OPTIMAL CONTROL PROBLEMS GOVERNED BY SEMILINEAR PARABOLIC EQUATIONS WITH LOW REGULARITY DATA

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Abstract. We study the existence of optimal controls for problems governed by semilinear parabolic equations. The nonlinearities in the state equation need not be monotone and the data need not be regular. In particular, the control may be any bounded Radon measure. Our examples include problems with nonlinear boundary conditions and parabolic systems.

1. Introduction

In [8] we developed a general existence and uniqueness theory for semilinear parabolic problems involving measures and low regularity data. The proofs were based on a generalized variation-of-constants formula in suitable extrapolated spaces and the Banach fixed point theorem. Other papers on this topic mostly use approximation of singular data by regular ones and, consequently, require a priori estimates (usually based on maximum principles) for the approximating solutions in order to solve the original problem. The approach in [8] is much simpler and more flexible. In particular, it can be easily used for problems with non-monotone nonlinearities and for systems. In [8] we also established stability estimates and compactness properties which play an important role in control theory.

It is the purpose of this paper to demonstrate the applicability of the results from [8] to optimal control problems with low regularity data. We restrict ourselves to the study of nonlinear model problems where the controls enter linearly. However, using the full strength of [8], one can also study feedback control problems where the control depends on the state variable in a nonlinear (and nonlocal) way. This will be the subject of a forthcoming paper.

Optimal control problems involving measures and low regularity data were studied before; see, for example, the recent papers [12], [14], [20] and the references therein. However, those studies are restricted to linear or monotone cases. Moreover, most of them solve the state equation by using the approximation procedure mentioned above so that the corresponding proofs are rather long and complicated.

Let us describe our results in more detail. In Section 3 we consider state problems of the form

$$\left.\begin{aligned} \partial_t y + \mathcal{A} y &= f(x,t,y,\nabla y) + u_Q, && x \in \Omega,\ t \in [0,T], \\ y &= 0, && x \in \Gamma_0,\ t \in [0,T], \\ \partial_{\boldsymbol{\nu}} y &= g(x,t,y) + u_\Sigma, && x \in \Gamma_1,\ t \in [0,T], \\ y(\cdot,0) &= y^0 && \text{in } \Omega, \end{aligned}\right\} \tag{1.1}$$

where, throughout this paper it is assumed that

$\Omega \subset \mathbb{R}^n$ is an open bounded set with a smooth boundary Γ, $n \geq 2$,

$\Gamma = \Gamma_0 \cup \Gamma_1$, where Γ_0 and Γ_1 are open as well as closed in Γ and disjoint .

Notice that either Γ_0 or Γ_1 may be empty. We set $Q := \Omega \times J$, $\Sigma := \Gamma \times J$, and $\Sigma_i := \Gamma_i \times J$, $i = 0, 1$, where $J := [0, T]$ with $T > 0$. The operator $\mathcal{A}$ is always of the form

$$\mathcal{A}y := -\nabla \cdot (\boldsymbol{a}\nabla y),$$

where

$\boldsymbol{a} = [a_{jk}] \in C^\infty(\bar{\Omega}, \mathbb{R}^{n\times n})$ is symmetric and uniformly positive definite.

Furthermore, $\partial_{\boldsymbol{\nu}}$ is the derivative with respect to the conormal $\boldsymbol{\nu} := \boldsymbol{a}\nu$, with ν being the outer unit normal on Γ. The maps f and g are nonlinear Carathéodory functions, y^0 is a bounded Radon measure in Ω, and the controls u_Q and u_Σ are bounded Radon measures in Q and Σ_1, respectively.

It should be observed that u_Q and u_Σ may be singular, e.g., point measures, in space and in time, whereas in most papers on control problems for parabolic equations with measures integrability with respect to the time variable is requested, at least if reasonably regular solutions are considered. An approach to control problems with measures, completely different from ours, is presented in the book by Fattorini [13] and generalized by Ahmed (see [1] for a survey and the references therein). These authors have to impose rather restrictive assumptions on the nonlinearities in order to construct solutions in a very weak setting. In particular, their methods do not seem to be applicable to the class of problems we treat in this paper. Furthermore, there is a large literature on impulsive evolution equations and their control (see the survey [1] and the references therein, for example). These are equations in which jumps may occur at prefixed points on the time axis. By an obvious choice of u_Q and u_Σ, such a situation is easily subsumed as a simple particular case in our general setting.

If f and g satisfy suitable growth conditions, then [8] guarantees the existence of a unique maximal (weak) solution of (1.1). Denoting by $\mathbb{U}_{\mathrm{ad}}$ the set of all admissible controls $u = (u_Q, u_\Sigma)$ and, setting

$$\mathbb{U}_{\mathrm{ad}}^G := \{u \in \mathbb{U}_{\mathrm{ad}} : \text{the solution } y = y(u) \text{ of (1.1) exists globally}\},$$

we prove an abstract existence theorem for optimal control problems of the form

$$\text{minimize } \mathbb{J}\bigl(y(u), u\bigr) \text{ over } u \in \mathbb{U}_{\mathrm{ad}}^G, \tag{1.2}$$

where $\mathbb{J}$ is the cost functional. This theorem requires (weak) compactness of the set $\mathbb{U}_{\mathrm{ad}}$ and lower semicontinuity of $\mathbb{J}$. In addition, in the general case we also require $\mathbb{U}_{\mathrm{ad}}^G \neq \emptyset$ and coercivity of $\mathbb{J}$ with respect to the state variable y. In the monotone case (for example, if $f(x,t,\xi,\eta)\xi \leq 0$ and $g(x,t,\xi)\xi \leq 0$ for all x,t,ξ,η) all solutions of (1.1) are global and solutions with bounded data are uniformly bounded so that these additional assumptions are not needed.

Given $u \in \mathbb{U}_{\mathrm{ad}}$, the abstract existence theorem in Section 3 requires $\mathbb{J}(\cdot, u)$ to be lower semicontinuous and coercive in a space Y which may depend on the nonlinearities f and g, and whose topology is somewhat complicated in the general case. In addition, there are many possibilities for the choice of Y. For this reason we consider in Section 4 model cases with power nonlinearities f, g and show how one can make simple choices for the space Y and the cost functional $\mathbb{J}$. For example, in the case of the problem

$$\left.\begin{aligned} \partial_t y - \Delta y &= |y|^{\lambda-1}y + u_Q && \text{in } Q, \\ y &= 0 && \text{on } \Sigma, \\ y(\cdot,0) &= y^0 && \text{in } \Omega, \end{aligned}\right\} \tag{1.3}$$

where $1 < \lambda < (n+2)/n$, a possible choice is $Y := L_p(Q)$ with $p \in [\lambda, (n+2)/n)$. Consequently, the optimal control problem (1.2) governed by (1.3) is solvable if $\mathbb{U}_{\rm ad}$ is (weakly) compact in the space of bounded Radon measures, $\mathbb{U}^G_{\rm ad} \neq \emptyset$, and $\mathbb{J}(y,u) = \int_Q |y|^p \, dx\, dt$, for example. Notice that the restriction $\lambda < (n+2)/n$ is necessary for the solvability of (1.3), see [10]. We also show how this restriction can be relaxed if u_Q is more regular (see Remark 4.7(i)).

Section 5 is devoted to various extensions and modifications. In particular we show how one can relax the lower semicontinuity and coercivity assumptions on $\mathbb{J}$ (see Example 5.1 and Theorem 5.5, respectively) and the compactness of $\mathbb{U}_{\rm ad}$ (see Example 5.2). In addition, we study several related control problems (e.g., maximizing the existence time of the solution y, control via initial data, etc.).

In Section 6 we consider optimal controls for problems governed by the system

$$\left.\begin{aligned} \partial_t y_1 - \Delta y_1 &= \kappa y_1 y_2 - b y_1 + u_1 + v_1 y_1 && \text{in } Q, \\ \partial_t y_2 - d\Delta y_2 &= a y_1 + u_2 && \text{in } Q, \end{aligned}\right\} \tag{1.4}$$

which is complemented by suitable boundary and initial conditions. Here $d \geq 0$, $a > 0$, κ, $b \in \mathbb{R}$, and u_1, v_1, u_2 are controls. System (1.4) (with $d = 0$ and $u_1 = v_1 = u_2 = 0$) was derived in [17] as a model for the dynamics of a nuclear reactor close to a stationary state. The state variables y_1 and y_2 correspond to the neutron flux and the temperature, respectively, and the constant κ represents the temperature feedback (cf. also [25]). Since this system (with $d \geq 0$, $\kappa > 0$ and $u_1 = v_1 = u_2 = 0$) possesses an interesting dynamics with possible blow-up in finite time, it became the object of study of many mathematical papers (see [11], [15], [21], [22], [27], [28] and the references therein). We consider the case $d = \kappa = 1$ and assume also that at most one of the controls u_1, v_1, u_2 is non-zero. Imposing suitable (low) regularity of these controls and lower semicontinuity and coercivity of the corresponding cost functional we prove the existence of optimal controls (see Theorems 6.1, 6.2, and 6.3). In the case $u_2 \neq 0$ we admit another control acting on the boundary $\partial\Omega$.

2. Preliminaries

First we introduce some further notation which will be used throughout this paper. We write $a \vee b := \max(a,b)$ and $a \wedge b := \min(a,b)$ for $a, b \in \mathbb{R}$. If $p \in (1,\infty)$, then p' is the dual exponent defined by $1/p + 1/p' = 1$. For $X \subset \mathbb{R}^n$ we denote by $\mathcal{M}(X)$ and $\mathcal{D}(X)$ the space of bounded Radon measures in X and of smooth functions with compact support in X, respectively. The symbols w and w^* are used to denote the weak and weak-star topology, respectively.

We write γ for the trace operator and set

$$\mathcal{B}y := \begin{cases} \gamma y & \text{on } \Gamma_0, \\ \partial_\nu y & \text{on } \Gamma_1. \end{cases} \tag{2.1}$$

Notice that problem (1.1) can now be written in the form

$$\left.\begin{aligned} \partial_t y + \mathcal{A}y &= f(x,t,y,\nabla y) + u_Q &&\text{in } Q,\\ \mathcal{B}y &= g(x,t,y) + u_\Sigma &&\text{in } \Sigma,\\ y(\cdot,0) &= y^0 &&\text{in } \Omega, \end{aligned}\right\} \tag{2.2}$$

where $g(x,t,y) := 0$ for $x \in \Gamma_0$ and u_Σ is extended by 0 on Σ_0.

Let $s \in [-2,2]$, $1 < q < \infty$, and $S_q := \{-2+1/q, -1+1/q, 1/q, 1+1/q\}$. We write $W_q^s := W_q^s(\Omega)$ for the usual Sobolev-Slobodeckii spaces; hence $W_q^0 = L_q$. We set

$$W_{q,\mathcal{B}}^s := \begin{cases} \{\, u \in W_q^s \ ;\ \mathcal{B}u = 0 \,\}, & 1+1/q < s \le 2,\\ \{\, u \in W_q^s \ ;\ \gamma u = 0 \text{ on } \Gamma_0 \,\}, & 1/q < s < 1+1/q,\\ \quad W_q^s, & 0 \le s < 1/q,\\ \quad (W_{q',\mathcal{B}}^{-s})', & -2 \le s < 0, \quad s \notin S_q, \end{cases}$$

where the dual space $(W_{q',\mathcal{B}}^{-s})'$ is determined by means of the standard L_q-duality pairing.

Consider the problem

$$\left.\begin{aligned} \partial_t y + \mathcal{A}y &= f &&\text{in } Q,\\ \mathcal{B}y &= g &&\text{on } \Sigma,\\ y(\cdot,0) &= y^0 &&\text{in } \Omega, \end{aligned}\right\} \tag{2.3}$$

where

$$y^0 \in \mathcal{M}(\Omega),\ f \in \mathcal{M}(Q), \text{ and } g \in \mathcal{M}(\Sigma) \text{ with } g = 0 \text{ on } \Sigma_0.$$

Assume that $s \in [0,2] \setminus S_q$ and $1 < p,q < \infty$. We recall from [8, p. 1059] that a **weak** $L_p(W_q^s)$ **solution** of (2.3) **on** $[0,t]$, where $0 < t \le T$, is a function $y \in L_{p,\mathrm{loc}}([0,t), W_{q,\mathcal{B}}^s)$ such that

$$\int_0^t \int_\Omega (-\partial_t\varphi + \mathcal{A}\varphi) y \, dx\, dt = \int_{\Omega\times[0,t]} \varphi \, df + \int_{\Gamma\times[0,t]} \varphi \, dg + \int_\Omega \varphi(0)\, dy^0 \tag{2.4}$$

for each $\varphi \in \mathcal{D}(\overline{\Omega} \times [0,t))$ satisfying $\mathcal{B}\varphi = 0$ on $\Gamma \times [0,t]$. It is **global** if $t = T$ and $y \in L_p((0,T), W_{q,\mathcal{B}}^s)$. Notice that, taking $\hat{f} := f + y^0 \otimes \delta_0$ instead of f, we can assume $y^0 = 0$.

