OPTIMAL CONTROL PROBLEMS WITH FINAL OBSERVATION
GOVERNED BY EXPLOSIVE PARABOLIC EQUATIONS

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Abstract. We study optimal controls problems with final observation. The
governing parabolic equations or systems involve superlinear nonlinearities and
their solutions may blow up in finite time. Our proof of the existence, regularity
and optimality conditions for an optimal pair is based on uniform a priori
estimates for the approximating solutions. Our conditions on the growth of the
nonlinearity are essentially optimal. In particular, we also solve a longstanding
open problem of J.L. Lions concerning singular systems.

1. Introduction

In his book [21], J.L. Lions studied several optimal control problems governed
by nonlinear parabolic equations of the form
\[
\partial_t y - \Delta y = y^\lambda + u, \quad x \in \Omega, \ t \in [0, T],
\]
where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \), \( \lambda \in \{2, 3\} \), \( u = u(x, t) \) is the control and \( y = y(x, t) \) is the state variable. Equation (1.1) is complemented by suitable boundary
and initial conditions, for example
\[
y = 0 \text{ on } \partial \Omega \times (0, T), \quad y(\cdot, 0) = y_0,
\]
where \( y_0 \in L_\infty(\Omega) \). If \( u \) is regular enough then the state problem (1.1)-(1.2)
possesses a unique strong solution \( y = y(u) \) defined on the maximal existence
interval \( J_u \) (see Section 2 for the definition of a strong solution). However, even
for smooth controls \( u \), the solution \( y(u) \) need not be global – the interval \( J_u \) need
not coincide with \([0, T]\). In this case, \( y(u) \) blows up at the time \( t(u) := \sup J_u \), i.e.
it develops a singularity and leaves its natural regularity class. After blow-up, the
solution either can be continued in a weak sense (the blow-up is incomplete [16])
or such continuation is not possible (the solution blows up completely [9]).

Let \( U_{ad} \) denote the set of admissible controls,
\[
U_{ad}^G := \{u \in U_{ad} : \text{the solution } y(u) \text{ is global}\},
\]
and \( J = J(y, u) \) be the cost functional. A standard way to solve the optimal control
problem
\[
\text{minimize } J(y(u), u) \text{ over } u \in U_{ad}^G
\]
is to consider controls \( u_k, k = 1, 2, \ldots \), such that \( (J(y(u_k), u_k)) \) is a minimizing
sequence for (1.3) and to show that a suitable subsequence of \( (y(u_k), u_k) \) converges
to an optimal pair \((y(u), u)\). Assume, for example, that \( U_{ad} \) is a (weakly closed)
subset of a reflexive Banach space. If \( J \) is coercive with respect to \( u \) (or \( U_{ad} \) is
bounded) then the sequence \( (u_k) \) is bounded and we may assume that \( u_k \to u \)
in the weak topology. Similarly, if \( J \) is coercive with respect to \( y \) (in a suitable space
of functions defined in $Q := \Omega \times [0,T]$, then the sequence $(y(u_k))$ is bounded and standard compactness results for the state problem enable us to pass to the limit in order to find a minimizer for (1.3).

If we consider problems with final observation (where $J$ depends just on $u$ and the final value $y(\cdot,T)$), then the coerciveness of $J$ provides us with a priori estimates for $u_k$ and final values of $y(u_k)$. However, such estimates are, in general, not sufficient for the uniform boundedness of solutions $y(u_k)$ on the whole interval $[0,T]$. Consequently, we have to find sufficient conditions on $\lambda$ and/or other parameters of the problem which guarantee a priori bounds for global solutions $y$ of (1.1)-(1.2) depending only on suitable norms of $u$ and $y(\cdot,T)$.

Let us discuss the question of a priori bounds for problems with final observation in the particular setting of [21, Section I.10]. Fix $N > 0$, $q \geq 1$, $y_d \in L_q(\Omega)$ and set

$$J(y, u) := \int_\Omega |y(x, T) - y_d(x)|^q \, dx + N \int_Q u^2(x, t) \, dx \, dt,$$

Assume also that $U_{ad} \subset L_2(Q)$ is closed and convex, and $U_{ad}^G \neq \emptyset$. If $\lambda = 2$, $q = 3$ and $n \leq 3$ then [21, Theorem I.10.1] and its proof guarantee the required boundedness for the solutions $y(u_k)$, hence the existence of an optimal pair $(y, u)$. If, in addition, $n \leq 2$, then optimality conditions for the optimal pair $(y, u)$ were derived in [21, Theorem I.10.3]. On the other hand, the existence of an optimal pair in the case $\lambda = 3$, $q = 4$ (or $\lambda = 2$, $q < 3$) and the optimality conditions for $n = 3$ were left as open problems, see [21, Remarks I.10.1, I.10.2 and I.10.4]. Our results give, in particular, positive answers to all those open problems. In fact, we consider an arbitrary dimension $n$, exponents $q \geq 2$, $\lambda > 1$ (where either $y^\lambda := |y|^{\lambda-1}y$ or $y^\lambda := |y|^\lambda$) and controls $u \in L_q(\Omega) \cap L_{r}(\Omega)$, $r \geq 2$, and find sufficient conditions on $q, \lambda$ and $r$ that guarantee the existence of optimal controls and the optimality conditions (see Section 2 for precise statements of our results).

We also show that many of our conditions are essentially optimal. In particular, if $U_{ad} \subset L_\infty([0,T], L_2(\Omega))$ then our sufficient conditions on $q$ and $\lambda$ guaranteeing the existence of optimal controls have the form

$$\lambda < \frac{n+2}{(n-2)_+} \quad \text{and} \quad q \in \left( (\lambda-1) \frac{n}{2}, \frac{2n}{(n-4)_+} \right),$$

where $a_+ := \max(a, 0)$ and $a/b_+ := \infty$ if $a > 0$ and $b \leq 0$. The upper bound for $q$ is required by the (low) regularity of controls $u$: it guarantees $y(u) \in C([0,T], L_q(\Omega))$ so that $J(y(u), u)$ is defined. If $q < (\lambda-1)n/2$ or $\lambda > (n+2)/(n-2)_+$ and $n \leq 10$ then we show that problem (1.3) need not be solvable even if the set $U_{ad}$ is a compact subset of $C^\infty(\Omega \times [0,T])$ and $U_{ad}^G \neq \emptyset$, see Remark 3.4. This nonexistence result is due to the fact that the set $U_{ad}^G \neq \emptyset$, need not be closed in $U_{ad}$: if $u_k \in U_{ad}^G$, $u_k \to u \in U_{ad}$, then the limiting solution $y(u)$ may blow-up at $t(u) < T$. The conditions on $q$ show the importance of a good choice of the cost functional in order to control the equation. On the other hand, if $\lambda > (n+2)/(n-2)_+$ then (a strong) solvability of our control problem cannot be guaranteed for any $q$.

The solvability of (1.3) with $U_{ad}$, $J$ as above was proved by Imanuvilov [18, Theorem 2.1] and Fursikov [15, Theorem 4.3] for $r = q = 2$ and any $\lambda > 1$ but their function $y(u)$ corresponding to the optimal control $u$ need not be a strong solution in our sense. In fact, the results of [18, 15] also apply to the example of Remark 3.4(i), where $y(u)$ blows up at $t(u) < T$ (but can be continued in a weak sense). This lack of regularity causes serious problems in establishing the optimality
conditions. In order to obtain these conditions, Imanuvilov and Fursikov have to assume \( q = 2 \geq (\lambda - 1)n/2 \), see [15, Theorem 5.1]. Note also that the proofs in [18, 15] substantially use the choice \( q = 2 \) hence require \( \lambda \leq 1 + 4/n \). In particular, if \( n = 3 \) then their method cannot be used in the case \( \lambda = 3, q = 4 \) mentioned above.

Our proof of a priori estimates is based on energy and perturbation arguments in [25, 27]. The same approach can be used for more general problems. For example, the case of general second-order elliptic operators and/or general nonlinearities with polynomial growth can be solved by adopting the proofs in [26]. Similarly, if one considers linear or nonlinear parabolic equations complemented by nonlinear Neumann boundary conditions of the form \( \partial_\nu y = y^\lambda \) or \( \partial_\nu y = y^\lambda + u \) then one can use estimates in [28] and [11].

In this paper we consider two modifications of the model problem (1.1)-(1.2): a problem with multiplicative control and a problem governed by a parabolic system.

In the case of multiplicative control we replace the state equation (1.1) by

\[
\partial_t y - \Delta y = y^\lambda + uy, \quad x \in \Omega, \ t \in [0,T],
\]

and we prove the required a priori bounds by using the energy and perturbation arguments mentioned above. This study is motivated by the fact that multiplicative controls often appear in the literature.

