BOUNDED H_∞-CALCULUS FOR ELLIPTIC OPERATORS

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Abstract. It is shown, in particular, that L_p-realizations of general elliptic systems on $\mathbb{R}^n$ or on compact manifolds without boundaries possess bounded imaginary powers, provided rather mild regularity conditions are satisfied. In addition, there are given some new perturbation theorems for operators possessing a bounded H_∞-calculus.

0. Introduction. It is the main purpose of this paper to prove — under mild regularity assumptions — that L_p-realizations of elliptic differential operators acting on vector valued functions over $\mathbb{R}^n$ or on sections of vector bundles over compact manifolds without boundaries possess bounded imaginary powers. In fact, we shall prove a more general result guaranteeing that, given any elliptic operator $\mathcal{A}$ with a sufficiently large zero order term such that the spectrum of its principal symbol is contained in a sector of the form $S_{\theta_0} := \{\, z \in \mathbb{C} \,;\, |\arg z| \le \theta_0 \,\} \cup \{0\}$ for some $\theta_0 \in [0, \pi)$, and given any bounded holomorphic function $f : \mathring{S}_\theta \to \mathbb{C}$ for some $\theta \in (\theta_0, \pi)$, we can define a bounded linear operator $f(\mathcal{A})$ on L_p, and an estimate of the form

$$\|f(\mathcal{A})\|_{\mathcal{L}(L_p)} \le c\, \|f\|_\infty$$

is valid, provided $\mu \ge 0$ is sufficiently large. This means that elliptic operators possess a bounded H_∞-calculus in the sense of McIntosh [16]. Choosing, in particular, $f(z) := z^{it}$ for $t \in \mathbb{R}$, it follows that $\mathcal{A}$ possesses bounded imaginary powers (cf. Section 2 below for more precise statements).

There are two main reasons for our interest in this problem. First, it is known (cf. [22], [24]) that the complex interpolation spaces $\big[E, D(A)\big]_\theta$ coincide with the domains of the fractional powers A^θ for $0 < \theta < 1$, provided A is a densely defined linear operator on the Banach space E possessing bounded imaginary powers. Second, by a result of Dore and Venni [10], the fact that A possesses bounded imaginary powers is intimately connected with 'maximal regularity results' for abstract evolution equations of the form $\dot{u} + Au = f(t)$. Both these results are of great use in the functional analytic approach to quasilinear parabolic evolution equations and,

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in particular, in applications of this theory to quasilinear parabolic systems. In this context it is important that we can handle elliptic operators whose coefficients possess little regularity only (cf. [2]).

Complex powers of elliptic operators on compact manifolds without boundaries have first been studied by Seeley [20] (also cf. [23] and the references given there). In [22] Seeley proved that L_p-realizations of elliptic boundary value problems possess bounded imaginary powers (also see [21] for some corrections). This latter result has been extended in [9] to guarantee a bounded H_∞-calculus. All these authors work in the C^∞-category.

In [19] it has been shown that a scalar second order elliptic operator on $\mathbb{R}^n$ possesses bounded imaginary powers, provided the top-order coefficients are Hölder continuous and asymptotically constant. In addition, these authors also consider the case of the Dirichlet problem on a bounded domain. Their approach relies on a general perturbation theorem and on commutator estimates. By means of an abstract result of Coifman and Weiss [4] it is possible to prove that second order elliptic operators under the usual coercive boundary conditions possess bounded imaginary powers in L_p under rather weak regularity assumptions for the coefficients. However, this method is restricted to second order operators and it does not give the optimal estimate as far as the angle θ is concerned. Estimates which are (almost) optimal in this sense are, however, important for applying the Dore-Venni theorem.

Our approach is completely different and closer, in spirit, to Seeley's original proof, since it relies on the theory of pseudo differential operators. As we are interested in weak regularity assumptions we have to deal with pseudo differential operators with nonsmooth symbols depending, in addition, upon parameters. For this we appropriately modify the technique of symbol smoothing of Kumano-go and Nagase [12].

The main results of this paper concerning elliptic systems are contained in Sections 9 and 10 below. In Section 9 we deal with elliptic systems on all of $\mathbb{R}^n$, where we generalize considerably the corresponding results of [19]. Observe that in the latter section we also prove a generation theorem for analytic semigroups which seems to be new in the given generality. In particular, it suffices that the top-order coefficients are uniformly continuous without any additional conditions at infinity. In order to guarantee that our elliptic operators possess a bounded H_∞-calculus we have to require that the top-order coefficients satisfy a suitable Dini condition. This is trivially true if they are uniformly Hölder continuous. In Section 10 we prove the corresponding results for elliptic systems on compact manifolds without boundaries.

Our approach extends to more general elliptic pseudo differential operators, whose symbols belong to the classes used in this paper. This is of interest since it is known that, in general, an elliptic pseudo differential operator can have bounded imaginary powers without possessing a bounded H_∞-calculus. However, since this paper is already rather long and technical, we do not include this generalization. We also do not consider the case of elliptic boundary value problems. This problem will be dealt with elsewhere.

Lastly, it should be pointed out that the question remains if general elliptic operators possess a bounded H_∞-calculus (or bounded imaginary powers) under the same weak regularity hypotheses for the top-order coefficients which guarantee the resolvent estimates.

Notations and Conventions Throughout this paper vector spaces are over $\mathbb{C}$, in general. If E and F are Banach spaces, $\mathcal{L}(E,F)$ is the Banach space of all bounded linear operators from E to F, and $\mathcal{L}(E) := \mathcal{L}(E,E)$. We denote by $\mathcal{L}\mathrm{is}(E,F)$ the open subset of all isomorphisms in $\mathcal{L}(E,F)$, and $\mathcal{L}\mathrm{aut}(E) := \mathcal{L}\mathrm{is}(E,E)$. If E is a vector subspace of F such that the natural inclusion $x \mapsto x$ belongs to $\mathcal{L}(E,F)$, that is, if E is continuously injected in F, we write $E \hookrightarrow F$. If in addition, E is dense in F, this is denoted by $E \overset{d}{\hookrightarrow} F$. Lastly, $E \doteq F$ means that $E \hookrightarrow F$ and $F \hookrightarrow E$ so that E and F coincide, except for equivalent norms.

Given a nonempty subset M of some vector space, $\dot{M} := M \backslash \{0\}$. We often write $[\dots]$ for $\{\, x \in X \ ;\ \dots \}$, where $\dots$ stand for definitions and relations, provided it is clear from the context which set X is being considered. For example, $[\,|\arg z| \le \vartheta] := \{\, z \in \dot{\mathbb{C}} \ ;\ |\arg z| \le \vartheta \,\}$. If A is a linear operator in E, we denote its domain by $\mathrm{dom}(A)$, its resolvent set by $\rho(A)$, and its spectrum by $\sigma(A)$.

We denote by c various constants which may differ from occurrence to occurrence but are always independent of the free variables of a given formula. If c depends on additional constants $\alpha, \beta, \dots$, we sometimes indicate this by writing $c(\alpha, \beta, \dots)$.

1. Operators of Positive Type. In this section we prove some simple qualitative estimates and perturbation results for operators of positive type. It is the main purpose of these considerations to show that the bounds do not depend upon the particular operators but only upon two constants appearing in the resolvent estimate. This fact will be crucial in later sections.

Let E be a Banach space. Given $K \ge 1$ and $\vartheta \in [0,\pi)$, a linear operator A in E is said to be **of type** (K,ϑ), in symbols:

$$A \in \mathcal{P}(K,\vartheta) := \mathcal{P}(E;K,\vartheta) \ ,$$

if it is densely defined, if

$$S_\vartheta := [\,|\arg z| \le \vartheta] \cup \{0\} \subset \rho(-A) \ ,$$

and if

$$(1+|\lambda|)\,\|(\lambda + A)^{-1}\| \le K \ , \qquad \lambda \in S_\vartheta \ . \tag{1.1}$$

Put

$$\mathcal{P}(\vartheta) := \mathcal{P}(E;\vartheta) := \bigcup_{K \ge 1} \mathcal{P}(K,\vartheta)$$

and note that, trivially

$$\mathcal{P}(K,\vartheta) \subset \mathcal{P}(L,\theta) \ , \qquad 1 \le K \le L \ , \quad 0 \le \theta \le \vartheta < \pi \ . \tag{1.2}$$

We say that A is **of positive type** if it belongs to

$$\mathcal{P} := \mathcal{P}(E) := \mathcal{P}(E;0) \ .$$

Suppose that $K \ge 1$ and $0 \le \vartheta < \pi$, and that $A \in \mathcal{P}(K,\vartheta)$. Given $\lambda_0 \in S_\vartheta$ and $\lambda \in \mathbb{C}$ satisfying

$$|\lambda - \lambda_0| \le (1 + |\lambda_0|)/(2K) \ , \tag{1.3}$$

it follows from $\lambda + A = (\lambda_0 + A)\big(1 + (\lambda - \lambda_0)(\lambda_0 + A)^{-1}\big)$ that $\lambda \in \rho(-A)$ and

$$\begin{aligned}\|(\lambda + A)^{-1}\| &\le \big\|\big[1 + (\lambda - \lambda_0)(\lambda_0 + A)^{-1}\big]^{-1}\big\| \, \|(\lambda_0 + A)^{-1}\| \le \frac{2K}{1 + |\lambda_0|} \\ &\le \frac{2K}{1 + |\lambda|} \, \frac{1 + |\lambda_0| + |\lambda - \lambda_0|}{1 + |\lambda_0|} \le \frac{2K}{1 + |\lambda|}\Big(1 + \frac{1}{2K}\Big) = \frac{2K + 1}{1 + |\lambda|} \ .\end{aligned} \tag{1.4}$$

Let

$$\vartheta_K := \vartheta + \frac{\pi - \vartheta}{2} \wedge \arcsin\frac{1}{2K} \tag{1.5}$$

and note that

$$\rho(-A) \supset \bigcup_{\lambda_0 \in S_\vartheta} \big[\, |\lambda - \lambda_0| \le |\lambda_0|/(2K)\big] \supset S_{\vartheta_K} \ .$$

Hence it follows from (1.4) that

$$\mathcal{P}(K, \vartheta) \subset \mathcal{P}(2K + 1, \vartheta_K) \ . \tag{1.6}$$

In particular, $\mathcal{P}(K, 0) \subset \mathcal{P}\big(2K + 1, \arcsin\big(1/(2K)\big)\big)$. Thus, in the following, we always assume without loss of generality that $\vartheta \in (0, \pi)$.

The following lemma shows that $\mathcal{P}$ is stable under suitable additive perturbations. Here and in the following,

$$s(\vartheta) := 1 + \Theta(\vartheta - \pi/2)\Big[\frac{1}{\sin(\pi - \vartheta)} - 1\Big] \ , \qquad 0 < \vartheta < \pi \ ,$$

where $\Theta(t) := 1$ for $t > 0$, and $\Theta(t) := 0$ for $t \le 0$.

Lemma 1.1. *Suppose that $A \in \mathcal{P}(K, \vartheta)$ for some $K \ge 1$ and $\vartheta \in (0, \pi)$.*

(i) *If B is a linear operator in E satisfying* $\mathrm{dom}(B) \supset \mathrm{dom}(A)$ *and*

$$\|B(\lambda + A)^{-1}\| \le \beta < 1 \ , \qquad \lambda \in S_\vartheta \ , \tag{1.7}$$

then $A + B \in \mathcal{P}\big((1 - \beta)^{-1}K, \vartheta\big)$.

(ii) *$\mu + A \in \mathcal{P}\big(K s(\vartheta), \vartheta\big)$ for $\mu \ge 0$.*

Proof. (i) It follows from (1.7) that $1 + B(\lambda + A)^{-1} \in \mathcal{L}\mathrm{aut}(E)$ and

$$\big\|\big[1 + B(\lambda + A)^{-1}\big]^{-1}\big\| \le (1 - \beta)^{-1} \ .$$

Thus we deduce from

$$\lambda + A + B = \big[1 + B(\lambda + A)^{-1}\big](\lambda + A) \tag{1.8}$$

that $S_\vartheta \subset \rho\big(-(A + B)\big)$ and

$$\|(\lambda + A + B)^{-1}\| \le (1 - \beta)^{-1} \, \|(\lambda + A)^{-1}\| \ , \qquad \lambda \in S_\vartheta \ .$$

Now the assertion is obvious.

(ii) This is a consequence of the fact that $\lambda \in S_\vartheta$ implies $|\lambda + \mu| \ge |\lambda| \sin(\pi - \vartheta)$ if $\pi/2 < \vartheta < \pi$, and $|\lambda + \mu| \ge |\lambda|$ if $0 < \vartheta \le \pi/2$. □

Remarks 1.2.

(a) Let $A \in \mathcal{P}(K,\vartheta)$ for some $K \geq 1$ and $\vartheta \in (0,\pi)$. Put

$$E_1 := E_1(A) := \big(\mathrm{dom}(A), \|A\cdot\|\big) \ .$$

Then E_1 is a Banach space such that $E_1 \overset{d}{\hookrightarrow} E_0 := E$. Since

$$A(\lambda+A)^{-1} = 1 - \lambda(\lambda+A)^{-1} \ ,$$

it follows that $(\lambda+A)^{-1} \in \mathcal{L}(E_0,E_1)$ and that

$$\|(\lambda+A)^{-1}\|_{\mathcal{L}(E_0,E_1)} \leq 1+K \ , \qquad \lambda \in S_\vartheta \ . \tag{1.9}$$

Hence Lemma 1.1 implies $A+B \in \mathcal{P}\big((1-\beta)^{-1}K,\vartheta\big)$, provided $B \in \mathcal{L}(E_1,E_0)$ and $\|B\| \leq \beta/(1+K)$.

(b) Suppose that $\alpha \in [0,1)$ and let $\mathfrak{F}_\alpha$ be an exact interpolation functor of exponent α, if $\alpha > 0$ (e.g., [3] or [24]). Put $E_\alpha := E_\alpha(A) := \mathfrak{F}_\alpha(E_0,E_1)$ if $\alpha > 0$. Then $E_1 \hookrightarrow E_\alpha \hookrightarrow E_0$, and (1.1) and (1.9) imply

$$\|(\lambda+A)^{-1}\|_{\mathcal{L}(E_0,E_\alpha)} \leq (1+K)(1+|\lambda|)^{\alpha-1} \ , \qquad \lambda \in S_\vartheta \ . \tag{1.10}$$

Suppose that $B \in \mathcal{L}(E_\alpha,E_0)$. Then

$$\|B(\lambda+\mu+A)^{-1}\| \leq (1+K)\,\|B\|\,(1+|\lambda+\mu|)^{\alpha-1} \ , \qquad \lambda \in S_\vartheta \ , \quad \mu \geq 0 \ .$$

If $0 < \vartheta \leq \pi/2$ then $|\lambda+\mu| \geq \mu$, and $|\lambda+\mu| \geq \mu\sin(\pi-\vartheta)$ if $\pi/2 < \vartheta < \pi$. Hence, given $\beta \in (0,1)$, we see that $\|B(\lambda+\mu+A)^{-1}\| \leq \beta < 1$, provided

$$\mu \geq \mu_0 := \Big[\Big(\frac{(1+K)\,\|B\|}{\beta}\Big)^{1/(1-\alpha)} - 1\Big]_+ s(\vartheta) \ ,$$

where $t_+ := t \vee 0$ for $t \in \mathbb{R}$. Thus Lemma 1.1 implies

$$\mu + A + B \in \mathcal{P}\big((1-\beta)^{-1}Ks(\vartheta),\vartheta\big)$$

for $\mu \geq \mu_0$. $\square$

2. H_∞-Calculus and Perturbation Theorems. This section is the centerpiece of the abstract part of this paper. First we review some basic facts about the H_∞-calculus as introduced by McIntosh (cf. [16]; for other approaches we refer to [5], [6]). Then we prove some perturbation theorems for operators possessing a bounded H_∞-calculus.

Given $\vartheta \in (0,\pi)$, we denote by $H_\infty(\vartheta)$ the Banach algebra of all bounded holomorphic functions $f : \mathring{S}_{\pi-\vartheta} \to \mathbb{C}$, equipped with the supremum norm. We also write $H(\vartheta)$ for the set of all $g \in H_\infty(\vartheta)$ such that there exist $c \geq 0$ and $s > 0$ with

$$|g(z)| \leq \frac{c\,|z|^s}{1+|z|^{2s}} \ , \qquad z \in \mathring{S}_{\pi-\vartheta} \ . \tag{2.1}$$

Let $K \geq 1$ and $\vartheta \in (0, \pi)$, and denote by $\Gamma := \Gamma(K, \vartheta)$ the negatively oriented boundary of $S_{\vartheta_K} \cup \big[\,|z| \leq 1/(2K)\big]$. Also put $-\Gamma := [-\lambda \in \Gamma]$ so that $-\Gamma$ is the positively oriented boundary of $S_{\pi-\vartheta_K} \cap \big[\,|z| \geq 1/(2K)\big]$. Then, given $A \in \mathcal{P}(K, \vartheta)$ and $g \in H(\vartheta)$, it follows from (1.3)–(1.6) that

$$g(A) := \frac{1}{2\pi i} \int_\Gamma g(-\lambda)(\lambda + A)^{-1}\, d\lambda = \frac{1}{2\pi i} \int_{-\Gamma} g(\lambda)(\lambda - A)^{-1}\, d\lambda \tag{2.2}$$

is a well-defined element of $\mathcal{L}(E)$. By Cauchy's theorem, Γ can be replaced by $\Gamma(K_1, \vartheta)$ for any $K_1 \geq K$. Put $h(z) := z(1+z)^{-2}$ and let

$$f(A) := \big[h(A)\big]^{-1}(fh)(A) \ , \qquad f \in H_\infty(\vartheta) \ , \quad A \in \mathcal{P}(K, \vartheta) \ .$$

It has been shown by McIntosh [16] that $f(A)$ is a well-defined linear operator in E and that this definition is consistent with the earlier one for $f \in H(\vartheta)$. In fact, the definition of $f(A)$ can even be extended to encompass unbounded holomorphic functions, and the resulting 'holomorphic function calculus' is uniquely determined by the requirement $f_0(A) = \mathrm{id}_E$ and $f_1(A) = A$ if $f_0 = 1$ and $f_1 = \mathrm{id}_{\mathbb{C}}$. In particular,

$$f(A) = A^{it} \quad \text{for} \quad A \in \mathcal{P} \quad \text{and} \quad f(\lambda) = \lambda^{it} \ , \quad t \in \mathbb{R} \ ,$$

where A^z are the well-known 'fractional powers' of A (e.g., [13], [14], [15], [24]). Note that, in general, $f(A)$ is not bounded, even if $f \in H_\infty(\vartheta)$.