Suppose that $k \in \mathbb{N}$, $q, p_1, p_2, \dots, p_k \in (1,\infty)$ and $s_1, s_2, \dots, s_k \in [0,2] \setminus S_q$. Set $\vec{p} := (p_1, p_2, \dots, p_k)$ and $\vec{s} := (s_1, s_2, \dots, s_k)$. We say that y is a **weak** $L_{\vec{p}}(W_q^{\vec{s}})$ **solution** of (2.3) **on** $[0,t]$ if it is a weak $L_{p_i}(W_q^{s_i})$ solution of (2.3) on $[0,t]$ for each $i = 1, 2, \dots, k$.

The differential operator $\mathcal{C} := 1 + \mathcal{A}$ defines an isomorphism between $W_{q,\mathcal{B}}^2$ and L_q and this isomorphism admits a unique extension to an isomorphism C_s between $W_{q,\mathcal{B}}^s$ and $W_{q,\mathcal{B}}^{s-2}$ for any $s \in [0,2] \setminus S_q$ (see [2]). Moreover, $-A := 1 - C_s$ generates a strongly continuous analytic semigroup $\{e^{-tA} \,;\, t \ge 0\}$ on $W_{q,\mathcal{B}}^r$ for r in $[-2,s] \setminus S_q$, and

$$(t \mapsto e^{-tA}x) \in C\big([0,T], W_{q,\mathcal{B}}^r\big) \cap C\big((0,T], W_{q,\mathcal{B}}^s\big) \tag{2.5}$$

with

$$\|e^{-tA}x\|_{W_{q,\mathcal{B}}^s} \le ct^{(r-s)/2}\, \|x\|_{W_{q,\mathcal{B}}^r}, \qquad 0 < t \le T, \tag{2.6}$$

for $x \in W^r_{q,\mathcal{B}}$ (cf. [2, Theorem 5.2] and [3, Theorem V.2.1.3]). Then, provided $1 < q < n/(n-2)$ and $0 \le s < 2 - n/q'$, (the weak form of) problem (2.3) is equivalent to the abstract evolution equation

$$\dot{y} + Ay = F \text{ in } [0,T], \qquad y(0) = y^0, \tag{2.7}$$

with $F := f + \gamma' g$, where

$$\gamma' : W_q^{s-1-1/q}(\Gamma) \to W^{s-2}_{q,\mathcal{B}}, \qquad s < 1 + 1/q,$$

is the dual of the trace operator from $W^{2-s}_{q',\mathcal{B}}$ into $W^{1+1/q-s}_{q'}(\Gamma)$ (see [8] for details).

A weak $L_p(W^s_q)$ solution y of (2.3) on $[0,t]$ is a **strong** $L_p(W^s_q)$ **solution** if

$$y \in W^1_{r,\mathrm{loc}}([0,t), W^{s-2}_{q,\mathcal{B}}) \cap L_{r,\mathrm{loc}}([0,t), W^s_{q,\mathcal{B}})$$

for some $r > 1$ and (2.7) is satisfied a.e. in $[0,t]$. If, in addition, $y \in C^\rho([0,t), W^s_{q,\mathcal{B}})$ for some $\rho \in [0,1)$ then y is called **strong** $C^\rho(W^s_q)$ **solution**. Similarly as above we define strong $L_{\vec{p}}(W^{\vec{s}}_q)$ or $C^{\vec{\rho}}(W^{\vec{s}}_q)$ solutions.

Let $X \in \{Q, \Sigma_1\}$. We write $\varphi \in \mathrm{Car}^1(X \times \mathbb{R}^m, \mathbb{R})$ if $\varphi :\ X \times \mathbb{R}^m \to \mathbb{R}$ is a Carathéodory function such that $\varphi(x,t,\cdot)$ is continuously differentiable for a.a. (x,t) in X. Let $r \ge 1$ and $\lambda_j \in (1,\infty)$ for $j = 0,1,\dots,4$. Assume that

$$f \in \mathrm{Car}^1\big(Q \times (\mathbb{R} \times \mathbb{R}^n), \mathbb{R}\big), \qquad g \in \mathrm{Car}^1(\Sigma_1 \times \mathbb{R}, \mathbb{R})$$

satisfy the growth conditions

$$\begin{aligned} &f_0 := f(\cdot,\cdot,0,0) \in L_r(J, L_1);\\ &|\partial_\xi f(x,t,\xi,\eta)| \le C\big(1 + |\xi|^{\lambda_0 - 1} + |\eta|^{\lambda_1 - 1}\big),\\ &|\partial_\eta f(x,t,\xi,\eta)| \le C\big(1 + |\xi|^{\lambda_2 - 1} + |\eta|^{\lambda_3 - 1}\big)\\ &\qquad\qquad \text{for } (x,t,\xi,\eta) \in \Omega \times J \times \mathbb{R} \times \mathbb{R}^n \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} &g_0 := g(\cdot,\cdot,0) \in L_r\big(J, L_1(\Gamma_1)\big);\\ &|\partial_\xi g(y,t,\xi)| \le C\big(1 + |\xi|^{\lambda_4 - 1}\big) \text{ for } (y,t,\xi) \in \Gamma_1 \times J \times \mathbb{R}. \end{aligned} \tag{2.9}$$

Let

$$\begin{aligned} &(\mu_Q, \nu_\Sigma) \in \mathcal{M}(Q) \times \mathcal{M}(\Sigma_1) && \text{if } r = 1,\\ &(\mu_Q, \nu_\Sigma) \in L_r(J, \mathcal{M}(\Omega) \times \mathcal{M}(\Gamma_1)) && \text{if } r > 1, \end{aligned} \tag{2.10}$$

and

$$y^0 \in \mathcal{M}(\Omega) \ \text{ if } r = 1, \qquad y^0 \in W^{2-2/r}_{q_0,\mathcal{B}} \ \text{ for some } q_0 > 1 \text{ if } r > 1. \tag{2.11}$$

Define numbers $\lambda^i_j := \lambda^i_j(n,r)$ for $j = 0,1,\dots,4$ and $i = 0,2$ by

$$\begin{aligned} &\lambda^i_0\big(2 + (n-2)r\big) = i + rn, && \lambda^i_1\big(2 + (n-1)r\big) = i + r(n+1),\\ &\lambda^i_2\big(2 + (n-2)r\big) = i + r(n-1), && \lambda^i_3\big(2 + (n-1)r\big) = i + rn,\\ &\lambda^i_4\big(2 + (n-2)r\big) = i + r(n-1), \end{aligned} \tag{2.12}$$

and assume that

$$\lambda_j < \lambda^2_j, \quad 0 \le j \le 4. \tag{2.13}$$

Observe that $\lambda^2_j > \lambda^0_j$ and $\lambda^2_j > 1 \vee \lambda^1_j$ for $0 \le j \le 4$, where

$$\lambda^1_0 := 1 \ \text{ if } n = 2,\ r = 1, \qquad \lambda^1_0 := n/(n-1) \ \text{ otherwise},$$
$$\lambda^1_1 := (n+1)/n, \quad \lambda^1_2 := \lambda^1_3 := 1, \quad \lambda^1_4 := 1/(n-1).$$

Therefore, increasing λ_j if necessary, we can assume that

$$\lambda_j > 1 \vee \lambda_j^0 \vee \lambda_j^1, \quad 0 \le j \le 4. \tag{2.14}$$

Set $\vec{\lambda} := (\lambda_0, \lambda_1, \dots, \lambda_4)$.

Proposition 2.1. *Under the assumptions above there exists $q^* > 1$ with the following property: if $q \in (1, q^*]$, then there exist $s_j \in [0, 2 - n/q')$, $0 \le j \le 4$, such that* (1.1) *has a unique maximal $L_{r\vec{\lambda}}(W_q^{\vec{s}})$ solution u, which is strong if $r > 1$.*

Proof. We set

$$d_0 := \frac{1}{\lambda_0}, \quad d_1 := \frac{n+1}{\lambda_1 n} - \frac{1}{n}, \quad d_2 := \frac{1}{\lambda_2 n'}, \quad d_3 := \frac{1}{\lambda_3} - \frac{1}{n}, \quad d_4 := \frac{1}{\lambda_4 n'},$$

and assume that we are not in the special case $n = 2$, $r = 1$. The hypothesis $\lambda_j > \lambda_j^1$ guarantees that the inequalities $1/n + d_j < 1$, $0 \le j \le 4$, are satisfied. Hence we can fix $q^* > 1$ such that $1/q^* \ge 1/n + d_j$, $0 \le j \le 4$, and $q^* \le q_0$ if $r > 1$. Now, fixing $q \in (1, q^*]$, we can choose s_j such that the square brackets in [8, (6.7)] vanish, that is,

$$\begin{gathered} s_0 := \frac{n}{q} - \frac{n}{\lambda_0}, \quad s_1 := \frac{n}{q} + 1 - \frac{n+1}{\lambda_1}, \\ s_2 := \frac{n}{q} - \frac{n-1}{\lambda_2}, \quad s_3 := \frac{n}{q} + 1 - \frac{n}{\lambda_3}, \quad s_4 := \frac{n}{q} - \frac{n-1}{\lambda_4}. \end{gathered}$$

The assumptions $\lambda_j > \lambda_j^0$ guarantee that $2/r > 2 - n/q' - s_j$.

In the special case $n = 2$, $r = 1$, we replace the condition $1/q^* \ge 1/n + d_0$ above with $1/q^* > 3/2 - 1/\lambda_0$ and we set $s_0 := 1$.

Now the assertion follows from [8, Theorem 6.1], Theorems A.5, and A.6 (also cf. [8, Remark 6.2(a)]). ☐

Remark 2.2. The solution u in Proposition 2.1 depends Lipschitz continuously on f, μ_Q, g and μ_Σ in suitable topologies, see Theorems A.5, A.6 and/or [8] for details. In particular, if $u = (u_Q, u_\Sigma) \in \mathcal{M}(Q) \times \mathcal{M}(\Sigma_1)$ and the solution $y(u)$ of (1.1) is global then there exists $\varepsilon > 0$ such that the solution $y(\tilde{u})$ is global for each $\tilde{u} \in \mathcal{M}(Q) \times \mathcal{M}(\Sigma_1)$ with $\|\tilde{u} - u\|_{\mathcal{M}(Q)\times\mathcal{M}(\Sigma_1)} < \varepsilon$. In addition,

$$\|y(\tilde{u}) - y(u)\|_Y \le c\|\tilde{u} - u\|_{\mathcal{M}(Q)\times\mathcal{M}(\Sigma_1)}, \tag{2.15}$$

where $Y = L_{r\vec{\lambda}}(J, W_{q,\mathcal{B}}^{\vec{s}}) := \bigcap_{j=1}^4 L_{r\lambda_j}(J, W_{q,\mathcal{B}}^{s_j})$, and the constants c, ε depend only on $\|y(u)\|_Y$. Finally, if $f = 0$ and $g = 0$ then the solution $y(u)$ of the (linear) problem (1.1) is global and satisfies

$$\|y(u)\|_{L_p(J, W_{q,\mathcal{B}}^s)} \le c\|u\|_{\mathcal{M}(Q)\times\mathcal{M}(\Sigma_1)} \tag{2.16}$$

for any $q \in (1, n/(n-2))$, $s \in [0, 2 - n/q')$ and $p \in (1, 2/(s + n/q'))$, see [6]. ☐

We close this section with a few comments concerning the identification of the space $\mathcal{M}(Q) \times \mathcal{M}(\Sigma_1)$ with $\mathcal{M}(Q \cup \Sigma_1)$. With $(u_Q, u_\Sigma) \in \mathcal{M}(Q) \times \mathcal{M}(\Sigma_1)$, we can associate the measure $u \in \mathcal{M}(Q \cup \Sigma_1)$ defined by

$$\int_{Q\cup\Sigma_1} \varphi\, du = \int_{Q\cup\Sigma_1} \varphi\, du_Q^0 + \int_{\Sigma_1} \gamma\varphi\, du_\Sigma, \qquad \varphi \in C_0(Q \cup \Sigma_1), \tag{2.17}$$

where u_Q^0 is the trivial extension of u_Q (cf. [8, Remark 4.2(b)]). The map

$$\mathcal{J} : \mathcal{M}(Q) \times \mathcal{M}(\Sigma_1) \to \mathcal{M}(Q \cup \Sigma_1), \quad (u_Q, u_\Sigma) \mapsto u \tag{2.18}$$

is linear, injective and continuous. On the other hand, if we take any bounded Radon measure $u \in \mathcal{M}(Q \cup \Sigma_1)$, then the completion of its restrictions to the Borel σ-algebras of Q and Σ_1 define bounded Radon measures u_Q and u_Σ in $\mathcal{M}(Q)$ and $\mathcal{M}(\Sigma_1)$ respectively. Obviously, $u = \mathcal{J}(u_Q, u_\Sigma)$. Hence $\mathcal{J}$ is a linear isomorphism and this enables us to identify the space $\mathcal{M}(Q) \times \mathcal{M}(\Sigma_1)$ with $\mathcal{M}(Q \cup \Sigma_1)$. Notice also that (2.4) with $t = T$ is equivalent to

$$\int_Q (-\partial_t \varphi + \mathcal{A}\varphi) y \, dx \, dt = \int_{Q \cup \Sigma_1} \varphi \, dF + \int_\Omega \varphi(0) dy^0$$

where $F = \mathcal{J}(f, g)$ (or $F = f + \gamma' g$ if we consider f and g as elements of $\mathcal{M}(J, W_{q,\mathcal{B}}^{\sigma-2})$ and $\mathcal{M}(J, W_q^{\sigma-1-1/q}(\Gamma_1))$ with $\sigma < 2 - n/q'$, respectively).