In Section 6 we investigate the existence of optimal controls for problems governed by the system

\[
\begin{align*}
\partial_t y_1 - \Delta y_1 &= \kappa y_1 y_2 - by_1 + u, & x \in \Omega, \ t \in [0,T], \\
\partial_t y_2 - d\Delta y_2 &= ay_1, & x \in \Omega, \ t \in [0,T],
\end{align*}
\]

which is complemented by suitable boundary and (nonnegative) initial conditions. Here \( d \geq 0, \kappa, a > 0 \ b \in \mathbb{R} \) and \( u \) is a nonnegative control. System (1.5) (with \( d = 0 \) and \( u = 0 \)) was derived in [19] as a model for the dynamics of a nuclear reactor close to a stationary state. The state variables \( y_1 \) and \( y_2 \) correspond to the neutron flux and the temperature, respectively, and the constant \( \kappa \) represents the temperature feedback (cf. also [29]). Since this system (with \( d \geq 0, \kappa > 0 \) and \( u = 0 \)) possesses an interesting dynamics with possible blow-up in finite time, it became the object of study of many mathematical papers (see [10], [17], [23], [24], [31], [32] and the references therein). We consider the case \( d = \kappa = 1 \) and study the corresponding optimal control problem with final observation. Since the energy arguments used in the case of equations (1.1) or (1.4) cannot be applied, we use a different approach to the proof of a priori bounds.

This paper is organized as follows. In Section 2 we formulate our main results (Theorems 2.3, 2.6, 2.8 and 2.10). Sections 3 and 4 are devoted to the proof of existence of optimal controls and optimality conditions, respectively, for the problem governed by the model equation (1.1). Problems governed by (1.4) and (1.5) are studied in Sections 5 and 6, respectively. In the Appendix we recall, for the reader’s convenience, from [6] the basic existence, uniqueness and stability results for semilinear parabolic equations which are the fundament for our investigations.

2. Main Results

First we introduce some notation which will be used throughout this paper. If \( a, b \in \mathbb{R} \) then we denote \( a \vee b := \max(a, b) \) and \( a \wedge b := \min(a, b) \). If \( p \in (1,\infty) \)
then \( p' \) is the dual exponent defined by \( 1/p + 1/p' = 1 \). For \( X \subset \mathbb{R}^n \) we write \( \mathcal{D}(X) \) for the space of smooth functions with compact support in \( X \). The symbols \( w \) and \( w^* \) are used to denote the weak and weak-star topology, respectively. By \( \Omega \) we mean an open bounded subset of \( \mathbb{R}^n \) having a smooth boundary \( \Gamma \). We also set \( Q := \Omega \times J \) and \( \Sigma := \Gamma \times J \), where \( J := [0, T] \) with a fixed \( T > 0 \). By \( B \) we denote one of the boundary operators \( \gamma, \partial_{\nu} \), where \( \gamma \) is the trace operator and \( \partial_{\nu} \) the derivative with respect to \( \nu \), the outer unit normal on \( \Gamma \).

Let \( s \in [-2, 2] \) and \( 1 < q < \infty \). We write \( W^s_q := W^s_q(\Omega) \) for the usual Sobolev-Slobodeckii spaces; hence \( W^0_q = L_q(\Omega) \). If \( B = \gamma \) then we set

\[
W^s_{q,B} := \begin{cases} 
  \{ u \in W^s_q \mid Bu = 0 \}, & 1/q < s \leq 2, \\
  W^s_q, & 0 \leq s < 1/q, \\
  (W^{-s}_{q,B})', & -2 \leq s < 0, \quad s \neq -1 + 1/q,
\end{cases}
\]

where \( X' \) denotes the dual space to \( X \). If \( B = \partial_{\nu} \) then

\[
W^s_{q,B} := \begin{cases} 
  \{ u \in W^s_q \mid Bu = 0 \}, & 1 + 1/q < s \leq 2, \\
  W^s_q, & 0 \leq s < 1 + 1/q, \\
  (W^{-s}_{q,B})', & -2 \leq s < 0, \quad s \neq -2 + 1/q.
\end{cases}
\]

In either case the dual spaces are determined by means of the standard \( L_q \)-duality pairing. We also set \( S_q := \{-2 + 1/q, -1 + 1/q, 1/q, 1 + 1/q\} \).

### Weak and Strong Solutions

Consider the problem

\[
\begin{aligned}
\partial_t y - \Delta y &= f & \text{in } Q, \\
By &= 0 & \text{on } \Sigma, \\
y(\cdot, 0) &= y^0 & \text{in } \Omega,
\end{aligned}
\]

where \( y^0 \in L_1(\Omega) \) and \( f \in L_1(\Omega) \).

**Definition 2.1.** Assume that \( s \in [0, 2] \setminus S_q \) and \( 1 < p, q < \infty \). A weak \( L_p(W^s_q) \)-solution of (2.1) on \( [0, t] \), \( 0 < t \leq T \), is a function \( y \in L_{p,\text{loc}}([0, t), W^s_{q,B}) \) such that

\[
\int_0^t \int_{\Omega} (-\partial_t \varphi - \Delta \varphi)y \, dx \, d\tau = \int_0^t \int_{\Omega} \varphi f \, dx \, d\tau + \int_{\Omega} \varphi(0)y^0 \, dx
\]

for any \( \varphi \in \mathcal{D}(\Omega \times [0, t]) \) satisfying \( B\varphi = 0 \) on \( \Gamma \times [0, t] \). It is global if \( t = T \) and \( y \in L_p((0, T), W^s_{q,B}) \).

The differential operator \( C := 1 - \Delta \) defines an isomorphism between \( W^2_{q,B} \) and \( L_q(\Omega) \) and this isomorphism admits a unique extension to an isomorphism \( C = C_s \) between \( W^2_{q,B} \) and \( W^{2-s}_{q,B} \) for any \( s \in [0, 2] \setminus S_q \) (see [1]). Moreover, \(-A := 1 - C \) generates a strongly continuous analytic semigroup \( \{e^{-tA} \mid t \geq 0\} \) on \( W^r_{q,B} \) for \( r \in [-2, s] \setminus S_q \), and

\[
(t \mapsto e^{-tA}x) \in C([0, T], W^r_{q,B}) \cap C((0, T), W^s_{q,B})
\]

with

\[
\|e^{-tA}x\|_{W^s_{q,B}} \leq ct^{(r-s)/2}\|x\|_{W^r_{q,B}}, \quad 0 < t \leq T,
\]

(2.3)
for \( x \in W_{q,B}^r \) (cf. [1, Theorem 5.2] and [2, Theorem V.2.1.3]). Then, provided 
\( 1 < q < n/(n-2) \) and \( 0 \leq s < 2-n/q' \), (the weak form of) problem (2.1) is equivalent to the abstract evolution equation
\[
\dot{y} + Ay = f \text{ in } [0,T], \quad y(0) = y^0, \tag{2.4}
\]
(see [6] for details).

**Definition 2.2.** A weak \( L_p(W_{q,s}^r) \)-solution \( y \) of (2.1) on \([0,t]\) is a strong \( L_p(W_{q,s}^r) \)-solution if 
\[
y \in W_{r,loc}^1([0,t), W_{q,s}^{r-2}) \cap L_{r,loc}([0,t), W_{q,s}^r)
\]
for some \( r > 1 \) and (2.4) is satisfied a.e. in \([0,t]\). If, in addition, \( y \in C^\rho([0,t), W_{q,s}^r) \) for some \( \rho \in (0,1) \) then \( y \) is called strong \( C^\rho(W_{q,s}^r) \)-solution.

**A Model Problem**

Now we are ready to formulate the main results of this paper. First consider the optimal control problem (1.3) for the model state equation
\[
\begin{align*}
\partial_t y - \Delta y &= |y|^{\lambda-1}y + u \quad \text{ in } Q, \\
By &= 0 \quad \text{ on } \Sigma,
\end{align*}
\tag{2.5}
\]
where \( B \in \{\gamma, \partial_\nu\} \). As already announced in the introduction, instead of the operator \(-\Delta \) and the model nonlinearity \( |y|^{\lambda-1}y \) we could handle a general second-order elliptic operator \( A \) and a general superlinear function \( f(x,y) \) satisfying suitable growth conditions (see [26] for details).

In the following theorem we consider cost functionals \( J \) which depend on the final value of \( y \) and which satisfy the coercivity condition
\[
J(y,u) \geq c_1 \|y(\cdot,T)\|_{L_q(\Omega)} - c_2, \tag{2.6}
\]
with positive constants \( c_1 \) and \( c_2 \).

**Theorem 2.3.** Let
\[
1 < \lambda < \frac{n+2}{(n-2)_+}, \tag{2.7}
\]
\[
q \in \left( \lambda - 1, \frac{n}{2}, \frac{2n}{n-4}_+ \right) \quad \text{and} \quad q \geq 2, \tag{2.8}
\]
Suppose that \( r \geq 2 \) satisfies
\[
\frac{1}{r} < 1 - \frac{n}{4} + \frac{n}{2q}, \tag{2.9}
\]
and
\[
r > \frac{\lambda + 1}{\lambda} \frac{\lambda n - (n+4)}{n+2 - \lambda(n-2)} - \frac{2}{\lambda}. \tag{2.10}
\]
Assume that \( y^0 \in W_{q,B}^2 \) and \( U_{ad} \) is a weakly compact subset of \( L_r(J, L_2(\Omega)) \). If \( u \in U_{ad} \) then (2.5) has a unique strong \( L_r(\Omega) \)-solution defined on the maximal existence interval \( J_u \).