Given $M \geq 1$ and $\vartheta \in (0, \pi)$, we say that A has a **bounded H_∞-calculus** and write

$$A \in \mathcal{H}_\infty(M, \vartheta) := \mathcal{H}_\infty(E; M, \vartheta) \ ,$$

provided $A \in \mathcal{P}(\vartheta)$ and $f(A) \in \mathcal{L}(E)$ with

$$\|f(A)\|_{\mathcal{L}(E)} \leq M\, \|f\|_\infty \ , \qquad f \in H_\infty(\vartheta) \ . \tag{2.3}$$

Moreover,

$$\mathcal{H}_\infty(\vartheta) := \mathcal{H}_\infty(E; \vartheta) := \bigcup_{M \geq 1} \mathcal{H}_\infty(M, \vartheta) \ .$$

Note that

$$A \in \mathcal{H}_\infty(M, \vartheta) \quad \Longrightarrow \quad \|A^{it}\| \leq M e^{(\pi-\vartheta)|t|} \ , \qquad t \in \mathbb{R} \ , \tag{2.4}$$

thanks to

$$|\lambda^{it}| = e^{-t \arg \lambda} \leq e^{(\pi-\vartheta)|t|} \ , \qquad \lambda \in \mathring{S}_{\pi-\vartheta} \ , \quad t \in \mathbb{R} \ .$$

Hence an operator of positive type has bounded imaginary powers if it possesses a bounded H_∞-calculus. It is known that the converse is not true.

The following lemma shows that in order to prove (2.3) it suffices to establish that estimate for $g \in H(\vartheta)$. Thus in deriving estimates for $g(A)$ we can deal with absolutely convergent Dunford-Taylor type integrals, which greatly simplifies our problem.

Lemma 2.1. *There exists $\kappa \geq 1$ such that the following is true: if $A \in \mathcal{P}(\vartheta)$ and there is $M \geq 1$ such that*

$$\|g(A)\|_{\mathcal{L}(E)} \leq M \, \|g\|_\infty \ , \qquad g \in H(\vartheta) \ , \tag{2.5}$$

then $A \in \mathcal{H}_\infty(\kappa M, \vartheta)$.

Proof. Following [9] we pick $g \in H(\vartheta)$ satisfying $\int_0^\infty g(t)\, dt/t = 1$ and put

$$g_j(z) := \int_{1/j}^{j} g(tz)\, dt/t \ , \qquad z \in \mathring{S}_{\pi-\vartheta} \ , \quad j \in \dot{\mathbb{N}} \ .$$

Then $g_j \in H(\vartheta)$ and there exists κ such that $\|g_j\|_\infty \leq \kappa$ for $j \in \dot{\mathbb{N}}$. Moreover, it is easily seen that $g_j \to 1$ as $j \to \infty$, uniformly on each one of the sets

$$[\varepsilon \leq |z| \leq 1/\varepsilon] \cap \mathring{S}_{\pi-\vartheta} \ , \qquad 0 < \varepsilon < 1 \ . \tag{2.6}$$

Thus, given $f \in H_\infty(\vartheta)$, it follows that $f_j := f g_j \in H(\vartheta)$, that $\|f_j\|_\infty \leq \kappa \, \|f\|_\infty$, and that $f_j \to f$ as $j \to \infty$, uniformly on each one of the sets (2.6). Since, thanks to (2.5),

$$\|f_j(A)\|_{\mathcal{L}(E)} \leq M \, \|f_j\|_\infty \leq \kappa M \, \|f\|_\infty \ , \qquad j \in \dot{\mathbb{N}} \ ,$$

we deduce from [16] that $f(A) \in \mathcal{L}(E)$ and $\|f(A)\|_{\mathcal{L}(E)} \leq \kappa M \, \|f\|_\infty$. □

On the basis of this lemma it is now easy to establish a number of important perturbation theorems. Throughout the remainder of this section κ denotes a fixed constant satisfying the assertion of Lemma 2.1.

Lemma 2.2. *Let F be a Banach space, let A and B be densely defined linear operators in E and in F, respectively, and let $C \in \mathcal{L}(E,F)$ and $D \in \mathcal{L}(F,E)$. Suppose that $S_\vartheta \subset \rho(-A) \cap \rho(-B)$ and*

$$(\lambda + B)^{-1} = C(\lambda + A)^{-1} D \ , \qquad \lambda \in S_\vartheta \ . \tag{2.7}$$

Also suppose that $A \in \mathcal{P}(E; K, \vartheta)$. Then

(i) *$B \in \mathcal{P}(F; K_1, \vartheta)$ with $K_1 := \|C\| \, \|D\| \, K$.*

(ii) *If $A \in \mathcal{H}_\infty(E; M, \vartheta)$ then $B \in \mathcal{H}_\infty(F; M_1, \vartheta)$ with $M_1 := \kappa \, \|C\| \, \|D\| \, M$.*

Proof. (i) Obvious.

(ii) If $A \in \mathcal{H}_\infty(E; M, \vartheta)$, it follows that (2.5) is true. From (2.2) and (2.7) we see that

$$g(B) = \frac{1}{2\pi i} \int_\Gamma g(-\lambda) C(\lambda + A)^{-1} D \, d\lambda = C g(A) D \ , \qquad g \in H(\vartheta) \ .$$

Hence $\|g(B)\|_{\mathcal{L}(E)} \leq \|C\| \, \|D\| \, M \, \|g\|_\infty$ for $g \in H(\vartheta)$. Now the assertion is a consequence of Lemma 2.1. □

Observe that Lemma 2.2 implies, in particular, that $\mathcal{P}(\vartheta)$ and $\mathcal{H}_\infty(\vartheta)$ are invariant under similarity transformations.

Next we prove a simple 'splitting lemma' which will greatly simplify our proofs that a given operator of positive type has a bounded H_∞-calculus.

Lemma 2.3. *Suppose that $A \in \mathcal{P}(K,\vartheta)$ and*

$$(\lambda + A)^{-1} = R(\lambda) + S(\lambda) \ , \qquad \lambda \in \Gamma := \Gamma(K,\vartheta) \ , \tag{2.8}$$

and put $R_g(\lambda) := g(-\lambda)R(\lambda)$ for $\lambda \in \Gamma$ and $g \in H(\vartheta)$. Also suppose that

$$R_g, S \in L_1\big(\Gamma, ds, \mathcal{L}(E)\big)$$

and

$$\|R_g\|_{L_1} \le M\,\|g\|_\infty \ , \qquad g \in H(\vartheta) \ , \tag{2.9}$$

ds denoting the 'arc-length measure'. Then $A \in \mathcal{H}_\infty\big(\kappa(M + \|S\|_{L_1}), \vartheta\big)$.

Proof. It follows from (2.2) and (2.8) that

$$g(A) = \frac{1}{2\pi i}\int_\Gamma R_g(\lambda)\,d\lambda + \frac{1}{2\pi i}\int_\Gamma g(-\lambda)S(\lambda)\,d\lambda \ , \qquad g \in H(\vartheta) \ .$$

Thus we infer from (2.9) that $\|g(A)\|_{\mathcal{L}(E)} \le (M + \|S\|_{L_1})\,\|g\|_\infty$ for $g \in H(\vartheta)$, and Lemma 2.1 implies the assertion. □

As a first application of this splitting lemma we show that $\mathcal{H}_\infty(\vartheta)$ is invariant under suitable 'lower order perturbations'.

Theorem 2.4. *Suppose that $A \in \mathcal{P}(K,\vartheta) \cap \mathcal{H}_\infty(M,\vartheta)$ and $0 \le \beta < 1$. Fix $K_1 \ge K$ and put $R(\lambda) := (\lambda + A)^{-1}$ and $\Gamma := \Gamma\big((1-\beta)^{-1}K_1, \vartheta\big)$. Let B be a linear operator in E satisfying*

(i) $\mathrm{dom}(B) \supset \mathrm{dom}(A)$*;*

(ii) $\|BR(\lambda)\| \le \beta < 1$ *for* $\lambda \in S_\vartheta$*;*

(iii) $\|RBR\|_{L_1(\Gamma,ds,\mathcal{L}(E))} \le \sigma < \infty$.

Then $A + B \in \mathcal{H}_\infty\big(\kappa\big(M + (1-\beta)^{-1}\sigma\big), \vartheta\big)$.

Proof. Lemma 1.1 implies $A + B \in \mathcal{P}\big((1-\beta)^{-1}K, \vartheta\big)$. From (1.8) we deduce that $(\lambda + A + B)^{-1} = R(\lambda) + S(\lambda)$ for $\lambda \in \Gamma$, where $S := -RBR[1 + BR]^{-1}$. Hence (ii) and (iii) imply

$$S \in L_1\big(\Gamma, ds, \mathcal{L}(E)\big) \quad \text{and} \quad \|S\|_{L_1} \le (1-\beta)^{-1}\sigma \ .$$

Now the assertion follows from Lemma 2.3 and the fact that $\Gamma(K,\vartheta)$ can be replaced by $\Gamma\big((1-\beta)^{-1}K_1, \vartheta\big)$. □

Corollary 2.5. *Suppose that $A \in \mathcal{P}(K,\vartheta) \cap \mathcal{H}_\infty(M,\vartheta)$. Then, given $\nu > 0$, there exists N such that*

$$\mu + A \in \mathcal{H}_\infty(N,\vartheta) \ , \qquad 0 \le \mu \le \nu \ .$$

Proof. Fix $\mu_1 \geq 0$ and put $A_1 := \mu_1 + A$. Then Lemma 1.1 implies $A_1 \in \mathcal{P}(K_1, \vartheta)$ with $K_1 := Ks(\vartheta)$. Suppose that $A_1 \in \mathcal{H}_\infty(M_1, \vartheta)$ for some $M_1 \geq 1$. Note that this is true if $\mu_1 = 0$.

Assume that $0 < \mu \leq 1/(6K_1) =: \nu_1$, put $B := \mu 1_E$ and $R_1(\lambda) := (\lambda + A_1)^{-1}$, and let $\Gamma_1 := \Gamma(2K_1, \vartheta)$. Then (1.3) and (1.4) imply $\|R_1(\lambda)\| \leq 3K_1(1 + |\lambda|)^{-1}$ for $\lambda \in \Gamma_1$. Hence B satisfies (i)–(iii) of Theorem 2.4 with $\beta := 1/2$ and $\sigma := 2K_1\rho$, where ρ is the $L_1(\Gamma_1, ds)$-norm of $(1 + |\cdot|)^{-2}$. Thus, thanks to Theorem 2.4,

$$\mu + \mu_1 + A = \mu + A_1 \in \mathcal{H}_\infty(M_2, \vartheta) \ ,$$

where $M_2 := \kappa(M_1 + 2\sigma)$. Now the assertion follows by induction starting with $\mu_1 := 0$, since ν can be reached in finitely many steps of length at most ν_1. □

The following perturbation theorem will be of particular importance in applications. Here we again use the notations of Remark 1.2(b).

Theorem 2.6. *Suppose that $A \in \mathcal{P}(K, \vartheta) \cap \mathcal{H}_\infty(M, \vartheta)$. Also suppose that $\alpha \in [0, 1)$ and that $\mathfrak{F}_\alpha$ is an exact interpolation functor of exponent α, if $\alpha > 0$. Lastly, let $B \in \mathcal{L}(E_\alpha, E_0)$ and put*

$$\mu_B := \Big[\big(5Ks(\vartheta)\,\|B\|\big)^{1/(1-\alpha)} - 1 \Big]_+ \ .$$

Then there exists a constant $N \geq 1$ such that $\mu_B + A + B \in \mathcal{H}_\infty(N, \vartheta)$.

Proof. Suppose first that $\beta := 4K\,\|B\| < 1$. Then, letting $R(\lambda) := (\lambda + A)^{-1}$, we deduce from (1.3)–(1.5), similarly as in (1.9) and (1.10), that

$$(1 + |\lambda|)^{\alpha - 1}\, \|R(\lambda)\|_{\mathcal{L}(E_0, E_\alpha)} \leq 4K \ , \qquad \lambda \in \Gamma := \Gamma\big((1 - \beta)^{-1}K, \vartheta\big) \ .$$

Hence $\|BR(\lambda)\| \leq 4K\,\|B\| = \beta < 1$ for $\lambda \in \Gamma \cup S_\vartheta$ and

$$\|R(\lambda)BR(\lambda)\| \leq 16K^2\,\|B\|\,(1 + |\lambda|)^{\alpha - 2}$$

for $\lambda \in \Gamma$. Now Theorem 2.4 implies $A + B \in \mathcal{H}_\infty(N, \vartheta)$ for some $N \geq 1$. Since $s(\vartheta) \geq 1$, this proves the assertion if $\mu_B = 0$.

Suppose now that $\mu := \mu_B > 0$ and put $\beta := 4/5$ and $\Gamma := \Gamma\big((1 - \beta)^{-1}Ks(\vartheta), \vartheta\big)$. Also let $A_0 := \mu + A$ and $R_0(\lambda) := (\lambda + A_0)^{-1}$. Then (1.4) implies

$$\|R_0(\lambda)\| \leq \frac{2K + 1}{1 + |\lambda + \mu|} \leq \frac{3Ks(\vartheta)}{1 + \mu} \ , \qquad \lambda \in \Gamma \ . \tag{2.10}$$

Moreover, from $AR_0(\lambda) = 1 - (\lambda + \mu)R_0(\lambda)$ and (2.10) it follows that

$$\|AR_0(\lambda)\| \leq 1 + 2K + 1 \leq 4Ks(\vartheta) \ , \qquad \lambda \in \Gamma \ . \tag{2.11}$$

Thus, by interpolation, $\|R_0(\lambda)\|_{\mathcal{L}(E_0, E_\alpha)} \leq 4Ks(\vartheta)(1 + \mu)^{\alpha - 1} = \beta/\|B\|$ for $\lambda \in \Gamma$, so that $\|BR_0(\lambda)\| \leq \beta < 1$ for $\lambda \in \Gamma \cup S_\vartheta$. Lemma 1.1(ii) implies $A_0 \in \mathcal{P}\big(Ks(\vartheta), \vartheta\big)$. Hence we deduce from (1.4) that $(1 + |\lambda|)\,\|R_0(\lambda)\| \leq 3Ks(\vartheta)$ for $\lambda \in \Gamma$. Thus we infer from (2.11) and interpolation that

$$\|R_0(\lambda)BR_0(\lambda)\| \leq \big[4Ks(\vartheta)\big]^2\,\|B\|\,(1 + |\lambda|)^{\alpha - 2} \ , \qquad \lambda \in \Gamma \ .$$

Now Theorem 2.4 and Corollary 2.5 entail $A_0 + B = \mu_B + A + B \in \mathcal{H}_\infty(\vartheta)$. □

Suppose that $A \in \mathcal{P}(E)$ and $\mathfrak{F}_\alpha$ is a real interpolation functor $(\cdot,\cdot)_{\alpha,p}$ for some $\alpha \in (0,1)$ and $p \in [1,\infty]$. Then it has been shown by Dore [8] that A possesses a bounded H_∞-calculus on E_α.

3. Approximation-Perturbations. In this section we prove a rather technical perturbation result, namely Proposition 3.2 below, which will, however, be most useful in 'patching together' differential operators on manifolds from their local representations.

Let $\boldsymbol{E} := (E_j)$ be a finite or infinite sequence of Banach spaces. If $\boldsymbol{x} := (x_j) \in \prod_j E_j$ then put

$$\|\boldsymbol{x}\|_{\ell_q(\boldsymbol{E})} := \begin{cases} \left(\sum_j \|x_j\|_{E_j}^q\right)^{1/q} , & 1 \le q < \infty , \\ \sup_j \|x_j\|_{E_j} , & q = \infty . \end{cases}$$

Then, given $q \in [1,\infty]$,

$$\ell_q(\boldsymbol{E}) := \left(\left\{ \boldsymbol{x} \in \textstyle\prod_j E_j \ ; \ \|\boldsymbol{x}\|_{\ell_q(\boldsymbol{E})} < \infty \right\},\ \|\cdot\|_{\ell_q(\boldsymbol{E})} \right)$$

is a Banach space. If $\boldsymbol{F} := (F_j)$ is a second finite or infinite sequence of Banach spaces over the same index set, $\mathcal{L}(\boldsymbol{E},\boldsymbol{F}) := \big(\mathcal{L}(E_j,F_j)\big)$.

Given $\boldsymbol{A} = (A_j) \in \ell_\infty\big(\mathcal{L}(\boldsymbol{E},\boldsymbol{F})\big)$, put $\mathbb{A}\boldsymbol{x} := (A_j x_j)$ for $\boldsymbol{x} = (x_j) \in \boldsymbol{E}$. Then it is obvious that

$$\mathbb{A} \in \mathcal{L}\big(\ell_q(\boldsymbol{E}), \ell_q(\boldsymbol{F})\big) \tag{3.1}$$

and that

$$\|\mathbb{A}\|_{\mathcal{L}(\ell_q(\boldsymbol{E}),\ell_q(\boldsymbol{F}))} \le \|\boldsymbol{A}\|_{\ell_\infty(\mathcal{L}(\boldsymbol{E},\boldsymbol{F}))} \tag{3.2}$$

for $1 \le q \le \infty$. Moreover, if

$$A_j \in \mathcal{L}\mathrm{is}(E_j, F_j) , \qquad j = 0,1,2,\dots , \tag{3.3}$$

it follows that $\mathbb{A}$ is bijective and

$$\mathbb{A}^{-1}\boldsymbol{y} = (A_j^{-1} y_j) , \qquad \boldsymbol{y} = (y_j) \in \boldsymbol{F} . \tag{3.4}$$

Thus

$$\mathbb{A}^{-1} \in \mathcal{L}\big(\ell_q(\boldsymbol{F}), \ell_q(\boldsymbol{E})\big) , \qquad 1 \le q \le \infty , \tag{3.5}$$

by the open mapping theorem.