In the subsequent sections we will often work with elements $u \in \mathcal{M}(Q \cup \Sigma_1)$, where $\mathcal{M}(Q \cup \Sigma_1)$ is endowed with the w^*-topology. In this case, we have to be careful in distinguishing between the couple (u_Q, u_Σ) and $u = \mathcal{J}(u_Q, u_\Sigma)$. In fact, if $u_{Q,k} = \delta_{(x_k,t)}$, where $x_k \in \Omega$ with $x_k \to x \in \Gamma_1$ and $t \in J$, and if $u_{\Sigma,k} = 0$ then

$$\begin{aligned} (u_{Q,k}, u_{\Sigma,k}) &\to (0,0) && \text{in } (\mathcal{M}(Q), w^*) \times (\mathcal{M}(\Sigma_1), w^*), \\ \mathcal{J}(u_{Q,k}, u_{\Sigma,k}) &\to \delta_{(x,t)} && \text{in } (\mathcal{M}(Q \cup \Sigma_1), w^*). \end{aligned}$$

Finally, each $u \in \mathcal{M}(Q \cup \Sigma_1)$ and $u_\Sigma \in \mathcal{M}(\Sigma_1)$ will be identified with the trivial extension $u^0 \in \mathcal{M}(\bar{Q})$ and $u_\Sigma^0 \in \mathcal{M}(\Sigma)$, respectively.

3. Abstract existence result

In this section we prove existence results for optimal control problems governed by the state equation (1.1). Examples of cost functionals satisfying our abstract assumptions will be given in the subsequent section.

We fix $r \geq 1$ and assume (2.8)-(2.14). We also fix $q > 1$ and $s_j \in [0, 2 - n/q')$, $0 \leq j \leq 4$, as in Proposition 2.1 and set

$$Y := L_{r\vec{\lambda}}(J, W_{q,\mathcal{B}}^{\vec{s}}) = \bigcap_{j=0}^{4} L_{r\lambda_j}(J, W_{q,\mathcal{B}}^{s_j}). \tag{3.1}$$

Finally, we use the identification of $\mathcal{M}(Q) \times \mathcal{M}(\Sigma_1)$ with $\mathcal{M}(Q \cup \Sigma_1)$, see (2.18).

First consider the case $r = 1$. Let

$$\mathcal{M} := \mathcal{M}(Q \cup \Sigma_1) \tag{3.2}$$

be endowed either with the strong topology or the w^*-topology. We assume that

$$\mathbb{U}_{\mathrm{ad}} \subset \mathcal{M} \ \text{ is sequentially compact} \tag{3.3}$$

and

$$\mathbb{J} : Y \times \mathcal{M} \to \mathbb{R} \ \text{ is sequentially lower semicontinuous.} \tag{3.4}$$

Given $u = (u_Q, u_\Sigma) \in \mathcal{M}$, Proposition 2.1 guarantees that problem (1.1) possesses a unique maximal $L_{r\vec{\lambda}}(W_q^{\vec{s}})$ solution $y = y(u)$. Set

$$\mathbb{U}_{\mathrm{ad}}^G := \{u \in \mathbb{U}_{\mathrm{ad}} : y(u) \text{ is global}\}.$$

We also assume either

$$f(x,t,\xi,\eta)\xi \leq 0 \quad \text{and} \quad g(x,t,\xi)\xi \leq 0 \quad \text{for all } x,t,\xi,\eta, \tag{3.5}$$

and

$$f \text{ is independent of the gradient variable } \eta, \tag{3.6}$$

or

$$\mathbb{U}_{\mathrm{ad}}^G \neq \emptyset \tag{3.7}$$

and

$$\lim_{\|y\|_Y \to \infty} \mathbb{J}(y,u) = +\infty, \quad \text{uniformly w.r.t. } u \in \mathbb{U}_{\mathrm{ad}}. \tag{3.8}$$

Then we have the following theorem.

Theorem 3.1. *Let* $r = 1$. *Given the above assumptions, the optimal control problem* (1.2)*, governed by* (1.1)*, has a solution.*

Proof. We know that (1.1) possesses a unique maximal weak $L_{r\vec{\lambda}}(W_q^{\vec{s}})$ solution $y = y(u)$ for any $u = (u_Q, u_\Sigma) \in \mathbb{U}_{\mathrm{ad}}$. In addition (see Remark 2.2), if $u_1 \in \mathbb{U}_{\mathrm{ad}}^G$ then $y(u_1) \in Y$ and there exist $\varepsilon, c > 0$, depending only on $\|y(u_1)\|_Y$, with the following property: if $u_2 \in \mathbb{U}_{\mathrm{ad}}$, $\|u_1 - u_2\|_{\mathcal{M}(Q\cup\Sigma_1)} \le \varepsilon$ then $u_2 \in \mathbb{U}_{\mathrm{ad}}^G$ and

$$\|y(u_1) - y(u_2)\|_Y \le c\|u_1 - u_2\|_{\mathcal{M}(Q\cup\Sigma_1)}. \tag{3.9}$$

First assume that $\mathcal{M}$ is endowed with the strong topology and (3.7), (3.8) are true. Let $u_k \in \mathbb{U}_{\mathrm{ad}}^G$ be such that $\mathbb{J}(y(u_k), u_k) \to \inf$. Thanks to (3.3), we may assume $u_k \to u$ in $\mathcal{M}$. Hypothesis (3.8) implies the boundedness of $y_k := y(u_k)$ in Y, i.e.,

$$\|y_k\|_Y < R \text{ for some } R > 0 \text{ and all } k.$$

This estimate and (3.9) show $u \in \mathbb{U}_{\mathrm{ad}}^G$ and $y_k \to y(u)$ in Y. Now we infer from (3.4) that $(y(u), u)$ is an optimal pair for problem (1.2).

Next assume (3.5), (3.6) instead of (3.7), (3.8). We will show that $\mathbb{U}_{\mathrm{ad}}^G = \mathbb{U}_{\mathrm{ad}}$ and

$$\|y(u)\|_Y \le C(1 + \|\hat{u}\|_{\mathcal{M}}), \qquad u \in \mathbb{U}_{\mathrm{ad}}, \tag{3.10}$$

where $\hat{u} := |u| + |y^0 \otimes \delta_0|$. Then the rest of the proof is the same as in the non-monotone case.

Let $z = z(u)$ be the solution of

$$\left.\begin{aligned} \partial_t z + \mathcal{A}z &= |u_Q| && \text{in } Q, \\ \mathcal{B}z &= |u_\Sigma| && \text{on } \Sigma, \\ z(\cdot,0) &= |y^0| && \text{in } \Omega. \end{aligned}\right\} \tag{3.11}$$

Due to [6, Theorem 4, Corollary 3 and Proposition 3], z is global, nonnegative and

$$\|z\|_Z \le C\|\hat{u}\|_{\mathcal{M}}, \qquad u \in \mathbb{U}_{\mathrm{ad}}, \tag{3.12}$$

with $Z := L_{\vec{p}}(J, W_{q,\mathcal{B}}^{\vec{\sigma}}))$, where

$$\vec{p} = (p_0, p_1) \in (1,\infty)^2, \quad \vec{\sigma} = (\sigma_0, \sigma_1) \in ([2 - n/q') \setminus S_q)^2$$

can be chosen in such a way that $L_{p_0}(J, W_{q,\mathcal{B}}^{\sigma_0})$ is embedded into $L_{\lambda_0}(J, L_{\lambda_0})$ and the space of traces of functions in $L_{p_1}(J, W_{q,\mathcal{B}}^{\sigma_1})$ into $L_{\lambda_4}(J, L_{\lambda_4}(\Gamma))$ (cf. (2.16)). Setting $F(w) := f(\cdot,\cdot,w)$ and $G(w) := g(\cdot,\cdot,w)$, we see that $z - y := z(u) - y(u)$ satisfies

$$\left.\begin{aligned} \partial_t(z-y) + \mathcal{A}(z-y) &= F(z) - F(y) + u_Q^*, && x \in \Omega,\ t \in J_u, \\ \mathcal{B}(z-y) &= G(z) - G(y) + u_\Sigma^*, && x \in \Gamma,\ t \in J_u, \\ (z-y)(\cdot,0) &= |y^0| - y^0 \ge 0, && \end{aligned}\right\}$$

where

$$u_Q^* := -F(z) + |u_Q| - u_Q \ge 0, \qquad u_\Sigma^* := -G(z) + |u_\Sigma| - u_\Sigma \ge 0$$

and J_u denotes the maximal existence interval of $y(u)$. Consequently, [8, Theorem 6.3] guarantees $z \geq y$ in J_u. Analogous arguments show $z \geq -y$, hence $|y| \leq z$. Thus (2.8), (2.9), and (3.12) imply

$$\|F(y)\|_{\mathcal{M}(Q_u)} + \|G(y)\|_{\mathcal{M}(\Sigma_u)} \leq C(1 + \|\hat{u}\|_{\mathcal{M}}),$$

where $Q_u := \Omega \times J_u$ and $\Sigma_u := \Sigma_1 \times J_u$. This estimate guarantees that the right-hand side of (1.1) is bounded in $\mathcal{M}(Q_u \cup \Sigma_u)$ by $C(1 + \|\hat{u}\|_{\mathcal{M}})$. Hence [6] implies $u \in \mathbb{U}^G_{\mathrm{ad}}$ and (3.10).

Now assume that $\mathcal{M}$ is endowed with the w^*-topology. Then we can assume $u_k \to u$ in $(\mathcal{M}(Q \cup \Sigma_1), w^*)$. As above, the boundedness of the sequence $(y(u_k))$ in Y follows either from the coercivity of $\mathbb{J}$ (in the general non-monotone case) or from the comparison with the corresponding linear problem (3.11) and the boundedness of u_k in $\mathcal{M}(Q \cup \Sigma_1)$ (if (3.5), (3.6) are true). Let

$$F = (\varphi, \psi) \in (\mathcal{M}(Q), \mathcal{M}(\Sigma_1)) \doteq \mathcal{M}(Q \cup \Sigma_1).$$

Due to [8, Proposition 17.1], the solution operator $S : \mathcal{M}(Q \cup \Sigma_1) \to Y : F \mapsto y$ for the linear problem

$$\left.\begin{aligned} \partial_t y + \mathcal{A}y &= \varphi && \text{in } Q, \\ \mathcal{B}y &= \psi && \text{on } \Sigma, \\ y(\cdot, 0) &= y^0, && \end{aligned}\right\} \tag{3.13}$$

is compact. Problem (1.1) can be written in the form (3.13) with

$$F = F(y, u) := \big(f(\cdot, \cdot, y, \nabla y) + u_Q, g(\cdot, \cdot, y) + u_\Sigma\big).$$

Since the set $\{F(y_k, u_k) : k = 1, 2, \dots\}$ is bounded in $\mathcal{M}(Q \cup \Sigma_1)$, the sequence $y_k = SF(y_k, u_k)$ is precompact in Y and we may assume $y_k \to y$ in Y. Now it is easy to pass to the limit in the weak formulation of (1.1) in order to show that (y, u) is an optimal pair. $\square$

Now consider the case $r > 1$. Similarly as above, we identify $\mathcal{M} \times \mathcal{M}(\Gamma_1)$ with $\mathcal{M}(\Omega \cup \Gamma_1)$ and assume that

$$\mathcal{M} := L_r(J, \mathcal{M}(\Omega \cup \Gamma_1)) \tag{3.14}$$

is endowed either with the strong topology or the w^*-topology in the dual space to $L_{r'}(J, C_0(\Omega \cup \Gamma_1))$. Then then we can repeat word by word the proof of Theorem 3.1 in order to prove the following theorem.