Assume \( U_{ad}^G \neq \emptyset \). Let (2.6) be true and assume that \( J \) can be written in the form \( J(y,u) = \mathbb{J}_T(y(\cdot,T),u) \), where \( \mathbb{J}_T : L_q(\Omega) \times (L_r(J, L_2(\Omega)), w) \to \mathbb{R} \) is lower semicontinuous. Then the optimal control problem (1.3) governed by (2.5) has a solution.
Remarks 2.4. (i) Theorem 2.3 remains true if we replace the nonlinearity \(|y|^{\lambda-1}y\) with \(|y|^\lambda\), see Remark 3.3.

(ii) Let \(\lambda, q, r\) satisfy (2.7)–(2.10), \(y^0 \in W^{1,\infty}_q\),
\[
\mathcal{J}(y, u) := \int_{\Omega} |y(x, T) - y_d(x)|^q \, dx + N \int_0^T \left( \int_{\Omega} u^2(x, t) \, dx \right)^{r/2} \, dt,
\]
where \(y_d \in L_q(\Omega), \, N \geq 0\), and let \(\mathbb{U}_{ad} \subset L_r(J, L_2(\Omega))\) be closed, convex and bounded. Then all assumptions of Theorem 2.3 are satisfied provided \(\mathbb{U}^{G}_{ad} \neq \emptyset\). In addition, if \(N > 0\) then \(\mathbb{U}_{ad}\) need not be bounded (we can replace the set \(\mathbb{U}_{ad}\) in problem (1.3) with \(\mathbb{U}_{ad} := \mathbb{U}_{ad} \cap B_R,\) where \(B_R\) is a large closed ball in \(L_r(J, L_2(\Omega))\)).

(iii) If \(r = 2\) then conditions (2.7)–(2.10) in Theorem 2.3 read
\[
1 < \lambda < \frac{3n + 8}{3n - 4}, \quad q \in \left(\frac{\lambda - 1}{2}, \frac{2n}{(n - 2)+}\right) \quad \text{and} \quad q \geq 2.
\]
In particular, if \(n \leq 3\) then we may choose \(\lambda = 3\) and \(q = 4\) (cf. the open problems of J.L. Lions mentioned above).

Example 2.5. Let \(\lambda, q, r, y^0, J, \mathbb{U}_{ad}\) be as in Remark 2.4(ii). Assume \(|y^0| \leq C_0\) for some \(C_0 \geq 0\) and \(\{u \in L_\infty(Q) : |u| \leq C_0^A\} \subset \mathbb{U}_{ad}\). Then \(\mathbb{U}^{G}_{ad} \neq \emptyset\), hence the optimal control problem (1.3) has a solution. In fact, the solution \(\tilde{y}\) of the linear problem
\[
\begin{align*}
\partial_t \tilde{y} - \Delta \tilde{y} &= 0 & \text{in } Q, \\
B \tilde{y} &= 0 & \text{on } \Sigma, \\
\tilde{y}(-, 0) &= y^0 & \text{in } \Omega,
\end{align*}
\]
satisfies \(|\tilde{y}| \leq C_0\) by the maximum principle, thus \(u := -|\tilde{y}|^{\lambda-1} \tilde{y} \in \mathbb{U}_{ad}\) (the function \(y := \tilde{y}\) is a global solution of (2.5)).

Optimality Conditions

In order to obtain the optimality conditions, we restrict ourselves to the case \(r = 2\) and we also fix
\[
\mathcal{J}(y, u) := \int_{\Omega} |y(x, T) - y_d(x)|^q \, dx + N \int_Q u^2(x, t) \, dx \, dt,
\]
where \(q > 1\), \(y_d \in L_q(\Omega)\) and \(N \geq 0\) are given. This particular choice of \(r\) and \(\mathcal{J}\) corresponds to the setting of J.L. Lions in [21].

Theorem 2.6. Let the assumptions of Theorem 2.3 be fulfilled and let, moreover, \(r = 2, \mathbb{U}_{ad}\) be convex and \(\mathcal{J}\) be as in (2.12). If \((y, u)\) is an optimal pair for problem (1.3) governed by (2.5) and \(p\) is the solution of
\[
\begin{align*}
-\partial_t p - \Delta p &= \lambda |y|^{\lambda-1} p & \text{in } Q, \\
B p &= 0 & \text{on } \Sigma, \\
p(\cdot, T) &= q |y(\cdot, T) - y_d|^{q-2} (y(\cdot, T) - y_d) & \text{in } \Omega,
\end{align*}
\]
then
\[
\int_Q (p + 2Nu)(v - u) \, dx \, dt \geq 0 \quad \text{for all } v \in \mathbb{U}_{ad}.
\]
Remarks 2.7. (a) The existence of an optimal pair \((y, u)\) in Theorem 2.6 is guaranteed by Theorem 2.3. The solvability of (2.13) follows from Lemma 4.1 and Remark 4.2 below.

(b) As in Remark 2.4(ii), in Theorem 2.6 we can allow \(U_{ad}\) to be any closed convex subset of \(L_2(Q)\) if \(N > 0\). \(\Box\)

Multiplicative Controls

Next we consider the optimal control problem (1.3) governed by the equation

\[
\begin{align*}
\partial_t y - \Delta y &= |y|^{\lambda-1}y + uy, & \text{in } Q, \\
B y &= 0, & \text{on } \Sigma, \\
y(\cdot, 0) &= y^0, & \text{in } \Omega,
\end{align*}
\]

(2.14)

where \(B \in \{\gamma, \partial \nu\} \).

Theorem 2.8. Let (2.6), (2.7) and (2.8) be satisfied, \(y^0 \in W_0^2, U_{ad} \subset L_\infty(Q)\) be \(w^*\)-sequentially compact and \(U_{ad}^c = \emptyset\). Assume that \(J\) can be written in the form

\(J(y, u) = J_T(y(\cdot, T), u)\), where \(J_T : L_q(\Omega) \times (L_\infty(Q), w^*) \to \mathbb{R}\) is sequentially lower semicontinuous. Then problem (1.3) governed by (2.14) has a solution.

Remark 2.9. Similarly as in Remark 2.4(ii) and Example 2.5, all assumptions of Theorem 2.8 concerning \(U_{ad}\) and \(J\) are satisfied if, for example, \(|y^0| \leq C_0, 0 \leq D_1 \leq C_0^{\lambda-1}, 0 \leq D_2, 0 \leq N, U_{ad} = \{u \in L_\infty(Q) : -D_1 \leq u \leq D_2\}\) and

\[
J(y, u) = \int_\Omega |y(x, T) - y_d(x)|^q \, dx + N\|u\|_{L_\infty(Q)}.
\]

Again, we may take \(D_1 = \infty\) and/or \(D_2 = \infty\) if \(N > 0\). \(\Box\)

Control of Systems

Finally, let us formulate our result concerning the parabolic system

\[
\begin{align*}
\partial_t y_1 - \Delta y_1 &= y_1 y_2 - by_1 + u, & \text{in } Q, \\
\partial_t y_2 - \Delta y_2 &= ay_1, & \text{in } Q, \\
B y_1 &= B y_2 = 0, & \text{on } \Sigma, \\
y_1(\cdot, 0) &= y_1^0, & \text{in } \Omega, \\
y_2(\cdot, 0) &= y_2^0, & \text{in } \Omega,
\end{align*}
\]

(2.15)

where \(a > 0, b \in \mathbb{R}, B \in \{\gamma, \partial \nu\}\), \(y_1^0, y_2^0 \geq 0, y_1^0, y_2^0 \in C^2(\bar{\Omega}), B y_1^0 = B y_2^0 = 0\),

\(u \in L_r(J, L_z^+(\Omega)), r, z > 1, \frac{1}{r} + \frac{n}{2z} < 1\). \(2.16\)

As usual, \(L_r(J, L_z^+(\Omega))\) is the set of positive functions in \(L_r(J, L_z(\Omega))\). The regularity assumption (2.17) guarantees that (2.15) possesses a unique strong solution (defined on the maximal existence interval \(J_u\)) and this solution is Hölder continuous both in \(x\) and \(t\).
Theorem 1.1 shows that there exists $L$ weakly in $L_\infty$ for any $L$. Assume that $|x|_J$ is global and $L$. Let $\gamma$ and $n \leq 2$. Assume that $U_{ad}$ is a compact set in $L_\infty(J, L_+^r(\Omega))$, $U_{ad}^G \neq \emptyset$, and $J$ can be written in the form $J(\gamma u) = J^2(y_1(T), u)$, where $J^2 : L_\infty(\Omega) \times L_\infty(J, L_+^r(\Omega)) \to \mathbb{R}$ is lower semicontinuous, $q \in [1, \infty]$ and $J(\gamma u) \geq c_1\|y_1(T)\|_{L_\infty(\Omega)} - c_2$. Then the optimal control problem (1.3) governed by (2.15) has a solution.

Remark 2.11. As above, we can easily find examples of $U_{ad}$ and $J$ satisfying the compactness and lower semicontinuity assumptions in Theorem 2.10. The assumption $U_{ad}^G \neq \emptyset$ is satisfied if, for example, $\mathcal{B} = \gamma$, $b \geq 0$, $0 \in U_{ad}$ and $y_1^0, y_2^0$ are small enough (e.g. in $L_\infty(\Omega)$). This is due to the fact that in this case, zero is an asymptotically stable equilibrium of (2.15) with $u = 0$. If $\mathcal{B} = \partial_x$, $y_1^0 = y_2^0 = 0$ and $0 \in U_{ad}$ then obviously $0 \in U_{ad}^G$.

