi) = t^{-r-|\alpha|}\partial_\xi^\alpha a(\xi,\eta), \qquad t > 0, \quad (\xi,\eta) \in (\mathbb{R}^n \times \mathsf{H})^\bullet.$$

Upon replacing (ξ,η) by (ξ^*,η^*) and t by $\Lambda(\xi,\eta)$, respectively, it follows that

$$|\partial^\alpha a_\eta(\xi)| \le \|a\|_{H_{-r}}\, \Lambda_\eta^{-r-|\alpha|}(\xi), \qquad |\alpha| \le n+1, \quad (\xi,\eta) \in (\mathbb{R}^n \times \mathsf{H})^\bullet. \tag{5.10}$$

This implies (5.8).

Suppose that $r > 0$. Then (5.8) and Lemma 5.3 guarantee that

$$\|a_\eta\|_{\mathcal{F}L_1} \le c\,\|a\|_{H_{-r}}\,, \qquad \eta \in \mathsf{H}, \quad |\eta| = 1. \tag{5.11}$$

Denote by σ_t dilation by $t > 0$, that is, $\sigma_t u(x) := u(x/t)$ for functions u on $\mathbb{R}^n$ and extended to distributions in the usual way (e.g., [6, Section 1]). Then

$$\mathcal{F}^{-1} \circ \sigma_t = t^n \sigma_{1/t} \circ \mathcal{F}^{-1}, \qquad \|\sigma_t u\|_1 = t^n\,\|u\|_1\,. \tag{5.12}$$

Note that $a_\eta = |\eta|^{-r}\,\sigma_{|\eta|} a_{\eta/|\eta|}$ for $\eta \in \dot{\mathsf{H}}$. Hence (5.12) implies

$$\mathcal{F}^{-1} a_\eta = |\eta|^{-r+n}\,\sigma_{1/|\eta|} \mathcal{F}^{-1} a_{\eta/|\eta|}$$

and

$$\|a_\eta\|_{\mathcal{F}L_1} = |\eta|^{-r}\,\|a_{\eta/|\eta|}\|_{\mathcal{F}L_1}$$

for $\eta \in \dot{\mathsf{H}}$. Now (5.9) follows from (5.11). $\square$

Suppose that $k := \big(y \mapsto k(\cdot, y)\big) \in BUC\big(L_1(\mathcal{L}(E))\big)$ and set

$$K_1 u(x) := \int k(x-y, y) u(y)\,dy, \quad K_\infty u(x) := \int k(x-y, x) u(y)\,dy \tag{5.13}$$

for $x \in \mathbb{R}^n$ and $u : \mathbb{R}^n \to E$, whenever these integrals exist.

Lemma 5.6. *$K_j \in \mathcal{L}(\mathfrak{F}_j)$ and*

$$\|K_j\|_{\mathcal{L}(\mathfrak{F}_j)} \le c\,\|k\|\,, \qquad k \in BUC\big(L_1(\mathcal{L}(E))\big),$$

for $j \in \{1, \infty\}$, where $\|\cdot\|$ is the norm in $BUC\big(L_1(\mathcal{L}(E))\big)$.

Proof If $u \in L_1$ then

$$\|K_1 u\|_1 \le \iint |k(x-y, y)|\,dx\,|u(y)|\,dy = \int \|k(\cdot, y)\|_1\,|u(y)|\,dy \le \|k\|\,\|u\|_1\,.$$

For $u \in BUC$ we find

$$|K_\infty u(x)| \le \int |k(x-y, x)|\,|u(y)|\,dy \le \int |k(x-y, x)|\,dy\,\|u\|_\infty \le \|k\|\,\|u\|_\infty$$

for $x \in \mathbb{R}^n$. Furthermore,

$$\begin{aligned} |K_\infty u(x) - K_\infty u(y)| &\le \int \big[k(z, x) - k(z, y)\big]\,|u(x-z)|\,dz \\ &\quad + \int |k(z, y)|\,|u(x-z) - u(y-z)|\,dz \\ &\le \|k(\cdot, x) - k(\cdot, y)\|_1\,\|u\|_\infty + \sup_{z \in \mathbb{R}^n} |u(x-z) - u(y-z)|\,\|k\| \end{aligned}$$

for $x, y \in \mathbb{R}^n$. Consequently, $K_\infty u \in BUC$ and the assertion follows. $\square$

Now we set

$$a(x,\xi,\eta) := \eta^m + \sigma\mathcal{A}(x,\xi), \quad b(x,\xi,\eta) := a(x,\xi,\eta)^{-1}$$

for $(x,\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathsf{H}$. It is obvious that

$$\bigl(x \mapsto a(x,\cdot,\cdot)\bigr) \in BUC^\rho(H_m). \tag{5.14}$$

Remark 3.1(a) implies

$$|b(x,\xi,\eta)| \le \kappa_0(|\xi|^m + |\eta|^m)^{-1}, \qquad (x,\xi,\eta) \in \mathbb{R}^n \times (\mathbb{R}^n \times \mathsf{H})^\bullet. \tag{5.15}$$

It is easily verified that, given $\alpha \in (\mathbb{N}^n)^\bullet$, the derivative $\partial_\xi^\alpha b$ is a sum of terms of the form

$$\pm b(\partial_\xi^{\alpha_1} a) b(\partial_\xi^{\alpha_2} a) \cdot \cdots \cdot b(\partial_\xi^{\alpha_j} a) b, \tag{5.16}$$

where this sum extends over all $j \in \{1,\ldots,|\alpha|\}$ and all $\alpha_1,\ldots,\alpha_j \in (\mathbb{N}^n)^\bullet$ satisfying $\alpha_1 + \cdots + \alpha_j = \alpha$. From this we easily deduce that

$$\bigl(x \mapsto b(x,\cdot,\cdot)\bigr) \in BUC^\rho(H_{-m}). \tag{5.17}$$

Consequently, Lemmas 5.5 and 5.3 imply that

$$k_\eta(x,y) := \mathcal{F}^{-1} b(y,\cdot,\eta)(x), \qquad (x,y,\eta) \in \mathbb{R}^n \times \mathbb{R}^n \times \dot{\mathsf{H}},$$

is well-defined and that $k_\eta := \bigl(y \mapsto k_\eta(\cdot,y)\bigr) \in BUC\bigl(L_1(\mathcal{L}(E))\bigr)$ with

$$\|k_\eta\| \le c\,|\eta|^{-m}, \qquad \eta \in \dot{\mathsf{H}}. \tag{5.18}$$

We set

$$K_1(\eta)u(x) := \int k_\eta(x-y,y)u(y)\,dy, \qquad x \in \mathbb{R}^n, \quad \eta \in \dot{\mathsf{H}}.$$

Then it follows from Lemma 5.6 that

$$K_1(\eta) \in \mathcal{L}(L_1), \quad \|K_1(\eta)\|_{\mathcal{L}(L_1)} \le c\,|\eta|^{-m}, \qquad \eta \in \dot{\mathsf{H}}. \tag{5.19}$$

We also denote by $\mathcal{A}_\pi$ the principal part $\sum_{|\alpha|=m} a_\alpha D^\alpha$ of $\mathcal{A}$.

Lemma 5.7. *There exist $r > 0$ and $T_1(\eta) \in \mathcal{L}(L_1)$ satisfying*

$$\|T_1(\eta)\|_{\mathcal{L}(L_1)} \le c\,|\eta|^{-r} \tag{5.20}$$

and

$$(\eta^m + \mathcal{A}_\pi)K_1(\eta) = 1_{L_1} + T_1(\eta)$$

for $\eta \in \dot{\mathsf{H}}$.