Theorem 3.2. *Theorem 3.1 remains true if we replace the assumptions* $r = 1$ *and* (3.2) *with* $r > 1$ *and* (3.14), *respectively.*

Remark 3.3. Let us show that assumption (3.6) in the above theorems can be removed if the data are more regular. Consider the case $r = 1$ (Theorem 3.1), the case $r > 1$ is similar. We will sketch the proof of the following:

Fix $\rho > 1$ *and assume that* (2.10)-(2.11) *are satisfied with* $r = \rho$ *and* (2.12)-(2.13) *hold with* $r = 1$. *Notice that then* (2.12)-(2.13) *are true with* $r = \rho$ *as well. Let* Y *be defined by* (3.1) *with* $r = 1$ *and* $\mathcal{M}$ *by* (3.14) *with* $r = \rho$. *Assume* (3.3), (3.4) *and* (3.5). *Then the optimal control problem* (1.2) *has a solution.*

In order to prove this statement, we just need to make the following modifications in the proof of Theorem 3.1:

1. Instead of estimate (3.12) we obtain

$$\|z\|_{C(J, L_1)} \leq C\|\hat{u}\|_{\hat{\mathcal{M}}}, \tag{3.15}$$

where $\hat{u} := (u, y^0)$ and $\hat{\mathcal{M}} := \mathcal{M} \times W^{2-2/\rho}_{q_0,\mathcal{B}}$ (recall that $\mathcal{M} = L_\rho(J, \mathcal{M}(\Omega \cup \Gamma_1))$).
2. We set $F(w) := f(\cdot, \cdot, w, \nabla w)$.
3. Fixing $\tilde{t} \in J$ and setting $\tilde{u}(\cdot, t) := u(\cdot, t-\tilde{t})$, $\tilde{y}(\cdot, t) := y(\cdot, t-\tilde{t})$ (cf. [8, Section 12]), the function $\tilde{y}$ is a maximal solution of (1.1) with u, y^0 and J replaced with $\tilde{u}, y(\tilde{t})$ and $[0, T - \tilde{t}]$, respectively. Since

$$\|y(\tilde{t})\|_{\mathcal{M}(\Omega)} \leq \|y(\tilde{t})\|_{L_1} \leq \|z(\tilde{t})\|_{L_1} \leq C\|\hat{u}\|_{\hat{\mathcal{M}}}$$

due to $|y| \leq z$ and (3.15), the local existence proof in [8, the proof of Theorem 1.1] guarantees that there exist $\varepsilon, C > 0$ (depending on $\|\hat{u}\|_{\hat{\mathcal{M}}}$ only) such that $\tilde{y}$ exists on $\tilde{J} := [0, \varepsilon \wedge (T - \tilde{t})]$ and satisfies $\|\tilde{y}\|_{L_{\bar{\lambda}}(\tilde{J}, W^{\bar{s}}_{q,\mathcal{B}})} \leq C$. This shows that y is global and its norm in Y is bounded by a constant depending on $\|\hat{u}\|_{\hat{\mathcal{M}}}$ only.

Notice that in the above statement it would be sufficient to assume (2.11) with $r = 1$ only. Then we just need to estimate $y(u)$ on $[0, \varepsilon]$ and replace (3.15) with the estimate $\|z\|_{C([\varepsilon,T],L_1)} \leq C\big(\varepsilon, \|(u, y^0)\|_{\mathcal{M}\times\mathcal{M}(\Omega)}\big)$. □

Remark 3.4. Theorem A.5 guarantees a stronger version of the stability estimates (3.9). In fact, the norm in $\mathcal{M}(Q \cup \Sigma_1)$ in (3.9) can be replaced by the weaker one in $\mathcal{M}(J, W^{\sigma-2}_{q,\mathcal{B}})$ for suitable $\sigma < 2 - n/q'$. This fact will be used in Example 5.2 below. □

Remark 3.5. In all our considerations in this section we can replace the space Y by any other function space for which the existence, stability and compactness results for (1.1) remain true. □

4. Examples

Power nonlinearity and linear boundary conditions

We begin with the following model state equation

$$\left.\begin{aligned} \partial_t y - \Delta y &= \kappa|y|^{\lambda-1}y + u_Q, && \text{in } Q,\\ \mathcal{B}y &= u_\Sigma, && \text{on } \Sigma,\\ y(\cdot,0) &= y^0 && \text{in } \Omega, \end{aligned}\right\} \tag{4.1}$$

where

$$\kappa \in \mathbb{R}, \quad y^0 \in \mathcal{M}(\Omega), \quad 1 < \lambda < (n+2)/n, \tag{4.2}$$

and $u := (u_Q, u_\Sigma) \in \mathcal{M}(\bar{Q})$ is the control with $u_\Sigma = 0$ on Σ_0. Let $p \in [\lambda, 1 + 2/n)$. Then choosing $k = 1$, $s_1 = 0$, $q = p$ and $\sigma_1 \in (2 - 2/p, 2 - n/p') \setminus S_q$, Theorem A.5 and Lemma A.7 show that problem (4.1) possesses a unique maximal weak $L_p(L_p)$ solution $y = y(u)$.

Let $\mathcal{M} := \mathcal{M}(Q \cup \Sigma_1)$ be endowed either with the strong topology or the w^*-topology and $Y := L_p(Q)$. The following theorem is a consequence of Theorem 3.1 and Remark 3.5.

Theorem 4.1. *Assume* (4.2) *and let $\mathcal{M}$ and Y be defined as above. Assume also* (3.3) *and* (3.4)*, and either $\kappa \leq 0$ or* (3.7) *and* (3.8)*. Then control problem* (1.2)*, governed by* (4.1)*, has a solution.*

Remark 4.2. (Unbounded sets of admissible controls) It is easily seen that given $y_d \in L_p(Q)$, the cost functionals

$$\mathbb{J}(u) := \int_\Omega |y - y_d|^p \, dx \, dt$$

and

$$\mathbb{J}(u) := \int_{\Omega} |y - y_d|^p \, dx \, dt + \|u\|_{\mathcal{M}(\bar{Q})} \tag{4.3}$$

satisfy all assumptions in Theorem 4.1. If $\mathbb{J}$ is defined by (4.3) and $\mathcal{M}$ is endowed with the w^*-topology, then we can take $\mathbb{U}_{\mathrm{ad}}$ to be any closed convex subset of $\mathcal{M}$ (for example, $\mathbb{U}_{\mathrm{ad}} := \mathcal{M}(Q)$ if $\mathcal{B} = \gamma$). This follows from the fact that $\mathbb{J}$ is coercive with respect to $u \in \mathcal{M}$ so that, in the optimal control problem (1.2), the set $\mathbb{U}_{\mathrm{ad}}$ can be replaced with $\tilde{\mathbb{U}}_{\mathrm{ad}} := \{u \in \mathbb{U}_{\mathrm{ad}} : \|u\|_{\mathcal{M}(\bar{Q})} \le R\}$, where $R > 0$ is large enough. □

Remark 4.3. (Existence of global solutions) Let $\kappa > 0$, $y^0 \in L_\infty$, and

$$\{u \in L_\infty(Q) : |u| \le \|y^0\|_{L_\infty}^{\lambda}\} \subset \mathbb{U}_{\mathrm{ad}}.$$

Then one can easily prove that (3.7) is true (see [9, Example 2.5] for details). In particular, if $\mathcal{B} = \gamma$ and $\mathbb{U}_{\mathrm{ad}} = \mathcal{M}(Q)$, and $\mathbb{J}$ is defined by (4.3), then the optimal control problem (1.2) governed by (4.1) is solvable. □

Remark 4.4. If $p = \lambda$ and the set $\mathcal{M}$ is endowed with the strong topology, then Theorem 4.1 is true for unbounded domains as well. □

Remark 4.5. If we replace in (4.1) the nonlinearity $|y|^{\lambda-1}y$ with the function$|\nabla y|^{\lambda}$, and if $1 < \lambda < (n+2)/(n+1)$, then we can repeat the considerations leading to Theorem 4.1. More precisely, set $Y := L_p(J, W^1_{p,\mathcal{B}})$ with $\lambda \le p < (n+2)/(n+1)$ and $\mathcal{M} := \mathcal{M}(Q \cup \Sigma_1)$. Then (1.2) has a solution provided (4.2), (3.3), (3.4), (3.7) and (3.8) are true. □

Remark 4.6. (Weakly sequentially compact control sets) Consider problem (4.1) with κ, y^0, and λ satisfying (4.2). Let $\Phi \subset \mathcal{M}(J)$ and $M \subset \mathcal{M}(\Omega \cup \Gamma_1)$ be w^*-sequentially compact. Set $\mathbb{U}_{\mathrm{ad}} = M \otimes \Phi$ and let $\mathcal{M} := \mathcal{M}(Q \cup \Sigma_1)$ be endowed with the w^*-topology. Then $\mathbb{U}_{\mathrm{ad}}$ is sequentially compact in $\mathcal{M}$. If (3.4) is true (and (3.7), (3.8) are satisfied if $\kappa > 0$) then Theorem 4.1 guarantees the solvability of (1.2).

Let $K \subset \bar{\Omega}$ be compact, $K \setminus \Gamma_0 \ne \emptyset$. Denote $M_X := \{\delta_x \,;\, x \in X\}$ and set

$$M := \begin{cases} M_{K \setminus \Gamma_0} \cup \{0\} & \text{if } K \cap \Gamma_0 \ne \emptyset, \\ M_K & \text{otherwise.} \end{cases}$$

Then M is sequentially compact in $(\mathcal{M}(\Omega \cup \Gamma_1), w^*)$. In fact, if $x_k \in K$ with $x_k \to x$, then $\delta_{x_k} \to \delta_x$ in $(\mathcal{M}(\Omega \cup \Gamma_1), w^*)$ provided x_k and x belong to $\Omega \cup \Gamma_1$. If $x_k \in \Omega \cup \Gamma_1$ and $x \in \Gamma_0$, then $\delta_{x_k} \to 0$ in $(\mathcal{M}(\Omega \cup \Gamma_1), w^*)$. Notice that M is not compact in $\mathcal{M}(\Omega \cup \Gamma_1)$ if $K \setminus \Gamma_0$ is infinite. □

Relaxing growth restrictions

Remarks 4.7. (i) The growth in Theorem 4.1 can be weakened or even removed if we assume more regularity on the control u and the initial data y^0. If, for example, we consider problem (4.1) with $u \in L_r(J, \mathcal{M}(\bar{\Omega}))$, $r > 1$, and $y^0 \in W^{2-n/\lambda'-2/r}_{\lambda,\mathcal{B}}$ then the condition for λ in (4.2) can be replaced with

$$1 < \lambda < \frac{rn + 2}{2 + (n-2)r} \tag{4.4}$$

(cf. (2.13) and the definition of λ_0^2). More precisely, hypothesis (4.4) and Theorem A.6 guarantee the unique solvability and stability estimates for solutions of (4.1) in a suitable space Y (in $Y = L_{r\lambda}(J, L_\lambda)$, for example). If we adapt the assumptions on $\mathbb{U}_{\rm ad}$ and $\mathbb{J}$ to this new space Y, then we also get the solvability of the corresponding optimal control problem.

Similarly, if we consider problem (4.1) with $u \in L_r(J, L_z(M))$, where M is a smooth d-dimensional manifold in Ω or Γ, and $d, r, z \geq 1$, then the condition for λ in (4.2) may be replaced by

$$1 < \lambda < \frac{2/r + d/z + n - d}{(2/r + d/z + n - d - 2)}. \tag{4.5}$$

The solvability of (4.1) follows from Theorem A.6 with the choice $k = 1$, $s_1 = s$, $\sigma_1 = \sigma$, $q > 1$, where s, σ, q satisfy

$$\begin{gathered} s, \sigma \notin S_q, \quad 0 \leq s < \sigma < 2 \quad \text{and} \quad \sigma - 2/r \notin S_q, \\ \sigma - s < \frac{2}{r}, \qquad p := r\lambda < \frac{2}{s - \sigma + 2/r}, \end{gathered} \tag{4.6}$$

$$2 - \sigma - \frac{n}{q'} > -\frac{d}{z'} \quad s \geq \frac{n}{q} - \frac{n}{\lambda z_0}, \quad \text{where} \quad z_0 := \frac{n}{d/z + n - d} \in [1, z]. \tag{4.7}$$

In fact, the abstract formulation of problem (4.1) is

$$\dot{y} + Ay = \kappa |y|^{\lambda - 1} y + \gamma_M' u \quad \text{in } J, \qquad y(0) = y^0,$$

where A is as in (2.7) and γ_M is the trace operator corresponding to the manifold M. Now (4.7) guarantees

$$W^s_{q,\mathcal{B}} \hookrightarrow L_{\lambda z_0}, \quad L_{z_0} \hookrightarrow W^{\sigma-2}_{q,\mathcal{B}} \quad \text{and} \quad L_z(M) \hookrightarrow W_q^{\sigma - 2 + (n-d)/q'}(M),$$

hence $\gamma_M' : L_z(M) \to W^{\sigma-2}_{q,\mathcal{B}}$ is continuous and the mapping

$$L_{r\lambda}(J, W^s_{q,\mathcal{B}}) \to L_r(J, W^{\sigma-2}_{q,\mathcal{B}}), \quad y \mapsto \kappa |y|^{\lambda-1} y + \gamma_M' u$$

is Lipschitz continuous. In order to see that the conditions in (4.6)–(4.7) can be satisfied, let us write them in the form

$$s \geq \frac{n}{q} - \frac{1}{\lambda}\Big(\frac{d}{z} + n - d\Big), \quad \sigma < 2 - \frac{n}{q'} + \frac{d}{z'}, \quad \frac{2}{r} > \sigma - s > \frac{2}{r\lambda'}$$

and notice that the difference between the upper bound for σ and the lower bound for s is greater than $2/(r\lambda')$, due to (4.5).