3. Solvability of the Model Problem

Proof of Theorem 2.3. Set $s := 0$, $q := 2\lambda$ and $p := r\lambda$. Since $r \geq 2$ and $1 < \lambda < \frac{n+2}{(n-2)r}$, there exists $\sigma \notin S_q$ satisfying

$$\frac{2}{r\lambda'} < \sigma < \frac{2}{r} \wedge \left(2 - \frac{n}{2\lambda'}\right).$$

Now Theorem A.1 guarantees the existence of a unique $L_{r\lambda}(L_{2\lambda})$-solution $y$ of (2.5) on the maximal existence interval $J_u$. Fixing $u \in U_{ad}$, this solution is global and $|y|^\lambda \in L_r(J, L_2(\Omega))$. The Sobolev maximal regularity for (2.5), [2, Theorem III.4.10.2] and interpolation theorems in [4] (also see [3, Theorem 3]) imply

$$y \in W^1_2(J, L_2(\Omega)) \cap L_r(J, W^2_2) \hookrightarrow C(J, W^3_{2,q_2}) \cap L_{r\lambda}(J, L_{2\lambda}(\Omega))$$

for any

$$z < 2 - \frac{n}{2} + \frac{q}{r},$$

where the embedding into $C(J, W^3_{2,q_2}) \cap L_{r\lambda}(J, L_{2\lambda}(\Omega))$ is compact.

Let $(y_k, u_k)$ be a minimizing sequence for problem (1.3). We may assume $u_k \to u$ weakly in $L_r(J, L_2(\Omega))$ and $\|u_k\|_{L_r(J, L_2(\Omega))} \leq C_r$. Part (a) of the proof of [6, Theorem 1.1] shows that there exists $t_0 > 0$ independent of $k$ such that

$$y_k$$

are uniformly bounded in $L_{r\lambda}([0, t_0], L_{2\lambda}(\Omega))$. (3.2)

Set $u_k(x, t) := 0$ for $t \in (T, 2T]$ and consider problem (2.5) with $J$ replaced by $[0, 2T]$. This problem possesses a unique $L_{r\lambda}(L_{2\lambda})$-solution $y_k$ defined on the maximal existence interval $J_{\bar{y}} \subset [0, 2T]$. The function $w_k(t) := y_k(T + t)$ is the $L_{r\lambda}(L_{2\lambda})$-solution of (2.5) with $u \equiv 0$, initial condition $w_k(0) = y_k(T)$ and the maximal existence interval $J_{\bar{w}} \subset [0, T]$. The boundedness of $J(y_k, u_k)$ implies a bound for $y_k(T)$ in $L_q(\Omega)$ and the well posedness of (2.5) in $L_q(\Omega)$, guaranteed by Lemma 3.1 below, shows the existence of $t_1 > 0$ such that $[0, t_1] \subset J_{\bar{w}}$ for any $k$. Consequently, all solutions $y_k$ can be continued on the interval $[T, T + t_1]$. Now Lemma 3.2 below implies $\|y_k(\tau)\|_{L_q(\Omega)} \leq C_q$ for any $\tau \in [0, T]$. Let $\tau^* = \tau^*(C_r, C_q)$ be from Lemma 3.1. Fixing $\delta \in (0, t_0 \wedge \tau^*)$ and using the last statement of Lemma 3.1 for $w_k(t) := y_k(t + \tau)$, $t \in [0, \tau^*)$, $\tau \in [t_0 - \delta, T - \tau^*]$. 

[...]

Proof of Theorem 2.10. Consider problem (2.15) with $a > 0$, $b \in \mathbb{R}$. Let (2.16), (2.17) be satisfied, where either $\mathcal{B} = \partial_x$ and $u \leq 3$ or $\mathcal{B} = \gamma$ and $n \leq 2$. Assume that $U_{ad}$ is a compact set in $L_\infty(J, L^r_+(\Omega))$, $U_{ad}^G \neq \emptyset$, and $J$ can be written in the form $J(y, u) = J_T(y_1(T), u)$, where

$$J_T : L_\infty(\Omega) \times L_\infty(J, L^r_+(\Omega)) \to \mathbb{R}$$

is lower semicontinuous, $q \in [1, \infty]$ and $J(y, u) \geq c_1\|y_1(T)\|_{L_\infty(\Omega)} - c_2$. Then the optimal control problem (1.3) governed by (2.15) has a solution.

Proof of Theorem 2.10. Consider problem (2.15) with $a > 0$, $b \in \mathbb{R}$. Let (2.16), (2.17) be satisfied, where either $\mathcal{B} = \partial_x$ and $u \leq 3$ or $\mathcal{B} = \gamma$ and $n \leq 2$. Assume that $U_{ad}$ is a compact set in $L_\infty(J, L^r_+(\Omega))$, $U_{ad}^G \neq \emptyset$, and $J$ can be written in the form $J(y, u) = J_T(y_1(T), u)$, where

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Proof of Theorem 2.10. Consider problem (2.15) with $a > 0$, $b \in \mathbb{R}$. Let (2.16), (2.17) be satisfied, where either $\mathcal{B} = \partial_x$ and $u \leq 3$ or $\mathcal{B} = \gamma$ and $n \leq 2$. Assume that $U_{ad}$ is a compact set in $L_\infty(J, L^r_+(\Omega))$, $U_{ad}^G \neq \emptyset$, and $J$ can be written in the form $J(y, u) = J_T(y_1(T), u)$, where

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we get a uniform bound for \( y_k \) in \( L_\lambda(\mathbb{R}_+; L_{2\lambda}(\Omega)) \). This bound and (3.2) show the boundedness of \( |y_k|^{\lambda-1}y_k \) in \( L_r(J, L_2(\Omega)) \). As in (3.1), we get that the sequence \( (|y_k|^{\lambda-1}y_k) \) is compact in \( L_r(J, L_2(\Omega)) \) and \( (y_k(T)) \) is compact in \( L_\eta(\Omega) \). Now it is easy to pass to the limit to get a solution of (1.3).

Let \( \lambda, q \) be as in Theorem 2.3 and let \( r \geq 2 \) satisfy (2.9). These assumptions guarantee that there exists \( s \notin S_q \) such that
\[
0 \vee \left( \frac{n}{q} \vee \frac{n}{\lambda}(2 + \frac{n}{q}) \right) < s < 2 \wedge \left( 2 + \frac{n}{q} - \frac{2}{r} \right) \wedge \left( 1 - \frac{n}{r} - \frac{n}{q} - 2 \right).
\] (3.3)

**Lemma 3.1.** Let \( \lambda, q \) be as in Theorem 2.3 and let \( r \geq 2 \) satisfy (2.9). Assume \( u \in L_r(J, L_2(\Omega)) \). Then problem (2.5) is well posed in \( L_\eta(\Omega) \). More precisely, if the norm of \( u \) in \( L_r(J, L_2(\Omega)) \) is bounded by a constant \( C_r \), \( y^0 \in L_\eta(\Omega) \), \( \|y^0\|_{L_\eta(\Omega)} \leq C_\eta \), and \( s \) satisfies (3.3), then there exists \( \tau^* = \tau^*(C_r, C_\eta) > 0 \) and a unique solution
\[
y \in C([0, \tau^*], L_\eta(\Omega)) \cap C((0, \tau^*], W^s_{q, \eta}).
\] (3.4)

In addition, this solution satisfies
\[
\|y(t)\|_{L_\eta(\Omega)} + t^{s/2}\|y(t)\|_{W^s_{q, \eta}} \leq C, \quad t \in (0, \tau^*],
\] (3.5)
where \( C \) depends only on \( s, C_r, C_\eta \) (and \( q, r, \lambda, \Omega \)). If \( \hat{q} \geq q \) satisfies
\[
\hat{q} < \frac{2n}{(n-4)_+} \quad \text{and} \quad \frac{1}{r} < 1 - \frac{n}{4} + \frac{n}{2\hat{q}}
\] (3.6)
then
\[
y \in C((0, \tau^*], L_\hat{q}(\Omega)) \] (3.7)
and
\[
\|y(t)\|_{L_\hat{q}(\Omega)} \leq C(\delta, \hat{q}, C_r, C_\eta), \quad t \in [\delta, \tau^*], \quad \delta \in (0, \tau^*).
\] (3.8)
Finally,
\[
y \in C([\delta, \tau^*], W^1_{2, \hat{q}}(\Omega)) \cap L_\lambda([\delta, \tau^*], L_{2\lambda}(\Omega))
\] (3.9)
for any \( \delta > 0 \) and the norm of \( y \) in this space can be bounded by \( C(\delta, C_r, C_\eta) \).