Proof First suppose that $a_\alpha \in BUC^\infty\big(\mathcal{L}(E)\big)$ for $|\alpha| = m$. Then, if $u \in L_1$,

$$a_\alpha D^\alpha\big(K_1(\eta)u\big) = D^\alpha\big(a_\alpha K_1(\eta)u\big) - \sum_{0<\beta\leq\alpha} \binom{\alpha}{\beta} D^\beta a_\alpha D^{\alpha-\beta}\big(K_1(\eta)u\big) \tag{5.21}$$

for $|\alpha| = m$. Given $\varphi \in \mathcal{D}_{\mathbb{C}}$, we see that

$$\begin{aligned}
D^\alpha\big(a_\alpha K_1(\eta)u\big)(\varphi) &= (-1)^{|\alpha|}\big(a_\alpha K_1(\eta)u\big)(D^\alpha\varphi) \\
&= (-1)^{|\alpha|}\int a_\alpha(x)\big(K_1(\eta)u\big)(x) D^\alpha\varphi(x)\,dx \\
&= (-1)^{|\alpha|}\iint a_\alpha(x)k_\eta(x-y,y)u(y)\,dy\,D^\alpha\varphi(x)\,dx \\
&= (-1)^{|\alpha|}\iint a_\alpha(y)k_\eta(x-y,y)u(y)\,dy\,D^\alpha\varphi(x)\,dx + R_\alpha(\eta)(\varphi) \\
&= (-1)^{|\alpha|}\int a_\alpha(y)k_\eta(\cdot-y,y)u(y)\,dy\,(D^\alpha\varphi) + R_\alpha(\eta)(\varphi),
\end{aligned} \tag{5.22}$$

where

$$R_\alpha(\eta)(\varphi) := (-1)^{|\alpha|}\iint \big[a_\alpha(x) - a_\alpha(y)\big]k_\eta(x-y,y)u(y)D^\alpha\varphi(x)\,dy\,dx.$$

Note that

$$\begin{aligned}
&\int |a_\alpha(y)| \int |k_\eta(x-y,y)|\,|D^\alpha\varphi(x)|\,dx\,|u(y)|\,dy \\
&\quad\leq \|a_\alpha\|_\infty\,\|D^\alpha\varphi\|_\infty \iint |k_\eta(x-y,y)|\,dx\,|u(y)|\,dy \\
&\quad\leq \|a_\alpha\|_\infty\,\|D^\alpha\varphi\|_\infty\,\|k_\eta\|\,\|u\|_1 < \infty.
\end{aligned}$$

Hence, by Fubini's theorem and denoting by τ_x translation,

$$\begin{aligned}
&(-1)^{|\alpha|}\int a_\alpha(y)k_\eta(\cdot-y,y)u(y)\,dy\,(D^\alpha\varphi) \\
&\quad= (-1)^{|\alpha|}\int a_\alpha(y)\big[\tau_y k_\eta(\cdot,y)(D^\alpha\varphi)\big]u(y)\,dy \\
&\quad= \int a_\alpha(y)\big[D^\alpha k_\eta(\cdot,y)\big](\tau_{-y}\varphi)u(y)\,dy.
\end{aligned} \tag{5.23}$$

Setting $b_\alpha(y,\xi,\eta) := \xi^\alpha b(y,\xi,\eta)$, we find

$$a_\alpha(y)D^\alpha k_\eta(\cdot,y) = a_\alpha(y)D^\alpha\mathcal{F}^{-1}b(y,\cdot,\eta) = \mathcal{F}^{-1}\big(a_\alpha(y)b_\alpha(y,\cdot,\eta)\big). \tag{5.24}$$

Since

$$\begin{aligned}
\big(K_1(\eta)u\big)(\varphi) &= \int\varphi(x)\int k_\eta(x-y,y)u(y)\,dy\,dx = \int k_\eta(\cdot-y,y)(\varphi)u(y)\,dy \\
&= \int\big[\mathcal{F}^{-1}b(y,\cdot,\eta)(\tau_{-y}\varphi)\big]u(y)\,dy,
\end{aligned}$$

we deduce from (5.23) and (5.24), setting $\mathbf{1}(x) := 1$ for $x \in \mathbb{R}^n$, that

$$
\begin{aligned}
\eta^m \big(K_1(\eta)u\big)(\varphi) &+ \sum_{|\alpha|=m} (-1)^{|\alpha|} \int a_\alpha(y) k_\eta(\cdot - y, y) u(y)\, dy\, (D^\alpha \varphi) \\
&= \int \big[\mathcal{F}^{-1}\big(a(y,\cdot,\eta) b(y,\cdot,\eta)\big)(\tau_{-y}\varphi)\big] u(y)\, dy \\
&= \int \big[\mathcal{F}^{-1}(1_E \otimes \mathbf{1})(\tau_{-y}\varphi)\big] u(y)\, dy \\
&= \int (1_E \otimes \delta)(\tau_{-y}\varphi) u(y)\, dy = \int \varphi(y) u(y)\, dy = u(\varphi).
\end{aligned}
\tag{5.25}
$$

We also find

$$
\begin{aligned}
(-1)^{|\alpha|} &\iint a_\alpha(x) k_\eta(x-y,y) u(y) D^\alpha \varphi(x)\, dy\, dx \\
&= \iint a_\alpha(x) D_x^\alpha k_\eta(x-y,y) u(y) \varphi(x)\, dy\, dx \\
&\quad + \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} \iint D^\beta a_\alpha(x) D_x^{\alpha-\beta} k_\eta(x-y,y) u(y)\varphi(x)\, dy\, dx \\
&= \iint a_\alpha(x) D_x^\alpha k_\eta(x-y,y) u(y) \varphi(x)\, dy\, dx \\
&\quad + \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} \big[D^\beta a_\alpha D^{\alpha-\beta} K_1(\eta) u\big](\varphi),
\end{aligned}
\tag{5.26}
$$

thanks to the fact that, given $\gamma \in \mathbb{N}^n$ with $|\gamma| < m$,

$$
D^\gamma k_\eta(\cdot, y) = \mathcal{F}^{-1} b_\gamma(y,\cdot,\eta) \in L_1\big(\mathcal{L}(E)\big),
$$

as follows from Lemma 5.3 since $b_\gamma(y,\cdot,\eta) \in \mathsf{S}_{-1}$ by (5.6) and (5.10). Thus the application of Fubini's theorem in (5.26) is justified. Now, by combining (5.21), (5.22), (5.25), and (5.26), we obtain

$$
\big[(\eta^m + \mathcal{A}_\pi) K_1(\eta) u\big](\varphi) = u(\varphi) + \big(T_1(\eta)u\big)(\varphi),
\tag{5.27}
$$

where

$$
T_1(\eta)u(\varphi) := \iint \big(\mathcal{A}_\pi(x, D_x) - \mathcal{A}_\pi(y, D_x)\big) k_\eta(x-y,y) u(y) \varphi(x)\, dy\, dx.
$$

Put

$$
\begin{aligned}
t_\eta(x,y) &:= \big(\mathcal{A}_\pi(x, D_x) - \mathcal{A}_\pi(y, D_x)\big) k_\eta(x-y,y) \\
&= \sum_{|\alpha|=m} \big(a_\alpha(x) - a_\alpha(y)\big) \mathcal{F}^{-1} b_\alpha(y,\cdot,\eta)(x-y).
\end{aligned}
$$

Note that $b_\alpha(y,\cdot,\cdot)\in H_0$, uniformly with respect to $y\in\mathbb{R}^n$. Hence $b_\alpha(y,\cdot,\eta)\in \mathsf{S}_0$ and (5.10) imply $\|b_\alpha(y,\cdot,\eta)\|_{\mathsf{S}_0}\le c$ for $|\alpha|=m$ and $|\eta|\ge 1$. Moreover, by (5.12),

$$\mathcal{F}^{-1}b_\alpha(y,\cdot,\eta)(x)=|\eta|^n\,\mathcal{F}^{-1}b_\alpha(y,\cdot,\eta/|\eta|)\,(|\eta|\,x),\qquad x,y\in\mathbb{R}^n.$$

Fix $r\in(0,\rho)$. Then it follows from Lemma 5.4 and $BUC^\rho\big(\mathcal{L}(E)\big)\hookrightarrow BUC^r\big(\mathcal{L}(E)\big)$ that

$$\begin{aligned}|t_\eta(x,y)|&\le c\,|x-y|^r\,|\eta|^n\,\big|\mathcal{F}^{-1}b_\alpha(y,\cdot,\eta/|\eta|)\,\big(|\eta|\,(x-y)\big)\big|\\&\le c(\varepsilon)\,|\eta|^{n-r}\,\big||\eta|\,(x-y)\big|^{r-n-\varepsilon}\end{aligned}$$

for $x,y\in\mathbb{R}^n$ and $0<\varepsilon<1$. Thus, fixing ε in $(0,r)$ if $|\eta|\,(x-y)\le 1$ and in $(r,1)$ otherwise, we see that

$$\iint|t_\eta(x-y)|\,|u(y)|\,dy\,dx\le\iint|t_\eta(x-y)|\,dx\,|u(y)|\,dy\le c\,|\eta|^{-r}\,\|u\|_1. \tag{5.28}$$

Hence, by Fubini's theorem,

$$T_1(\eta)u(\varphi)=\iint t_\eta(x,y)u(y)\,dy\,\varphi(x)\,dx.$$

This being true for all $\varphi\in\mathcal{D}_{\mathbb{C}}$, it follows that

$$T_1(\eta)u=\int t_\eta(\cdot,y)u(y)\,dy.$$

Thus (5.28) implies that $T_1(\eta)\in\mathcal{L}(L_1)$ and that (5.20) is satisfied. Since (5.27) is true for every test function φ and every $u\in L_1$, the assertion has been shown under the additional assumption that $a_\alpha\in BUC^\infty$ for $|\alpha|=m$.