Note that if $u \in L_r(J, L_z)$ with $r, z > 1$ then our solutions belong to the space $L_{r\lambda}(J, L_{z\lambda})$. However, if $\kappa > 0$ then it is sufficient to assume the coercivity of $\mathbb{J}$ in $L_\alpha(Q)$ with $\alpha > \lambda$; see the proof of [19, Theorem I.5.1], for example.

Notice also that if $2/r + d/z < d + 2 - n$ then the solution y is Hölder continuous and we can replace u by $uf(y)$, for example, where $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous. In fact, in this case we can choose s close to 0, $q > n/s$ and $\sigma \in (s + 2/r, 2 - n + d - d/z)$, so that $y \in C^\rho(\bar{Q})$ for some $\rho > 0$ due to Theorem A.6.

(ii) If we consider monotone nonlinearities ($\kappa < 0$) and we are interested only in the unique solvability of the state problem then conditions (4.4) and (4.5) can be still weakened, see [8, Theorem 7.7] or [12], [20]. However, the solvability of the optimal control problem seems to require some additional assumptions on λ (cf. [20, Remark 3.1]). □

Example of Droniou and Raymond

Example 4.8. In [12] the authors study problem (4.1) with $\kappa = -1$, and u in $\mathbb{U}_{\mathrm{ad}} = \{\delta_x\} \otimes \Phi$, where $x \in \Omega$ is fixed and Φ is a closed convex subset of $L_r(J)$ for some $r > 1$. As mentioned in Remarks 4.7, the regularity of u and the monotonicity of the nonlinearity allow them to weaken the growth condition. Their cost functional $\mathbb{J}$ is of the form

$$\mathbb{J}(y, \delta_x \otimes \varphi) = \int_\Omega |y(T) - y_d|^q \, dx + \beta \int_0^T |\varphi|^r \, dt, \qquad 1 < q < \frac{n}{2/r + n - 2},$$

where $\beta \geq 0$. If Φ is unbounded, then one has to assume $\beta > 0$. This enables one to set $\mathbb{U}_{\mathrm{ad}} := \delta_x \otimes (\Phi \cap \mathbb{B}_c)$, where $\mathbb{B}_c$ denotes a large ball in $L_r(J)$, cf. Remark 4.2.

Assume $y^0 \in W_{\lambda,\mathcal{B}}^{2-n/\lambda'-2/r} \cap W_{q,\mathcal{B}}^{\sigma-2/r}$ for some $\sigma \in (2/r, 2 - n/q')$ (this regularity assumption can be relaxed). Let $\mathcal{M} := \mathcal{M}(\Omega \cup \Gamma_1)$ be equipped with the w^*-topology, and let $M \subset \mathcal{M}$ be sequentially compact. Let Φ and $\mathbb{J}$ be as above, and put $\mathbb{U}_{\mathrm{ad}} := M \otimes \Phi$. Let us show that, assuming (4.4), we can easily solve this more general problem and, in some cases, we can also allow $\kappa > 0$.

Theorem A.6 guarantees the existence of a unique maximal strong $L_{r\lambda}(L_\lambda)$ solution $y(u)$ of (4.1). In addition, if $y = y(u)$ is global then considering (4.1) as a linear problem with the right-hand side in $L_r(J, \mathcal{M}(\Omega \cup \Gamma_1))$, Theorem A.6 guarantees $y \in C(J, L_q)$ and

$$y \in L_r(J, W_{q,\mathcal{B}}^{\tilde\sigma}) \cap W_r^1(J, W_{q,\mathcal{B}}^{\tilde\sigma-2}) \quad \text{for any } \tilde\sigma < \sigma. \tag{4.8}$$

In particular, $\mathbb{J}(y(u), u)$ is well defined for any $u \in \mathbb{U}_{\mathrm{ad}}^G$.

Let (y_k, u_k) be a minimizing sequence for (1.2). Since $\mathbb{U}_{\mathrm{ad}}$ is sequentially compact in $(L_r(J, \mathcal{M}(\Omega \cup \Gamma_1)), w^*)$, we may assume $u_k \to u$ in this space.

First assume $\kappa \leq 0$. Comparison with problem (3.11), with u and z replaced by u_k and z_k, respectively, yields, similarly as in (3.12),

$$\|y_k\|_{L_{r\lambda}(J,L_\lambda)} \leq \|z_k\|_{L_{r\lambda}(J,L_\lambda)} \leq C(\|u_k\|_{L_r(J,\mathcal{M}(\bar\Omega))} + \|y^0\|_{W_{\lambda,\mathcal{B}}^{2-n/\lambda'-2/r}}) \leq C,$$

hence $f_k := |y_k|^{\lambda-1} y_k$ are uniformly bounded in $L_r(J, L_1)$. Due to [8, Proposition 17.2], the sequence (y_k) is compact in $L_{r\lambda}(J, L_\lambda)$, so that we may assume $y_k \to y$ in $L_{r\lambda}(J, L_\lambda)$. Consequently, $f_k \to f := |y|^{\lambda-1} y$ in $L_r(J, L_1)$. The functions y_k solve the problem

$$\int_Q (-\partial_t \varphi - \Delta\varphi) y_k \, dx \, dt = \kappa \int_Q \varphi |y_k|^{\lambda-1} y_k \, dx \, dt + \int_{Q \cup \Sigma_1} \varphi \, du_k + \int_\Omega \varphi(0) y^0 \, dx$$

for any $\varphi \in \mathcal{D}(\overline{\Omega} \times [0, T))$ satisfying $\mathcal{B}\varphi = 0$. Passing to the limit in this identity we obtain $y = y(u)$. Theorem A.6 guarantees that the sequence (y_k) is bounded in the space appearing in (4.8). Since that space is compactly embedded in $C(J, L_q)$ if $\tilde\sigma$ is close to σ (cf. [6, Theorem 3]), we may assume $y_k(T) \to y(T)$ in L_q. Now the weak-star lower semicontinuity of $\mathbb{J}$ with respect to u concludes the proof.

If $\kappa > 0$ then, in addition to the above hypotheses, we assume

$$\lambda \leq q, \ y^0 \in C^2(\bar\Omega), \ \mathcal{B}y^0 = 0, \ y^0 \geq 0, \ \Delta y^0 + \kappa (y^0)^\lambda \geq 0,$$

and that each $\varphi \in \Phi$ is nondecreasing and nonnegative. First let us show that any solution $y(u)$ of (4.1) is increasing in time. This is obviously true if $u = 0$, due to the classical maximum principle. Now [8, Theorem 6.3] guarantees $y(u) \geq y(0)$, hence, in particular, $y(u)(\cdot, t) \geq y^0$ for any $t \in J$. Using [8, Theorem 6.3] for the solution $y(u)$ and $z(x,t) := y(u)(x,t+\tau)$, $\tau > 0$, we obtain $y(u)(\cdot,t+\tau) \geq y(u)(\cdot,t)$ whenever $t, t+\tau \in J$. The boundedness of $\mathbb{J}(y_k,u_k)$ implies a bound for $y_k(T)$ in L_q and the monotonicity of y_k in t gives a uniform bound for y_k in $L_\infty(J,L_q)$. Now we can proceed as in the case $\kappa \leq 0$. □

Nonlinear boundary conditions

Now consider a model problem with nonlinearities appearing on the boundary, namely

$$\left.\begin{aligned} \partial_t y - \Delta y &= u_Q, && \text{in } Q, \\ \partial_\nu y &= \kappa |y|^{\lambda-1} y + u_\Sigma, && \text{on } \Sigma, \\ y(\cdot,0) &= y^0 && \text{in } \Omega, \end{aligned}\right\} \tag{4.9}$$

where

$$\kappa \in \mathbb{R}, \quad y^0 \in \mathcal{M}(\Omega), \quad 1 < \lambda < (n+1)/n, \tag{4.10}$$

and $u := (u_Q, u_\Sigma) \in \mathcal{M}(Q) \times \mathcal{M}(\Sigma)$ is the control. Set $p := \lambda$ and fix q in $(1,(n+1)/n]$. Condition (4.10) guarantees that we can choose $s, \sigma \in (1/q, 2) \setminus S_q$ such that

$$s - \frac{n}{q} \geq -\frac{n-1}{\lambda}, \qquad \sigma < 2 - \frac{n}{q'} \qquad \text{and} \qquad \sigma - s > 2 - \frac{2}{\lambda}.$$

These conditions imply, in particular, $W_q^{s-1/q}(\Gamma) \hookrightarrow L_\lambda(\Gamma)$ and $p < 2/(s-\sigma+2)$. Consequently, Theorem A.5 and Lemma A.7 guarantee the existence of a unique maximal $L_p(W_q^s)$ solution $y = y(u)$ of (4.9) and the stability estimates

$$\|y(u_1) - y(u_2)\|_{L_p(J,W^s_{q,\partial_\nu})} \leq c\|u_1 - u_2\|_{\mathcal{M}(\bar{Q})}. \tag{4.11}$$

Notice also that $W^s_{q,\mathcal{B}} \hookrightarrow L_{\hat{p}}$ for $\hat{p} := pn/(n-1)$. Repeating word by word the proof of Theorem 3.1 we see that the following theorem is true.

Theorem 4.9. *Let $\mathcal{M}$ be the space $\mathcal{M}(\bar{Q})$ endowed either with the strong topology or the w^*-topology. Assume that*

$$\mathbb{J} : L_p(J, L_{\hat{p}}) \times L_p(\Sigma) \times \mathcal{M} \to \mathbb{R}$$

is lower semicontinuous and

$$\mathbb{U}_{\mathrm{ad}} \ \textit{is sequentially compact in } \mathcal{M}.$$

If $\kappa > 0$ assume also that $\mathbb{U}^G_{\mathrm{ad}} \neq \emptyset$ and

$$\mathbb{J}(y,z,u) \geq c_1\|z\|_{L_p(\Sigma)} - c_2. \tag{4.12}$$

Then the optimal control problem:

$$\textit{minimize } \mathbb{J}\big(y(u), \gamma y(u), u\big) \ \textit{over } u \in \mathbb{U}^G_{\mathrm{ad}}$$

has a solution.

Similarly as in Remark 4.7(i), one can weaken the growth condition in Theorem 4.9 if the data are more regular.

5. Modifications and extensions

Non lower semicontinuous cost functionals

First we consider cost functionals which are not lower semicontinuous in the topologies mentioned above.

Example 5.1. Assume that $K \subset \Gamma$ is compact and consider the problem

$$\left.\begin{aligned} \partial_t y - \Delta y &= \kappa|y|^{\lambda-1}y, && \text{in } Q,\\ \partial_\nu y &= u, && \text{on } \Sigma,\\ y(\cdot,0) &= y^0 && \text{in } \Omega, \end{aligned}\right\} \tag{5.1}$$

where $\kappa \in \mathbb{R},\ 1 < \lambda < (n+2)/n,\ y^0 \in \mathcal{M}(\Omega)$, and

$$u \in \mathbb{U}_{\mathrm{ad}} := M_K \otimes \Phi,\ M_K := \{\delta_x\,;\, x \in K\} \text{ and } \Phi \subset \mathcal{M}(J) \text{ is } w^*\text{-compact.}$$

Let $x_0 \in \bar\Omega \setminus K,\ \ z_d \in \mathbb{R},\ \ y_d \in L_\lambda(Q)$ and

$$\mathbb{J}(y,u) = |y(x_0,T) - z_d| + C_0 \int_Q |y - y_d|^\lambda \,dx\,dt,$$

where $C_0 \ge 0$, $C_0 > 0$ if $\kappa > 0$. Problem (5.1) is a special case of (4.1). Hence it possesses a unique maximal weak $L_\lambda(L_\lambda)$ solution $y = y(u)$. Assuming $\mathbb{U}^G_{\mathrm{ad}} \ne \emptyset$ if $\kappa > 0$, we will prove that the optimal control problem (1.2) has a solution.