**Proof.** The proof of the first part is an easy modification of [8, Theorem 4.1]. In fact, let \( X \) be the Banach space of all functions
\[
y \in C([0, \tau^*], L_\eta(\Omega)) \cap C((0, \tau^*], W^s_{q, \eta})
\]
for which
\[
\|y\|_X := \sup_{t \in (0, \tau^*)} (\|y(t)\|_{L_\eta(\Omega)} + t^{s/2}\|y(t)\|_{W^s_{q, \eta}}) < \infty.
\]
Then it is sufficient to use the Banach fixed point theorem for the mapping
\[
K_y(t) = e^{-At}y^0 + \int_0^t e^{-A(t-\tau)}(|y(\tau)|^{\lambda-1}y(\tau) + u(\tau)) \, d\tau
\]
in a large closed ball \( B \) of \( X \) with radius \( R \), where \( A \) is as in (2.4). For example, assume that \( y \in B \) and denote by \( \| \cdot \|_s \) the norm in \( W^s_{q, \eta} \). Fixing \( s \) satisfying (3.3) there exists
\[
z \in (1, q] \quad \text{such that} \quad \lambda \left( \frac{n}{q} - s \right) \vee \lambda \left( \frac{2}{r} + \frac{n}{2} - 2 \right) < \frac{n}{z} < 2 + \frac{n}{q} - s\lambda.
\] (3.10)
Choose \(\sigma_1 \in (s\lambda, 2 + n/q - n/z)\) and \(\sigma_2 \in (s + 2/r, 2 + n/q - n/2)\), \(\sigma_1, \sigma_2 \notin S_q\). Then we have \(L^2(\Omega) \hookrightarrow W^{\sigma_1, 2}_{q, \mathcal{B}}\) and \(L^2(\Omega) \hookrightarrow W^{\sigma_2, 2}_{q, \mathcal{B}}\), hence it follows from (2.3) that

\[
\|t^{s/2}Ky(t)\|_s \leq C(C_q) + C t^{s/2} \int_0^t (t - \tau)^{(\sigma_1 - s)/2 - 1} \|y(\tau)\|^{1 - \epsilon} \|y(\tau)\|_{\sigma_1 - 2} d\tau
\]

\[
+ C t^{s/2} \int_0^t (t - \tau)^{(\sigma_2 - s)/2 - 1} \|u(\tau)\|_{\sigma_2 - 2} d\tau
\]

\[
\leq C(C_q) + C \epsilon t^{s/2} \int_0^t (t - \tau)^{(\sigma_1 - s)/2 - 1} \|y(\tau)\|_s d\tau
\]

\[
+ C \epsilon t^{s/2} \int_0^t (t - \tau)^{(\sigma_2 - s)/2 - 1} \|u(\tau)\|_{L_2(\Omega)} d\tau
\]

\[
\leq C(C_q) + C R \epsilon t^{s/2} \int_0^t (t - \tau)^{(\sigma_1 - s)/2 - 1/2} \tau^{-s/2} d\tau
\]

\[
+ CC_{\epsilon}t^{s/2} \int_0^t (t - \tau)^{1/2} d\tau \right)^{1/2},
\]

which shows \(t^{s/2}\|Ky(t)\|_s < R/2\) if \(R = R(C_q)\) is large enough and \(t = t(R, C_q)\) is small enough. Similar arguments show the same bound for \(\|Ky(t)\|_{L_2(\Omega)}\) and the fact that \(K\) is a contraction. Obviously, the fixed point of \(K\) is a solution of our problem. Uniqueness of this solution in the class (3.4) can be proved in the same way as in [7, pp. 295–296].

We have \(W^{\sigma_1, 2}_{q, \mathcal{B}} \hookrightarrow L_q(\Omega)\) whenever \(n/q_1 > n/q - s\). Due to the upper bound for \(s\) in (3.3), \(q_1\) is restricted by the conditions

\[
\frac{n}{q_1} > -2 + \frac{2}{r} + \frac{n}{2} \quad \text{and} \quad \frac{n}{q_1} > \frac{n}{q} - \varepsilon(q), \tag{3.11}
\]

where

\[
\varepsilon(q) := \frac{2}{\lambda} + \frac{n}{4q} + 2 - \frac{2}{r} - \frac{n}{2} > 0.
\]

Let \(\hat{q} \geq q\) satisfy (3.6). If \(n/\hat{q} > n/q - \varepsilon(q)\) then \(W^{\sigma_1, 2}_{q, \mathcal{B}} \hookrightarrow L_q(\Omega)\) since the second inequality in (3.6) guarantees that the first condition in (3.11) is satisfied with \(q_1 = \hat{q}\). Consequently, (3.7) and (3.8) follow from (3.4) and (3.5). If \(n/\hat{q} \leq n/q - \varepsilon(q)\) then we fix \(q_1 > q\) satisfying (3.11) (this is possible due to (2.9)). Now the first part of the Lemma with \(q\) replaced by \(q_1\) (and \(t = 0\) replaced by \(t = \delta_1\), where \(\delta_1 > 0\) is small) implies \(y \in C(\delta_1, \tau^*)\), \(W^{\sigma_1, 2}_{q_1, \mathcal{B}}\). Similarly as above, \(W^{\sigma_1, 2}_{q_1, \mathcal{B}} \hookrightarrow L_{q_2}(\Omega)\), where

\[
\frac{n}{q_2} > -2 + \frac{2}{r} + \frac{n}{2} \quad \text{and} \quad \frac{n}{q_2} > \frac{n}{q_1} - \varepsilon(q_1).
\]

Repeating this bootstrapping argument finitely many times, we obtain (3.7) and (3.8).

It remains to prove (3.9) and the corresponding bound. Fix \(\delta \in (0, \tau^*)\) and set

\[
t_0 := \delta/2, \quad J_0 := [t_0, \tau^*], \quad \text{and} \quad J^* := [\delta, \tau^*].
\]

Taking \(R > 1\) large and \(\bar{q}\) close to its upper bound, we have \(\bar{q} > \lambda\) and \(|y|^{\lambda} \in L_R(J_0, L_{\bar{q}/\lambda}(\Omega))\). Set

\[
f_1 := |y|^{\lambda - 1}y \quad \text{and} \quad f_2 := u.
\]

Writing

\[
y = y_1 + y_2 + y_3, \quad \text{where} \quad \mathcal{B}y_i = 0, \quad i = 1, 2, 3, \quad \text{and}
\]

\[
y_1 = Y_1 + Y_2, \quad \text{where} \quad \mathcal{B}Y_1 = \mathcal{B}Y_2 = 0.
\]

Then we have

\[
W^{\sigma_1, 2}_{q, \mathcal{B}} \hookrightarrow L_q(\Omega)
\]

and

\[
L_q(\Omega) \hookrightarrow W^{\sigma_1, 2}_{q, \mathcal{B}}.
\]

Thus (3.9) is satisfied and the corresponding bound follows from (3.4) and (3.5).
\[ \begin{align*}
    \partial_t y_1 - \Delta y_1 &= f_1 \quad \text{in } \Omega \times J_0, \quad y_1(t_0) = 0, \\
    \partial_t y_2 - \Delta y_2 &= f_2 \quad \text{in } \Omega \times J_0, \quad y_1(t_0) = 0, \\
    \partial_t y_3 - \Delta y_3 &= 0 \quad \text{in } \Omega \times J_0, \quad y_1(t_0) = g(t_0),
\end{align*} \]

(3.12)

the maximal Sobolev regularity implies

\[ y_1 \in W^1_R(J_0, L_{q/(\lambda)}(\Omega)) \cap L_R(J_0, W^{2}_{q/(\lambda), B}) \leftrightarrow C(J_0, W^1_{2, B}) \]

since we can take \( \hat{q} > 2\lambda/n/(n + 2) \) and \( R \) arbitrarily large. Similar arguments guarantee \( y_2 \in C(J_0, W^1_{2, B}) \) and \( y_3 \in C(J^*, W^1_{2, B}) \), hence \( y \in C(J^*, W^1_{2, B}) \) (and the corresponding estimate in this space is valid).

Choose \( k > 1 \) such that \( \hat{q} > (\lambda - 1/k)n/2 \) and fix \( m \in \mathbb{N} \) such that \( k^m \hat{q} > 2\lambda \).

Choose also \( R > r\lambda^{m+1} \). Set \( t_i := \delta/2 + i\delta/(2m + 2) \), \( J_i := [t_i, t^*] \) and \( \hat{q}_i := k^i \hat{q} \), \( i = 1, 2, \ldots, m \). Notice that \( y_{2, y_3} \in L_{r\lambda}(J_1, L_{2\lambda}(\Omega)) \) and \( W^2_{q/(\lambda), B} \hookrightarrow L_{\hat{q}_i}(\Omega) \), hence \( y_1 \in L_R(J_1, L_{\hat{q}_i}(\Omega)) \). Consequently, \( |y|^{\lambda} \) can be written in the form

\[ |y|^\lambda = \hat{f}_1 + \hat{f}_2, \quad \hat{f}_1 \in L_{r\lambda}(J_1, L_{\hat{q}_i}(\lambda)), \quad \hat{f}_2 \in L_r(J_1, L_{2\lambda}(\Omega)). \]

Writing \( y = \tilde{y}_1 + \tilde{y}_2 + \tilde{y}_3 \), where \( B\tilde{y}_i = 0 \), \( i = 1, 2, 3 \), and \( \tilde{y}_1, \tilde{y}_2, \tilde{y}_3 \) satisfy (3.12) with \( f_1, f_2, J_0, t_0 \) replaced by \( \tilde{f}_1, \tilde{f}_2, J_1, t_1 \), respectively, we obtain as above \( \tilde{y}_2, \tilde{y}_3, \tilde{y}_1 \in L_{r\lambda}(J_2, L_{2\lambda}(\Omega)) \) and \( \tilde{y}_1 \in L_{r\lambda}(J_1, L_{\hat{q}_i}(\Omega)) \). Repeating this argument \( m \) times we get

\[ y \in L_{r\lambda}(J^*, L_{2\lambda}(\Omega)) + L_{r\lambda}(J^*, L_{\hat{q}_i}(\Omega)) = L_{r\lambda}(J^*, L_{2\lambda}(\Omega)) \]

(and the corresponding estimates), which concludes the proof.