In the general case, we approximate a_α by elements from $BUC^\infty\big(\mathcal{L}(E)\big)$ and derive the desired result in the obvious way. □

We fix a nonnegative smooth function ψ on $\mathbb{R}^n$ having its support in the unit ball and satisfying $\int\psi\,dx=1$. Then we set $\psi_\varepsilon(x):=\varepsilon^{-n}\psi(x/\varepsilon)$ for $x\in\mathbb{R}^n$ and $\varepsilon>0$. We also set $\delta:=1/(1+\rho)$ and

$$a^\delta(\cdot,\xi,\eta):=\psi_{\Lambda^{-\delta}(\xi,\eta)}*a(\cdot,\xi,\eta),\qquad(\xi,\eta)\in(\mathbb{R}^n\times\mathsf{H})^\bullet.$$

Then

$$a^\delta(\cdot,\xi,\eta)=\Lambda^m(\xi,\eta)\psi_{\Lambda^{-\delta}(\xi,\eta)}*a(\cdot,\xi^*,\eta^*).$$

From (5.14) and well-known properties of convolutions it follows that

$$\lim_{\varepsilon\to0}\psi_\varepsilon*a(\cdot,\xi^*,\eta^*)=a(\cdot,\xi^*,\eta^*)\quad\text{in }BUC\big(\mathcal{L}(E)\big),$$

uniformly with respect to $(\xi,\eta)\in(\mathbb{R}^n\times\mathsf{H})^\bullet$. Thus, since $\Lambda^{-\delta}(\xi,\eta)\le|\eta|^{-\delta}$, we infer from Lemma 3.3 that there exits $\eta_0>0$ such that

$$d_0(\cdot,\xi,\eta):=\left[\psi_{\Lambda^{-\delta}(\xi,\eta)}*a(\cdot,\xi^*,\eta^*)\right]^{-1}$$

is well-defined for $\xi\in\mathbb{R}^n$ and $|\eta|\ge\eta_0$, belongs to $BUC\big(\mathcal{L}(E)\big)$, and satisfies

$$\|d_0(\cdot,\xi,\eta)\|_\infty\le 2\,\|b(\cdot,\xi^*,\eta^*)\|_\infty\le 2\kappa_0, \tag{5.29}$$

uniformly with respect to $(\xi,\eta)\in\mathbb{R}^n\times\mathsf{H}$ with $|\eta|\ge\eta_0$. Hence

$$d(\cdot,\xi,\eta):=\left[a^\delta(\cdot,\xi,\eta)\right]^{-1}\in BUC\big(\mathcal{L}(E)\big) \tag{5.30}$$

is also well-defined with $\|d(\cdot,\xi,\eta)\|_\infty\le 2\kappa_0\Lambda^{-m}(\xi,\eta)$ for $\xi\in\mathbb{R}^n$ and $\eta\in\mathsf{H}$ with $|\eta|\ge\eta_0$.

Finally, we set

$$e:=(a-a^\delta)d+\sum_{0<|\beta|\le m}\frac{1}{\beta!}\,\partial_\xi^\beta a D_x^\beta d, \tag{5.31}$$

so that e is defined for $x,\xi\in\mathbb{R}^n$ and $\eta\in\mathsf{H}$ with $|\eta|\ge\eta_0$.

Lemma 5.8. *Put $r:=\rho\delta/2$. Then*

(i) $\big(x\mapsto d(x,\cdot,\eta)\big)\in BUC(\mathcal{F}L_1)$ *and*

$$\sup_{x\in\mathbb{R}^n}\|d(x,\cdot,\eta)\|_{\mathcal{F}L_1}\le c\,|\eta|^{-m}, \qquad |\eta|\ge\eta_0.$$

(ii) $\big(x\mapsto e(x,\cdot,\eta)\big)\in BUC(\mathcal{F}L_1)$ *and*

$$\sup_{x\in\mathbb{R}^n}\|e(x,\cdot,\eta)\|_{\mathcal{F}L_1}\le c\,|\eta|^{-r}, \qquad |\eta|\ge\eta_0.$$

Proof Set $\widetilde{\eta}:=\eta/|\eta|$ and $\varepsilon_\eta:=(|\eta|\,\Lambda_1)^{-\delta}$, where $\Lambda_1(\xi)=\Lambda(\xi,1)=\Lambda(\xi,\widetilde{\eta})$. Then

$$a^\delta(\cdot,\xi,\eta)=|\eta|^m\,\psi_{\varepsilon_\eta(\xi/|\eta|)}*a(\cdot,\xi/|\eta|,\widetilde{\eta}), \qquad \xi\in\mathbb{R}^n, \quad \eta\in\dot{\mathsf{H}}.$$

Hence

$$d(\cdot,\xi,\eta)=|\eta|^{-m}\left[\psi_{\varepsilon_\eta(\xi/|\eta|)}*a(\cdot,\xi/|\eta|,\widetilde{\eta})\right]^{-1}, \qquad \xi\in\mathbb{R}^n, \quad |\eta|\ge\eta_0.$$

Thus we infer from (5.12) that

$$\|d(x,\cdot,\eta)\|_{\mathcal{F}L_1}=|\eta|^{-m}\left\|\left[\psi_{\varepsilon_\eta}*a(x,\cdot,\widetilde{\eta})\right]^{-1}\right\|_{\mathcal{F}L_1}, \qquad x\in\mathbb{R}^n, \quad |\eta|\ge\eta_0,$$

where convolution is taken with respect to x, of course. Consequently, Lemma 5.3 guarantees that it suffices to show that

$$\big(x \mapsto \big[\psi_{\varepsilon_\eta} * a(x,\cdot,\widetilde{\eta})\big]^{-1}\big) \in BUC(\mathsf{S}_{-m}) \tag{5.32}$$

with

$$\sup_{x\in\mathbb{R}^n} \big\| \big[\psi_{\varepsilon_\eta} * a(x,\cdot,\widetilde{\eta})\big]^{-1}\big\|_{\mathsf{S}_{-m}} \le c, \qquad |\eta| \ge \eta_0. \tag{5.33}$$

First note that

$$\psi_{\varepsilon_\eta(\xi)} * a(\cdot,\xi,\widetilde{\eta}) = \Lambda_1^m(\xi)\psi_{\varepsilon_\eta(\xi)} * a(\cdot,\xi^*,\widetilde{\eta}^*),$$

and the arguments leading to (5.29) imply

$$\begin{aligned} \big\| \big[\psi_{\varepsilon_\eta(\xi)} * a(\cdot,\xi,\widetilde{\eta})\big]^{-1}\big\|_\infty &= \Lambda_1^{-m}(\xi)\, \big\| \big[\psi_{\varepsilon_\eta(\xi)} * a(\cdot,\xi^*,\widetilde{\eta}^*)\big]^{-1}\big\|_\infty \\ &\le c\Lambda_1^{-m}(\xi) \end{aligned} \tag{5.34}$$

for $\xi \in \mathbb{R}^n$ and $|\eta| \ge \eta_0$.