In the following, we write $\boldsymbol{E} \hookrightarrow \boldsymbol{F}$ if $E_j \hookrightarrow F_j$ for each j and

$$\boldsymbol{i} = (i_j) \in \ell_\infty\big(\mathcal{L}(\boldsymbol{E},\boldsymbol{F})\big) ,$$

where $i_j : E_j \hookrightarrow F_j$ is the natural injection. If, in addition, $E_j \overset{d}{\hookrightarrow} F_j$ for each j, we write $\boldsymbol{E} \overset{d}{\hookrightarrow} \boldsymbol{F}$. It is easily verified that

$$\boldsymbol{E} \hookrightarrow \boldsymbol{F} \quad \Longrightarrow \quad \ell_q(\boldsymbol{E}) \hookrightarrow \ell_q(\boldsymbol{F}) , \qquad 1 \le q \le \infty , \tag{3.6}$$

and

$$\boldsymbol{E} \overset{d}{\hookrightarrow} \boldsymbol{F} \quad \Longrightarrow \quad \ell_q(\boldsymbol{E}) \overset{d}{\hookrightarrow} \ell_q(\boldsymbol{F}) , \qquad 1 \le q \le \infty . \tag{3.7}$$

Lemma 3.1. *Let $K, M \geq 1$ and $\vartheta \in (0, \pi)$ be given constants. Suppose that $\boldsymbol{E} \overset{d}{\hookrightarrow} \boldsymbol{F}$ and $\boldsymbol{A} \in \ell_\infty\big(\mathcal{L}(\boldsymbol{E}, \boldsymbol{F})\big)$.*

(i) *If $A_j \in \mathcal{P}(F_j; K, \vartheta)$ for each j then $\mathbb{A} \in \mathcal{P}\big(\ell_q(\boldsymbol{F}); K, \vartheta\big)$ for $1 \leq q \leq \infty$*

(ii) *If, in addition, $A_j \in \mathcal{H}_\infty(F_j; M, \vartheta)$ for each j then $\mathbb{A} \in \mathcal{H}_\infty\big(\ell_q(\boldsymbol{F}); \kappa M, \vartheta\big)$ for $1 \leq q \leq \infty$ and some $\kappa \geq 1$.*

Proof. It follows from (3.1) and (3.7) that $\mathbb{A}$ is densely defined. Now (i) is an easy consequence of (3.3)–(3.5). The same formulas easily imply that $g(\mathbb{A})\boldsymbol{x} = \big(g(A_j)x_j\big)$ for $\boldsymbol{x} \in \boldsymbol{F}$ and $g \in H(\vartheta)$. Now (ii) follows from (3.2) and Lemma 2.1. □

Let E be a Banach space and suppose that

$$\varphi_{E,j} \in \mathcal{L}(E_j, E) \quad \text{and} \quad \psi_{E,j} \in \mathcal{L}(E, E_j) \tag{3.8}$$

satisfy

$$\sum_j \varphi_{E,j}\psi_{E,j} = 1_E \qquad \text{in } \mathcal{L}_s(E) \ , \tag{3.9}$$

where $\mathcal{L}_s(E)$ is the vector space $\mathcal{L}(E)$ equipped with its strong topology. Put

$$r_E\boldsymbol{x} := \sum_j \varphi_{E,j}x_j \ , \qquad \boldsymbol{x} = (x_j) \in \boldsymbol{E} \ , \tag{3.10}$$

and

$$r_E^c x := (\psi_{E,j}x) \ , \qquad x \in E \ . \tag{3.11}$$

Note that, thanks to (3.9),

$$r_E r_E^c = 1_E \ . \tag{3.12}$$

If there exists $q \in [1, \infty]$ such that

$$r_E \in \mathcal{L}\big(\ell_q(\boldsymbol{E}), E\big) \quad \text{and} \quad r_E^c \in \mathcal{L}\big(E, \ell_q(\boldsymbol{E})\big) \tag{3.13}$$

then $\big(\boldsymbol{E}, (\varphi_{E,j}), (\psi_{E,j})\big)$ is said to be an **ℓ_q-approximation system** for E.

We denote by $[\cdot, \cdot]_\alpha$, $0 < \alpha < 1$, the complex interpolation functor of exponent α, and put $[X, Y]_0 := X$ and $[X, Y]_1 := Y$ whenever (X, Y) is an interpolation couple of Banach spaces. We also put

$$[\boldsymbol{E}, \boldsymbol{F}]_\alpha := \big([E_j, F_j]_\alpha\big)$$

and recall that $[\boldsymbol{E}, \boldsymbol{F}]_\alpha = [\boldsymbol{F}, \boldsymbol{E}]_{1-\alpha}$. Also note that

$$\big[\ell_q(\boldsymbol{E}), \ell_q(\boldsymbol{F})\big]_\alpha \doteq \ell_q\big([\boldsymbol{E}, \boldsymbol{F}]_\alpha\big) \ , \qquad 1 \leq q < \infty \ , \tag{3.14}$$

(e.g., Theorem 1.18.1 in [24]).

We introduce now the following assumption:

$$\left.\begin{array}{rl}
\text{(i)} & E \text{ and } F \text{ are Banach spaces with } E \overset{d}{\hookrightarrow} F. \\
\text{(ii)} & 1 \le q < \infty, \text{ and } \big(\boldsymbol{E}, (\varphi_{E,j}), (\psi_{E,j})\big) \text{ and } \big(\boldsymbol{F}, (\varphi_{F,j}), (\psi_{F,j})\big) \text{ are} \\
& \ell_q\text{-approximation systems for } E \text{ and } F, \text{ respectively, such that} \\
& \boldsymbol{E} \overset{d}{\hookrightarrow} \boldsymbol{F}, \quad \varphi_{E,j} \subset \varphi_{F,j}, \text{ and } \psi_{E,j} \subset \psi_{F,j}. \\
\text{(iii)} & A \in \mathcal{L}(E,F) \text{ and } \boldsymbol{A} = (A_j) \in \ell_\infty\big(\mathcal{L}(\boldsymbol{E},\boldsymbol{F})\big). \\
\text{(iv)} & 0 < \alpha \le 1 \text{ and } B \in \mathcal{L}\big([E,F]_\alpha, \ell_q(\boldsymbol{F})\big) \text{ such that} \\
& \qquad \psi_{F,j} A = A_j \psi_{E,j} + B_j \ , \\
& \text{where } B_j x := (Bx)_j \text{ for } j = 0,1,2,\ldots\ . \\
\text{(v)} & C_j \in \mathcal{L}\big([E_j,F_j]_\alpha, F\big) \text{ such that } A\varphi_{E,j} = \varphi_{F,j} A_j + C_j. \\
& \text{Moreover, letting} \\
& \qquad C\boldsymbol{x} := \sum_j C_j x_j \ , \qquad \boldsymbol{x} = (x_j) \in [\boldsymbol{E},\boldsymbol{F}]_\alpha \ , \\
& \text{it follows that } C \in \mathcal{L}\big(\ell_q([\boldsymbol{E},\boldsymbol{F}]_\alpha), F\big).
\end{array}\right\} \tag{3.15}$$

Then we can prove the following ‘approximation-perturbation’ result.

Proposition 3.2. *Let* (3.15) *be satisfied and let* $K, M \ge 1$ *and* $\vartheta \in (0,\pi)$. *Suppose that* $A_j \in \mathcal{P}(F_j; K, \vartheta) \cap \mathcal{L}\mathrm{is}(E_j, F_j)$ *and* $\|A_j^{-1}\|_{\mathcal{L}(F_j,E_j)} \le K$ *for each* j. *Then there are constants* $N \ge 1$ *and* $\mu \ge 0$ *such that*

(i) $\mu + A \in \mathcal{P}(F; N, \vartheta) \cap \mathcal{L}\mathrm{is}(E,F)$ *and* $\|\mu + A\|_{\mathcal{L}(E,F)} + \|(\mu + A)^{-1}\|_{\mathcal{L}(F,E)} \le N$.

(ii) $\mu + A \in \mathcal{H}_\infty(F; N, \vartheta)$ *if* $A_j \in \mathcal{H}_\infty(F_j; M, \vartheta)$ *for each* j.

Proof. It follows from (3.15(ii)) that $r_F \supset r_E$ and $r_F^c \supset r_E^c$. Hence we can omit the indices in the following. Then we deduce from (3.13) that

$$r \in \mathcal{L}\big(\big[\ell_q(\boldsymbol{E}), \ell_q(\boldsymbol{F})\big]_\alpha, [E,F]_\alpha\big) \ .$$

Thus

$$Br \in \mathcal{L}\big(\big[\ell_q(\boldsymbol{E}), \ell_q(\boldsymbol{F})\big]_\alpha, \ell_q(\boldsymbol{F})\big) \ . \tag{3.16}$$

Suppose that $A_j \in \mathcal{P}(F_j; K, \vartheta) \cap \mathcal{L}\mathrm{is}(E_j, F_j)$ for each j. Then it follows from (3.1), (3.5), and Lemma 3.1 that

$$\mathbb{A} \in \mathcal{L}\mathrm{is}\big(\ell_q(\boldsymbol{E}), \ell_q(\boldsymbol{F})\big) \cap \mathcal{P}\big(\ell_q(\boldsymbol{F}); K, \vartheta\big) \ . \tag{3.17}$$

Hence, letting

$$\mu_B := \Big[(4K\,\|Br\|)^{1/\alpha} - 1\Big]_+ s(\vartheta) \ ,$$

we deduce from Remark 1.2(b) that

$$\mu_B + \mathbb{A} + Br \in \mathcal{P}\big(\ell_q(\boldsymbol{F}); 2Ks(\vartheta), \vartheta\big) \ . \tag{3.18}$$

Consequently, (3.13) implies

$$L(\lambda + \mu_B) := r(\lambda + \mu_B + \mathbb{A} + Br)^{-1} r^c \in \mathcal{L}(F,E) \ , \qquad \lambda \in S_\vartheta \ . \tag{3.19}$$

Note that, thanks to (3.15(iv)) and (3.12),

$$r^c(\lambda+\mu_B+A)=(\lambda+\mu_B+\mathbb{A})r^c+B=(\lambda+\mu_B+\mathbb{A}+Br)r^c\ . \tag{3.20}$$

Hence it follows that

$$L(\lambda+\mu_B)(\lambda+\mu_B+A)=rr^c=1_E\ ,\qquad \lambda\in S_\vartheta\ . \tag{3.21}$$

From (3.14) and (3.15(v)) we deduce that $C\in\mathcal{L}\big(\big[\ell_q(\boldsymbol{E}),\ell_q(\boldsymbol{F})\big]_\alpha,F\big)$. Thus, thanks to (3.13), $r^cC\in\mathcal{L}\big(\big[\ell_q(\boldsymbol{E}),\ell_q(\boldsymbol{F})\big]_\alpha,\ell_q(\boldsymbol{F})\big)$. Consequently, letting

$$\mu_C:=\Big[(4K\,\|r^cC\|)^{1/\alpha}-1\Big]_+ s(\vartheta)\ ,$$

we infer from (3.17) and Remark 1.2(b) that

$$\lambda+\mu_C+\mathbb{A}+r^cC\in\mathcal{L}\mathrm{is}\big(\ell_q(\boldsymbol{E}),\ell_q(\boldsymbol{F})\big)\ ,\qquad \lambda\in S_\vartheta\ .$$

Hence, thanks to (3.13),

$$R(\lambda+\mu_C):=r[\lambda+\mu_C+\mathbb{A}+r^cC]^{-1}r^c\in\mathcal{L}(F,E)\ ,\qquad \lambda\in S_\vartheta\ . \tag{3.22}$$

Observe that (3.15(v)) and (3.12) imply

$$(\lambda+\mu_C+A)r=r(\lambda+\mu_C+\mathbb{A})+C=r(\lambda+\mu_C+\mathbb{A}+r^cC)\ .$$

Thus

$$(\lambda+\mu_C+A)R(\lambda+\mu_C)=rr^c=1_F\ ,\qquad \lambda\in S_\vartheta\ . \tag{3.23}$$

Put $\mu_0:=\mu_B\vee\mu_C$. Then we infer from (3.19)–(3.23) that $S_\vartheta\subset\rho\big(-(\mu_0+A)\big)$ and

$$(\lambda+\mu_0+A)^{-1}=L(\lambda+\mu_0)=R(\lambda+\mu_0)\ ,\qquad \lambda\in S_\vartheta\ .$$

If $A_j\in\mathcal{P}(F_j;K,\vartheta)\cap\mathcal{H}_\infty(F_j;M,\vartheta)$ for each j, Lemma 3.1 guarantees that

$$\mathbb{A}\in\mathcal{H}_\infty\big(\ell_q(\boldsymbol{F});\kappa M,\vartheta\big)$$

for some $\kappa\geq 1$. Consequently, from (3.16), (3.17), and Theorem 2.6 we infer the existence of $N\geq 1$ and $\mu\geq\mu_0$ such that $\mu+\mathbb{A}+Br\in\mathcal{H}_\infty\big(\ell_q(\boldsymbol{F});N,\vartheta\big)$. Note that (3.12) and (3.20) imply $(\mu+A)^{-1}=r(\mu+\mathbb{A}+Br)^{-1}r^c$. Hence the assertions follow from Lemma 2.2. □

4. Finite-Dimensional Spectral Estimates. In this section we derive some easy technical estimates related to the spectrum of a matrix. In addition, we introduce spaces of uniformly continuous functions whose continuity is dominated by a given modulus of continuity. The results of this section will be needed in later sections to obtain uniform estimates on which we can base perturbation arguments.

Throughout the remainder of this paper we denote by $H:=(H,|\cdot|)$ a finite-dimensional Banach space, and $N:=\dim H$.

Lemma 4.1. *There exists a positive constant* c_N *such that*

$$|a^{-1}| \le c_N\, |a|^{N-1}\, r^{-N}$$

for all $r > 0$ *and all* $a \in \mathcal{L}(H)$ *satisfying* $\sigma(a) \subset [\,|z| \ge r]$.

Proof. This follows, for example, from Cramer's rule (e.g., I.(4.12) in [11]). □

As an easy consequence of Lemma 4.1 we obtain the following quantitative form of the well-known upper semicontinuity of the spectrum.

Lemma 4.2. *Suppose that* $r, M \in \dot{\mathbb{R}}^+$ *and put*

$$K := c_N(2M+1)^{N-1} r^{-N} \quad \text{and} \quad \delta := (1 \wedge 1/K)/2\ .$$

Then, given $a, a_0 \in \mathcal{L}(H)$ *satisfying* $|a_0| \le M$ *and* $|a - a_0| \le \delta$, *it follows that*

$$\sigma(a) \subset \big[\operatorname{dist}\big(z, \sigma(a_0)\big) < r\big]\ .$$

Proof. Suppose that $|\lambda| \ge M + 1$. Then $|a| \le |a_0| + |a - a_0| \le M + \delta < M + 1$ shows that $\lambda \in \rho(a)$. Thus assume that $|\lambda| \le M + 1$ and $\operatorname{dist}\big(\lambda, \sigma(a_0)\big) \ge r$. Then Lemma 4.1 implies that $|(\lambda - a_0)^{-1}| \le K$. Consequently,

$$|(a_0 - a)(\lambda - a_0)^{-1}| \le \delta K \le 1/2\ ,$$

which guarantees that $1 + (a_0 - a)(\lambda - a_0)^{-1} \in \mathcal{L}\mathrm{aut}(H)$. Hence we deduce from

$$\lambda - a = \big[1 + (a_0 - a)(\lambda - a_0)^{-1}\big](\lambda - a_0)$$

that $\lambda - a \in \mathcal{L}\mathrm{aut}(H)$, that is, $\lambda \in \rho(a)$. □

Of course, the precise form of the constants K and δ is of no particular importance. What is important, however, is the fact that these constants depend upon r and M only and not upon the individual operators a and a_0.

Let $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ be a **modulus of continuity**, that is, an increasing function which is continuous at 0 and vanishes there, is positive elsewhere, and satisfies $\omega(2t) \le c\omega(t)$ for $t > 0$. Notice that these assumptions imply that for every positive c_1 there exists a positive c_2 so that $\omega(c_1 t) \le c_2 \omega(t)$ for $t > 0$. Then we define the **ω-seminorm** $[a]_\omega$ of $a : \mathbb{R}^n \to \mathcal{L}(H)$ by

$$[a]_\omega := \sup\Big\{ \frac{|a(x) - a(y)|}{\omega(|x-y|)}\ ;\ x \ne y \Big\}\ .$$

We denote by

$$BUC(\omega) := BUC\big(\mathbb{R}^n, \mathcal{L}(H); \omega\big)$$

the Banach space of all $a \in BUC\big(\mathbb{R}^n, \mathcal{L}(H)\big)$ satisfying

$$\|a\|_{C(\omega)} := \|a\|_\infty + [a]_\omega < \infty\ ,$$

where $BUC\big(\mathbb{R}^n, \mathcal{L}(H)\big)$ is the Banach space of all bounded and uniformly continuous functions from $\mathbb{R}^n$ to $\mathcal{L}(H)$ equipped with the maximum norm $\|\cdot\|_\infty$. Of course, if $\omega(t) = t^\rho$ for some $\rho \in (0,1)$ and all $t \geq 0$, we write $\|\cdot\|_{C^\rho}$ and BUC^ρ for $\|\cdot\|_{C(\omega)}$ and $BUC(\omega)$, respectively. Note that $BUC\big(\mathbb{R}^n, \mathcal{L}\mathrm{aut}(H); \omega\big)$ is the open subset of $BUC(\omega)$ consisting of all $a \in BUC(\omega)$ such that $a(x) \in \mathcal{L}\mathrm{aut}(H)$ for each $x \in \mathbb{R}^n$.

Given a smooth function ψ on $\mathbb{R}^n$, we put $\psi_\varepsilon(x) := \varepsilon^{-n}\psi(\varepsilon^{-1}x)$ for $\varepsilon > 0$ and $x \in \mathbb{R}^n$. We fix now such a function φ which, in addition, is nonnegative, has support in the unit ball, and satisfies $\int \varphi\, dx = 1$. Then $\{\, \varphi_\varepsilon \;;\; \varepsilon > 0 \,\}$ is a mollifier.

Using these notations we can prove an invertibility result for mollified $\mathcal{L}(H)$-valued maps.