**Lemma 3.2.** Let \( \lambda,q,r \) be as in Theorem 2.3. Let \( t_1 > 0, \ u \in L_r([0, T + t_1], L_2(\Omega)) \) and let its norm in this space be bounded by a positive constant \( C_\ast \). Assume that \( y \) is a global solution of (2.5) (with \( J \) replaced by \([0, T + t_1]\)) and \( y^0 \in W^2_q(\Omega), \ ||y^0||_{W^2_q(\Omega)} \leq C_q \). Then there exists a constant \( C = C(C_\ast, C_q, t_1) \) such that \( ||y(t)||_{L_q(\Omega)} \leq C \) for any \( t \in [0, T] \).

**Proof.** The proof is a modification of the proof of the main result in [25] (cf. also [26] and [27, the proof of Theorem 5.1]).

All our constants (and bounds) in this proof may change from line to line and may depend on \( C_\ast, C_q, t_1 \). First we deduce from Lemma 3.1 and the beginning of the proof of Theorem 2.3 that \( y \in C([0, T + t_1], W^1_{2, B}) \) and there exists \( \tau > 0 \) such that

\[ y \text{ is bounded in } C([0, \tau], \ L_q(\Omega)) \text{ by a constant } C = C(C_\ast, C_\tau). \]  

(3.13)

Denote

\[ V(t) = \frac{1}{2} \int_{\Omega} |\nabla y(x,t)|^2 \, dx - \frac{1}{\lambda + 1} \int_{\Omega} |y(x,t)|^{\lambda + 1} \, dx. \]

If \( u \) is smooth then

\[ V'(t) = - \int_{\Omega} (\partial_t y)^2 \, dx + \int_{\Omega} u \partial_t y \, dx \leq \frac{1}{2} \int_{\Omega} u^2 \, dx - \frac{1}{2} \int_{\Omega} (\partial_t y)^2 \, dx, \]

hence

\[ V(\tau_2) - V(\tau_1) \leq C - \frac{1}{2} \int_{\tau_1}^{\tau_2} \int_{\Omega} (\partial_t y)^2 \, dx \, dt. \]  

(3.14)

Now let \( u \) be general. Approximating \( u \) by smooth functions \( u_k \) we see that (3.14) remains true for any \( u \in L_r([0, T + t_1], L_2(\Omega)) \).
We will show that \( V(t) \) is bounded for \( t \in [0, T] \). The upper estimate for \( V(t) \) follows immediately from (3.14). To prove the lower estimate we assume on the contrary that \( V(t_0) \leq - (C + K) \) for some \( t_0 \in [0, T] \), where \( C \) is from (3.14) and \( K \gg 1 \). Then (3.14) guarantees \( V(t) \leq - K \) for all \( t \geq t_0 \). Multiplying the equation in (2.5) by \( y \) and integrating over \( \Omega \) we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega y^2 \, dx = -2V(t) + c_1 \int_\Omega |y|^{\lambda+1} \, dx + \int_\Omega u y \, dx \\
\geq K + c_2 \left( \int_\Omega y^2 \, dx \right)^{(\lambda+1)/2} - C_2 \int_\Omega u^2 \, dx,
\]

(3.15)

where the inequality is true for all \( t \geq t_0 \). Denote \( Y(t) = \int_{t_0}^t \int_\Omega y^2 \, dx \, dt \). Then integrating estimate (3.15) we get

\[
Y' \geq c_3 Y^{(\lambda+1)/2} + 2K(t - t_0) - C_3.
\]

Let \( K \geq 10C_3/t_1 \). Integrating the inequality \( Y' \geq 2K(t - t_0) - C_3 \) on \([t_0, t_0 + t_1/2] \) we obtain

\[
Y(t_0 + t_1/2) \geq K \frac{t_1^2}{4} - C_3 \frac{t_1}{2} \geq K \frac{t_1^2}{5}.
\]

We also have

\[
Y' \geq c_3 Y^{(\lambda+1)/2} \quad \text{for} \quad t \geq t_0 + \frac{t_1}{2}.
\]

Since the solution of the equation \( Z'(t) = c_3 Z^{(\lambda+1)/2}(t) \) for \( t \geq 0 \), \( Z(0) = K \frac{t_1^2}{5} \), blows up at \( t < t_1/2 \) if \( K \) is large enough, the function \( Y(t) \geq Z(t - t_0 - t_1/2) \) blows up at some \( t < T + t_1 \) which yields a contradiction. Hence \( K \) has to be bounded by a constant depending only on \( c_3, C_3, t_1 \) and \( \lambda \). Consequently, \( V \) is bounded on \([0, T] \) and (3.14) provides a bound for \( y \) in the space \( W_2^1([0, T], L_3(\Omega)) \).

If \( \lambda < 1 + 4/n \) then Lemma 3.1 with \( q \) replaced by \( \tilde{q} := 2 \) and \( q \) replaced by \( q \) guarantees a bound for \( y \) in \( L_\infty([a, T], L_q(\Omega)) \) which (together with (3.13)) implies the assertion.

Let \( \lambda \geq 1 + 4/n \). Since \( y \) is bounded in \( W_2^1([0, T], L_2(\Omega)) \) \( \hookrightarrow \) \( L_\infty([0, T], L_2(\Omega)) \), we have

\[
\int_0^T \| y \|_{L_2(\Omega)}^2 \, dt \leq C \int_0^\infty \| u \|_{L_2(\Omega)}^2 \, dt \leq C, \quad z \leq r.
\]

Using this bound and the boundedness of \( V \) on \([0, T] \), we obtain from the equality in (3.15)

\[
\int_0^T \| y(t) \|^2_{L_3(\Omega)} \, dt \leq C \left( 1 + \int_0^T \| \partial_y y(t) \|_{L_2(\Omega)} \, dt \right).
\]

In particular, if \( z = 2 \) then this estimate, the bound for \( y \) in \( W_2^1([0, T], L_2(\Omega)) \) and

\[
\| \partial_y y(t) \|_{L_1(\Omega)} \leq \| \partial_y y(t) \|_{L_1(\Omega)} \| y(t) \|_{L_2(\Omega)} \| y(t) \|_{L_2(\Omega)} \leq C \| \partial_y y \|_{L_2(\Omega)}
\]

guarantee a uniform bound for \( y \) in

\[
X_z := L_{2(\lambda+1)}([0, T], L_{\lambda+1}(\Omega)).
\]

Interpolating between the bound of \( y \) in \( X_z \) and in \( W_2^1([0, T], L_2(\Omega)) \) yields a bound in \( L_\infty([0, T], L_m(\Omega)) \) provided

\[
m < \lambda + 1 - \frac{\lambda - 1}{z + 1}
\]

(3.16)
and assuming the estimate in $X_z > 0$ in order to get these estimates for some $z > 2$. Replacing $u$ by $y$, $p$ by $\lambda$, $q$ by $z$, $\tilde{q}$ by $\tilde{z}$, and $\lambda$ by $m$ in [25], denoting

$$\lambda_1 := (\lambda + 1)/\lambda, \quad \theta := \frac{\lambda + 1}{m - 2}\frac{m}{\lambda - 1}, \quad \beta := \frac{2}{(1 - \theta)\tilde{z}}$$

and assuming the estimate in $X_z$ for some $z \geq 2$, we get for $\tilde{z} > z$

$$\int_0^T \|y(t)\|_{L_{\lambda_1 + 1}(\Omega)}^\tilde{z} dt \leq C\left(1 + \int_0^T \|\partial_t y(t)\|_{L_{\lambda_1 + 1}(\Omega)}^\tilde{z} dt\right)$$

$$\leq C\left(1 + \int_0^T \|\partial_t y(t)\|_{L_{m + 1}(\Omega)}^\tilde{z} dt\right)$$

$$\leq C\left(1 + \int_0^T \|\partial_t y(t)\|_{L_{\lambda_1 + 1}(\Omega)}^{\theta\tilde{z}} \|\partial_t y(t)\|_{L_{2}(\Omega)}^{(1 - \theta)\tilde{z}} dt\right)$$

$$\leq C\left(1 + \left(\int_0^T \|\partial_t y(t)\|_{L_{\lambda_1 + 1}(\Omega)}^{\theta\tilde{z}} dt\right)^{1/\beta'}\right)^{1/\beta'}$$

provided $\tilde{z} \leq r$ and

$$u \in L_{\theta\beta'\tilde{z}}(J, L_{\lambda_1}(\Omega)). \quad (3.17)$$

Recall from [25] that the bootstrap condition $\theta \beta' \leq \lambda_1$ is satisfied if $m$ is chosen close to its upper bound and $\tilde{z}$ is close to $z$. For such $m$ and $\tilde{z}$, one can even check that $\theta \beta' < (\lambda + 1)r/((\lambda r + 2)$ provided $\tilde{z} < r$. Consequently, $\theta \beta' \tilde{z} \vee \tilde{z} < r$ (and (3.17) is true) whenever $\tilde{z} < (\lambda r + 2)/(\lambda + 1)$. Hence, we obtain a bound for $y$ in $X_z$ for any

$$z < (\lambda r + 2)/(\lambda + 1). \quad (3.18)$$

Recall that this guarantees a bound in $L_\infty([0, T], L_m(\Omega))$ for any $m$ satisfying (3.16). Using (2.10) we can find $z$ satisfying (3.18) and $m \in ((\lambda - 1)n/2, q]$ such that (3.16) is true. Now we can use Lemma 3.1 with $q$ replaced by $m$ and $\tilde{q}$ replaced by $q$ to get a bound for $y$ in $L_\infty([\tau, T], L_q(\Omega))$ which (together with (3.13)) concludes the proof. \[\square\]

**Remark 3.3.** We announced in Remark 2.4(i) that Theorem 2.3 remains true if we replace the nonlinearity $|y|^{\lambda - 1}y$ with $|y|^\lambda$. Let us sketch the proof of this statement.