Second, it is not difficult to verify (cf. the proof of [9, Lemma 5.1]) that

$$\partial_{\xi^j}\big(\psi_{\varepsilon_\eta} * a(x,\cdot,\widetilde{\eta})\big) = (\psi_1)_{\varepsilon_\eta} * a_1(x,\cdot,\widetilde{\eta}) + \psi_{\varepsilon_\eta} * \partial_{\xi^j} a(x,\cdot,\widetilde{\eta}),$$

where ψ_1 belongs to $\mathcal{D}_{\mathbb{C}}$, has its support in the unit ball, and is independent of η, and where $a_1(\cdot,\xi,\widetilde{\eta}) := -\delta\xi^j\Lambda_1^{-2}(\xi)a(\cdot,\xi,\widetilde{\eta})$. Clearly,

$$\|a_1(\cdot,\xi,\widetilde{\eta})\|_\infty + \|\partial_{\xi^j} a(\cdot,\xi,\widetilde{\eta})\|_\infty \le c\Lambda_1^{m-1}(\xi)$$

and, consequently,

$$\big\|\partial_{\xi^j}\big(\psi_{\varepsilon_\eta} * a(\cdot,\xi,\widetilde{\eta})\big)\big\|_\infty \le c\Lambda_1^{m-1}(\xi).$$

Proceeding by induction, it follows that

$$\big(x \mapsto \Lambda_1^{-m+|\alpha|}(\xi)\partial_\xi^\alpha\big(\psi_{\varepsilon_\eta} * a(\cdot,\xi,\widetilde{\eta})\big)(x)\big) \in BUC\big(\mathcal{L}(E)\big)$$

for $|\alpha| \le n+1$, uniformly with respect to $\xi \in \mathbb{R}^n$ and $|\eta| \ge \eta_0$. From this (5.32) and (5.33) are obtained by employing (5.34) and the analogue of (5.16).

(ii) By modifying the arguments of step (i) in the obvious way we find that

$$\big(x \mapsto \Lambda_1^{-m+|\alpha|-\delta\,|\beta|}(\xi,\eta)\partial_\xi^\alpha\partial_x^\beta a^\delta(x,\xi,\eta)\big) \in BUC\big(\mathcal{L}(E)\big)$$

and

$$\big(x \mapsto \Lambda_1^{m+|\alpha|-\delta\,|\beta|}(\xi,\eta)\partial_\xi^\alpha\partial_x^\beta d(x,\xi,\eta)\big) \in BUC\big(\mathcal{L}(E)\big)$$

for $|\alpha| \le n+1$ and $|\beta| \le m$, uniformly with respect to $\xi \in \mathbb{R}^n$ and $|\eta| \ge \eta_0$ (cf. the proofs of [9, Lemmas 5.1 and 5.3]).

Similarly, (5.14) and the proof of [9, Lemma 5.2] show that

$$\big(x \mapsto \Lambda^{-m+|\alpha|+2r}(\xi,\eta)\partial_\xi^\alpha(a-a^\delta)\,(x,\xi,\eta)\big) \in BUC\big(\mathcal{L}(E)\big)$$

for $|\alpha| \le n+1$, uniformly with respect to $\xi \in \mathbb{R}^n$ and $|\eta| \ge \eta_0$. From this and Leibniz' rule we infer that

$$\big(x \mapsto \Lambda^{2r+|\alpha|}(\xi,\eta)\partial_\xi^\alpha e(x,\xi,\eta)\big) \in BUC\big(\mathcal{L}(E)\big)$$

for $|\alpha| \le n+1$, uniformly with respect to $\xi \in \mathbb{R}^n$ and $|\eta| \ge \eta_0$. Now assertion (ii) follows from Lemma 5.3. □

We set $k_{\infty,\eta} := \mathcal{F}^{-1}d(y,\cdot,\eta)(x)$ for $x,y \in \mathbb{R}^n$ and

$$K_\infty(\eta)u(x) := \int k_{\infty,\eta}(x-y,x)u(y)\,dy, \qquad x \in \mathbb{R}^n, \quad u \in L_\infty,$$

for $\eta \in \mathsf{H}$ with $|\eta| \ge \eta_0$. Since Lemma 5.8(i) implies that $y \mapsto k_{\infty,\eta}(\cdot,y)$ belongs to $BUC\big(L_1(\mathcal{L}(E))\big)$ and

$$\sup_{y\in\mathbb{R}^n} \|k_{\infty,\eta}(\cdot,y)\|_1 \le c\,|\eta|^{-m}, \qquad |\eta| \ge \eta_0,$$

it follows from Lemma 5.6 that $K_\infty(\eta) \in \mathcal{L}(BUC)$ with

$$\|K_\infty(\eta)\|_{\mathcal{L}(BUC)} \le c\,|\eta|^{-m}, \qquad \eta \in \mathsf{H}, \quad |\eta| \ge \eta_0. \tag{5.35}$$

The following lemma is an analogue for BUC to Lemma 5.7.

Lemma 5.9. *There exist $r, \eta_0 > 0$ and $T_\infty \in \mathcal{L}(BUC)$ satisfying*

$$\|T_\infty(\eta)\|_{\mathcal{L}(BUC)} \le c\,|\eta|^{-r} \tag{5.36}$$

and

$$(\eta^m + \mathcal{A}_\pi)K_\infty(\eta) = 1_{BUC} + T_\infty(\eta) \tag{5.37}$$

for $\eta \in \mathsf{H}$ with $|\eta| \ge \eta_0$.

Proof Suppose that $u \in BUC$ and $\varphi \in \mathcal{D}_\mathbb{C}$. Similarly as in the proof of Lemma 5.7, we deduce that

$$\big[(\eta^m + \mathcal{A}_\pi)K_\infty(\eta)u\big](\varphi) = u(\varphi) + \iint t_\eta(x-y,x)u(y)\varphi(x)\,dx\,dy, \qquad |\eta| \ge \eta_0,$$

for $\varphi \in \mathcal{D}_\mathbb{C}$, where

$$t_\eta(x,y) := \mathcal{F}^{-1}e(y,\cdot,\eta)(x).$$

Lemma 5.8(ii) implies $\big(y \mapsto t_\eta(\cdot, y)\big) \in BUC\big(L_1(\mathcal{L}(E))\big)$ and

$$\sup_{y\in\mathbb{R}} \|t_\eta(\cdot, y)\|_1 \le c\,|\eta|^{-r}, \qquad |\eta| \ge \eta_0.$$

Hence, defining $T_\infty(\eta)$ by

$$T_\infty(\eta)u(x) := \int t_\eta(x-y, x)u(y)\,dy, \qquad u \in BUC, \quad |\eta| \ge \eta_0,$$

it follows from Lemma 5.6 that $T_\infty(\eta) \in \mathcal{L}(BUC)$ and (5.36) is true. Consequently,

$$\iint |t_\eta(x-y, x)|\,|u(y)|\,dy\,|\varphi(x)|\,dx \le \|u\|_\infty \sup_{y\in\mathbb{R}^n} \|t_\eta(\cdot, y)\|_1\,\|\varphi\|_1$$

and Fubini's theorem guarantee that

$$\iint t_\eta(x-y, x)u(y)\varphi(x)\,dx\,dy = \int \big(T_\infty(\eta)u\big)(x)\varphi(x)\,dx = \big(T_\infty(\eta)u\big)(\varphi).$$

This proves

$$\big[(\eta^m + \mathcal{A}_\pi)K_\infty(\eta)u\big](\varphi) = u(\varphi) + \big(T_\infty(\eta)u\big)(\varphi)$$

for $\varphi \in \mathcal{D}_{\mathbb{C}}$ and $u \in BUC$ and for $\eta \in \mathsf{H}$ with $|\eta| \ge \eta_0$. Hence (5.37) is also true. $\square$

After these preparations we can prove the main result of this section. For this we denote by $\mathcal{A}_\infty^0$ the C_0-realization of $\mathcal{A}_\infty$.

Theorem 5.10. *There exist $\kappa \ge 1$ and $\omega > 0$ such that*

$$\|(\lambda + \mathcal{A}_q)^{-1}\|_{\mathcal{L}(\mathfrak{F}_q)} + \|(\lambda + \mathcal{A}_\infty^0)^{-1}\|_{\mathcal{L}(C_0)} \le c/|\lambda|, \qquad 1 \le q \le \infty, \tag{5.38}$$

for $\lambda \in \omega + \Sigma_\vartheta$, whenever $\mathcal{A}$ is (κ_0, ϑ)-elliptic and satisfies $\sum_{|\alpha|\le m} \|a_\alpha\|_{\rho,\infty} \le M$.