Lemma 4.3. *Suppose that $\varepsilon_0, M \in \dot{\mathbb{R}}^+$ and ω is a modulus of continuity. Then there exist constants c and κ such that, given any*

$$a \in BUC\big(\mathbb{R}^n, \mathcal{L}\mathrm{aut}(H); \omega\big)$$

satisfying

$$\|a\|_\infty + \|a^{-1}\|_\infty \leq M \quad \text{and} \quad [a]_\omega \leq \kappa \;, \tag{4.1}$$

it follows that $\varphi_\varepsilon * a \in C^\infty\big(\mathbb{R}^n, \mathcal{L}\mathrm{aut}(H)\big)$ and

$$\|(\varphi_\varepsilon * a)^{-1}\|_\infty \leq c \;, \qquad 0 < \varepsilon \leq \varepsilon_0 \;.$$

Proof. Note that

$$\varphi_\varepsilon * a(x) - a(x) = \int_{|y|\leq 1} \varphi(y)\big[a(x - \varepsilon y) - a(x)\big]\, dy$$

implies

$$\|\varphi_\varepsilon * a - a\|_\infty \leq \omega(\varepsilon)[a]_\omega \leq \omega(\varepsilon_0)[a]_\omega \tag{4.2}$$

for $0 < \varepsilon \leq \varepsilon_0$. From (4.1) we deduce that $\sigma\big(a(x)\big) \subset \big[\,|z| \geq 1/M\big]$ for $x \in \mathbb{R}^n$. Hence (4.1), (4.2), and Lemma 4.2 guarantee the existence of κ such that $[a]_\omega \leq \kappa$ implies

$$\sigma\big(\varphi_\varepsilon * a(x)\big) \subset \big[\,|z| \geq 1/(M+1)\big] \;, \qquad x \in \mathbb{R}^n \;, \quad 0 < \varepsilon \leq \varepsilon_0 \;.$$

Now the assertion follows from (4.1) and Lemma 4.1. □

5. Estimates for Symbols. Below we derive technical estimates for matrix-valued symbols, that is, functions from $\mathbb{R}^n \times \mathbb{R}^k$ to $\mathcal{L}(H)$, which are positively homogeneous in the 'Fourier variable' $\xi \in \mathbb{R}^k$ and possess only little regularity in the 'space variable' $x \in \mathbb{R}^n$. We use a variant of the technique of 'symbol smoothing' introduced by Kumano-go and Nagase in [12] and subsequently applied by Nagase in many papers dealing with boundedness properties of pseudo differential operators with non-regular symbols. By a simple trick the results of this section will be applied in Sections 7 and 8 below to the case of parameter-dependent symbols.

Let $n, \ell \in \dot{\mathbb{N}}$ be fixed and put $k := n + \ell$. Denote by $\zeta := (\xi, \eta)$ the general point of $\mathbb{R}^k = \mathbb{R}^n \times \mathbb{R}^\ell$ and put

$$\zeta^* := \zeta / |\zeta| \ , \qquad \zeta \in \dot{\mathbb{R}}^k \ ,$$

where $|\cdot|$ is the euclidean norm.

We fix now $m, K \in \dot{\mathbb{R}}^+$ and a modulus of continuity ω. Then we assume that

$$a : \mathbb{R}^n \times \mathbb{R}^k \to \mathcal{L}(H)$$

has the following properties:

$$\left.\begin{array}{l} \bullet \quad a(\cdot, \zeta) \in BUC(\omega) \text{ for } \zeta \in \mathbb{R}^k \ ; \\ \bullet \quad a(x, \cdot) \text{ is positively homogeneous of degree } m \text{ for } x \in \mathbb{R}^n \ ; \\ \bullet \quad a(x, \cdot, \eta) \in C^{n+2}\big(\mathbb{R}^n, \mathcal{L}(H)\big) \text{ for } (x, \eta) \in \mathbb{R}^n \times \mathbb{R}^\ell \text{ and} \\ \qquad \max\limits_{|\alpha| \le n+2} \|\partial_\xi^\alpha a(\cdot, \zeta^*)\|_\infty \le K \ , \qquad \zeta \in \dot{\mathbb{R}}^k \ . \end{array}\right\} \tag{5.1}$$

Then, given $\delta \in (0,1)$, we put

$$a^\delta(\cdot, \zeta) := \varphi_{|\zeta|^{-\delta}} * a(\cdot, \zeta) \ , \qquad \zeta \in \dot{\mathbb{R}}^k \ , \tag{5.2}$$

where φ is the function introduced in Section 4.

Lemma 5.1. *There exists a constant c such that, given any $a \colon \mathbb{R}^n \times \dot{\mathbb{R}}^k \to \mathcal{L}(H)$ satisfying conditions* (5.1)*, it follows that*

$$\|\partial_x^\beta \partial_\xi^\alpha a^\delta(\cdot, \zeta)\|_\infty \le cK \, |\zeta|^{m - |\alpha| + \delta|\beta|} \ , \qquad \zeta \in \dot{\mathbb{R}}^k \ ,$$

for $|\alpha| \vee |\beta| \le n + 2$.

Proof. Given $\varepsilon > 0$, it is obvious that

$$\partial_x^\beta \varphi_\varepsilon = \varepsilon^{-|\beta|} (\partial^\beta \varphi)_\varepsilon \ . \tag{5.3}$$

On the other hand,

$$\begin{aligned} \partial_\varepsilon \varphi_\varepsilon &= -n\varepsilon^{-1} \varphi_\varepsilon - \varepsilon^{-2} \varepsilon^{-n} \textstyle\sum_j x^j \partial_j \varphi(\varepsilon^{-1} \cdot) \\ &= -\varepsilon^{-1} \big(n\varphi_\varepsilon + \big(\textstyle\sum_j x^j \partial_j \varphi\big)_\varepsilon \big) =: \varepsilon^{-1} (\varphi_1)_\varepsilon \ . \end{aligned} \tag{5.4}$$

Note that φ_1 is smooth on $\mathbb{R}^n$ and has its support in the unit ball.

Now, letting $\varepsilon := |\cdot|^{-\delta}$ and observing

$$\partial_{\xi^j} |\zeta|^{-\delta} = -\delta \xi^j \, |\zeta|^{-\delta - 2} \ , \qquad \zeta \in \dot{\mathbb{R}}^k \ , \quad 1 \le j \le n \ , \tag{5.5}$$

it follows that

$$\partial_{\xi^j} a^\delta = (\varphi_1)_{|\cdot|^{-\delta}} * a_j + \varphi_{|\cdot|^{-\delta}} * \partial_{\xi^j} a \ ,$$

where $a_j := -\delta a \, |\cdot|^{-2} \, \xi^j$ for $1 \le j \le n$. Note that a_j and $\partial_{\xi^j} a$ are positively homogeneous of degree $m - 1$ in ζ.

Given $\alpha \in \mathbb{N}^n$ with $|\alpha| \le n + 2$, by induction it is easily verified that $\partial_\xi^\alpha (\varphi_{|\cdot|^{-\delta}} * a)$ is a finite linear combination of terms of the form $\psi_{|\cdot|^{-\delta}} * b$, where ψ is a smooth function on $\mathbb{R}^n$ with support in the unit ball, and where $b \colon \mathbb{R}^n \times \dot{\mathbb{R}}^k \to \mathcal{L}(H)$ is such that $b(x, \cdot)$ is positively homogeneous of degree $m - |\alpha|$ and

$$\|b(\cdot, \zeta^*)\|_\infty \le cK \ .$$

Since the coefficients of these linear combinations are independent of a, the assertion follows. □

Lemma 5.2. *There exists a constant c such that, given any $a\colon \mathbb{R}^n \times \mathbb{R}^k \to \mathcal{L}(H)$ satisfying conditions* (5.1) *and*

$$\max_{|\alpha|\le n+2} \sup_{|\zeta|=1} \bigl|\partial_\xi^\alpha \bigl[a(x,\zeta) - a(y,\zeta)\bigr]\bigr| \le K\omega(|x-y|) \tag{5.6}$$

for $x, y \in \mathbb{R}^n$, it follows that

$$\bigl\|\partial_\xi^\alpha \bigl[a(\cdot,\zeta) - a^\delta(\cdot,\zeta)\bigr]\bigr\|_\infty \le cK\omega(|\zeta|^{-\delta})\,|\zeta|^{m-|\alpha|} \ , \qquad \zeta \in \dot{\mathbb{R}}^k \ ,$$

for $|\alpha| \le n+2$.

Proof. Let $\varepsilon := |\cdot|^{-\delta}$. Then

$$b(x,\cdot) := a(x,\cdot) - a^\delta(x,\cdot) = \int \varphi_\varepsilon(y)\bigl[a(x,\cdot) - a(x-y,\cdot)\bigr]\,dy$$

implies

$$\partial_\xi^\alpha b(x,\cdot) = \sum_{\beta\le\alpha} \tbinom{\alpha}{\beta} \int \partial_\xi^\beta \varphi_\varepsilon(y) \partial_\xi^{\alpha-\beta}\bigl[a(x,\cdot) - a(x-y,\cdot)\bigr]\,dy \ .$$

From (5.3) and (5.4) it follows by induction that $\partial_\xi^\beta \varphi_\varepsilon$ is a finite linear combination of terms of the form $\varepsilon^{-|\beta|} e\psi_\varepsilon$, where ψ is smooth on $\mathbb{R}^n$ with support in the unit ball and $e\colon \dot{\mathbb{R}}^k \to \mathbb{R}$ is positively homogeneous of degree zero and bounded on $|\zeta| = 1$. Thus $\partial_\xi^\alpha b$ is a finite linear combination of terms of the form

$$|\cdot|^{-|\beta|}\, e \int_{|y|\le 1} \psi(y) \partial_\xi^{\alpha-\beta}\bigl[a(x,\cdot) - a(x - |\cdot|^{-\delta}\, y,\cdot)\bigr]\,dy \ . \tag{5.7}$$

Since $\partial_\xi^{\alpha-\beta} a(x,\cdot)$ is positively homogeneous of degree $m - |\alpha| + |\beta|$ for $x \in \mathbb{R}^n$, we deduce from (5.6) that (5.7) can be estimated by

$$cK\omega(|\zeta|^{-\delta})\,|\zeta|^{m-|\alpha|}$$

for $x \in \mathbb{R}^n$ and $\zeta \in \dot{\mathbb{R}}^k$. Now the assertion is obvious. $\square$

By combining Lemmas 5.1 and 5.2 with Lemma 4.3 we can prove the following estimates.

Lemma 5.3. *Suppose that $\sigma_0, m, K, K_{-1} \in \dot{\mathbb{R}}^+$ and that ω is a modulus of continuity. Then there exist constants κ and c such that, given any*

$$a\colon \mathbb{R}^n \times \dot{\mathbb{R}}^k \to \mathcal{L}(H)$$

satisfying conditions (5.1), (5.6), $a(x,\zeta) \in \mathcal{L}\mathrm{aut}(H)$ *for* $(x,\zeta) \in \mathbb{R}^n \times \dot{\mathbb{R}}^k$ *and*

$$\sup_{|\zeta|=1} \|a^{-1}(\cdot,\zeta)\|_\infty \le K_{-1} \ , \tag{5.8}$$

and

$$\sup_{|\zeta|=1} \left[a(\cdot,\zeta)\right]_\omega \le \kappa \ ,$$

it follows that $a^\delta : \mathbb{R}^n \times \dot{\mathbb{R}}^k \to \mathcal{L}\mathrm{aut}(H)$ *and*

$$\text{(i)} \qquad \|\partial_x^\beta \partial_\xi^\alpha (a^\delta)^{-1}(\cdot,\zeta)\|_\infty \le c\,|\zeta|^{-m-|\alpha|+\delta|\beta|} \ , \qquad |\alpha| \vee |\beta| \le n+2 \ ,$$

$$\text{(ii)} \qquad \left\|\partial_\xi^\alpha \big((a-a^\delta)(a^\delta)^{-1}\big)(\cdot,\zeta)\right\|_\infty \le c\omega(|\zeta|^{-\delta})\,|\zeta|^{-|\alpha|} \ , \qquad |\alpha| \le n+2 \ ,$$

$$\text{(iii)} \qquad \left\|\partial_\xi^\alpha \left[\partial_\xi^\beta a\,\partial_x^\beta (a^\delta)^{-1}\right](\cdot,\zeta)\right\|_\infty \le c\,|\zeta|^{-|\alpha|+(\delta-1)|\beta|} \ , \qquad |\alpha|+|\beta| \le n+2 \ ,$$

$$\text{(iv)} \qquad \left\|\partial_\xi^\alpha \left[a^{-1}-(a^\delta)^{-1}\right](\cdot,\zeta)\right\|_\infty \le c\omega(|\zeta|^{-\delta})\,|\zeta|^{-m-|\alpha|} \ , \qquad |\alpha| \le n+2 \ ,$$

for $|\zeta| \ge \sigma_0$.

Proof. Suppose that

$$b \in C^{n+2}\big(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{L}\mathrm{aut}(H)\big) \ .$$

Then, given $\alpha,\beta \in \mathbb{N}^n$ with $|\alpha| \vee |\beta| \le n+2$, it is easily verified that $\partial_x^\beta \partial_\xi^\alpha b^{-1}$ can be represented as a finite linear combination of terms of the form

$$b^{-1}(\partial_x^{\beta_1}\partial_\xi^{\alpha_1} b)b^{-1}\cdots b^{-1}(\partial_x^{\beta_r}\partial_\xi^{\alpha_r} b)b^{-1} \ ,$$

where $\alpha_1+\cdots+\alpha_r=\alpha$ and $\beta_1+\cdots+\beta_r=\beta$ with $\alpha_i,\beta_i \in \mathbb{N}^n$. Hence (i) follows from Lemmas 4.3 and 5.1.

Thanks to Leibniz' rule,

$$\partial_\xi^\alpha\left[(a-a^\delta)(a^\delta)^{-1}\right] = \sum_{\gamma\le\alpha} \tbinom{\alpha}{\gamma}\partial_\xi^\gamma(a-a^\delta)\partial_\xi^{\alpha-\gamma}(a^\delta)^{-1} \ .$$

Hence (ii) is a consequence of (i) and Lemma 5.2.

Again by Leibniz' rule,

$$\partial_\xi^\alpha\left[\partial_\xi^\beta a\,\partial_x^\beta (a^\delta)^{-1}\right] = \sum_{\gamma\le\alpha} \tbinom{\alpha}{\gamma}\partial_\xi^{\beta+\gamma} a\,\partial_x^\beta\partial_\xi^{\alpha-\gamma}(a^\delta)^{-1} \ .$$

Therefore we infer (iii) from (i) and the fact that $\partial_\xi^{\beta+\gamma}a$ is positively homogeneous of degree $m-|\beta|-|\gamma|$ in $\zeta \in \dot{\mathbb{R}}^k$.

Lastly, note that

$$a^{-1}-(a^\delta)^{-1} = a^{-1}(a^\delta - a)(a^\delta)^{-1}$$

and that a^{-1} is positively homogeneous of degree $-m$ in $\zeta \in \dot{\mathbb{R}}^k$. Thus assertion (iv) is an easy consequence of Leibniz' rule and (ii). □

We fix now $\theta_0 \in [0,\pi)$ and suppose that

$$b : \mathbb{R}^k \to \mathcal{L}(H)$$

has the following properties:

$$\left.\begin{array}{l}\bullet\quad b \text{ is positively homogeneous of degree } m\ ;\\ \bullet\quad b(\cdot,\eta)\in C^{n+2}\big(\mathbb{R}^n,\mathcal{L}(H)\big) \text{ for } \eta\in\mathbb{R}^\ell\ ;\\ \bullet\quad \sigma\big(b(\zeta^*)\big)\subset S_{\theta_0} \text{ for } \zeta\in\dot{\mathbb{R}}^k\ ;\\ \bullet\quad \max\limits_{|\alpha|\le n+2}\sup\limits_{|\zeta|=1}|\partial_\xi^\alpha b(\zeta)|+\sup\limits_{|\zeta|=1}|b^{-1}(\zeta)|\le K\ .\end{array}\right\} \tag{5.9}$$

We also suppose that $\theta_0<\theta<\pi$ and that there exists $\varepsilon>0$ such that

$$g\colon \mathring{S}_\theta\to\mathbb{C} \tag{5.10}$$

is holomorphic and satisfies

$$|z|^\varepsilon\, g(z)\to 0 \qquad \text{as} \quad z\to\infty \quad \text{in } \mathring{S}_\theta\ . \tag{5.11}$$

Let $\vartheta:=\pi-\theta$ and $\Gamma:=\Gamma(K,\vartheta)$ and put

$$g(b)(\zeta):=\frac{1}{2\pi i}\int_\Gamma g(-\lambda)\big(\lambda+b(\zeta)\big)^{-1}\,d\lambda\ , \qquad \zeta\in\dot{\mathbb{R}}^k\ . \tag{5.12}$$

The following lemma implies, in particular, that $g(b)(\zeta)\in\mathcal{L}(H)$ is well-defined.

Lemma 5.4. *Suppose that $m,K\in\dot{\mathbb{R}}^+$ and $0\le\theta_0<\theta<\pi$. Then there exists a constant c such that, given $b\colon\mathbb{R}^k\to\mathcal{L}(H)$ satisfying* (5.9)*, and given a holomorphic function g satisfying* (5.10) *and* (5.11) *for some $\varepsilon>0$, it follows that*

$$g(b)(\cdot,\eta)\in C^{n+2}\big(\mathbb{R}^n,\mathcal{L}(H)\big) \tag{5.13}$$

and

$$|\zeta|^{|\alpha|}\,|\partial_\xi^\alpha g(b)(\zeta)|\le c\sup\big\{\,|g(\lambda)|\ ;\ \lambda\in\mathring{S}_\theta\cap\big[\,|z|\ge|\zeta|^m/(2K)\big]\,\big\} \tag{5.14}$$

for $\zeta\in\dot{\mathbb{R}}^k$ and $|\alpha|\le n+2$.

Proof. Observe that

$$\sigma\big(-b(\zeta)\big)=-\,|\zeta|^m\,\sigma\big(b(\zeta^*)\big)\ , \qquad \zeta\in\dot{\mathbb{R}}^k\ . \tag{5.15}$$

Also note that, thanks to (5.9),

$$\sigma\big(-b(\zeta^*)\big)\subset[\,|\arg z|\ge\pi-\theta_0]\cap[1/K\le|z|\le K]\ . \tag{5.16}$$

Let Σ_R be the positively oriented boundary of

$$[\,|\arg z|\ge\vartheta_K]\cap[1/(2K)\le|z|\le R]\ , \qquad R\ge K+1\ ,$$

and put $\Sigma:=\Sigma_{K+1}$. Then (5.16) implies the existence of $\rho:=\rho(K,\theta)>0$ such that

$$\sigma\big(\lambda+b(\zeta^*)\big)\subset[\,|z|\ge\rho]\ , \qquad \lambda\in\Sigma\ , \quad \zeta\in\dot{\mathbb{R}}^k\ .$$

Hence we deduce from Lemma 4.1 and from (5.9) the existence of a constant c such that

$$\left|\left(\lambda+b(\zeta^*)\right)^{-1}\right| \le c\ , \qquad \lambda\in\Sigma\ , \quad \zeta\in\dot{\mathbb{R}}^k\ , \tag{5.17}$$

for all b under consideration.

Given $t>0$, let $t\Sigma$ be the curve obtained from Σ by the dilatation $\lambda\mapsto t\lambda$. Then $|\zeta|^m\,\Sigma$ is a positively oriented contour which, thanks to (5.15), contains $\sigma\bigl(-b(\zeta)\bigr)$ in its interior. Hence

$$G(b)(\zeta) := \frac{1}{2\pi i}\int_{|\zeta|^m\Sigma} g(-\lambda)\bigl(\lambda+b(\zeta)\bigr)^{-1}\,d\lambda\ , \qquad \zeta\in\dot{\mathbb{R}}^k\ , \tag{5.18}$$

is well-defined.