In fact, choosing $p \in (\lambda, (n+2)/n)$, one can repeat the arguments from the proof of Theorem 3.1 in order to find a sequence (y_k, u_k) such that $u_k \to u$ in $(\mathcal{M}(\bar Q), w^*)$, $y_k = y(u_k) \to y = y(u)$ in $L_p(Q)$ and $\mathbb{J}(y_k,u_k) \to \inf$. Now $\mathbb{J}(y_k,u_k) \to \mathbb{J}(y,u)$ follows from the interior regularity of the solutions y_k on the complement of the set

$$\big\{(x,t) \in Q\,;\, t = 0 \text{ or } x \in K\big\},$$

cf. [8, Remark 5.17(b)]. To be more more precise, assume, for simplicity, $x_0 \in \Omega$. Then there exists $\varepsilon > 0$ such that the ε-neighbourhood $B_\varepsilon(x_0)$ of x_0 is a subset of Ω. Set $Q_j := B_{2^{-j}\varepsilon}(x_0) \times (T - 2^{-j}T, T)$. The interior parabolic L_p-estimates (see [18, Chapter 7]) imply

$$\|y_k - y\|_{W^{2,1}_{p/\lambda}(Q_1)} \le C\|y_k - y\|_{L_p(Q_0)} \to 0.$$

Hence $y_k \to y$ in $L_{p_1}(Q_1)$, where the size of $p_1 > p$ is restricted by the condition $1/p_1 \ge \lambda/p - 2/(n+2)$, which guarantees $W^{2,1}_{p/\lambda}(Q_1) \hookrightarrow L_{p_1}(Q_1)$. Repeating this estimate finitely many times we obtain $y_k \to y$ in $L_{p_m}(Q_m)$ for some $m \in \mathbb{N}$ and $p_m > n/2 + 1$. The next iteration gives $y_k \to y$ in $C(\overline{Q_{m+1}})$ and, in particular, $y_k(x_0,T) \to y(x_0,T)$. □

Controls supported on submanifolds

Example 5.2. We consider problem (5.1) with $n \ge 2,\ \ \kappa \in \mathbb{R},\ \ \lambda > 1,$ and $y^0 \in C^2(\bar\Omega)$. Given a continuous function $\psi : [0,1] \to \Gamma$, we define the measure $\delta_{\langle\psi\rangle} \in \mathcal{M}(\Gamma)$ by

$$\langle \delta_{\langle\psi\rangle}, \varphi\rangle := \int_0^1 \varphi(\psi(\theta))\,d\theta, \qquad \varphi \in C(\Gamma),$$

and we consider controls u of the form $u = \delta_{\langle\psi\rangle} \otimes 1$. (Notice that $\delta_{\langle\psi\rangle}$ is the Dirac distribution supported by the image of ψ if $\psi \in W^1_1$ is injective and $|\psi'| = 1$ a.e.) One can think of $\delta_{\langle\psi\rangle}$ as a heating wire which should be positioned on Γ in an optimal way.

Assume that $\Psi \subset C([0,1],\Gamma)$ is compact in $L_1([0,1],\Gamma)$, $\mathbb{U}_{\rm ad} = \{\delta_{\langle\psi\rangle}\otimes 1\,;\,\psi\in\Psi\}$, $\mathbb{U}_{\rm ad}^G \neq \emptyset$. Consider one of the following possibilities

(i) $\lambda < (n+2)/n$, $\mathbb{J}(y,u) = \int_Q |y-y_d|^\lambda\,dx\,dt$,

(ii) $\lambda < n/(n-2)$, $\mathbb{J}(y,u) = \sup_{t\in J}\int_\Omega |y(t)-y_d(t)|^\lambda\,dx$,

(iii) $\lambda < n/(n-2)$, $\kappa \le 0$, $\mathbb{J}(y,u) = \int_\Omega |y(T)-y_d|^\lambda\,dx$,

where $y_d \in L_\lambda(Q)$, $y_d \in C(J,L_\lambda)$ and $y_d \in L_\lambda$, respectively. We will show that in any of these three cases, the optimal control problem (1.2) has a solution.

Note that similar problems (with $\kappa = 0$) were studied in [16] under more restrictive assumptions on the class of functions ψ. Let us also remark that we could also consider wires ψ whose positions at time t depend on the solution y.

First notice that problem (5.1) can be formulated in the abstract form

$$\dot y + Ay = \kappa|y|^{\lambda-1}y + \gamma' u \quad \text{in } J, \qquad y(0) = y^0$$

(cf. (2.7)).

(i) Set $p := q := \lambda$, and $s := 0$, choose $\sigma \in (2-2/\lambda, 2-n/\lambda')$ and fix $\alpha < 1$ with $\alpha \in (0, 2-n/\lambda'-\sigma)$. Then, given $u \in \mathbb{U}_{\rm ad}$, we obtain a unique maximal $L_p(L_p)$ solution $y(u)$ and this solution satisfies the following stability estimates (see Theorem A.5 and cf. Remark 3.4): if $u_1 \in \mathbb{U}_{\rm ad}^G$ then there exist $\varepsilon, C > 0$ depending on $\|y(u_1)\|_{L_p(Q)}$ such that $u_2 \in \mathbb{U}_{\rm ad}^G$ and

$$\|y(u_1) - y(u_2)\|_{L_p(Q)} \le C\|\gamma'(u_1-u_2)\|_{\mathcal{M}(J,W_{q,\mathcal{B}}^{\sigma-2})} \tag{5.2}$$

for any $u_2 \in \mathbb{U}_{\rm ad}$ with $\|\gamma'(u_1-u_2)\|_{\mathcal{M}(J,W_{q,\mathcal{B}}^{\sigma-2})} \le \varepsilon$.

Let (y_k,u_k) be a minimizing sequence for $\mathbb{J}$, $u_k = \delta_{\langle\psi_k\rangle}\otimes 1$. Then $\|y_k\|_{L_p(Q)}$ stays bounded, due to the coercivity of $\mathbb{J}$, and we may also assume that $\psi_k \to \psi$ in $L_1([0,1],\Gamma)$ and a.e. in $[0,1]$, where $\psi\in\Psi$. Denote $u = \delta_{\langle\psi\rangle}\otimes 1$. Then

$$\begin{aligned}
\|\gamma'(u_k-u)\|_{\mathcal{M}(J,W_{q,\mathcal{B}}^{\sigma-2})} &\le C\|\gamma'(u_k-u)\|_{L_\infty(J,W_{q,\mathcal{B}}^{\sigma-2})} = C\|\gamma'(\delta_{\langle\psi_k\rangle}-\delta_{\langle\psi\rangle})\|_{W_{q,\mathcal{B}}^{\sigma-2}}\\
&\le C\|\delta_{\langle\psi_k\rangle}-\delta_{\langle\psi\rangle}\|_{W_q^{\sigma-1-1/q}(\Gamma)} \le C\|\delta_{\langle\psi_k\rangle}-\delta_{\langle\psi\rangle}\|_{[C^\alpha(\Gamma)]'}\\
&\le C\sup_{\|\varphi\|_{C^\alpha(\Gamma)}\le 1}\Big|\int_0^1\big(\varphi(\psi_k(\theta))-\varphi(\psi(\theta))\big)\,d\theta\Big|\\
&\le C\int_0^1 |\psi_k(\theta)-\psi(\theta)|^\alpha\,d\theta \to 0,
\end{aligned}$$

due to the Lebesgue theorem. Consequently, $u \in \mathbb{U}_{\rm ad}^G$ and $y(u_k)\to y(u)$ in $L_p(Q)$. Now it is easy to see that $(y(u),u)$ is an optimal pair.

(ii) The proof is the same as in (i); we only have to replace the space of the solutions $L_p(J,L_p)$ by $C(J,L_p)$ and the space of the right-hand sides $\mathcal{M}(J,W_{q,\mathcal{B}}^{\sigma-2})$ by $L_r(J,W_{q,\mathcal{B}}^{\sigma-2})$ with r large enough.

(iii) As in Theorem 3.1, the assumption $\kappa\le 0$ guarantees an easy a priori bound for the minimizing sequence (y_k,u_k), hence the coercivity of $\mathbb{J}$ is superfluous and we can proceed as above. □

Maximizing the existence time

Consider problem (4.1) with $\Gamma_1 = \emptyset$. Let (4.2) be true, $\mathbb{U}_{\rm ad}\subset\mathcal{M}(Q)$ be compact, and $y(u)$ denote the solution of (4.1) defined on the maximal existence interval J_u.

Then the same arguments as in the proof of Theorem 3.1 show that, given $C > 0$, the problem

$$\text{maximize } \; T_C := \sup\{t \in J\,;\ \|y(u)\|_{L_p(\Omega\times[0,t])} \leq C\} \;\text{ over } u \in \mathbb{U}_{\mathrm{ad}} \tag{5.3}$$

has a solution. In fact, let $u_k \in \mathbb{U}_{\mathrm{ad}}$, $\|y(u_k)\|_{L_p(\Omega\times[0,T_k])} \leq C$ and $T_k \to T_C$. We may assume $u_k \to u$ in $\mathcal{M}(Q)$. Choose $\tilde{T} < T_C$ and set $\tilde{J} := [0, \tilde{T}]$. Then $y(u_k)$ are global solutions of (4.1) with J replaced by $\tilde{J}$ and $\|y(u_k)\|_{L_p(\Omega\times\tilde{J})} \leq C$ for k large enough. Now the stability estimates (3.9) guarantee $\tilde{J} \subset J_u$ and $\|y(u)\|_{L_p(\Omega\times\tilde{J})} \leq C$. Passing to the limit as $\tilde{T} \to T_C$ we obtain $[0, T_C] \subset J_u$ and $\|y(u)\|_{L_p(\Omega\times[0,T_C])} \leq C$.

Control via initial data

Example 5.3. Consider the problem

$$\left.\begin{aligned} \partial_t y - \Delta y &= \kappa |y|^{\lambda-1} y && \text{in } Q, \\ \mathcal{B} y &= 0 && \text{on } \Sigma, \\ y(\cdot, 0) &= y^0 && \text{in } \Omega, \end{aligned}\right\} \tag{5.4}$$

where κ , y^0, and λ satisfy (4.2). Now y^0 plays the role of the control:

$$u = y^0 \otimes \delta_0 \in \mathbb{U}_{\mathrm{ad}} := \mathcal{M}(\Omega) \otimes \{\delta_0\}$$

(cf. [19, Section I.12.1]).

First assume that $p \in [\lambda, (n+2)/n)$, $y_d \in L_p(Q)$, and

$$\mathbb{J}(y, u) = \int_Q |y - y_d|^p \, dx \, dt + \|y^0\|_{\mathcal{M}(\Omega)}.$$

Then Theorem 4.1 and Remark 4.2 guarantee the solvability of (1.2).

Next assume that $q > 1 \vee n(\lambda - 1)/2$, $y_d \in L_q$, and

$$\mathbb{J}(y, u) = \int_\Omega |y(x, T) - y_d(x)|^q \, dx + \|y^0\|_{\mathcal{M}(\Omega)}.$$

Notice that $\mathbb{J}$ is well defined for any global solution y since y is a classical solution of (5.4) for $t > 0$. Let $(y_k, y_k^0 \otimes \delta_0)$ be a minimizing sequence for problem (1.2). Then the sequence (y_k^0) is bounded in $\mathcal{M}(\Omega)$ and the solutions y_k are global. In addition, the first part of the proof of [8, Theorem 1.1] guarantees that the restriction of y_k to $Q_\tau := \Omega \times [0, \tau]$ is bounded in $L_\lambda(Q_\tau)$ for suitable $\tau > 0$. Consequently, we may find $\tau_k \in [\tau/3, 2\tau/3]$ such that $\|y_k(\cdot, \tau_k)\|_{L_\lambda} \leq C_1$, and we can also assume that $\tau_k \to \tau_\infty$ as $k \to \infty$. Due to $\lambda > n(\lambda - 1)/2$, problem (5.4) is well posed in L_λ in the following sense: There exist $C_2 > 0$ and $\tilde{\tau} \in (0, \tau/6)$ such that the solution y of (5.4) exists for $t \leq 3\tilde{\tau}$ and satisfies $\|y(\cdot, t)\|_{L_\infty} \leq C_2$ for all $t \in [\tilde{\tau}, 3\tilde{\tau}]$ whenever $\|y^0\|_{L_\lambda} \leq C_1$ (see [26]). Hence, for k large enough, we obtain $\|y_k(\cdot, t_0)\|_{L_\infty} \leq C_2$, where $t_0 := \tau_\infty + 2\tilde{\tau} \leq \tau$. Since $y_k(\cdot, T)$ are uniformly bounded in L_q and (5.4) is well posed in L_q (in the sense mentioned above), we can continue the solutions y_k on an interval $[T, T + \delta]$, where $\delta > 0$ does not depend on k. Now [24] implies a uniform L_∞ bound for y_k on $\Omega \times [t_0, T]$. Hence the sequence y_k is bounded in $L_\lambda(Q)$. Due to this bound, the solvability of (1.2) follows exactly as in the proof of Theorem 3.1. □

Final observation

We have seen in Examples 4.8 and 5.3 that the coercivity assumption (3.8) can sometimes be relaxed so that we can consider non-monotone problems with final observation (where $\mathbb{J}$ depends on the control u and the final value of y only). Let us consider problem (4.1) with $\mathcal{B} = \gamma$, $u_\Sigma = 0$ and $\kappa = 1$. If $u_Q \in L_r(J, L_2)$ for some $r \geq 2$ and

$$\mathbb{J}(y,u) \geq c_1 \|y(\cdot,T)\|_{L_q} - c_2 \tag{5.5}$$

for some $q > n(\lambda - 1)/2$, then (assuming suitable lower semicontinuity of $\mathbb{J}$, compactness of $\mathbb{U}_{\mathrm{ad}}$, and $\mathbb{U}_{\mathrm{ad}}^G \neq \emptyset$), [9, Theorem 2.3] provides conditions on λ (depending on r and n) which guarantee the solvability of the optimal control problem (1.2). In addition, those conditions are essentially optimal. Unfortunately, the proof of [9, Theorem 2.3] (and also the a priori bounds used in Example 5.3) are based on energy estimates which (more or less) require the solution $y(\cdot,t)$ to belong to the space $W_2^1(\Omega)$. If, for example, $x \in \Omega$ and $u_\Omega = \delta_x \otimes 1$, then y will not possess this regularity. However, under some additional assumptions one can still use energy methods in order to get the necessary bounds. In order to formulate a typical result of this type, we need the following definition.