Proof (i) Suppose that $q = 1$ and $a_\alpha = 0$ for $|\alpha| < m$. Then (5.19) and Lemmas 5.7 and 3.3 imply that there exists $\kappa_1 \ge 1$ such that, setting $\eta := \lambda^{1/m}$,

$$R_1(\lambda) := K_1(\eta)\big(1 + T_1(\eta)\big)^{-1} \in \mathcal{L}(L_1)$$

is well-defined for $\lambda \in \omega_1 + \Sigma_\vartheta$ and satisfies

$$\|R_1(\lambda)\|_{\mathcal{L}(L_1)} \le \kappa_1/|\lambda|, \qquad \lambda \in \omega_1 + \Sigma_\vartheta,$$

and also $(\lambda + \mathcal{A})R_1(\lambda)u = u$ for $u \in L_1$. Thus $R_1(\lambda)(L_1) \subset D(\mathcal{A}_1)$ and $R_1(\lambda)$ is a right inverse of $(\lambda + \mathcal{A}_1)$. Since we know already that $\lambda \in \rho(-\mathcal{A}_1)$ for $\lambda \in \omega_1 + \Sigma_\vartheta$, that part of assertion (5.38) concerning $q = 1$ is true if $a_\alpha = 0$ for $|\alpha| < m$.

(ii) Next suppose that $q = \infty$ and $a_\alpha = 0$ for $|\alpha| < m$. Similarly as in (i), we deduce from (5.35) and Lemmas 5.9 and 3.3 that there exist $\kappa_\infty \geq 1$ and $\omega_\infty > 0$ such that

$$(\lambda + \mathcal{A}_\infty)^{-1} = K_\infty(\eta)\big(1 + T_\infty(\eta)\big)^{-1} \in \mathcal{L}(BUC)$$

for $\lambda = \eta^m \in \omega_\infty + \Sigma_\vartheta$, and that

$$\|(\lambda + \mathcal{A}_\infty)^{-1}\|_{\mathcal{L}(BUC)} \leq \kappa_\infty/|\lambda|, \qquad \lambda \in \omega_\infty + \Sigma_\vartheta.$$

(iii) Suppose that $q \in \{1, \infty\}$ and set $\omega_0 := \omega_1 \vee \omega_\infty$. Note that $\mathcal{A} - \mathcal{A}_\pi$ belongs to $\mathcal{L}(B^{m-1}_{q,1}, \mathfrak{F}_q)$ and that (3.8) implies

$$\|\mathcal{A} - \mathcal{A}_\pi\|_{\mathcal{L}(B^{m-1}_{q,1}, \mathfrak{F}_q)} \leq c(M).$$

Since $u \mapsto \|(\omega_q + \mathcal{A}_\pi)u\|_{\mathfrak{F}_q}$ is equivalent to the graph norm of $\mathcal{A}_{\pi,q}$, we infer from $\mathcal{A}_{\pi,q}(\lambda + \mathcal{A}_{\pi,q})^{-1} = 1 - \lambda(\lambda + \mathcal{A}_{\pi,q})^{-1}$ and the validity of (5.38) for $\mathcal{A}_{\pi,q}$ that

$$\|(\lambda + \mathcal{A}_{\pi,q})^{-1}\|_{\mathcal{L}(\mathfrak{F}_q, D(\mathcal{A}_{\pi,q}))} \leq c, \qquad \lambda \in \omega_0 + \Sigma_\vartheta. \tag{5.39}$$

By interpolating (5.38) (for $\mathcal{A}_{\pi,q}$) and (5.39) we obtain from Proposition 5.1 that

$$\|(\lambda + \mathcal{A}_{\pi,q})^{-1}\|_{\mathcal{L}(\mathfrak{F}_q, B^{m-1}_{q,1})} \leq c\,|\lambda|^{-1/m}, \qquad \lambda \in \omega + \Sigma_\vartheta.$$

Now Lemma 3.3 guarantees the existence of $\omega \geq \omega_0$ such that the part of (5.38) referring to $q \in \{1, \infty\}$ is true for $\lambda \in \omega + \Sigma_\vartheta$.

(iv) Denote by $[\cdot,\cdot]_\theta$ the complex interpolation functors for $0 < \theta < 1$ and recall that $L_p = [L_1, L_\infty]_{1-1/p}$ for $1 < p < \infty$ (e.g., [11, Theorem 5.11] or [27, Theorem 1.18.6.1]). Hence the part of (5.38) concerning $q \in (1, \infty)$ follows by interpolation from (iii).

(v) It is clear that $\mathcal{A}^0_\infty$ is the C_0-realization of $\mathcal{A}_{\mathring{B}^0_\infty}$. Thus Theorem 4.4 implies that $(\lambda + \mathcal{A})^{-1}(C_0) \subset C_0$ for $\lambda \in \omega + \Sigma_\vartheta$. Now the assertion concerning $\mathcal{A}^0_\infty$ is clear. $\square$

Corollary 5.11. *If $\mathcal{A}$ is normally elliptic and has coefficients in $BUC^\rho\big(\mathcal{L}(E)\big)$ for some $\rho > 0$ then $-\mathcal{A}$ generates a strongly continuous analytic semigroup on L_p for $1 \leq p < \infty$, on BUC, and on C_0.*

The idea for the proof of (5.38) for the operator $\mathcal{A}_1$ is taken from [15], where the scalar version of Lemma 5.7 is derived. Our proof is somewhat more complicated since we have to build on the theory of vector-valued distributions and cannot employ duality arguments.

It should also be mentioned that Corollary 5.11, as far as L_1 is concerned, coincides in the case $E = \mathbb{C}^N$ with Guidetti's generation theorem [15, Theorem 1.7].

Note that, formally,

$$K_\infty(\eta)u = (2\pi)^{-n} \int e^{i\,x\cdot\xi} d(x,\xi,\eta)\widehat{u}(\xi)\, d\xi.$$

Thus $K_\infty(\eta)$ is a pseudodifferential operator. Since it has an operator-valued symbol and we are considering an arbitrary Banach space E, no general theory is available to which we could refer. Of course, the 'natural' candidate for the construction of a parametrix for $\eta^m + \mathcal{A}$ is obtained by replacing d by b. Since this symbol is not smooth we had to use the technique of symbol smoothing (see [9] for references).

If $E = \mathbb{C}$ (or $\mathbb{C}^N$) and $\mathcal{A}$ is normally elliptic, it has been shown by several authors that $-\mathcal{A}$ generates an analytic semigroup on BUC (cf. [24], [25], [18, Sections 3.1.2 and 3.2] and the bibliographic remarks in Lunardi's book). All these proofs rely on L_p-estimates (of Agmon-Douglis-Nirenberg type). Hence they cannot be generalized to the infinite-dimensional setting (see below). It should also be remarked that our proof is much simpler than the ones previously known (in the finite-dimensional setting). In addition, (5.4) is a rather precise inclusion result for $D(\mathcal{A}_\infty)$.

In the finite-dimensional case it is well-known that

$$D(\mathcal{A}_p) \doteq W_p^m, \qquad 1 < p < \infty. \tag{5.40}$$

This relies on the fact that Mikhlin's multiplier theorem holds for L_p-spaces if $1 < p < \infty$. In the case of operator-valued symbols such a theorem cannot be true, in general, as has been shown by Pisier (cf. [17]). Thus it cannot be expected that (5.40) is true, in general, if E is infinite-dimensional.

Recently, it has been shown by Weis [31] and Štrkalj and Weis [26] that operator-valued Fourier multiplier theorems are valid on L_p-spaces for $1 < p < \infty$ if the Banach spaces and the symbols belong to suitably restricted classes. It would be interesting to investigate whether these results can be used to prove that (5.40) is true under these assumptions.

Appendix

A1. Besov spaces

For $\varphi \in \mathcal{D}_{\mathbb{C}}$ we denote by $\varphi(D) \in \mathcal{L}(\mathcal{S}')$ the Fourier multiplier operator defined by $\varphi(D)u := \mathcal{F}^{-1}(\varphi\widehat{u}) = \mathcal{F}^{-1}\varphi * u$.