Let $\zeta_0\in\dot{\mathbb{R}}^k$ be fixed. The upper semicontinuity of the spectrum implies the existence of a neighborhood U of ζ_0 in $\dot{\mathbb{R}}^k$ such that $|\zeta_0|^m\,\Sigma$ contains $\sigma\bigl(-b(\zeta)\bigr)$ for each $\zeta\in U$ in its interior. Thus, thanks to Cauchy's theorem, we can replace $|\zeta|^m\,\Sigma$ by the fixed contour $|\zeta_0|^m\,\Sigma$ as long as $\zeta\in U$. From this we easily deduce that

$$G(b)(\cdot,\eta)\in C^{n+2}\bigl(\mathbb{R}^n,\mathcal{L}(H)\bigr)\ , \qquad \eta\in\dot{\mathbb{R}}^\ell\ ,$$

and that

$$\partial_\xi^\alpha G(b)(\zeta) = \frac{1}{2\pi i}\int_{|\zeta|^m\Sigma} g(-\lambda)\partial_\xi^\alpha\bigl(\lambda+b(\zeta)\bigr)^{-1}\,d\lambda$$

for $\zeta\in\mathbb{R}^n\times\dot{\mathbb{R}}^\ell$ and $|\alpha|\le n+2$.

Recall that $\partial_\xi^\alpha(\lambda+b)^{-1}=c_\alpha(\lambda,\cdot)(\lambda+b)^{-1}$, where c_α is a finite linear combination of terms of the form

$$(\lambda+b)^{-1}(\partial_\xi^\beta b)(\lambda+b)^{-1}(\partial_\xi^\gamma b)\cdots(\lambda+b)^{-1}(\partial_\xi^\sigma b) \tag{5.19}$$

with $\beta+\gamma+\cdots+\sigma=\alpha$. From the positive homogeneity of b it follows that

$$\bigl(\lambda+b(\zeta)\bigr)^{-1} = |\zeta|^{-m}\left(|\zeta|^{-m}\,\lambda+b(\zeta^*)\right)^{-1}\ , \qquad \zeta\in\dot{\mathbb{R}}^k\ ,$$

and, in turn, that $c_\alpha(\lambda,\zeta)=|\zeta|^{-|\alpha|}\,c_\alpha(|\zeta|^{-m}\,\lambda,\zeta^*)$ for $\zeta\in\dot{\mathbb{R}}^k$. Thus

$$|\zeta|^{|\alpha|}\,\partial_\xi^\alpha G(b)(\zeta) = \frac{1}{2\pi i}\int_\Sigma g(-\,|\zeta|^m\,\lambda)c_\alpha(\lambda,\zeta^*)\bigl(\lambda+b(\zeta^*)\bigr)^{-1}\,d\lambda$$

for $\zeta\in\dot{\mathbb{R}}^k$. Now we infer from (5.9), (5.17), and (5.19) that

$$|\zeta|^{|\alpha|}\,|\partial_\xi^\alpha G(b)(\zeta)| \le c\sup\bigl\{\,|g(\lambda)|\ ;\ \lambda\in\mathring{S}_\theta\cap\bigl[\,|z|\ge|\zeta|^m/(2K)\bigr]\,\bigr\}$$

for $\zeta\in\dot{\mathbb{R}}^k$ and $|\alpha|\le n+2$.

Fix $\psi\in[-\pi+\theta,\pi-\theta]$ and put $d(t,\zeta):=|t|^m\,e^{i\psi}+b(\zeta)$ for $(t,\zeta)\in(\mathbb{R}\times\mathbb{R}^k)^{\boldsymbol{\cdot}}$. From (5.9) we deduce the existence of a constant $r:=r(K,\theta)>0$ such that

$$\sigma\bigl(d(t,\zeta)\bigr)\subset\bigl[\,|z|\ge r\bigr]\ , \qquad |t|^2+|\zeta|^2=1\ .$$

Hence Lemma 4.1 and the fact that d is positively homogeneous of degree m guarantee

$$\left|\left(\lambda+b(\zeta)\right)^{-1}\right|\le c(|\zeta|)(1+|\lambda|)^{-1}\ ,\qquad |\arg\lambda|\le\pi-\theta\ ,\quad \zeta\in\dot{\mathbb{R}}^k\ . \tag{5.20}$$

Thanks to Cauchy's theorem we can replace the contour Σ in (5.18) by Σ_R for any $R\ge K+1$. Thus, letting $R\to\infty$, we infer from (5.11), (5.18), and (5.20) that $G(b)(\zeta)=g(b)(\zeta)$ for $\zeta\in\dot{\mathbb{R}}^k$. This proves the lemma. □

6. Pseudo Differential Operators. Let $a\in C\left(\mathbb{R}^n\times\mathbb{R}^n,\mathcal{L}(H)\right)$ such that $a(x,\cdot)$ is polynomially bounded for each $x\in\mathbb{R}^n$. Then we define the pseudo differential operator

$$Op(a):\mathcal{S}(\mathbb{R}^n,H)\to BC(\mathbb{R}^n,H)$$

with symbol a by

$$Op(a)u(x):=(2\pi)^{-n}\int e^{i\langle x,\xi\rangle}a(x,\xi)\widehat{u}(\xi)\,d\xi\ ,\qquad x\in\mathbb{R}^n\ ,$$

where $\widehat{u}$ denotes the Fourier transform of u and $\mathcal{S}(\mathbb{R}^n,H)$ is the Schwartz space of rapidly decreasing smooth H-valued functions on $\mathbb{R}^n$.

In order to guarantee that $Op(a)$ extends to a continuous linear map of $L_p(\mathbb{R}^n,H)$ into itself for $1<p<\infty$, we introduce the following symbol classes. Suppose that $\delta\in[0,1)$ and put

$$\overline{n}:=[n/2]+1\ ,$$

where $[t]$ is the integer part of $t\in\mathbb{R}^+$. Then $\mathcal{S}_\delta$ is the set of all

$$a\in C^{\overline{n},2\overline{n}}\left(\mathbb{R}^n\times\mathbb{R}^n,\mathcal{L}(H)\right)$$

satisfying

$$\|a\|_{\mathcal{S}_\delta}:=\max_{\substack{|\alpha|\le 2\overline{n}\\ |\beta|\le\overline{n}}}\ \sup_{\xi\in\mathbb{R}^n}\langle\xi\rangle^{|\alpha|-\delta|\beta|}\,\|\partial_x^\beta\partial_\xi^\alpha a(\cdot,\xi)\|_\infty<\infty\ ,$$

where $\langle\xi\rangle:=(1+|\xi|^2)^{1/2}$ for $\xi\in\mathbb{R}^n$. We equip $\mathcal{S}_\delta$ with the norm $\|\cdot\|_{\mathcal{S}_\delta}$ so that it becomes a Banach space.

Let ω be a modulus of continuity satisfying the Dini condition

$$\int_0^1\frac{\omega(t)}{t}\,dt<\infty\ . \tag{6.1}$$

Then we denote by $\mathcal{S}_\delta(\omega)$ the set of all $a\in C^{0,n+1}\left(\mathbb{R}^n\times\mathbb{R}^n,\mathcal{L}(H)\right)$ such that

$$\|a\|_{\mathcal{S}_\delta(\omega)}:=\max_{|\alpha|\le n+1}\ \sup_{\xi\in\mathbb{R}^n}\frac{\langle\xi\rangle^{|\alpha|}}{\omega\left(\langle\xi\rangle^{-\delta}\right)}\,\|\partial_\xi^\alpha a(\cdot,\xi)\|_\infty<\infty\ .$$

We give this space the norm $\|\cdot\|_{\mathcal{S}_\delta(\omega)}$ so that it becomes a Banach space too.

The introduction of these multiplier spaces is justified by the following

Theorem 6.1. *Suppose that $1 < p < \infty$ and put $L_p := L_p(\mathbb{R}^n, H)$. Also suppose that ω satisfies* (6.1). *Then*

$$Op \in \mathcal{L}\big(\mathcal{S}_\delta, \mathcal{L}(L_p)\big) \cap \mathcal{L}\big(\mathcal{S}_\delta(\omega), \mathcal{L}(L_p)\big) \ .$$

Proof. The assertion for the symbol class $\mathcal{S}_\delta$ follows from the results and techniques in [18] and [25]. As for the symbol class $\mathcal{S}_\delta(\omega)$, we refer to [17] and [18]. □

7. Homogeneous Elliptic Operators on $\mathbb{R}^n$. It is the main purpose of this section to prove that an elliptic operator on $\mathbb{R}^n$, acting on vector valued functions, that is, an elliptic system on $\mathbb{R}^n$, is an operator of positive type, provided the symbol does not contain $(-\infty, 0)$ in its spectrum and the coefficients are nearly constant (matrices). These results are of auxiliary nature and will be used in subsequent sections.

We fix now $m \in \dot{\mathbb{N}}$ and $p \in (1, \infty)$ arbitrarily. Given $s \in \mathbb{R}^+$, we denote by

$$W_p^s := \big(W_p^s(\mathbb{R}^n, H), \|\cdot\|_{s,p}\big)$$

the usual Sobolev-Slobodeckii spaces of order s of H-valued functions on $\mathbb{R}^n$. We also put $D_j := -i\,\partial_j$ for $1 \le j \le n$.

By a differential operator on $\mathbb{R}^n$ we mean a linear differential operator of order m,

$$\mathcal{A} := \sum_{|\alpha| \le m} a_\alpha D^\alpha \ , \tag{7.1}$$

with $\mathcal{L}(H)$-valued coefficients

$$a_\alpha : \mathbb{R}^n \to \mathcal{L}(H) \ , \qquad \alpha \in \mathbb{N}^n \ , \quad |\alpha| \le m \ .$$

We associate with $\mathcal{A}$ its principal symbol

$$\mathcal{A}_\pi(x, \xi) := \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha \ , \qquad (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \ .$$

Then, given $M > 0$ and $\theta_0 \in [0, \pi]$, we say $\mathcal{A}$ is **uniformly (M, θ_0)-elliptic** if

$$\max_{|\alpha| = m} \|a_\alpha\|_\infty \le M \tag{7.2}$$

and

$$\sigma\big(\mathcal{A}_\pi(x, \xi)\big) \subset \dot{S}_{\theta_0} \quad \text{and} \quad \big|\big[\mathcal{A}_\pi(x, \xi)\big]^{-1}\big| \le M \ , \qquad x \in \mathbb{R}^n \ , \quad |\xi| = 1 \ . \tag{7.3}$$

Throughout the remainder of this section we assume that

$$\left.\begin{aligned} &\omega \text{ is a modulus of continuity,} \\ &a_\alpha \in BUC(\omega) = BUC\big(\mathbb{R}^n, \mathcal{L}(H); \omega\big) \ , \qquad |\alpha| = m \ , \\ &\text{and } a_\alpha = 0 \text{ for } |\alpha| < m, \end{aligned}\right\} \tag{7.4}$$

that is, $\mathcal{A} = \sum_{|\alpha| = m} a_\alpha D^\alpha$ is homogeneous of degree m. Note that, given any $b \in BUC\big(\mathbb{R}^n, \mathcal{L}(H)\big)$, there exists a modulus of continuity ω such that $b \in BUC(\omega)$.

Given $\theta \in (\theta_0, \pi)$ and $\psi \in [-\pi + \theta, \pi - \theta]$, put

$$a_\psi(x, \zeta) := |\rho|^m e^{i\psi} + |\sigma|^m + \mathcal{A}_\pi(x, \xi) \ , \qquad \zeta := (\xi, \eta) \in \mathbb{R}^k := \mathbb{R}^n \times \mathbb{R}^2 \ , \tag{7.5}$$

where we write $\eta := (\rho, \sigma)$ for the general point of $\mathbb{R}^2$.

Lemma 7.1. *Let $\mathcal{A}$ be uniformly (M,θ_0)-elliptic. There are constants $K := K(M)$ and $K_{-1} := K_{-1}(M,\theta)$ such that the maps $a_\psi : \mathbb{R}^n \times \mathbb{R}^k \to \mathcal{L}(H)$ satisfy condition* (5.1) *and* (5.8)*, uniformly with respect to $|\psi| \le \pi - \theta$.*

Proof. The validity of (5.1) is obvious. To prove (5.8) observe that

$$\sigma\big(\mathcal{A}_\pi(x,\xi)\big) \subset S_{\theta_0} \cap [\,|z| \ge |\xi|^m/M]\ , \qquad (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n\ .$$

This implies the existence of a constant $r := r(M) > 0$ such that

$$\sigma\big(a_\psi(x,\zeta)\big) \subset \big[\,|z| \ge r\sin(\theta-\theta_0)\big]$$

for $x \in \mathbb{R}^n$, $|\zeta| = 1$, and $|\psi| \le \pi - \theta$. Hence $a_\psi : \mathbb{R}^n \times \dot{\mathbb{R}}^k \to \mathcal{L}\mathrm{aut}(H)$ and Lemma 4.1 guarantees the existence of $K_{-1} := K_{-1}(M,\theta)$ such that (5.8) is true, uniformly with respect to $|\psi| \le \pi - \theta$. □

We consider now first the case of constant coefficients $a_\alpha \in \mathcal{L}(H)$ and prove the following basic

Lemma 7.2. *Let $M, \mu \in \dot{\mathbb{R}}^+$ and $\theta \in (\theta_0, \pi)$ be fixed. Then there exist constants c and $K \ge 1$ such that $\mu + \mathcal{A} \in \mathcal{P}(L_p; K, \pi - \theta)$ and*

$$\|\mu + \mathcal{A}\|_{\mathcal{L}(W_p^m, L_p)} + \|(\lambda + \mu + \mathcal{A})^{-1}\|_{\mathcal{L}(L_p, W_p^m)} \le c\ , \qquad \lambda \in S_{\pi-\theta}\ , \tag{7.6}$$

for all homogeneous (M,θ_0)-elliptic operators $\mathcal{A}$ with constant coefficients.

Proof. If $\mathcal{A}$ has constant coefficients, its principal symbol and, consequently, a_ψ are independent of $x \in \mathbb{R}^n$. Hence $a_\psi^\delta = a_\psi$ and we deduce from Lemmas 7.1 and 5.3(i) that

$$|\partial_\xi^\alpha a_\psi^{-1}(\zeta)| \le c\,|\zeta|^{-m-|\alpha|}\ , \qquad |\alpha| \le 2\overline{n}\ , \quad |\zeta| \ge \mu^{1/m}\ , \tag{7.7}$$

uniformly with respect to $|\psi| \le \pi - \theta$. Note that

$$|\zeta|^2 \ge (|\rho|^2 + |\sigma|^2) \vee (|\sigma|^2 + |\xi|^2) \tag{7.8}$$

and that

$$(1 \wedge \sigma)\langle\xi\rangle \le (|\xi|^2 + \sigma^2)^{1/2} \le (1 \vee \sigma)\langle\xi\rangle\ , \qquad \xi \in \mathbb{R}^n\ , \quad \sigma > 0\ . \tag{7.9}$$

Thus, letting $\lambda := |\rho|^m e^{i\psi}$ and $\sigma := \mu^{1/m}$, we deduce from (7.7)–(7.9) that

$$\langle\xi\rangle^{|\alpha|}\big|\partial_\xi^\alpha\big(\lambda + \mu + \mathcal{A}_\pi(\xi)\big)^{-1}\big| \le c(1+|\lambda|)^{-1}\ , \qquad |\alpha| \le 2\overline{n}\ , \quad \lambda \in S_{\pi-\theta}\ ,$$

where c depends upon M, μ, and θ only. Observe that this implies

$$(\lambda + \mu + \mathcal{A}_\pi)^{-1} \in \mathcal{S}_0$$

and that there exists $c := c(M,\mu,\theta)$ such that

$$|(\lambda + \mu + \mathcal{A}_\pi)^{-1}|_{\mathcal{S}_0} \le c(1+|\lambda|)^{-1}\ , \qquad \lambda \in S_{\pi-\theta}\ .$$

Thus, since

$$\lambda + \mu + \mathcal{A} = Op(\lambda + \mu + \mathcal{A}_\pi)$$

and since $\mathcal{A}$ has constant coefficients, it is an easy consequence of Theorem 6.1 (or Mikhlin's multiplier theorem, of course) that there exists $K := K(M, \mu, \theta) \geq 1$ with $\mu + \mathcal{A} \in \mathcal{P}(K, \pi - \theta)$.

It is obvious that $\mathcal{A} \in \mathcal{L}(W_p^m, L_p)$ and that its norm is bounded by a constant depending on M only.

We infer from (7.7), (7.9), and Leibniz' rule that

$$\langle \xi \rangle^{|\alpha|} \bigl| \partial_\xi^\alpha \bigl[\langle \xi \rangle^m a_\psi^{-1}(\zeta) \bigr] \bigr| \leq c \ , \qquad |\alpha| \leq 2\overline{n} \ , \quad |\zeta| \geq \mu^{1/m} \ , \quad |\psi| \leq \pi - \theta \ .$$

From this and Theorem 6.1 (or again by Mikhlin's theorem) it follows that

$$\|(1 - \Delta)^{m/2} (\lambda + \mu + \mathcal{A})^{-1}\|_{\mathcal{L}(L_p)} \leq c \ , \qquad \lambda \in S_{\pi - \theta} \ ,$$

where $c := c(M, \mu, \theta)$. Since $(1 - \Delta)^{m/2} \in \mathcal{L}\mathrm{is}(W_p^m, L_p)$, we see that (7.6) is true. □

It is now easy to prove the main result of this section, namely

Proposition 7.3. *Suppose that $M, \mu \in \dot{\mathbb{R}}^+$ and $\theta \in (\theta_0, \pi)$. Then there exist constants $c, K \geq 1$ and $\beta > 0$ such that, given any homogeneous uniformly (M, θ_0)-elliptic operator $\mathcal{A}$ with coefficients in $BUC\bigl(\mathbb{R}^n, \mathcal{L}(H)\bigr)$ and satisfying*

$$\max_{|\alpha| = m} \|a_\alpha - a_\alpha(y)\|_\infty \leq \beta \tag{7.10}$$

for some $y \in \mathbb{R}^n$, it follows that $\mu + \mathcal{A} \in \mathcal{P}(L_p; K, \pi - \theta)$ and

$$\|\mu + \mathcal{A}\|_{\mathcal{L}(W_p^m, L_p)} + \|(\mu + \mathcal{A})^{-1}\|_{\mathcal{L}(L_p, W_p^m)} \leq c \ . \tag{7.11}$$

Proof. Write $\mathcal{A} = \mathcal{A}(y) + \mathcal{B}$, where

$$\mathcal{A}(y) := \sum_{|\alpha| = m} a_\alpha(y) D^\alpha$$

and note that $\mathcal{B} \in \mathcal{L}(W_p^m, L_p)$ with $\|\mathcal{B}\| \leq \max_{|\alpha|=m} \|a_\alpha - a_\alpha(y)\|_\infty$. Now the first part of the assertion follows from Lemmas 7.2 and 1.1.