Definition 5.4. Let $\lambda \in (1, n/(n-2))$. We say that the set $M \subset \mathcal{M}(\Omega)$ is **λ-admissible** if there exists $c_M > 0$ with the following property: If $u \in M$ then the elliptic problem

$$-\Delta z = |z|^{\lambda-1} z + u \quad \text{in } \Omega, \qquad z = 0 \quad \text{on } \Gamma, \tag{5.6}$$

has a solution satisfying $\|z\|_{L_\lambda} \leq c_M$. □

The results of [7] guarantee that a small ball in $\mathcal{M}(\Omega)$ centered at zero or the set $M := \{u\,;\, 0 \leq u \leq u_0\}$, where u_0 is a positive measure for which (5.6) has a positive solution, are λ-admissible.

Theorem 5.5. *Let $\Omega \subset \mathbb{R}^n$ be smooth and bounded. Consider problem* (4.1) *with $\mathcal{B} = \gamma$, $u_\Sigma = 0$ and $\kappa = 1$. Let*

$$\lambda < n/(n-2) \text{ and } q \in (1 \vee (\lambda-1)n/2, n/(n-2)).$$

Assume that $M \subset \mathcal{M}(\Omega)$ is w^*-sequentially compact and λ-admissible. Let

$$\mathbb{U}_{\mathrm{ad}} = M \otimes \{1\},\ \mathbb{U}_{\mathrm{ad}}^G \neq \emptyset,\ \text{and } y^0 \in W_{q,\mathcal{B}}^{\tilde{s}}$$

with $\tilde{s} > 0 \vee (n/q - n/\lambda)$. If $\mathbb{J} : C(J, L_q) \times (\mathbb{U}_{\mathrm{ad}}, w^*) \to \mathbb{R}$ is sequentially lower semicontinuous and satisfies (5.5) *then problem* (1.2) *has a solution.*

Proof. Fix $s \in (0 \vee n/q - n/\lambda, \tilde{s} \wedge (2 - n/q'))$, $s \notin S_q$. Then Theorem A.6 (with $\sigma \in (s + 2/r, (\tilde{s} + 2/r) \wedge (2 - n/q')) \setminus S_q$ and r large enough) guarantees the existence of $\rho > 0$ such that (4.1) possesses a unique maximal strong $C^\rho(W_q^s)$ solution. In fact, exactly as in [9, Lemma 3.1] one can show that problem (4.1) is well posed in L_q in the following sense: If $y^0 \in L_q$ then there exists $\tau = \tau(\|y^0\|_{L_q}) > 0$ such that the solution exists on the time interval $[0,\tau]$ and belongs to

$$C([0,\tau], L_q) \cap C((0,\tau], W_{q,\mathcal{B}}^s) \cap L_p((0,\tau), W_{q,\mathcal{B}}^s),$$

provided $s \geq 0 \vee n/q - n/\lambda$, $s\lambda < 2 - n/q'$ and $ps < 2$.

Let (y_k, u_k) be a minimizing sequence for problem (1.2). Since u_k is bounded in $\mathcal{M}(\Omega)$, we may assume $u_k \to u$ in $(\mathcal{M}(\Omega), w^*)$. The boundedness of $\mathbb{J}(y_k, u_k)$ implies a bound for $y_k(T)$ in L_q, and the well posedness of (4.1) in L_q, mentioned above, guarantees that the solution $y_k(u_k)$ can be continued on some interval $[T, T+$

$t_1]$, where $t_1 > 0$ does not depend on k. Now a straightforward modification of the a priori estimates in [23] guarantees a uniform bound for y_k in $L_\infty(J, L_\lambda)$. Finally, [6, Proposition 2(iii)] implies a uniform bound for y_k in $C^\rho(J, W^s_{q,\mathcal{B}})$ and the assertion follows in the same way as in the proof of [9, Theorem 2.3]. □

6. Problems governed by a parabolic system

As already announced in the introduction, in this section we consider optimal control problems governed by system (1.4) with $\kappa = d = 1$. Recall that (in the nuclear reactor model derived in [17]) y_1 denotes the neutron flux and y_2 the temperature.

We first control the neutron flux. Denoting by u and v the controls, we study the problem

$$\left.\begin{aligned} \partial_t y_1 - \Delta y_1 &= y_1 y_2 - b y_1 + u + v y_1, && \text{in } Q,\\ \partial_t y_2 - \Delta y_2 &= a y_1, && \text{in } Q,\\ \mathcal{B}_1 y_1 &= 0, && \text{on } \Sigma,\\ \mathcal{B}_2 y_2 &= -c y_2, && \text{on } \Sigma,\\ y_1(\cdot,0) &= y_1^0, &&\\ y_2(\cdot,0) &= y_2^0, && \end{aligned}\right\} \tag{6.1}$$

where $n \le 3$, $\mathcal{B}_i \in \{\gamma, \partial_\nu\}$, $i = 1,2$, $a > 0$, $b, c \in \mathbb{R}$, $c = 0$ if $\mathcal{B}_2 y = \gamma$, and

$$y_1^0, y_2^0 \in C^2(\bar{\Omega}), \qquad \mathcal{B}_1 y_1^0 = 0, \quad \mathcal{B}_2 y_2^0 = -c y_2^0. \tag{6.2}$$

(These regularity and compatibility conditions on y_1^0, y_2^0 can be considerably relaxed.)

The case $v = 0$

First assume $v = 0$ and choose

$$p \in \Big(\frac{2(n+2)}{n+4}, \frac{n+2}{n}\Big). \tag{6.3}$$

As we shall see below, problem (6.1) possesses for each $u \in \mathcal{M}(Q)$ a unique maximal weak $L_p(L_p) \times L_{p'}(L_{p'})$ solution $(y_1(u), y_2(u))$.

Theorem 6.1. *Assume $v = 0$ and (6.3). Let $\mathbb{U}_{\rm ad} \subset \mathcal{M}(Q)$ be compact,*

$$\mathbb{J} : L_p(Q) \times L_{p'}(Q) \times \mathcal{M}(Q) \to \mathbb{R}$$

lower semicontinuous,

$$\mathbb{J}(y_1, y_2, u) \ge c_1 \|y_1\|_{L_p(Q)} - c_2, \tag{6.4}$$

and $\mathbb{U}^G_{\rm ad} \neq \emptyset$. Then the optimal control problem

$$\textit{minimize } \mathbb{J}\big(y_1(u), y_2(u), u\big) \textit{ over } u \in \mathbb{U}^G_{\rm ad} \tag{6.5}$$

has a solution.

Proof. Assumption (6.3) guarantees the existence of $\sigma_1 \in (0,2) \setminus S_p$ and σ_2 in $(0,2) \setminus S_{p'}$ satisfying $\sigma_2 - 2/p \notin S_{p'}$,

$$-n > \sigma_1 - 2 - \frac{n}{p}, \quad -\frac{n}{p} > \sigma_2 - 2 - \frac{n}{p'}, \tag{6.6}$$

and

$$p < \frac{2}{2-\sigma_1}, \quad p' < \frac{2}{2/p-\sigma_2}, \quad p < \frac{2}{\sigma_2}. \tag{6.7}$$

In fact, the inequalities in (6.6)–(6.7) are equivalent to

$$2-\frac{2}{p} < \sigma_1 < 2+\frac{n}{p}-n, \qquad \frac{4}{p}-2 < \sigma_2 < \Big(n+2-\frac{2n}{p}\Big) \wedge \frac{2}{p}$$

and, due to (6.3), the lower bounds for σ_1, σ_2 in these conditions lie below the corresponding upper bounds. Now (6.6) guarantees that $L_1 \hookrightarrow \mathcal{M}(\Omega) \hookrightarrow W^{\sigma_1-2}_{p,\mathcal{B}_1}$, $L_p \hookrightarrow W^{\sigma_2-2}_{p',\mathcal{B}_2}$, and the mapping

$$\begin{gathered} L_p(J,L_p) \times L_{p'}(J,L_{p'}) \to \mathcal{M}(J,W^{\sigma_1-2}_{p,\mathcal{B}_1}) \times L_p(J,W^{\sigma_2-2}_{p',\mathcal{B}_2}), \\ (y_1,y_2) \mapsto (y_1y_2 - by_1 + u, ay_1) \end{gathered}$$

is uniformly Lipschitz continuous on bounded sets. These facts and (6.7) enable us to use [8, Theorem 10.1] with $s_1 = s_2 = 0$, $p_1 = q_1 = p$, $p_2 = q_2 = p'$, $r_1 = 1$ and $r_2 = p$, in order to get the existence of a unique maximal weak $L_p(L_p) \times L_{p'}(L_{p'})$ solution $y = (y_1, y_2)$ of (6.1) for any $u \in \mathbb{U}_{\mathrm{ad}}$. That theorem also implies the corresponding stability estimates: given $u \in \mathbb{U}^G_{\mathrm{ad}}$, there exist $\varepsilon, c > 0$ depending on $\|y(u)\|_{L_p(Q)\times L_{p'}(Q)}$ such that

$$\tilde u \in \mathbb{U}^G_{\mathrm{ad}} \quad \text{and} \quad \|y(u)-y(\tilde u)\|_{L_p(Q)\times L_{p'}(Q)} \le c\|u-\tilde u\|_{\mathcal{M}(Q)}$$

for any $\tilde u \in \mathbb{U}_{\mathrm{ad}}$ satisfying $\|u-\tilde u\|_{\mathcal{M}(Q)} \le \varepsilon$ (cf. (3.9)).

The rest of the proof is the same as the corresponding part of the proof of Theorem 3.1. □

The case $u = 0$

Now consider the case $u = 0$ and

$$v \in L_r(J, L_z(M)), \qquad r, z > 1, \quad \frac{2}{r} + \frac{d}{z} < d + 2 - n, \tag{6.8}$$

where M is a smooth d-dimensional manifold in Ω. Notice that this assumption requires $d \ge n-1$. We shall see that in this case problem (6.1) possesses for each v a unique maximal strong solution $(y_1(v), y_2(v))$ and this solution belongs to $C(\bar Q) \times C(\bar Q)$ if it is global.

Theorem 6.2. *Assume $u = 0$ and (6.8). Let $\mathbb{U}_{\mathrm{ad}} \subset L_r(J, L_z(M))$ be compact, $\mathbb{J} : C(\bar Q) \times C(\bar Q) \times L_r(J, L_z(M)) \to \mathbb{R}$ be lower semicontinuous,*

$$\mathbb{J}(y_1, y_2, v) \ge c_1\|y_1\|_{C(\bar Q)} - c_2 \tag{6.9}$$

and $\mathbb{U}^Q_{\mathrm{ad}} \ne \emptyset$. Then the optimal control problem

$$\text{minimize } \mathbb{J}\big(y_1(v), y_2(v), v\big) \text{ over } v \in \mathbb{U}^G_{\mathrm{ad}} \tag{6.10}$$

has a solution.