We fix a real-valued radial $\psi \in \mathcal{D}_{\mathbb{C}}$ satisfying $\psi(\xi) = 1$ for $|\xi| \le 1$ and having its support in $[\,|\xi| \le 3/2\,] := \{\, \xi \in \mathbb{R}^n \ ;\ |\xi| \le 3/2 \,\}$. Then we set $\psi_0 := \psi$ and $\psi_k(\xi) := \psi(2^{-k}\xi) - \psi(2^{1-k}\xi)$ for $\xi \in \mathbb{R}^n$ and $k \in \dot{\mathbb{N}}$. Note that

$$\operatorname{supp}(\psi_k) \subset [2^{k-1} \le |\xi| \le 3 \cdot 2^{k-1}], \qquad k \in \dot{\mathbb{N}}, \tag{A1.1}$$

and

$$\sum_{k=0}^{m} \psi_k(\xi) = \psi(2^{-m}\xi), \qquad \xi \in \mathbb{R}^n, \quad m \in \mathbb{N}. \tag{A1.2}$$

Let $E := (E, |\cdot|)$ be a Banach space. For $s \in \mathbb{R}$ and $p, q \in [1, \infty]$, the Besov space $B^s_{p,q} := B^s_{p,q}(\mathbb{R}^n, E)$ is defined to consist of all $u \in \mathcal{S}'$ satisfying

$$\|u\|_{B^s_{p,q}} := \left\| \left(2^{sk} \left\|\psi_k(D)u\right\|_p\right) \right\|_{\ell_q} < \infty.$$

It is a Banach space with the norm $\|\cdot\|_{B^s_{p,q}}$, and different choices of ψ lead to equivalent norms.

If $s > 0$ then $B^s_{p,q}$ can be renormed as follows: Denote by $\triangle_h := \tau_{-h} - 1$ the difference operator for $h \in \mathbb{R}^n$. If $0 < s \le 1$ and $p, q \in [1, \infty]$ then set

$$[u]_{s,p,q} := \left\| \, |h|^{-s} \, \|\triangle_h^{[s]+1} u\|_p \, \right\|_{L_q^*},$$

where $L_q^* := L_q(\mathbb{R}^n, dh/|h|^n; \mathbb{R})$. Then, given $s \in (0, \infty)$ and $p, q \in [1, \infty]$,

$$\|\cdot\|_{B^s_{p,q}} \sim \|\cdot\|_p + \sum_{|\alpha| \le [s]_-} [\partial^\alpha \cdot\,]_{s-[s]_-,p,q}, \tag{A1.3}$$

where $[s]_- := [s]$ if $s \notin \mathbb{N}$ and $[s]_- := s - 1$ for $s \in \dot{\mathbb{N}}$. For details we refer to [5, Chapter VII] (also see [29, Section III.15]).

Following Yamazaki [32] we now prove an important criterion for $v \in \mathcal{S}'$ to belong to $B^s_{p,q}$. It is based on the following lemma which we include for completeness.

A1.1 Lemma. *Suppose that $s > 0$ and $1 \le q \le \infty$. Then*

$$\left\| \left(2^{js} \sum_{k=j}^{\infty} a_k\right) \right\|_{\ell_q} \le c \left\|(2^{js} a_j)\right\|_{\ell_q}$$

and

$$\left\| \left(2^{-js} \sum_{k=0}^{j} a_k\right) \right\|_{\ell_q} \le c \left\|(2^{-js} a_j)\right\|_{\ell_q}$$

for every sequence (a_j) in $\mathbb{C}$.

Proof [32, Lemma 3.8]. □

Thanks to the multiplier theorems for vector-valued Besov spaces of [6] we can now extend [32, Theorems 3.6(1) and (3.7(1)] to our setting by following Yamazaki's arguments.

A1.2 Proposition. *Fix $s \in \mathbb{R}$ and $p, q \in [1, \infty]$. Let (v_j) be a sequence in $\mathfrak{F}_p$ satisfying $\left\| 2^{js} \|v_j\|_p \right\|_{\ell_q} < \infty$. Also suppose that* either

$$\mathrm{supp}(\widehat{v_j}) \subset [2^{j-3} \leq |\xi| \leq 2^{j+1}], \qquad j \in \dot{\mathbb{N}}. \tag{A1.4}$$

or

$$\mathrm{supp}(\widehat{v_j}) \subset [\,|\xi| \leq 2^{j+3}], \qquad j \in \mathbb{N}. \tag{A1.5}$$

Then $v := \sum_{j=0}^{\infty} v_j$ exists in $\mathcal{S}'$, belongs to $B^s_{p,q}$, and satisfies

$$\|v\|_{B^s_{p,q}} \leq c \left\| (2^{js} \|v_j\|_p) \right\|_{\ell_q}, \tag{A1.6}$$

where c is independent of (v_j), and $s > 0$ if (A1.5) *is presupposed.*

Proof (i) Suppose that $v = \sum_{j=0}^{\infty} v_j$ exists in $\mathcal{S}'$.
If (A1.3) is satisfied then (A1.1) and [6, Proposition 4.5] (with $a := 1$) imply

$$2^{ks} \|\psi_k(D)v\|_p \leq 2^{ks} \sum_{j=k-3}^{k+3} \|\psi_j(D)v_j\|_p \leq c \sum_{j=k-3}^{k+3} 2^{js} \|v_j\|_p$$

for $k \in \mathbb{N}$. Hence

$$\|v\|_{B^s_{p,q}} \leq c \left\| (2^{js} \|v_j\|_p) \right\|_{\ell_q}. \tag{A1.7}$$

If (A1.5) is true then it follows from [6, Proposition 4.5] that

$$2^{ks} \|\psi_k(D)v\|_p \leq c 2^{ks} \sum_{k-4 \leq j < \infty} \|v_j\|_p.$$

Thus, if $s > 0$ then Lemma A1.1 guarantees that (A1.7) holds in this case also.
(ii) Fix $t < s$ and note that

$$\left\| (2^{jt} \|v_j\|_p) \right\|_{\ell_1} \leq c \left\| (2^{js} \|v_j\|_p) \right\|_{\ell_q}.$$

Hence we can apply the results of (i), with (s, p) replaced by $(t, 1)$, to the partial sums $v^{(m)} := \sum_{j=0}^{m} v_j$ for $m \in \mathbb{N}$. Then we obtain from (A1.7) that

$$\|v^{(m)} - v^{(\ell)}\|_{B^t_{p,1}} \leq c \sum_{j=\ell+1}^{m} 2^{jt} \|v_j\|_p\,, \qquad 0 \leq \ell < m < \infty.$$

This shows that the series $\sum_j v_j$ converges in $B^t_{p,1}$, hence in $\mathcal{S}'$, which proves everything. $\square$

A2. Point-Wise Multipliers

Let E_0, E_1, and E_2 be Banach spaces and suppose that

$$E_1 \times E_2 \to E_0, \quad (e_1, e_2) \mapsto e_1 \bullet e_2 \tag{A2.1}$$

is a multiplication, that is, a continuous bilinear map.

Given $(u_1, u_2) \in C(E_1) \times L_{p,\mathrm{loc}}(E_2)$ for some $p \in [1, \infty]$, the point-wise product $u_1 \bullet u_2$ (induced by (A2.1)) is well-defined by

$$u_1 \bullet u_2(x) := u_1(x) \bullet u_2(x), \qquad \text{a.a. } x \in \mathbb{R}^n. \tag{A2.2}$$

Furthermore, the map

$$BUC(E_1) \times L_p(E_2) \to L_p(E_0), \quad (u_1, u_2) \mapsto u_1 \bullet u_2$$

is a multiplication, the **point-wise multiplication** induced by (A2.1). Henceforth we express this fact by writing

$$BUC(E_1) \bullet L_p(E_2) \hookrightarrow L_p(E_0), \qquad 1 \le p \le \infty. \tag{A2.3}$$

If $s, t > 0$ and $p, q \in [1, \infty]$ then

$$BUC^s(E_1) \times B^t_{p,q}(E_2) \hookrightarrow BUC(E_1) \times L_p(E_2)$$

by (1.9), (1.11), and (1.12). Thus point-wise multiplication (A2.3) restricts to $BUC^s(E_1) \times B^t_{p,q}(E_2)$ as a multiplication with values in $L_p(E_0)$, that is,

$$BUC^s(E_1) \bullet B^t_{p,q}(E_2) \hookrightarrow L_p(E_0), \qquad s, t > 0, \quad p, q \in [1, \infty]. \tag{A2.4}$$

Of course, the image space in (A2.4) is not optimal. In fact, a much better result is true.