It is clear that $\mu + \mathcal{A} \in \mathcal{L}(W_p^m, L_p)$ with an estimate for its norm depending upon M only. If β is chosen so small that $\bigl\| \mathcal{B}\bigl(\mu + \mathcal{A}(y)\bigr)^{-1} \bigr\|_{\mathcal{L}(L_p)} \leq 1/2$, it follows from (1.8) that

$$\|(\mu + \mathcal{A})^{-1}\|_{\mathcal{L}(L_p, W_p^m)} \leq 2 \bigl\| \bigl(\mu + \mathcal{A}(y)\bigr)^{-1} \bigr\|_{\mathcal{L}(L_p, W_p^m)} \ .$$

Thus (7.11) is also a consequence of Lemma 7.2. □

We will remove the smallness condition (7.10) and admit lower order terms in Section 9 below.

8. Bounded H_∞-Calculus Under Smallness Conditions. By requiring a little more regularity for the coefficients of the differential operator $\mathcal{A}$ considered in the preceding section we shall now show that $\mu+\mathcal{A}$ has a bounded H_∞-calculus for any $\mu>0$. These results are again of auxiliary character.

We fix now two moduli of continuity ω_j satisfying

$$\int_0^1 \frac{\omega_j(t)}{t}\,dt<\infty\ ,\qquad j=1,2\ . \tag{8.1}$$

We also fix $M,\mu\in\dot{\mathbb{R}}^+$ and $0\le\theta_0<\theta<\pi$. Then we denote by

$$\mathcal{A}:=\sum_{|\alpha|=m} a_\alpha D^\alpha \tag{8.2}$$

an arbitrary uniformly (M,θ_0)-elliptic operator with coefficients

$$a_\alpha\in BUC(\omega_1\omega_2)\ ,\qquad |\alpha|=m\ . \tag{8.3}$$

Note that $\omega_1\omega_2$ is a modulus of continuity too.

Lemma 8.1. *There are constants $K\ge 1$ and $\beta,\kappa\in\dot{\mathbb{R}}^+$ such that*

$$\mu+\mathcal{A}\in\mathcal{P}(K,\pi-\theta) \tag{8.4}$$

and

$$(\lambda+\mu+\mathcal{A})^{-1}=R(\lambda)+S(\lambda)\ ,\qquad \lambda\in\Gamma:=\Gamma(K,\pi-\theta)\ , \tag{8.5}$$

where

$$R(\lambda):=Op\big((\lambda+\mu+\mathcal{A}_\pi^\delta)^{-1}\big)$$

and

$$S\in L_1\big(\Gamma,ds,\mathcal{L}(L_p)\big)\ , \tag{8.6}$$

provided

$$\max_{|\alpha|=m}\|a_\alpha-a_\alpha(y)\|_\infty\le\beta \tag{8.7}$$

for some $y\in\mathbb{R}^n$ and

$$\max_{|\alpha|=m}[a_\alpha]_{\omega_1\omega_2}\le\kappa\ . \tag{8.8}$$

Proof. Put $\theta_1:=(\theta_0+\theta)/2$ and define $a_\psi:\mathbb{R}^n\times\mathbb{R}^k\to\mathcal{L}(H)$ for $|\psi|\le\pi-\theta_1$ by (7.5). Let $\delta\in(0,1)$ be fixed and define a_ψ^δ by (5.2). It follows from Lemmas 7.1 (with θ replaced by θ_1) and 5.3 that there exists $\kappa>0$ such that (8.8) implies

$$a_\psi^\delta:\mathbb{R}^n\times\dot{\mathbb{R}}^k\mapsto\mathcal{L}\mathrm{aut}(H)\ ,\qquad |\psi|\le\pi-\theta_1\ ,$$

and that the estimates (i)–(iv) of Lemma 5.3 are valid, uniformly with respect to $|\psi|\le\pi-\theta_1$ and with ω replaced by $\omega_1\omega_2$.

Given $\eta = (\rho, \sigma) \in \dot{\mathbb{R}}^2$ and $|\psi| \le \pi - \theta_1$, it is easily verified that

$$\begin{aligned}(|\rho|^m e^{i\psi} + |\sigma|^m + \mathcal{A}) Op\big([a_\psi^\delta(\cdot,\cdot,\eta)]^{-1}\big)u \\ = u + Op\big(b_\psi(\cdot,\cdot,\eta)\big)u + Op\big(r_\psi(\cdot,\cdot,\eta)\big)u\end{aligned} \tag{8.9}$$

for $u \in \mathcal{S}$, where $b_\psi := (a_\psi - a_\psi^\delta)[a_\psi^\delta]^{-1}$ and

$$r_\psi := \sum_{0<|\beta|\le m} \frac{1}{\beta!} \partial_\xi^\beta a_\psi D_x^\beta [a_\psi^\delta]^{-1} \ .$$

Given $\sigma_0 \in (0,1]$, we deduce from Lemma 5.3(ii) the existence of a constant c such that

$$|\zeta|^{|\alpha|} \, \|\partial_\xi^\alpha b_\psi(\cdot,\zeta)\|_\infty \le c\omega_1\omega_2(|\zeta|^{-\delta}) \ , \qquad |\alpha| \le 2\overline{n} \ , \quad |\zeta| \ge \sigma_0 \ , \quad |\psi| \le \pi - \theta_1 \ ,$$

where, of course, $\zeta := (\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^2 = \mathbb{R}^k$. Hence (7.8) and (7.9) imply

$$\langle\xi\rangle^{|\alpha|} \omega_1\big(\sigma_0^{-\delta}\langle\xi\rangle^{-\delta}\big)^{-1} \, \|\partial_\xi^\alpha b_\psi(\cdot,\zeta)\|_\infty \le c\omega_2(|\eta|^{-\delta})$$

for $\xi \in \mathbb{R}^n$ and $\eta = (\rho,\sigma) \in \mathbb{R}^2$ with $\sigma \ge \sigma_0$, and for $|\psi| \le \pi - \theta_1$. Define a modulus of continuity $\widetilde{\omega}_1$ by $\widetilde{\omega}_1(t) := \omega_1(\sigma_0^{-\delta} t)$. Then it follows that $b_\psi(\cdot,\cdot,\eta) \in \mathcal{S}_\delta(\widetilde{\omega}_1)$ and

$$\|b_\psi(\cdot,\cdot,\eta)\|_{\mathcal{S}_\delta(\widetilde{\omega}_1)} \le c\omega_2(|\eta|^{-\delta}) \tag{8.10}$$

for $\eta = (\rho,\sigma) \in \mathbb{R}^2$ with $\sigma \ge \sigma_0$, and for $|\psi| \le \pi - \theta_1$.

Similarly, putting $\delta_0 := (1-\delta)/2$, we infer from Lemma 5.3(iii) that

$$|\zeta|^{|\alpha|+\delta_0} \, \|\partial_\xi^\alpha r_\psi(\cdot,\zeta)\|_\infty \le c\,|\zeta|^{-(1-\delta)/2} \ , \qquad |\alpha| \le 2\overline{n} \ , \quad |\zeta| \ge \sigma_0 \ , \quad |\psi| \le \pi - \theta_1 \ .$$

Thus, letting $\omega_0(t) := t$ for $t \ge 0$, we see from (7.8) and (7.9) that $r_\psi(\cdot,\cdot,\eta) \in \mathcal{S}_{\delta_0}(\omega_0)$ and

$$\|r_\psi(\cdot,\cdot,\eta)\|_{\mathcal{S}_{\delta_0}(\omega_0)} \le c\,|\eta|^{-(1-\delta)/2} \tag{8.11}$$

for $\eta = (\rho,\sigma) \in \mathbb{R}^2$ with $\sigma \ge \sigma_0$, and for $|\psi| \le \pi - \theta_1$.

Proposition 7.3 guarantees the existence of $\beta > 0$ and $K_0 \ge 1$ such that

$$\mu/2 + \mathcal{A} \in \mathcal{P}(K_0, \pi - \theta_1) \ ,$$

provided (8.7) is satisfied. Thus, thanks to (1.2) and Lemma 1.1(ii), we can find $K \ge K_0$ such that (8.4) is true, that

$$\mu/2 + \mathcal{A} \in \mathcal{P}(K, \pi - \theta_1) \subset \mathcal{P}(K, \pi - \theta) \ , \tag{8.12}$$

and that

$$\big[|\arg z| \le (\pi - \theta)_K\big] \cup \big[|z| \le 1/(2K)\big] \subset -\mu/2 + S_{\pi-\theta_1} \ . \tag{8.13}$$

Hence $\Gamma + \mu/2 \subset S_{\pi-\theta_1}$, where $\Gamma := \Gamma(K, \pi - \theta)$. Thus (8.12) and the trivial decomposition $\lambda + \mu/2 + \mu/2 + \mathcal{A} = \lambda + \mu + \mathcal{A}$ imply

$$(1 + |\lambda|)\, \|(\lambda + \mu + \mathcal{A})^{-1}\|_{\mathcal{L}(L_p)} \le c \ , \qquad \lambda \in \Gamma \ , \tag{8.14}$$

thanks to the fact that

$$|\lambda + \mu/2| \geq \begin{cases} |\lambda| \sin\theta_1 & \text{if} \quad 0 < \theta_1 < \pi/2 \ , \\ |\lambda| & \text{if } \pi/2 \leq \theta_1 < \pi \ . \end{cases} \tag{8.15}$$

Given $\lambda \in \Gamma$, it follows from (8.13) that there exists a unique pair (r, ψ) with $r > 0$ and $|\psi| \leq \pi - \theta_1$ satisfying

$$\lambda + \mu/2 = |\rho|^m \, e^{i\psi} \ . \tag{8.16}$$

Thus, letting

$$\sigma := \sigma_0 := (\mu/2)^{1/m} \ , \qquad \eta := (\rho, \sigma) \ , \tag{8.17}$$

it follows from (8.10) and Theorem 6.1 that

$$T_1(\lambda) := Op\big(b_\psi(\cdot, \cdot, \eta)\big) \in \mathcal{L}(L_p) \ , \qquad \lambda \in \Gamma \ ,$$

and that

$$\|T_1(\lambda)\|_{\mathcal{L}(L_p)} \leq c\omega_2\big(\big[\, |\lambda + \mu/2|^{2/m} + (\mu/2)^{2/m}\big]^{-\delta/2}\big) \ , \qquad \lambda \in \Gamma \ , \tag{8.18}$$

thanks to the fact that ω_0 and $\widetilde{\omega}_1$ satisfy the Dini condition (6.1). Similarly, (8.11) implies

$$T_2(\lambda) := Op\big(r_\psi(\cdot, \cdot, \eta)\big) \in \mathcal{L}(L_p) \ , \qquad \lambda \in \Gamma \ ,$$

and

$$\|T_2(\lambda)\|_{\mathcal{L}(L_p)} \leq c\big[\, |\lambda + \mu/2|^{2/m} + (\mu/2)^{2/m}\big]^{-(1-\delta)/4} \ , \qquad \lambda \in \Gamma \ . \tag{8.19}$$

Finally, let

$$R(\lambda) := Op\big(\big[a_\psi^\delta(\cdot, \cdot, \eta)\big]^{-1}\big) \ , \qquad \lambda \in \Gamma \ , \tag{8.20}$$

where λ and η satisfy (8.16) and (8.17). Then we infer from (8.9) that

$$(\lambda + \mu + \mathcal{A})R(\lambda) = 1 + T_1(\lambda) + T_2(\lambda) \ , \qquad \lambda \in \Gamma \ .$$

Thus, putting

$$S(\lambda) := -(\lambda + \mu + \mathcal{A})^{-1}\big(T_1(\lambda) + T_2(\lambda)\big) \ , \qquad \lambda \in \Gamma \ ,$$

we obtain (8.5). From (8.14), (8.15), (8.18), and (8.19) it follows that

$$\|S(\lambda)\|_{\mathcal{L}(L_p)} \leq c(1 + |\lambda|)^{-1}\big[\widetilde{\omega}_2\big((1 + |\lambda|)^{-\delta/m}\big) + (1 + |\lambda|)^{-(1-\delta)/(2m)}\big] \tag{8.21}$$

for $\lambda \in \Gamma$, where $\widetilde{\omega}_2(t) := \omega_2(\alpha t)$ for a suitable $\alpha := \alpha(\mu, \theta) > 0$. Note that

$$\int_{\alpha^{m/\delta}}^{\infty} \frac{\widetilde{\omega}_2(t^{-\delta/m})}{t} \, dt = \frac{m}{\delta} \int_0^1 \frac{\omega_2(t)}{t} \, dt < \infty \ . \tag{8.22}$$

Hence (8.21) implies (8.6). □

After these preparations we can prove that $\mu + \mathcal{A} \in \mathcal{H}_\infty(L_p; \pi - \theta)$ if (8.7) and (8.8) are satisfied.

Proposition 8.2. *Suppose that* (8.7) *and* (8.8) *are satisfied. Then there are constants* $N \geq 1$ *and* $\beta, \kappa \in \dot{\mathbb{R}}^+$ *such that* $\mu + \mathcal{A} \in \mathcal{H}_\infty(L_p; N, \pi - \theta)$.

Proof. The assertion follows from Lemmas 2.3 and 8.1, provided we show that

$$\Big\| \int_\Gamma g(-\lambda) R(\lambda)\, d\lambda \Big\|_{\mathcal{L}(L_p)} \leq c\, \|g\|_\infty \ , \qquad g \in H(\pi - \theta) \ . \tag{8.23}$$

Define $\eta = (\rho, \sigma)$ and ψ by (8.16) and (8.17) for $\lambda \in \Gamma$ and put

$$r(\lambda, x, \xi) := \big[a_\psi^\delta(x, \xi, \eta)\big]^{-1} \ , \qquad (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \ , \quad \lambda \in \Gamma \ ,$$

Then, thanks to (8.20) and Theorem 6.1, the estimate (8.23) is valid, provided we can show that

$$\Big[(x, \xi) \mapsto \int_\Gamma g(-\lambda) r(\lambda, x, \xi)\, d\lambda =: h_g(x, \xi)\Big] \in \mathcal{S}_\tau$$

for some $\tau \in [0, 1)$ and the norm can be estimated by $c\, \|g\|_\infty$ for $g \in H(\pi - \theta)$.

Recall that δ has been fixed arbitrarily in $(0, 1)$. Thus we can assume that $\delta < 1/\overline{n}$. Then we deduce from Lemma 5.3(i) and from (7.8), (7.9), and (8.15) that

$$\begin{aligned} \langle \xi \rangle^{|\alpha| - \tau|\beta|}\, \|\partial_x^\beta \partial_\xi^\alpha r(\lambda, \cdot, \xi)\|_\infty &\leq c\, |\zeta|^{-m - |\alpha| + \delta|\beta|}\, \langle \xi \rangle^{|\alpha| - \tau|\beta|} \\ &\leq c(1 + |\lambda|)^{-1 - (\tau - \delta)/m} \end{aligned} \tag{8.24}$$

for $\lambda \in \Gamma,\ \ 1 \leq |\alpha| \leq 2\overline{n},\ \ 1 \leq |\beta| \leq \overline{n}$, and $\delta < \tau < 1/\overline{n}$.

From (7.5) and (5.3) we easily infer that

$$\|\partial_x^\beta a_\psi^\delta(\cdot, \zeta)\|_\infty \leq c\, |\zeta|^{\delta|\beta|}\, |\xi|^m \ , \qquad \zeta \in \dot{\mathbb{R}}^k \ ,$$

for $1 \leq |\beta| \leq \overline{n}$ and $|\psi| \leq \pi - \theta$. Hence (cf. the proof of Lemma 5.3(i))

$$\big\|\partial_x^\beta \big(a_\psi^\delta(\cdot, \zeta)\big)^{-1}\big\|_\infty \leq c\, |\zeta|^{-m + \delta|\beta|}\, (|\xi|/|\zeta|)^m$$

for $|\zeta| \geq \sigma_0$ and $1 \leq |\beta| \leq \overline{n}$, and for $|\psi| \leq \pi - \theta$. This implies, thanks to (7.8), (7.9), and (8.15), the estimate

$$\begin{aligned} \langle \xi \rangle^{-\tau|\beta|}\, \|\partial_x^\beta r(\lambda, \cdot, \xi)\|_\infty &\leq c\, |\zeta|^{-m + (\delta - \tau)|\beta|}\, \big(\langle \xi \rangle / |\zeta|\big)^{m - \tau|\beta|} \\ &\leq c(1 + |\lambda|)^{-1 - (\tau - \delta)/m} \end{aligned} \tag{8.25}$$

for $\lambda \in \Gamma,\ \ 1 \leq |\beta| \leq \overline{n}$, and $\delta < \tau < 1/\overline{n}$. Note that (8.24) and (8.25) entail

$$\sup_{\xi \in \mathbb{R}^n} \langle \xi \rangle^{|\alpha| - \tau|\beta|}\, \|\partial_x^\beta \partial_\xi^\alpha h_g(\cdot, \xi)\|_\infty \leq c\, \|g\|_\infty \ , \qquad |\alpha| \leq 2\overline{n} \ , \quad 1 \leq |\beta| \leq \overline{n} \ ,$$

for $g \in H(\pi - \theta)$. Hence it remains to estimate $\partial_\xi^\alpha h_g(\cdot, \xi)$ for $|\alpha| \leq 2\overline{n}$.

Put

$$r_1(\lambda, x, \xi) := \big[a_\psi^\delta(x, \xi, \eta)\big]^{-1} - \big[a_\psi(x, \xi, \eta)\big]^{-1} \ .$$

Then it follows from Lemma 5.3(iv) and (7.8), (7.9), and (8.15) that

$$\langle\xi\rangle^{|\alpha|}\,\|\partial_\xi^\alpha r_1(\lambda,\cdot,\xi)\|_\infty \le c(1+|\lambda|)^{-1}\widetilde{\omega}_2\bigl((1+|\lambda|)^{-\delta/m}\bigr)$$

for $|\alpha|\le 2\overline{n}$ and $\lambda\in\Gamma$, where $\widetilde{\omega}_2$ has been defined in (8.21). From this and (8.22) we infer that

$$\sup_{\xi\in\mathbb{R}^n}\langle\xi\rangle^{|\alpha|}\Bigl\|\partial_\xi^\alpha\Bigl[\int_\Gamma g(-\lambda)r_1(\lambda,\cdot,\xi)\,d\lambda\Bigr]\Bigr\|_\infty \le c\,\|g\|_\infty$$

for $|\alpha|\le 2\overline{n}$ and $g\in H(\pi-\theta)$. Thus, thanks to (7.5), (8.16), and (8.17), it remains to prove that

$$\sup_{\xi\in\mathbb{R}^n}\langle\xi\rangle^{|\alpha|}\Bigl\|\partial_\xi^\alpha\Bigl[\int_\Gamma g(-\lambda)\bigl(\lambda+\mu+\mathcal{A}_\pi(\cdot,\xi)\bigr)^{-1}\,d\lambda\Bigr]\Bigr\|_\infty \le c\,\|g\|_\infty \tag{8.26}$$

for $|\alpha|\le 2\overline{n}$ and $g\in H(\pi-\theta)$.