Since $y$ satisfies

$$\partial_t y - \Delta y = |y|^{\lambda - 1}y + u \geq u \quad \text{in } Q,$$

the parabolic maximum principle implies $y \geq y_L$, where $y_L$ is the solution of the linear problem

$$\begin{align*}
\partial_t y_L - \Delta y_L &= u \quad \text{in } Q, \\
y_L &= 0 & \text{on } \Sigma, \\
y_L(\cdot, 0) &= y^0 & \text{in } \Omega.
\end{align*}$$

Using the same arguments as in (3.1) we see that $y_L \in L_r(\lambda, L_{2\lambda}(\Omega))$ and that the norm of $y_L$ in this space can be bounded by the norm of $u$ in $L_r(\lambda, L_2(\Omega))$ and a suitable norm of $y^0$. Notice that

$$|y|^{\lambda} = |y|^{\lambda - 1}y + 2|y|^{\lambda - 1}y^-,$$
where \( y^- := -\min(0, y) \) is bounded above by \( |y_L| \), hence \( 2|y^-|^{\lambda-1}y^- \) is bounded in \( L_r(J, L_0(\Omega)) \). Consequently, replacing \( u \) with \( \bar{u} := u + 2|y^-|^{\lambda-1}y^- \), we can repeat word by word the proof of Theorem 2.3. \( \Box \)

**Optimality of the Growth Bounds**

**Remarks 3.4.**

(i) Consider problem (2.5) with \( \Omega \) being the unit ball in \( R^n \), \( n \geq 3 \), \( B = \gamma \) and \( \lambda > (n+2)/(n-2) \). If \( n > 10 \) then assume also

\[
\lambda < 1 + 4 \frac{n-4 + 2\sqrt{n-1}}{(n-2)(n-10)}.
\]

(3.19)

Choose a smooth radial, radially decreasing function \( \psi : \Omega \to R^+ \) satisfying

\[
\psi(0) > 0 \quad \text{and} \quad \psi(x) = 0 \quad \text{for} \ x \in \Gamma \quad \text{and denote by} \ w_\alpha \quad \text{the (classical) solution of (2.5) with} \ u = 0 \quad \text{and} \ y^0 = \alpha \psi, \ \alpha \geq 0 . \quad \text{We deduce from [22] and an obvious modification of [20]} \]

that there exists \( \alpha^* > 0 \) with the following property: if \( \alpha < \alpha^* \) then \( w_\alpha(t) \) exists for all \( t \in R^+ \) and \( w_\alpha(t) \to 0 \) as \( t \to \infty \); if \( \alpha > \alpha^* \) then this solution blows up in finite time completely.

From now on fix \( y^0 = \alpha^* \psi \). Let \( y_k \) be the solution of (2.5) with \( u = 0 \) and the nonlinearity \( y^k \) replaced by \( \min(y^k, k) \), \( k = 1, 2, \ldots \). Then \( y_k \) are globally defined classical solutions, \( y_{k+1} \geq y_k \). Set \( y^*(t) = \lim_{k \to \infty} y_k(t) \). The results in [22] and [16] guarantee that \( y^* \in L_{p,loc}([0, \infty), L_p(\Omega)) \) is a weak solution solution of (2.5) with \( u = 0 \) and there exists \( T^* \in (0, \infty) \) such that \( y^* \) is a classical solution on \( (0, T^*) \) but it blows up at \( t = T^* \) in the \( L_\infty(\Omega) \)-norm. In particular, \( w_{\alpha^*} = y^*|_{[0, T^*)} \). Next [12] shows that \( y^* \) is a classical solution for all \( t \) except for finitely many points \( T_0 = T^* < T_1 < \cdots < T_k \). Choose \( T > T^* \) such that \( T \neq T_j \) for any \( j \) and let \( y_d(x) := y^*(x, T) \). Choose also \( 0 < t_1 < t_2 < T^* \) and a smooth function \( U : \Omega \times [0, T] \to [0, \infty) \) with support \( K_U \subset \Omega \times [t_1, t_2] \), \( K_U \neq \emptyset \), and denote by \( y^*_j \) the solution of (2.5) with \( u = \beta U \) and \( y^0 = \alpha^* \psi \).

Since \( y^* > 0 \) in \( K_U \) and \( y^* \to y^* \) uniformly in \( K_U \) as \( \beta \to 0^+ \), fixing \( b > 0 \) small we have \( |y^*_{\beta} - y^*| = |y^*_{\beta} - y^*| - bU \geq 0 \) in \( K_U \). Consequently, the maximum principle implies \( y^*_{\beta} \geq 0 \). Choose \( \beta \in (0, b] \). Since \( y^*(t_2) - y^*_{\beta}(t_2) \) belongs to the interior of the positive cone in \( C^\prime(\Omega) \) and \( w_\alpha(t_2) \to w_{\alpha^*}(t_2) = y^*(t_2) \) in \( C^\prime(\Omega) \) as \( \alpha \to \alpha^* \), there exists \( \alpha < \alpha^* \) such that \( y^*_{\beta}(t_2) \leq w_\alpha(t_2) \). Now the maximum principle implies \( y^*_{\beta}(t) \leq w_\alpha(t) \) for any \( t \geq t_2 \) and \( y^*_{\beta} \geq y^*_{\beta} \geq 0 \) for any \( t \geq 0 \), hence \( y^*_{\beta} \) is a global nonnegative classical solution. On the other hand, if \( \beta \geq 0 \) then \( y^*_{\beta} \) blows up at finite time \( T_\beta \leq T^* \) in the \( L_\infty(\Omega) \)-norm and, consequently, in \( L_q(\Omega) \)-norm for any \( q > n(\lambda - 1)/2 \) (cf. [14], [30]).

Let \( U_{ad} = \{ \beta U : \beta \in [-b, b] \} \). Fix \( q > n(\lambda - 1)/2 \) and set \( J(y, u) = \int_\Omega |y(T) - y_d|^q dx \). The above arguments show that \( y^*_\beta \) is a global \( L_\infty(L_q(\Omega)) \)-solution of (2.5) if and only if \( \beta < 0 \). Moreover, \( \beta \to J(y^*_\beta, \beta U) \) is decreasing on \( [-b, 0) \). Hence the optimal control problem (1.3) does not have a solution with \( y \in L_\infty(J, L_q(\Omega)) \).

(ii) Consider problem (2.5) with \( \Omega \) being the unit ball in \( R^n \), \( B = \gamma \) and let \( 1 \leq q < (\lambda-1)n/2 \). Then there exists a smooth radial positive function \( y^0 \) such that the solution \( y(t) \) of (2.5) with \( u = 0 \) blows up at \( t = T \) in the \( L_\infty \)-norm and satisfies \( \partial_T y \geq 0 \), \( y_d := y(\cdot, T) \in L_q(\Omega) \) (see [14]). Let \( U \) be a smooth nonnegative function with support \( K \subset \{(x, t) : |x| < 1/2\} \), \( K \neq \emptyset \), and \( u_c := cU \). Then there exists \( \varepsilon > 0 \) such that the solution \( y(t) \) of (2.5) with \( u \) replaced by \( \varepsilon u \) remains positive.
Let \( U_{ad} = \{ u_c \mid c \in [-\varepsilon, 0] \} \) and
\[
\mathcal{J}(y, u) = \int_{\Omega} |y(x, T)|^q \, dx - \int_{\Omega} y_0^q \, dx.
\]
Then \((y(u_{-1/k}), u_{-1/k})\), \( k \geq k_0 \), is obviously a minimizing sequence for the control problem (1.3) but \( y(u_0) \) is not a (classical) global solution of (2.5).

\[\square\]

4. Proof of the Optimality Conditions

We start with the following technical lemma concerning linear problems.