A2.1 Proposition. *Suppose that $0 < t < s$. Then*

$$BUC^s(E_1) \bullet B^t_{p,q}(E_2) \hookrightarrow B^t_{p,q}(E_0), \qquad 1 \le p, q \le \infty. \tag{A2.5}$$

Proof This has been shown in [2, Remark 4.2(b)] as a consequence of much more general results, under the assumption that E_0, E_1, and E_2 are finite-dimensional. That proof is, in contrast to those by other authors (cf. [22], [28], [16], [20] and the references therein), solely based on (A1.3), embedding properties (1.9) and (1.10), and on Hölder's inequality. Hence it applies to the general, possibly infinite-dimensional, case also. $\square$

Now we turn to $t < 0$. Since we do not want to put any conditions on the underlying Banach spaces E_j, we cannot use duality arguments as done in the finite-dimensional case in [2]. For this reason we employ the theory of paramultiplication following Yamazaki [32], Johnson [16], and Runst-Sickel [20].

We know from [6, Theorem 2.1] that there exists a unique hypocontinuous bilinear map, called point-wise multiplication too,

$$\mathcal{O}_M(E_1) \times \mathcal{S}'(E_2) \to \mathcal{S}'(E_0), \quad (u_1, u_2) \mapsto u_1 \bullet u_2 \tag{A2.6}$$

such that

$$(\varphi_1 \otimes e_1) \bullet (\varphi_2 \otimes e_2) = \varphi_1 \varphi_2 \otimes (e_1 \bullet e_2), \qquad \varphi_j \otimes e_j \in \mathcal{D}_{\mathbb{C}} \otimes E_j, \quad j = 1, 2.$$

Since it coincides on $(\mathcal{D}_{\mathbb{C}} \otimes E_1) \times (\mathcal{D}_{\mathbb{C}} \otimes E_2)$ with the one introduced in (A2.2), the two definitions of point-wise multiplication are consistent.

Given $u_j \in \mathcal{S}'(E_j)$, we set

$$S_k u_j := \psi_k(D) u_j, \quad S^k u_j := \sum_{\ell=0}^{k} S_\ell u_j, \qquad k \in \mathbb{N},$$

and

$$\pi(u_1 \bullet u_2) := \lim_{k \to \infty} S^k u_1 \bullet S^k u_2,$$

whenever this limit exists in $\mathcal{S}'(E_0)$. Since $S^k u_j \in \mathcal{O}_M(E_j)$ (cf. [6, Theorem 3.1]), the point-wise products $S^k u_1 \bullet S^k u_2$ are well-defined.

First we show that π coincides with $(u_1, u_2) \mapsto u_1 \bullet u_2$ on $BUC^\infty(E_1) \times \mathcal{S}(E_2)$.

A2.2 Lemma. *$\pi(u, v) = u \bullet v$ for $u \in BUC^\infty(E_1)$ and $v \in \mathcal{S}(E_2)$.*

Proof From (A1.2) it follows that $S^k u = \mathcal{F}^{-1}(\sigma_{2^k}\psi) * u$. Hence (5.12) implies

$$\|S^k u\|_\infty \le \|\mathcal{F}^{-1}\sigma_{2^k}\psi\|_1 \, \|u\|_\infty = \|\psi\|_1 \, \|u\|_\infty \,, \qquad k \in \mathbb{N}. \tag{A2.7}$$

Thanks to $\partial^\alpha S^k u = S^k \partial^\alpha u$, we thus obtain

$$\|S^k u\|_{\ell,\infty} \le \|\psi\|_1 \, \|u\|_{\ell,\infty} \,, \qquad k, \ell \in \mathbb{N}.$$

This shows that $\{\, S^k u \ ; \ k \in \mathbb{N} \}$ is bounded in $BUC^\infty(E_1)$, hence in $\mathcal{O}_M(E_1)$.

It is not difficult to see that $\sigma_{2^k}\psi \to \mathbf{1}$ in $\mathcal{O}_M(\mathbb{C})$ (cf. [6, pp. 19/20]). Hence $(\sigma_{2^k}\psi)w \to w$ in $\mathcal{S}(E)$ [resp. $\mathcal{S}'(E)$] if $w \in \mathcal{S}(E)$ [resp. $\mathcal{S}'(E)$] (cf. [6, Theorem 2.1]). Consequently, $S^k w = \mathcal{F}^{-1}\big((\sigma_{2^k}\psi)\widehat{w}\big) \to w$ in $\mathcal{S}(E)$ if $w \in \mathcal{S}(E)$, and in $\mathcal{S}'(E)$ if $w \in \mathcal{S}'(E)$. Now

$$S^k u \bullet S^k v - u \bullet v = S^k u \bullet (S^k v - v) + (S^k u - u) \bullet v$$

and the hypocontinuity of (A2.6) imply the assertion. □

For $u_j \in \mathcal{S}'(E_j)$, we put $S_{-1}u_j := 0$ and

$$\pi_1(u_1,u_2) := \sum_{j=2}^{\infty} S^{j-2}u_1 \bullet S_j u_2,$$

$$\pi_2(u_1,u_2) := \sum_{j=0}^{\infty} (S_{j-1}u_1 \bullet S_j u_2 + S_j u_1 \bullet S_j u_2 + S_j u_1 \bullet S_{j-1}u_2),$$

$$\pi_3(u_1,u_2) := \sum_{j=2}^{\infty} S_j u_1 \bullet S^{j-2}u_2,$$

provided these series converge in $\mathcal{S}'(E_0)$. Note that

$$\pi = \pi_1 + \pi_2 + \pi_3. \tag{A2.8}$$

Also note that

$$\operatorname{supp}\bigl(\mathcal{F}(S^{j-2}u_1 \bullet S_j u_2)\bigr) \cup \operatorname{supp}\bigl(\mathcal{F}(S_j u_1 \bullet S^{j-2}u_2)\bigr) \subset [2^{j-3} \le |\xi| \le 2^{j+1}] \tag{A2.9}$$

and, setting $T_j(u_1,u_2) := S_{j-1}u_1 \bullet S_j u_2 + S_j u_1 \bullet S_j u_2 + S_j u_1 \bullet S_{j-1}u_2$,

$$\operatorname{supp}\bigl(\mathcal{F}T_j(u_1,u_2)\bigr) \subset [\,|\xi| \le 2^{j+3}] \tag{A2.10}$$

for $j \ge 2$ (cf. [6, Remark 3.2(e)]).

A2.3 Lemma. *Suppose that $0 < -t < s$ and $p,q \in [1,\infty]$. Then*

$$\pi : BUC^s(E_1) \times B^t_{p,q}(E_2) \to B^t_{p,q}(E_2)$$

is well-defined, bilinear, and continuous.

Proof Thanks to (A2.8), (1.6), and (1.9) it suffices to show that

$$\|\pi_j(u,v)\|_{B^t_{p,q}(E_0)} \le c\,\|u\|_{B^s_\infty(E_1)}\,\|v\|_{B^t_{p,q}(E_2)}, \qquad j=1,2,3, \tag{A2.11$_j$}$$

for $u \in B^s_\infty(E_1)$ and $v \in B^t_{p,q}(E_2)$ and that $\pi_j(u,v)$ exists in $\mathcal{S}'(E_0)$, since we can assume that $s \notin \mathbb{N}$.

Thus we suppose that $u \in B^s_\infty(E_1)$ and $v \in B^t_{p,q}(E_2)$ and follow arguments from the proof of [16, Theorem 5.1].

(i) From (A2.7) we infer that

$$\begin{aligned}\bigl\|(2^{jt}\,\|S^{j-2}u \bullet S_j v\|_p)\bigr\|_{\ell_q} &\le \bigl\|(\|S^j u\|_\infty)\bigr\|_{\ell_\infty}\,\bigl\|(2^{jt}\,\|S_j v\|_p)\bigr\|_{\ell_q}\\ &\le c\,\|u\|_\infty\,\|v\|_{B^t_{p,q}} \le c\,\|u\|_{B^s_\infty}\,\|v\|_{B^t_{p,q}}.\end{aligned}$$

Thanks to (A2.9), we can apply Proposition A1.2 to deduce that $\pi_1(u,v)$ exists in $\mathcal{S}'(E_0)$ and satisfies $(\text{A2.11})_1$.