Let

$$b(x,\xi,t):=|t|^m+\mathcal{A}_\pi(x,\xi)\ ,\qquad (x,\xi,t)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}\ ,$$

and note that the bracket in (8.26) equals $2\pi i\,g\bigl(b(x,\cdot,\cdot)\bigr)(\cdot,\mu^{1/m})$, where we use the notation (5.12). Thus (8.26) is a consequence of Lemmas 7.1 and 5.4. □

9. Elliptic Operators on $\mathbb{R}^n$. In this section we consider general elliptic systems on $\mathbb{R}^n$ and prove the fundamental resolvent estimates and the existence of a bounded H_∞-calculus under weak continuity conditions for the coefficients.

Let $Q:=(-1,1)^n$ be the open unit ball in $(\mathbb{R}^n,|\cdot|_\infty)$ and let $\{\,\tau_x\ ;\ x\in\mathbb{R}^n\,\}$ be the translation group in $L_{1,\mathrm{loc}}\bigl(\mathbb{R}^n,\mathcal{L}(H)\bigr)$, that is,

$$\tau_x a:=a(\cdot-x)\ ,\qquad x\in\mathbb{R}^n\ ,\quad a\in L_{1,\mathrm{loc}}\bigl(\mathbb{R}^n,\mathcal{L}(H)\bigr)\ .$$

Then, given $p\in[1,\infty]$, the function $a\in L_{1,\mathrm{loc}}\bigl(\mathbb{R}^n,\mathcal{L}(H)\bigr)$ belongs to L_p **locally uniformly** if

$$\|a\|_{p,\mathrm{unif}}:=\sup_{x\in\mathbb{Z}^n}\|\tau_x a\|_{L_p(Q,\mathcal{L}(H))}<\infty\ .$$

We put

$$L_{p,\mathrm{unif}}\bigl(\mathbb{R}^n,\mathcal{L}(H)\bigr):=\Bigl(\bigl\{\,a\in L_{1,\mathrm{loc}}\bigl(\mathbb{R}^n,\mathcal{L}(H)\bigr)\ ;\ \|a\|_{p,\mathrm{unif}}<\infty\,\bigr\},\ \|\cdot\|_{p,\mathrm{unif}}\Bigr)$$

for $1\le p\le\infty$. Note that $L_{p,\mathrm{unif}}$ is a Banach space, and $L_{\infty,\mathrm{unif}}\doteq L_\infty$. Also note that

$$L_p\bigl(\mathbb{R}^n,\mathcal{L}(H)\bigr)\hookrightarrow L_{p,\mathrm{unif}}\bigl(\mathbb{R}^n,\mathcal{L}(H)\bigr)\hookrightarrow L_{q,\mathrm{unif}}\bigl(\mathbb{R}^n,\mathcal{L}(H)\bigr)$$

for $1\le q\le p\le\infty$.

Let $\varepsilon\in(0,1]$ be fixed and let (U_j) be an enumeration of the open covering

$$\bigl\{\,(\varepsilon/2)(z/2+Q)\ ;\ z\in\mathbb{Z}^n\,\bigr\}$$

of $\mathbb{R}^n$ such that $j \geq k$ implies $|x_j|_\infty \geq |x_k|_\infty$, where x_j is the center of the cube U_j. Note that the covering (U_j) has **finite multiplicity**, that is, there exists $\ell \in \mathbb{N}$ such that no point of $\mathbb{R}^n$ is contained in more than ℓ cubes of the sequence (U_j).

Observe that

$$\varphi_j(x) := (2/\varepsilon)(x - x_j) \ , \qquad x \in \mathbb{R}^n \ ,$$

is a smooth diffeomorphism from U_j onto Q. Let π be a smooth function with support in Q being equal to one on $(1/2)Q$. Then each

$$\pi_j := (\pi \circ \varphi_j)\big(\textstyle\sum_k (\pi \circ \varphi_k)^2\big)^{-1/2} \ , \qquad j \in \mathbb{N} \ , \tag{9.1}$$

is smooth, has its support in U_j, and

$$\sum_j \pi_j^2 = 1 \ . \tag{9.2}$$

Let $p \in [1, \infty]$ be fixed, put

$$E := E_j := W_p^m \ , \quad F := F_j := L_p \ , \qquad j \in \mathbb{N} \ ,$$

and let $\boldsymbol{E} := (E_j)$ and $\boldsymbol{F} := (F_j)$. Given $X \in \{E, F\}$, denote by $\varphi_{X,j} := \psi_{X,j}$ the multiplication operator $u \mapsto \pi_j u$ on X.

Lemma 9.1. *$\big(\boldsymbol{X}, (\varphi_{X,j}), (\psi_{X,j})\big)$ is an ℓ_p-approximation system for $X \in \{E, F\}$. Moreover, $\boldsymbol{E} \overset{d}{\hookrightarrow} \boldsymbol{F}$, $\varphi_{E,j} \subset \varphi_{F,j}$, and $\psi_{E,j} \subset \psi_{F,j}$.*

Proof. It is easily seen that (3.8) and (3.9) are true and that the second part of the assertion is valid. Hence it remains to prove (3.13).

Observe that, given $\alpha \in \mathbb{N}^n$,

$$\|\partial^\alpha \pi_j\|_\infty \leq c(\alpha) \ , \qquad j \in \mathbb{N} \ .$$

Thus, thanks to the finite multiplicity of the covering (U_j), given $\alpha \in \mathbb{N}^n$ and $q \in [1, \infty)$, there exists a constant c such that

$$\Big|\sum_j (\partial^\alpha \pi_j) u_j(x)\Big|^q \leq c \sum_j |u_j(x)|^q$$

and

$$\sum_j |(\partial^\alpha \pi_j) u(x)|^q \leq c\, |u(x)|^q$$

for $u_j, u \in L_p$ and a.a. $x \in \mathbb{R}^n$. From this and Leibniz' rule we infer that, given $k \in \mathbb{N}$,

$$\Big\|\sum_j \pi_j u_j\Big\|_{k,p} \leq c\, \|\boldsymbol{u}\|_{\ell_p(\boldsymbol{W}_p^k)} \ , \qquad \boldsymbol{u} = (u_j) \in \ell_p(\boldsymbol{W}_p^k) \ ,$$

and

$$\|(\pi_j u)\|_{\ell_p(\boldsymbol{W}_p^k)} \leq c\, \|u\|_{k,p} \ , \qquad u \in W_p^k \ ,$$

where $\boldsymbol{W}_p^k := (Y_j)$ with $Y_j := W_p^k$. Now the assertion is obvious. □

Corollary 9.2. *Suppose that* $s \in \mathbb{R}^+$ *and* $1 \le p < \infty$. *Then*

$$u \mapsto \left(\sum_j \|\pi_j u\|_{s,p}^p\right)^{1/p} \tag{9.3}$$

is an equivalent norm for $W_p^s(\mathbb{R}^n, H)$.

Proof. Let $m \in \dot{\mathbb{N}}$ satisfy $m > s$. Then it follows from Lemma 9.1 that

$$r \in \mathcal{L}\big(\ell_p(\boldsymbol{E}), E\big) \cap \mathcal{L}\big(\ell_p(\boldsymbol{F}), F\big)$$

and

$$r^c \in \mathcal{L}\big(E, \ell_p(\boldsymbol{E})\big) \cap \mathcal{L}\big(F, \ell_p(\boldsymbol{F})\big) \ .$$

Let $(\cdot,\cdot)_{s/m}$ be the complex interpolation functor $[\,\cdot\,,\,\cdot\,]_{s/m}$ if $s \in \dot{\mathbb{N}}$, and the real interpolation functor $(\cdot,\cdot)_{s/m,p}$ if $s \in \mathbb{R}^+ \setminus \mathbb{N}$. Then it is well-known that

$$(F, E)_{s/m} \doteq W_p^s \ , \qquad 0 < s < m \ .$$

Moreover,

$$\big(\ell_p(\boldsymbol{F}), \ell_p(\boldsymbol{E})\big)_{s/m} \doteq \ell_p\big((\boldsymbol{F}, \boldsymbol{E})_{s/m}\big) \ , \qquad 0 < s < m \ ,$$

(e.g., Theorem 1.18.1 in [24]. Thus $(\boldsymbol{F}, \boldsymbol{E})_{s/m} \doteq \boldsymbol{G} := (G_j)$ with $G_j := W_p^s$ for $j \in \mathbb{N}$. Consequently,

$$r \in \mathcal{L}\big(\ell_p(\boldsymbol{G}), W_p^s\big) \ , \quad r^c \in \mathcal{L}\big(W_p^s, \ell_p(\boldsymbol{G})\big) \ ,$$

and (3.12) implies that $r^c r \in \mathcal{L}\big(\ell_p(\boldsymbol{G})\big)$ is a projection onto $\mathrm{im}(r^c)$ having $\ker(r)$ as kernel. Hence

$$\ell_p(\boldsymbol{G}) = \mathrm{im}(r^c) \oplus \ker(r)$$

and $r^c \in \mathcal{L}\mathrm{is}\big(W_p^s, \mathrm{im}(r^c)\big)$. Now the assertion is an obvious consequence of the definition of r^c. □

We fix now $m \in \dot{\mathbb{N}}$ and $p \in (1, \infty)$. Then we put

$$p_\alpha := n/(m - |\alpha|) \qquad \text{if} \quad m - n/p \le |\alpha| < m \ . \tag{9.4}$$

Then we prove the following continuity theorem for linear differential operators of 'lower order'.

Lemma 9.3. *Suppose that* $q_\alpha > p_\alpha$ *for* $m - n/p \le |\alpha| < m$ *and* $q_\alpha := p$ *otherwise, and that*

$$\mathcal{B} := \sum_{|\alpha| \le m-1} b_\alpha D^\alpha \ ,$$

with $b_\alpha \in L_{q_\alpha,\mathrm{unif}}\big(\mathbb{R}^n, \mathcal{L}(H)\big)$. *Then* $\mathcal{B} \in \mathcal{L}(W_p^s, L_p)$ *for some* $s \in (m-1, m)$ *and*

$$\|\mathcal{B}\|_{\mathcal{L}(W_p^s, L_p)} \le c(s) \max_{|\alpha| \le m-1} \|b_\alpha\|_{q_\alpha,\mathrm{unif}} \ .$$

Proof. Choose a smooth function χ with support in Q and being constantly equal to 1 on the support of π. Then define (χ_j) by replacing in the construction of (π_j) the function π by χ.

By Sobolev's imbedding theorem we know that, given $|\alpha| \le m-1 < s < m$,

$$W_p^{s-|\alpha|} \hookrightarrow L_{r_\alpha} , \qquad \frac{1}{p} \ge \frac{1}{r_\alpha} \ge \Big(\frac{1}{p} - \frac{s-|\alpha|}{n}\Big)_+ , \tag{9.5}$$

where the second inequality sign is strict if $s = |\alpha| + n/p$ (e.g., Section 2.8 in [24]). It is easily seen that we can choose $s \in (m-1, m)$ so that (9.5) is true if we put

$$\frac{1}{r_\alpha} := \frac{1}{p} - \frac{1}{q_\alpha} , \qquad |\alpha| \le m-1 . \tag{9.6}$$

Thus, given $u \in W_p^s$, it follows from the fact that χ_j equals one on the support of π_j, from (9.5), and from (9.6) that

$$\begin{aligned}\|\pi_j b_\alpha D^\alpha u\|_p = \|\pi_j b_\alpha D^\alpha(\chi_j u)\| &\le \|\boldsymbol{\pi}\|_{\ell_\infty(\boldsymbol{F})} \|b_\alpha\|_{q_\alpha,\mathrm{unif}} \|D^\alpha(\chi_j u)\|_{r_\alpha} \\ &\le c\, \|\boldsymbol{\pi}\|_{\ell_\infty(\boldsymbol{F})} \|b_\alpha\|_{q_\alpha,\mathrm{unif}} \|\chi_j u\|_{s,p}\end{aligned}$$

for $|\alpha| \le m-1$ and $j \in \mathbb{N}$. Now the assertion is a consequence of Corollary 9.2. □

After these preparations we can prove the following fundamental resolvent estimates for uniformly (M,θ_0)-elliptic operators on $\mathbb{R}^n$ under rather mild assumptions on the coefficients.

Theorem 9.4. *Suppose that* $1 < p < \infty$, $m \in \dot{\mathbb{N}}$, $M > 0$, *and* $0 \le \theta_0 < \theta < \pi$, *and let* ω *be a modulus of continuity. Also suppose that* $q_\alpha := p$ *if* $|\alpha| < m - n/p$, *and* $q_\alpha > p_\alpha$ *otherwise. Then there exist constants* $c, K \ge 1$ *and* $\mu > 0$ *such that, given any uniformly* (M,θ_0)*-elliptic operator*

$$\mathcal{A} = \sum_{|\alpha| \le m} a_\alpha D^\alpha \tag{9.7}$$

whose coefficients satisfy

$$a_\alpha \in \begin{cases} BUC\big(\mathbb{R}^n, \mathcal{L}(H); \omega\big) & \text{if } |\alpha| = m , \\ L_{q_\alpha,\mathrm{unif}}\big(\mathbb{R}^n, \mathcal{L}(H)\big) & \text{if } |\alpha| \le m-1 , \end{cases} \tag{9.8}$$

and

$$\max_{|\alpha| \le m-1} \|a_\alpha\|_{q_\alpha,\mathrm{unif}} + \max_{|\alpha| = m} \|a_\alpha\|_{C(\omega)} \le M , \tag{9.9}$$

it follows that

$$\mu + \mathcal{A} \in \mathcal{P}(L_p; K, \pi - \theta) \cap \mathcal{L}\mathrm{is}(W_p^m, L_p)$$

and

$$\|\mu + \mathcal{A}\|_{\mathcal{L}(W_p^m, L_p)} + \|(\mu + \mathcal{A})^{-1}\|_{\mathcal{L}(L_p, W_p^m)} \le c .$$

Proof. Given $y \in \mathbb{R}^n$, put

$$r_\varepsilon(y) := \begin{cases} y & \text{if } |y|_\infty \le \varepsilon \ , \\ \varepsilon y/|y|_\infty & \text{if } |y|_\infty > \varepsilon \ , \end{cases}$$

Then r_ε is the radial retraction in $(\mathbb{R}^n, |\cdot|_\infty)$ onto the closed ε-ball $\varepsilon\overline{Q}$. Hence r_ε is uniformly Lipschitz continuous (cf. Lemma 19.8 in [1]). Put

$$a_{\alpha,j,\varepsilon} := a_\alpha\bigl(x_j + \tau_{x_j} r_\varepsilon(\cdot)\bigr) \ , \qquad |\alpha| = m \ , \quad j \in \mathbb{N} \ .$$

where x_j is the center of U_j. Then

$$a_{\alpha,j,\varepsilon} \in BUC\bigl(\mathbb{R}^n, \mathcal{L}(H)\bigr)$$

and

$$\|a_{\alpha,j,\varepsilon} - a_{\alpha,j,\varepsilon}(x_j)\|_\infty \le \sup_{|y-z|_\infty \le \varepsilon} |a_\alpha(y) - a_\alpha(z)| \le \Bigl(\max_{|\alpha|=m} [a_\alpha]_\omega\Bigr)\omega(\sqrt{n}\varepsilon) \tag{9.10}$$

for $|\alpha| = m$ and $j \in \mathbb{N}$. Note that each

$$\mathcal{A}_{j,\varepsilon} := \sum_{|\alpha|=m} a_{\alpha,j,\varepsilon} D^\alpha \ , \qquad j \in \mathbb{N} \ ,$$

is a homogeneous uniformly (M, θ_0)-elliptic operator whose coefficients belong to $BUC\bigl(\mathbb{R}^n, \mathcal{L}(H)\bigr)$.

Let $\sigma > 0$ be fixed. Then Proposition 7.3 and (9.10) imply the existence of constants $c, K \ge 1$ and $\varepsilon_0 \in (0,1]$ such that, putting $A_j := \sigma + \mathcal{A}_{j,\varepsilon_0}$,

$$A_j \in \mathcal{P}(L_p; K, \pi - \theta) \cap \mathcal{L}\text{is}(W_p^m, L_p) \tag{9.11}$$

and

$$\|A_j\|_{\mathcal{L}(W_p^m, L_p)} + \|A_j^{-1}\|_{\mathcal{L}(L_p, W_p^m)} \le c \tag{9.12}$$

for $j \in \mathbb{N}$. Note that, having fixed $\varepsilon = \varepsilon_0$, the covering (U_j) and the functions π_j are now fixed too.

Let

$$A := \sigma + \sum_{|\alpha|=m} a_\alpha D^\alpha \in \mathcal{L}(E, F) \tag{9.13}$$

and $\boldsymbol{A} := (A_j) \in \ell_\infty\bigl(\mathcal{L}(\boldsymbol{E}, \boldsymbol{F})\bigr)$. Then

$$\|A\|_{\mathcal{L}(E,F)} \vee \|\boldsymbol{A}\|_{\ell_\infty(\mathcal{L}(\boldsymbol{E},\boldsymbol{F}))} \le \sigma + M \ .$$

Given $u \in E$,

$$A(\pi_j u) = \pi_j A u + \sum_{|\alpha|=m} a_\alpha \sum_{\beta<\alpha} \binom{\alpha}{\beta} D^{\alpha-\beta}\pi_j D^\beta u \ . \tag{9.14}$$

Put

$$B_j := -\sum_{|\alpha|=m} a_\alpha \sum_{\beta<\alpha} \tbinom{\alpha}{\beta} D^{\alpha-\beta}\pi_j D^\beta u =: \sum_{|\alpha|\le m-1} b_{j,\alpha} D^\alpha u = \sum_{|\alpha|\le m-1} \chi_j b_{j,\alpha} D^\alpha u \ ,$$

where χ_j is smooth with support in U_j and equals one on $\mathrm{supp}(\pi_j)$. Then it follows from $[E,F]_{1/m} \doteq W_p^{m-1}$ (cf. the proof of Lemma 9.1) that

$$B := (B_j) \in \mathcal{L}\big([E,F]_{1/m}, \ell_p(\boldsymbol{F})\big)$$

with

$$\|B\| \le c \max_{|\alpha|=m} \|a_\alpha\|_\infty \le cM \ . \tag{9.15}$$

Moreover, (9.14) implies

$$\psi_{F,j} A = A_j \psi_{E,j} + B_j \ , \qquad j \in \mathbb{N} \ , \tag{9.16}$$

since $A(\pi_j\,\cdot\,) = A_j(\pi_j\,\cdot\,)$, thanks to the fact that $a_\alpha|U_j = a_{\alpha,j,\varepsilon_0}|U_j$ for $j\in\mathbb{N}$.