Proof. Assume that $q > 1$ and $\sigma \in (0,2) \setminus S_q$ satisfy

$$\sigma - 2/r \notin S_q \quad \text{and} \quad 2 - \sigma - n/q' > -d/z'.$$

Then $\gamma'_M : L_z(M) \to W^{\sigma-2}_{q,\mathcal{B}_1}$ is continuous (cf. Remark 4.7(i)). We look for solutions $y_i \in C^\rho([0,t], W^s_{q,\mathcal{B}_i})$, $i = 1,2$, with some $\rho \ge 0$ and $s \in (n/q, \sigma) \setminus S_q$. Since $W^s_{q,\mathcal{B}} \hookrightarrow C(\bar\Omega)$ in this case, it is easy to see that the right-hand sides in (6.1) belong to $L_r([0,t], W^{\sigma-2}_{q,\mathcal{B}_i})$, $i = 1,2$. Now the existence of y_1, y_2 satisfying the corresponding stability estimates follows from [8, Theorem 10.1], provided we find ρ, s, σ satisfying the above conditions and $2\rho < \sigma - s - 2/r$. The existence of such ρ, s, σ is an easy consequence of (6.8). The rest of the proof is clear. $\square$

Boundary controls

Finally we control the temperature. This problem can be written in the form

$$\left.\begin{aligned} \partial_t y_1 - \Delta y_1 &= y_1 y_2 - b y_1, && \text{in } Q,\\ \partial_t y_2 - \Delta y_2 &= a y_1 + u_Q, && \text{in } Q,\\ \mathcal{B}_1 y_1 &= 0, && \text{on } \Sigma,\\ \mathcal{B}_2 y_2 &= -c y_2 + u_\Sigma, && \text{on } \Sigma,\\ y_1(\cdot,0) &= y_1^0, &&\\ y_2(\cdot,0) &= y_2^0, && \end{aligned}\right\} \tag{6.11}$$

where $n, \mathcal{B}_1, \mathcal{B}_2, a, b, c, y_1^0, y_2^0$ are as above, and

$$u := (u_Q, u_\Sigma) \in L_r(J, \mathcal{M}(\Omega) \times \mathcal{M}(\Gamma)) \quad \text{with} \quad u_\Sigma = 0 \text{ if } \mathcal{B}_2 = \gamma$$

is the control. Assume $r > n-1$ and fix $p_1 \geq r' \vee r$, $q_1 > n/2$. We show that problem (6.11) possesses for each u a unique maximal (strong) solution $(y_1(u), y_2(u))$, and $y_1(u) \in L_{p_1}(J, L_{q_1})$ if it is global.

Theorem 6.3. *Assume that* $\mathbb{U}_{\mathrm{ad}} \subset L_r(J, \mathcal{M} \times \mathcal{M}(\Gamma))$ *is compact,*

$$\mathbb{J} : L_{p_1}(J, L_{q_1}) \times L_r(J, \mathcal{M} \times \mathcal{M}(\Gamma)) \to \mathbb{R}$$

is lower semicontinuous,

$$\mathbb{J}(y_1, u) \geq c_1 \|y_1\|_{L_{p_1}(J, L_{q_1})} - c_2 \tag{6.12}$$

and $\mathbb{U}_{\mathrm{ad}}^G \neq \emptyset$. *Then the optimal control problem*

$$\textit{minimize } \mathbb{J}\big(y_1(u), u\big) \textit{ over } u \in \mathbb{U}_{\mathrm{ad}}^G \tag{6.13}$$

has a solution.

Proof. Choose

$$q_2 \in \Big(q_1' \vee \frac{n}{2}, \frac{n}{n-2}\Big) \qquad \text{and} \qquad p_2 \in \Big(p_1', \frac{2}{2/r - 2 + n/q_2'}\Big)$$

such that

$$\frac{1}{p_2} + \frac{n}{2q_2} < 1. \tag{6.14}$$

Notice that this choice is possible due to $q_1 > n/2$, $p_1 \geq r'$ and $r > 2/(4-n)$. Choose also $\sigma_2 \in (2/r - 2/p_2, 2 - n/q_2') \setminus S_{q_2}$ such that $\sigma - 2/r \notin S_{q_2}$. Then $u_Q + \gamma' u_\Sigma \in L_r(J, W_{q_2,\mathcal{B}_2}^{\sigma_2 - 2})$. Let $p, q > 1$ be defined by

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \qquad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Denote $J_t = [0,t]$. Given $y_1 \in L_{p_1}(J_t, L_{q_1})$ and $y_2 \in L_{p_2}(J_t, L_{q_2})$, we have $y_1 y_2 \in L_p(J_t, L_q)$.

Due to (6.14) we may choose $\sigma_1 \in (0, 2/p) \setminus S_{q_1}$ such that $\sigma_1 - 2/p \notin S_{q_1}$,

$$p_1 < \frac{2}{2/p - \sigma_1} \qquad \text{and} \qquad -\frac{n}{q} > \sigma_1 - 2 - \frac{n}{q_1}.$$

Consequently, $L_q \hookrightarrow W^{\sigma_1-2}_{q_1,\mathcal{B}_1}$ and the map

$$L_{p_1}(J_t, L_{q_1}) \times L_{p_2}(J_t, L_{q_2}) \to L_p(J_t, W^{\sigma_1-2}_{q_1,\mathcal{B}_1}) : y_1 \mapsto y_1 y_2 - b y_1$$

is Lipschitz continuous. Now [8, Theorem 10.1] (with $s_1 := s_2 := 0$, $r_1 := p$ and $r_2 := r$) guarantees the existence of a unique maximal strong $L_{p_1}(L_{q_1}) \times L_{p_2}(L_{q_2})$ solution $y = (y_1, y_2)$ of (6.11) satisfying the corresponding stability estimates.

The rest of the proof is the same as in the proof of Theorem 3.1. □

Remarks 6.4. (i) It is clear from the proof of Theorem 6.3 that similar assertion can be proved also for cost functionals $\mathbb{J}$ depending on both y_1 and y_2.

(ii) If $n = 1$ then we could work with $u_Q \in \mathcal{M}(Q)$ and $u_\Sigma \in \mathcal{M}(\Sigma)$. This follows from the proof of Theorem 6.3 and the results in [8].

(iii) As in Theorem 3.1, we could work with the w^*-toplogy instead of the norm topology in the space of controls. □

Appendix: The Basic Existence, Uniqueness, and Stability Theorem for Semilinear Problems

For the reader's convenience we collect here the main existence, uniqueness and stability results for weak and strong solutions of the semilinear problem

$$\dot{y} + Ay = F(y) \text{ in } [0,T], \qquad y(0) = y^0, \tag{A.1}$$

where $A = A_s = C_s - 1$, with C_s being the isomorphism between $W^s_{q,\mathcal{B}}$ and $W^{s-2}_{q,\mathcal{B}}$ mentioned in Section 2. Analogous results are true in the case of systems.

We write $C^{1-}_b(Y, X)$ for the space of all maps from Y into X which are uniformly Lipschitz continuous on bounded sets. If X and Y are paces of functions defined on $[0,T]$, then $F : X \to Y$ is said to possess the Volterra property if, given any $u \in X$ and $t \in (0,T)$, the restriction of $F(u)$ to $[0,t]$ depends on the values of $u\,|\,[0,t]$ only. We fix $k \in \mathbb{N}$ and set $\vec{p} := (p_1, p_2, \dots p_k)$, $\vec{s} := (s_1, s_2, \dots s_k)$ and $\vec{\rho} := (\rho_1, \rho_2, \dots, \rho_k)$.

In the first theorem, we consider weak solutions and assume $y^0 = 0$. The general case, $y^0 \in W^{\sigma-2}_{q,\mathcal{B}}$, can be reduced to this special case by taking $\hat{F}(y) = F(y) + y^0 \otimes \delta_0$.

Theorem A.5. *Assume*

$$s_1, s_2, \dots, s_k, \sigma \notin S_q, \quad 0 \le s_i < \sigma < 2, \; i = 1, 2, \dots, k, \tag{A.2}$$

and suppose that $1 \le p_i < 2/(s_i - \sigma + 2)$, $i = 1, 2, \dots, k$, $y^0 = 0$. *Denote*

$$Y_t = L_{\vec{p}}([0,t], W^{\vec{s}}_{q,\mathcal{B}}) := \bigcap_{i=1}^{k} L_{p_i}([0,t], W^{s_i}_{q,\mathcal{B}}), \qquad X_t := \mathcal{M}([0,t], W^{\sigma-2}_{q,\mathcal{B}}).$$

Let $F \in C^{1-}_b(Y_T, X_T)$ *have the Volterra property. Then problem* (2.3) *has a unique maximal weak* $L_{\vec{p}}(W^{\vec{s}}_q)$ *solution* $y(F)$ *defined on the maximal existence interval* $[0, t(F))$. *If* $y(F) \in Y_{t(F)}$ *or* $F(y(F)) \in X_{t(F)}$, *then* $y(F)$ *is global.*

The map

$$F \mapsto y(F) \in L_{\vec{p},\mathrm{loc}}([0, t(F)), W^{\vec{s}}_{q,\mathcal{B}})$$

is Lipschitz continuous in the following sense: Set $t = t(F)$ *if* $y(F)$ *is global and fix* $t < t(F)$ *otherwise. Let* $\omega_1 > 0$, *and let* $\omega_2 : \mathbb{R}^+ \to \mathbb{R}^+$ *be an increasing function* *such that*

$$\left.\begin{aligned} \|F(0)\|_{X_T} &\le \omega_1, \\ \|F(y_1)-F(y_2)\|_{X_T} &\le \omega_2(R)\|y_1-y_2\|_{Y_T}, \end{aligned}\right\} \tag{A.3}$$

for any $R>0$ and $y_1, y_2 \in Y_T$ whose norms are bounded by R. Fix $R > \|y(F)\|_{Y_t}$. Then there exist positive constants ε, c (depending only on R, t, ω_1, ω_2) with the following property: If $\tilde F \in C_b^{1-}(Y_T, X_T)$ has the Volterra property, satisfies (A.3) *and*

$$\sup_{\|y\|_{Y_T}\le R} \|(F-\tilde F)(y)\|_{X_T} \le \varepsilon,$$

then $t \le t(\tilde F)$, $y(\tilde F) \in Y_t$ and

$$\|y(F) - y(\tilde F)\|_{Y_t} \le c \sup_{\|y\|_{Y_T}\le R} \|(F-\tilde F)(y)\|_{X_T}.$$

Proof. See [8, Theorem 3.2, Remark 2.4(d)] and [6, Theorem 7(i)]. □

Theorem A.6. *Assume* (A.2) *and suppose that*

$$r>1,\ r \ne 2/(\sigma - s_i), i=1,2,\dots,k,\ \sigma - 2/r \notin S_q,\ y^0 \in W_{q,\mathcal{B}}^{\sigma-2/r}.$$

Denote $X_t := L_r([0,t], W_{q,\mathcal{B}}^{\sigma-2})$. For any $i=1,2,\dots,k$,

if $r < 2/(\sigma - s_i)$ fix $p_i \in [1, 2/(s_i - \sigma + 2/r))$ and set $Y_t^i := L_{p_i}([0,t], W_{q,\mathcal{B}}^{s_i})$,
if $r > 2/(\sigma - s_i)$ fix $\rho_i \in [0, (\sigma - s_i - 2/r)/2)$ and set $Y_t^i := C^{\rho_i}([0,t], W_{q,\mathcal{B}}^{s_i})$.

Set $Y_t := \bigcap_{i=1}^k Y_t^i$. Let $F \in C_b^{1-}(Y_T, X_T)$ have the Volterra property. Then (2.3) *has a unique maximal strong solution $y(y^0,F)$ defined on the maximal existence interval $[0, t(y^0,F))$, and $y(y^0,F) \in Y_t$ for any $t < t(y^0,F))$. If $y(y^0,F) \in Y_{t(y^0,F)}$ or $F(y(y^0,F)) \in X_{t(y^0,F)}$, then $y(y^0,F)$ is global.*

The map $(y^0,F) \mapsto y(y^0,F)$ is Lipschitz continuous in an analogous sense as in Theorem A.5 (see [8] *for details).*

If $y = y(y^0,F)$ is global, then

$$y \in L_r(J, W_{q,\mathcal{B}}^{\tilde\sigma}) \cap W_r^1(J, W_{q,\mathcal{B}}^{\tilde\sigma-2}) \tag{A.4}$$

for any $\tilde\sigma < \sigma$, and the norm of y in this space can be estimated by a constant $C = C(\|F(y)\|_{X_T}, \|y^0\|_{W_{q,\mathcal{B}}^{\sigma-2/r}})$.

Proof. See [8, Theorems 3.3, 3.4, Remark 2.4(d)] and [6, Theorems 5, 7(ii)]. □

The following lemma guarantees that we can apply the above theorems to problems with measure valued right-hand sides provided $\sigma < 2 - n/q'$.

Lemma A.7. *Suppose that $q>1$ and $\sigma \in [0, 2-n/q') \setminus S_q$. Then*

(i) $\mathcal{M}(\Omega) \hookrightarrow W_{q,\mathcal{B}}^{\sigma-2}$ if $\mathcal{B} = \gamma$.
(ii) $\mathcal{M}(\bar\Omega) \hookrightarrow W_{q,\mathcal{B}}^{\sigma-2}$ if $\mathcal{B} = \partial_\nu$.

Proof. See [8, Lemma 4.1(iii)]. □

Notice also that if $\sigma < 2 - n/q'$ then $\mathcal{M}(\Gamma) \hookrightarrow W_q^{\sigma-1-1/q}(\Gamma)$ so that $\gamma' g$ is well defined for $g \in \mathcal{M}(\Gamma)$, where $\gamma' : W_q^{\sigma-1-1/q}(\Gamma) \to W_{q,\mathcal{B}}^{\sigma-2}$ is the dual of the trace operator $\gamma : W_{q'}^{1+1/q-\sigma}(\Gamma) \to W_{q',\mathcal{B}}^{2-\sigma}$.

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