(ii) To estimate π_2 we observe that

$$\begin{aligned}\left\|(2^{j(s+t)}\,\|S_{j-1}u \bullet S_j v\|_p)\right\|_{\ell_q} &\le c\left\|(2^{js}\,\|S_j u\|_\infty)\right\|_{\ell_\infty}\left\|(2^{jt}\,\|S_j v\|_p)\right\|_{\ell_q}\\ &= c\,\|u\|_{B^s_\infty}\,\|v\|_{B^t_{p,q}}.\end{aligned}$$

Similar estimates for $S_j u \bullet S_j v$ and $S_j u \bullet S_{j-1} v$ imply

$$\left\|(2^{j(s+t)}\,\|T_j(u,v)\|_p)\right\|_{\ell_q} \le c\,\|u\|_{B^s_\infty}\,\|v\|_{B^t_{p,q}}.$$

Hence (A2.10), the fact that $s+t>0$, and Proposition A1.2 imply that $\pi_2(u,v)$ exists in $\mathcal{S}'(E_0)$ and satisfies $(\mathrm{A2.11})_2$, thanks to $B^{s+t}_{p,q} \hookrightarrow B^t_{p,q}$.

(iii) Similarly as in (i),

$$\begin{aligned}\left\|(2^{jt}\,\|S_j u \bullet S^{j-2} v\|_p)\right\|_{\ell_q} &\le \left\|(\|S_j u\|_\infty)\right\|_{\ell_\infty}\left\|(2^{jt}\,\|S^{j-2} v\|_p)\right\|_{\ell_q}\\ &\le \|u\|_{B^0_\infty}\left\|\Bigl(2^{jt}\sum_{k=0}^{j-2}\|S_j v\|_p\Bigr)\right\|_{\ell_q} \le c\,\|u\|_{B^s_\infty}\,\|v\|_{B^t_{p,q}},\end{aligned}$$

where the last inequality sign follows from Lemma A1.1, thanks to $t<0$. Now, once again, (A2.9) and Proposition A1.2 imply that $\pi_3(u,v)$ exists in $\mathcal{S}'(E_0)$ and satisfies $(\mathrm{A2.11})_3$. $\square$

After these preparations we can extend Proposition A2.1 to certain negative values of t.

A2.4 Proposition. *Suppose that $s,t \in \mathbb{R}$ with $|t| < s$, and $p,q \in [1,\infty]$. Then there exists a unique continuous bilinear map*

$$BUC^s(E_1) \times B^t_{p,q}(E_2) \to B^t_{p,q}(E_0),$$

denoted by $(u,v) \mapsto u \bullet v$ and called point-wise multiplication, which coincides with the point-wise product (A2.3) *whenever (u,v) belongs to $BUC(E_1) \cap L_r(E_2)$ for some $r \in [1,\infty]$.*

Proof (i) If $t>0$ then this follows from Proposition A2.1.

(ii) Suppose that $t<0$. Fix $\sigma \in (0,s)\backslash\mathbb{N}$ and $\tau \in (-s,t)$ and note that

$$BUC^s(E_1) \hookrightarrow buc^\sigma(E_1) \hookrightarrow BUC^\sigma(E_1), \quad B^t_{p,q}(E_2) \hookrightarrow B^\tau_{p,1}(E_1)$$

by (1.9). Thus, given $(u,v) \in BUC^s(E_1) \times B^t_{p,q}(E_2)$, there exists a sequence (u_j,v_j) in $BUC^\infty(E_1) \times \mathcal{S}(E_2)$ converging in $BUC^\sigma(E_1) \times B^\tau_{p,1}(E_2)$ towards (u,v). Hence Lemmas A2.2 and A2.3 imply

$$u_j \bullet v_j = \pi(u_j,v_j) \to \pi(u,v)$$

in $B^{\tau}_{p,1}(E_0)$, hence in $\mathcal{S}'(E_0)$. This being true for every such sequence $\big((u_j,v_j)\big)$, the product

$$u \bullet v := \pi(u,v)$$

is well-defined for $(u,v) \in BUC^s(E_1) \times B^t_{p,q}(E_2)$ (and, in particular, independent of ψ). This proves the assertion in this case.

(iii) If $t = 0$ then we obtain the desired result from (i) and (ii) by interpolation, thanks to (1.19). $\square$

Henceforth we use the notations introduced in (A2.3) in the more general case of Proposition A2.4 and in related situations also. Then we can prove the main result of this section.

A2.5 Theorem. *Suppose that $s,t \in \mathbb{R}$ with $|t| < s$, and $p,q \in [1,\infty]$. Then*

$$BUC^s(E_1) \bullet \mathcal{B}^t_{p,q}(E_2) \hookrightarrow \mathcal{B}^t_{p,q}(E_0)$$

for $\mathcal{B} \in \{B, b, \mathring{B}\}$.

Proof If $\mathcal{B} = B$ then this is a restatement of Proposition A2.4. Using [6, Theorem 2.1], it is not difficult to verify that, given $\mathfrak{G} \in \{BUC^\infty, \mathcal{S}\}$,

$$u \bullet v \in \mathfrak{G}(E_0) \qquad \text{for} \quad u \in BUC^\infty(E_1), \quad v \in \mathfrak{G}(E_2). \tag{A2.12}$$

Fix $\sigma \in (|t|, s)$ and $(u,v) \in BUC^s(E_1) \times \mathcal{B}^t_{p,q}(E_2)$, where $\mathcal{B} \in \{b, \mathring{B}\}$. Then there exits a sequence (u_j,v_j) in $BUC^\infty(E_1) \times \mathfrak{G}(E_2)$, where $\mathfrak{G} := BUC^\infty$ if $\mathcal{B} = b$ and $\mathfrak{G} := \mathcal{S}$ if $\mathcal{B} = \mathring{B}$, converging in $BUC^\sigma(E_1) \times \mathcal{B}^t_{p,q}(E_2)$ towards (u,v). Since $u_j \bullet v_j$ lies in $\mathfrak{G}(E_0)$ by (A2.12) and

$$BUC^\sigma(E_1) \bullet B^t_{p,q}(E_2) \hookrightarrow B^t_{p,q}(E_0)$$

by what we already know, it follows that $u \bullet v \in \mathcal{B}^t_{p,q}(E_0)$. $\square$

A2.6 Remark. Suppose that $s \in \mathbb{R}^+ \backslash \mathbb{N}$. Then

$$BUC^s(E_1) \bullet BUC^s(E_2) \hookrightarrow BUC^s(E_0)$$

and

$$buc^s(E_1) \bullet buc^s(E_2) \hookrightarrow buc^s(E_0)$$

as well as

$$buc^s(E_1) \bullet C^s_0(E_2) \hookrightarrow C^s_0(E_0).$$

Proof This follows, as in the scalar case, by a direct investigation of the Hölder norms and an approximation argument similar to the one used in the preceding proof. $\square$

In the scalar case, Theorem A2.5 is known, of course, and contained as a special case in much more general results (cf. [28], [16, Theorem 6.6(1b) and (1d)] and [20, Theorem 4.4.3.2]). Triebel's proof uses duality arguments, and the one in [20] relies on Triebel-Lizorkin spaces. Although the latter spaces can be defined in the infinite-dimensional setting (see [29, Section III.15]), they are not too useful since this scale is not known to contain the L_p-spaces for $1 < p < \infty$ unless restrictive conditions on E are imposed. Consequently, the techniques of Triebel and Runst-Sickel do not extend to infinite-dimensional spaces. As mentioned above, our arguments are those of Johnson [16], except that we have to invoke the full strength of Schwartz' theory of vector-valued distributions in order to define point-wise products of smooth vector-valued functions and vector-valued distributions.

It should also be mentioned that the investigations by Triebel, Sickel, and Johnson for the scalar case show that Theorem A2.5 is optimal (except for the possibility of having $s = |t|$ in some cases).

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