Let $C_j := -B_j$ for $j\in\mathbb{N}$ and note that $C_j \in \mathcal{L}\big([E_j,F_j]_{1/m}, F\big)$. It is easily verified that

$$\Big(\boldsymbol{u} \mapsto C\boldsymbol{u} := \sum_j C_j u_j\Big) \in \mathcal{L}\big(\ell_p([\boldsymbol{E},\boldsymbol{F}]_{1/m}), F\big)$$

with

$$\|C\| \le c \max_{|\alpha|=m} \|a_\alpha\|_\infty \le cM \ . \tag{9.17}$$

Similarly as above, we deduce from (9.14) that $A\varphi_{E,j} = \varphi_{F,j}A_j + C_j$ for $j\in\mathbb{N}$. Thus the assertion follows from (9.11), (9.12), Lemma 9.1, and Proposition 3.2, provided $\mathcal{A}$ is homogeneous of degree m. The general case is now an easy consequence of Remark 1.2(b) and Lemma 9.3. □

Corollary 9.5. *Suppose that $\theta_0 < \pi/2$. Then $\mathcal{A}$ is the negative infinitesimal generator of a strongly continuous analytic semigroup on $L_p(\mathbb{R}^n, H)$.*

Although elliptic operators on $\mathbb{R}^n$ have been studied by many authors, Theorem 9.4 and Corollary 9.5 seem to be new in this generality. Previous generation theorems require much stronger 'conditions at infinity' for the coefficients (e.g., [19]). It should also be observed that the resolvent estimates of Theorem 9.4 are uniform with respect to the class of uniformly (M,θ_0)-elliptic operators satisfying (9.9).

In the above theorem ω can be an arbitrary modulus of continuity. We restrict now the class of admissible moduli to be able to prove that $\mu+\mathcal{A}$ possesses a bounded H_∞-calculus.

Theorem 9.6. *Let the hypotheses of Theorem 9.4 be satisfied and suppose that*

$$\int_0^1 \frac{\omega^{1/3}(t)}{t}\,dt < \infty \ . \tag{9.18}$$

Then there are constants $N\ge 1$ and $\mu>0$ such that

$$\mu + \mathcal{A} \in \mathcal{H}_\infty(L_p; N, \pi-\theta)$$

for each uniformly (M,θ_0)-elliptic operator $\mathcal{A}$ on $\mathbb{R}^n$ satisfying (9.7)–(9.9).

Proof. Let $\omega_j := \omega^{1/3}$ for $j = 1, 2, 3$. Then

$$\frac{|a_\alpha(x) - a_\alpha(y)|}{\omega_1\omega_2(|x-y|)} \leq \omega_3(|x-y|)[a_\alpha]_\omega$$

and an easy calculation using the growth properties of the moduli of continuity give

$$\max_{|\alpha|=m} [a_{\alpha,j,\varepsilon}]_{\omega_1\omega_2} \leq c\omega_3(\varepsilon) \max_{|\alpha|=m} [a_\alpha]_\omega \leq cM\omega_3(\varepsilon) \ .$$

Hence we can assume that $\varepsilon_0 \in (0,1]$ has been chosen such that the operators A_j satisfy (8.7) and (8.8) for each j, where β and κ are the constants of Proposition 8.2 and $y := x_j$. Proposition 8.2 guarantees the existence of $N \geq 1$ such that

$$A_j \in \mathcal{H}_\infty(L_p; N, \pi - \theta) \ , \qquad j \in \mathbb{N} \ .$$

Thus the assertion follows from (9.11)–(9.17), Lemma 9.1, and Proposition 3.2, provided $\mathcal{A}$ is homogeneous of degree m. The general case is then a consequence of Lemma 9.3 and Theorem 2.6. □

Corollary 9.7. *Let the hypotheses of Theorem* 9.4 *be satisfied and suppose that* (9.18) *is true. Then there exist constants* $\mu > 0$ *and* $M \geq 1$ *such that*

$$\|(\mu + A)^{it}\|_{\mathcal{L}(L_p(\mathbb{R}^n,H))} \leq Me^{\theta|t|} \ , \qquad t > 0 \ ,$$

for each uniformly (M,θ_0)*-elliptic operator* $\mathcal{A}$ *on* $\mathbb{R}^n$ *satisfying* (9.7)–(9.9).

Observe that condition (9.18) is satisfied if $\omega(t) = t^\rho$ for some $\rho \in (0,1)$, that is, if the top-order coefficients of $\mathcal{A}$ are bounded and uniformly Hölder continuous. Hence Corollary 9.7 extends considerably the corresponding result in [19].

10. Elliptic Operators on Compact Manifolds. In this section we show that elliptic operators on compact manifolds without boundary, acting on sections of vector bundles and possessing continuous coefficients, are of positive type. If the top-order coefficients are Hölder continuous, we prove the existence of a bounded H_∞-calculus.

Let X be a compact n-dimensional C^m-manifold without boundary for some $m \in \dot{\mathbb{N}}$, and let $G := (G, \pi, X)$ be a complex C^m-vector bundle over X of rank N with fiber H. By a trivializing coordinate system (κ, χ_κ) for G we mean a chart κ of X with domain X_κ together with a trivializing map

$$\pi^{-1}(X_\kappa) \to X_\kappa \times H \ , \quad g \mapsto \big(\pi(g), \chi_\kappa(g)\big)$$

over X_κ for G. Given a section u of G, its local representation u_κ with respect to (κ, χ_κ) is defined by

$$u_\kappa := \chi_\kappa \circ u \circ \kappa^{-1} \ .$$

Then, given $s \in [0, m]$ and $p \in (1, \infty)$, we denote by $W_p^s(X, G)$ the vector space of all sections u of G such that

$$\varphi u_\kappa \in W_p^s\big(\kappa(X_\kappa), H\big)$$

for each C^m-function φ with compact support in $\kappa(X_\kappa) \subset \mathbb{R}^n$ and each trivializing coordinate system (κ, χ_κ) for G, where sections coinciding almost everywhere (cf. Section 16.22.2 in [7]) have been identified. This space is topologized by the family of seminorms

$$u \mapsto \|\varphi u_\kappa\|_{s,p} \ ,$$

and $L_p(X,G) := W_p^0(X,G)$.

Choose a finite atlas $\mathfrak{K}$ of trivializing coordinate systems for G and a C^m-partition of unity $\{\, \tau_\kappa \ ; \ \kappa \in \mathfrak{K} \,\}$ on X subordinate to $\{\, X_\kappa \ ; \ \kappa \in \mathfrak{K} \,\}$. Then it is well-known and easily seen that

$$\|u\|_{s,p} := \Big(\sum_{\kappa \in \mathfrak{K}} \|(\tau_\kappa \circ \kappa^{-1}) u_\kappa\|_{s,p}^p \Big)^{1/p}$$

is a norm on $W_p^s(X,G)$ inducing the topology and that $W_p^s(X,G)$ is a Banach space with respect to this norm.

Let

$$\mathcal{A} \colon W_p^m(X,G) \to L_p(X,G)$$

be a linear differential operator of order m with continuous coefficients and let

$$\mathcal{A}_\pi : T^*(X) \to \mathrm{End}(G)$$

be its principal symbol (e.g., Section 23.29 in [7]). Then, given $\theta_0 \in [0,\pi)$, the operator $\mathcal{A}$ is θ_0**-elliptic** provided

$$\sigma\big(\mathcal{A}_\pi(\xi_x^*)\big) \subset \dot{S}_{\theta_0} \ , \qquad \xi_x^* \in \big[T_x^*(X)\big]^{\cdot} \ , \quad x \in X \ .$$

Using these notations we can prove the following

Theorem 10.1. *Suppose that $\mathcal{A}$ is θ_0-elliptic for some $\theta_0 \in [0,\pi)$. Then, given $p \in (1,\infty)$ and $\theta \in (\theta_0,\pi)$, there exists $\mu > 0$ such that*

$$\mu + \mathcal{A} \in \mathcal{P}\big(L_p(X,G); \pi - \theta\big) \cap \mathcal{L}\mathrm{is}\big(W_p^m(X,G), L_p(X,G)\big) \ .$$

Proof. We can (and will) assume that $\overline{Q} \subset \kappa(X_\kappa)$ and $\mathrm{supp}(\tau_\kappa \circ \kappa^{-1}) \subset Q$ for each $\kappa \in \mathfrak{K}$.

Let (cf. Section 17.13 in [7])

$$\mathcal{A}_\kappa := \sum_{|\alpha| \le m} a_{\kappa,\alpha} D^\alpha$$

be the local representation of $\mathcal{A}$ with respect to the trivializing coordinate system (κ, χ_κ). Recall that r_1 is the radial retraction in $(\mathbb{R}^n, |\cdot|_\infty)$ onto $\overline{Q}$ and put

$$\mathcal{A}_\kappa^0 := \sum_{|\alpha| \le m} (a_{\kappa,\alpha} \circ r_1) D^\alpha \ .$$

Note that

$$\mathcal{A}^0_{\kappa,\pi}(\mathbb{R}^n \times S^{n-1}) = \mathcal{A}_{\kappa,\pi}(\overline{Q} \times S^{n-1})$$

and that the spectra of the operators $\mathcal{A}_{\kappa,\pi}(x,\xi)$, $(x,\xi) \in \overline{Q} \times S^{n-1}$, are contained in a compact subset of $\dot{S}_{\theta_0}$, thanks to the upper semicontinuity of the spectrum. Hence there exists $M \geq 1$ such that each $\mathcal{A}^0_\kappa$ is a uniformly (M,θ_0)-elliptic linear differential operator on $\mathbb{R}^n$ whose coefficients belong to $BUC\bigl(\mathbb{R}^n, \mathcal{L}(H)\bigr)$. Thus Theorem 9.4 guarantees the existence of $\mu > 0$ such that

$$\mu + \mathcal{A}^0_\kappa \in \mathcal{P}\bigl(L_p(\mathbb{R}^n, H); \pi - \theta\bigr) \cap \mathcal{L}\mathrm{is}\bigl(W^m_p(\mathbb{R}^n, H), L_p(\mathbb{R}^n, H)\bigr) \tag{10.1}$$

for $\kappa \in \mathfrak{K}$.

Let

$$E := W^m_p(X,G) \ , \quad F := L_p(X,G)$$

and note that $E \overset{d}{\hookrightarrow} F$. Also let

$$E_\kappa := W^m_p\bigl(\kappa^{-1}(X_\kappa), H\bigr) \ , \quad F_\kappa := L_p\bigl(\kappa^{-1}(X_\kappa), H\bigr) \ , \qquad \kappa \in \mathfrak{K} \ ,$$

and

$$\boldsymbol{E} := (E_\kappa)_{\kappa \in \mathfrak{K}} \ , \quad \boldsymbol{F} := (F_\kappa)_{\kappa \in \mathfrak{K}} \ ,$$

where we fix an arbitrary enumeration of $\mathfrak{K}$. Put

$$\psi_\kappa(u) := (\tau_\kappa \circ \kappa^{-1}) u_\kappa \ , \qquad u \in L_p(X,G) \ , \quad \kappa \in \mathfrak{K} \ .$$

For each $\kappa \in \mathfrak{K}$ choose a C^m-function σ_κ on X with support contained in $\kappa^{-1}(Q)$ and such that $\sigma_\kappa | \operatorname{supp}(\tau_\kappa) = 1$. Define $\varphi_\kappa : F_\kappa \to F$ by

$$\chi_\kappa \circ \varphi_\kappa(v) := \sigma_\kappa(v \circ \kappa) \ , \qquad v \in F_\kappa \ , \quad \kappa \in \mathfrak{K} \ .$$

It is not difficult to verify that

$$\varphi_k \in \mathcal{L}(E_\kappa, E) \cap \mathcal{L}(F_\kappa, F) \ ,$$

that

$$\psi_k \in \mathcal{L}(E, E_\kappa) \cap \mathcal{L}(F, F_\kappa) \ ,$$

and that

$$\sum_\kappa \varphi_\kappa \psi_\kappa = 1 \ .$$

Since the atlas $\mathfrak{K}$ is finite, it is clear that $\bigl(\boldsymbol{E}, (\varphi_{E,\kappa}), (\psi_{E,\kappa})\bigr)$ and $\bigl(\boldsymbol{F}, (\varphi_{F,\kappa}), (\psi_{F,\kappa})\bigr)$ are ℓ_p-approximation systems for E and F, respectively. Thus conditions (i) and (ii) of (3.15) are satisfied. Moreover, putting $A := \mathcal{A}$ and $A_\kappa := \mathcal{A}^0_\kappa$ (where we mean the obvious restriction, of course), condition (iii) of (3.15) is satisfied too.

Let

$$B_\kappa u := -\sum_{|\alpha| \leq m} a_{\kappa,\alpha} \sum_{\beta < \alpha} \binom{\alpha}{\beta} D^{\alpha-\beta}(\tau_\kappa \circ \kappa^{-1}) D^\beta u_\kappa \ , \qquad u \in E \ .$$

Since $[E,F]_{1/m} \doteq W_p^{m-1}(X,G)$ it follows that

$$B := (B_\kappa) \in \mathcal{L}\big([E,F]_{1/m}, \ell_p(\boldsymbol{F})\big) \ .$$

Thanks to $(\mathcal{A}u)_\kappa = \mathcal{A}_\kappa u_\kappa$ and the fact that $\mathcal{A}_\kappa v = \mathcal{A}_\kappa^0 v$ if $v \in E_\kappa$ has its support in Q, we see that

$$\psi_\kappa A = A_\kappa \psi_\kappa + B_\kappa \ , \qquad \kappa \in \mathfrak{K} \ .$$

Thus condition (iv) of (3.15) is satisfied.

Lastly, note that $\big[\mathcal{A}\varphi_\kappa(v)\big]_\kappa = \mathcal{A}_\kappa\big((\sigma_\kappa \circ \kappa^{-1})v\big)$ for $v \in E_\kappa$ implies

$$\big[A\varphi_\kappa(v)\big]_\kappa = \big[\varphi_\kappa(A_\kappa v)\big]_\kappa + \widetilde{C}_\kappa v \ ,$$

where

$$\Big[v \mapsto \widetilde{C}_\kappa v := \sum_{|\alpha| \le m} a_{\kappa,\alpha} \sum_{\beta < \alpha} \tbinom{\alpha}{\beta} D^{\alpha-\beta}(\sigma_\kappa \circ \kappa^{-1}) D^\beta v\Big] \in \mathcal{L}\big([E_\kappa, F_\kappa]_{1/m}, F_\kappa\big)$$

For each $\kappa \in \mathfrak{K}$ choose a C^m-function $\widetilde{\sigma}_\kappa$ on X with support contained in $\kappa^{-1}(Q)$ and such that $\widetilde{\sigma}_\kappa | \operatorname{supp}(\sigma_\kappa) = 1$. Define $\widetilde{\varphi}_\kappa : F_\kappa \to F$ by

$$\chi_\kappa \circ \widetilde{\varphi}_\kappa(v) := \widetilde{\sigma}_\kappa(v \circ \kappa) \ , \qquad v \in F_\kappa \ , \quad \kappa \in \mathfrak{K} \ .$$

Then, letting

$$C_\kappa := \big[v \mapsto \widetilde{\varphi}_\kappa(\widetilde{C}_\kappa v)\big] \ , \in \mathcal{L}\big([E_\kappa, F_\kappa]_{1/m}, F\big) \ ,$$

we find that

$$A\varphi_\kappa = \varphi_\kappa A_\kappa + C_\kappa \ , \qquad \kappa \in \mathfrak{K} \ .$$

Hence the last condition of (3.15) is satisfied too. Now the assertion follows from (10.1) and Proposition 3.2(i). □

Corollary 10.2. *If $\theta_0 < \pi/2$ then $-\mathcal{A}$ generates a strongly continuous analytic semigroup on $L_p(X,G)$.*

In order to show that $\mu + \mathcal{A}$ has a bounded H_∞-calculus we have to impose more regularity. Namely, we suppose that

$$G \text{ is a } C^{m \vee 2}\text{-vector bundle} \ . \tag{10.2}$$

Thus $T^*(X)$ is at least a C^1-manifold and it makes sense to assume that

$$\begin{aligned} &\text{there exists } \varepsilon \in (0,1) \text{ such that} \\ &\mathcal{A}_\pi \in C^\varepsilon\big(T^*(X), \operatorname{End}(G)\big) \ . \end{aligned} \tag{10.3}$$

Of course, the definition of Hölder continuous sections is similar to the definition of sections in W_p^s given above.

Theorem 10.3. *Suppose that $\mathcal{A}$ is θ_0-elliptic for some $\theta_0 \in [0,\pi)$ and that conditions* (10.2) *and* (10.3) *are satisfied. Then, given $p \in (1,\infty)$ and $\theta \in (\theta_0,\pi)$, there exists $\mu > 0$ such that*

$$\mu + \mathcal{A} \in \mathcal{H}_\infty\big(L_p(X,G); \pi - \theta\big) \ .$$

Proof. Using the notations of the preceding proof, it follows that the top-order coefficients of $\mathcal{A}_\kappa^0$ are uniformly ε-Hölder continuous on $\mathbb{R}^n$. Thus, letting $\omega(t) := t^\varepsilon$ for $t \geq 0$, Theorem 9.6 guarantees that $\mu + A_\kappa \in \mathcal{H}_\infty(F_\kappa; \pi - \theta)$ for some $\mu > 0$ and each $\kappa \in \mathfrak{K}$. Now the assertion follows from Proposition 3.2 and Theorem 10.1. □

Corollary 10.4. *Given the hypotheses of Theorem* 10.3, *there exist* $\mu > 0$ *and* $M \geq 1$ *such that*

$$\|(\mu + A)^{it}\|_{\mathcal{L}(L_p(X,G))} \leq M e^{\theta |t|} , \qquad t \in \mathbb{R} .$$

For simplicity, we have restricted our considerations to the case of boundariless compact manifolds. It is not difficult to extend our results to noncompact manifolds without boundary which are suitably 'uniformly regular at infinity'.

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