**Lemma 4.1.** Suppose that \( \beta > 2 \vee (n + 2)/2 \) and \( 2 \leq q < 2n/(n - 2)_+ \). Given \( a \in L_{\beta}(Q), \ u \in L_2(Q) \) and \( y^0 \in L_q(\Omega) \), problem
\[
\begin{align*}
\partial_t y - \Delta y &= ay + u \quad \text{in } Q, \\
By &= 0 \quad \text{on } \Sigma, \\
y(t, 0) &= y^0 \quad \text{in } \Omega,
\end{align*}
\]
has a unique solution
\[
y \in C([0, T], L_q(\Omega)) \cap C((0, T], L_q(\Omega)) \cap L_2(Q).
\]
The map
\[
L_{\beta}(Q) \times L_2(Q) \times L_q(\Omega) \to L_2(Q) \times L_q(\Omega), \quad (a, u, y^0) \mapsto (y, y(T)),
\]
is analytic and bounded on bounded sets.

**Proof.**
(i) Writing (4.1) in the abstract form
\[
\dot{y} + Ay = ay + u \quad \text{in } (0, T], \quad y(0) = y^0,
\]
and denoting \( U(t) := e^{-tA} \), we see that we have to prove the unique solvability of
\[
y = U * (ay) + U * u + Uy^0
\]
in appropriate spaces.
(ii) Fix \( s \in [0, 1) \setminus \{1/q^*\} \) such that \( q \leq 2n/(n - 2s)_+ \). Then \( W_{q', B}^s \hookrightarrow L_2(\Omega) \).
Hence we infer from (2.3) (with \( q \) replaced by \( q' \) and \( r := 0 \)) that
\[
\|U(t)y^0\|_{L_2(\Omega)} \leq c\|U(t)y^0\|_{W_{q', B}^s} \leq ct^{-s/2}\|y^0\|_{L_{q', s}(\Omega)}, \quad 0 < t \leq T.
\]
Since \( s < 1 \) it follows that
\[(y^0 \mapsto Uy^0) \in L(L_{q'}(\Omega), L_2((0, T), L_2(\Omega))) = L(L_{q'}(\Omega), L_2(Q)),\]
where \( L(X, Y) \) denotes the space of continuous linear operators from \( X \) to \( Y \).
(iii) It is easy to see that
\[(u \mapsto U * u) \in L(L_2(Q)).\]
(iv) Put \( 1/r := 1/\beta + 1/2 < 1 \) and note that
\[
L_\gamma(\Omega) \hookrightarrow W_{2, B}^{-2+\gamma} \quad \text{if} \quad 1/2 \geq 1/r + (\gamma - 2)/n,
\]
that is, if \( 0 \leq \gamma \leq 2 - n/\beta \).
(v) For \( m \in \mathbb{R} \) we write \( L_{2,m}(Q) \) for \( L_2(Q) \) endowed with the equivalent norm
\[
y \mapsto \left( \int_0^T e^{-2mt}\|y(t)\|_{L_2(Q)}^2 \, dt \right)^{1/2}.
\]
From (iv), Hölder’s inequality, and (2.3) (with $q = 2$ and $r := \gamma - 2$) we infer that

\[
\|U \ast (ay)(t)\|_{L^2(\Omega)} \leq c \int_0^t (t - \tau)^{\gamma/2 - 1} \|a(\tau)\|_{L^\beta(\Omega)} \|y(\tau)\|_{L^2(\Omega)} \, d\tau
\]

\[
= ce^{mt} \int_0^t (t - \tau)^{\gamma/2 - 1} e^{-m(t - \tau)} \|a(\tau)\|_{L^\beta(\Omega)} e^{-m\tau} \|y(\tau)\|_{L^2(\Omega)} \, d\tau.
\]

Thus, by Young’s inequality for convolutions (cf. the proof of [5, Lemma 3]), followed by Hölder’s inequality,

\[
\|U \ast (ay)\|_{L^2,\infty(\Omega)} \leq \|a\|_{L^\beta(\Omega)} \|y\|_{L^2,\infty(\Omega)},
\]

where

\[
I(m) := \left( \int_0^T t^{(\gamma/2-1)\beta'} e^{-\beta' mt} \, dt \right)^{1/\beta'},
\]

provided $\gamma > 2/\beta$. Such a choice is possible by (iv), thanks to $2/\beta < 2 - n/\beta$.

(vi) For $a \in L_\beta(\Omega)$ set $T_a(y) := U \ast (ay)$. Then (v) implies

\[
(a \mapsto (1 - T_a)^{-1})
\]

and

\[
\|T_a\|_{\mathcal{L}(L^2,\infty(\Omega))} \leq c I(m) \|a\|_{L^\beta(\Omega)}.
\]

Note that, by Lebesgue’s theorem, $I(m) \to 0$ as $m \to \infty$. Thus, given $R > 0$, there exists $m := m_R > 0$ such that $\|T_a\|_{\mathcal{L}(L^2,\infty(\Omega))} \leq 1/2$ for all $a \in L_\beta(\Omega)$ satisfying $\|a\|_{L_\beta(\Omega)} \leq R$. Consequently, $1 - T_a$ has a bounded inverse on $L^2,\infty(\Omega)$, and the map $a \mapsto (1 - T_a)^{-1}$ is analytic for $\|a\|_{L^\beta(\Omega)} \leq R$. Hence, by (4.2),

\[
y = (1 - T_a)^{-1} (U \ast u + Uy^0) \in L^2(\Omega)
\]

for $\|a\|_{L^\beta(\Omega)} \leq R$, thanks to (ii) and (iii), and the map

\[
L_\beta(\Omega) \times L^2(\Omega) \times L_{q'}(\Omega) \to L^2(\Omega), \quad (a, u, y^0) \mapsto y
\]

is analytic and bounded on bounded sets.

(vii) Let

\[
q' \leq q_1 \leq 2 \leq q_2 \leq q \quad \text{with} \quad \frac{1}{n} > \frac{1}{q_1} - \frac{1}{q_2}.
\]

Choose $s$ such that

\[
1 + n \left( \frac{1}{2} - \frac{1}{q_1} \right) > s > n \left( \frac{1}{2} - \frac{1}{q_2} \right).
\]

Then there exists $\xi \in (1/2, 1)$ such that $2 - 2\xi + n(1/2 - 1/q_1) > s$. This choice of $s, \xi$ guarantees

\[
W_{2q_2}^{s,\mathcal{B}} \hookrightarrow L^2_{q_2}(\Omega)
\]

and

\[
L^1_{q_1}(\Omega) \hookrightarrow W_{s,\mathcal{B}}^{-2+2\xi}.
\]

(viii) Let $q_1, q_2, s, \xi$ be as in (vii). For $m \in \mathbb{R}$ we denote by $C_{1-\xi,m}((0,T], L_{q_2}(\Omega))$ the Banach space of all $v \in C((0,T], L_{q_2}(\Omega))$ such that $\sup_{0 < t \leq T} t^{1-\xi} \|v(t)\|_{L_{q_2}(\Omega)} < \infty$, endowed with the norm

\[
\|v\|_{C_{1-\xi,m}} := \sup_{0 < t \leq T} t^{1-\xi} e^{-mt} \|v(t)\|_{L_{q_2}(\Omega)}.
\]
Hence there exists \( c \) such that
\[
\|U \ast u(t)\|_{L_{q_2}(\Omega)} \leq c \|U \ast u(t)\|_{W^{2,q}_2} \leq c \int_0^t (t - \tau)^{-s/2} \|u(\tau)\|_{L_2(\Omega)} d\tau
\]
for \( 0 < t \leq T \). In particular,
\[
(u \mapsto U \ast u) \in L(L_2(\Omega), C_{1-\xi,m}([0,T], L_{q_2}(\Omega))).
\]

(ix) Let \( q_1, q_2, s, \xi \) be as in (vii). Using (2.3) we get
\[
\frac{1}{r} = 1 - \frac{n + 2}{\beta} > s - \frac{n}{\beta} \left( \frac{1}{2} - \frac{1}{q_2} \right).
\]
Then there exists \( \eta > 1/\beta \) such that
\[
2 - \frac{n + 2}{\beta} > s - \frac{n}{\beta} \left( \frac{1}{2} - \frac{1}{q_2} \right).
\]
Hence
\[
L_{r}(\Omega) \hookrightarrow W^{2,2+2\eta}_{2,\beta},
\]
where \( 1/r := 1/\beta + 1/q_2 \). With this choice it follows that
\[
e^{-mt}\|U \ast (ay)(t)\|_{L_{q_2}(\Omega)} \leq ce^{-mt}\|U \ast (ay)(t)\|_{W^{2,q}_2}
\]
\[
\leq ce^{-mt} \int_0^t (t - \tau)^{\eta - 1} \|a(\tau)\|_{L_\beta(\Omega)} \|y(\tau)\|_{L_{q_2}(\Omega)} d\tau
\]
\[
\leq c \int_0^t (t - \tau)^{\eta - 1} e^{-m(t - \tau)} \|a(\tau)\|_{L_\beta(\Omega)} \|y\|_{C_{1-\xi,m}} d\tau
\]
for \( 0 < t \leq T \). Thus, by Hölder’s inequality,
\[
t^{1-\xi}e^{-mt}\|U \ast (ay)(t)\|_{L_{q_2}(\Omega)} \leq cK(t, m)\|a\|_{L_\beta(\Omega)}\|y\|_{C_{1-\xi,m}},
\]
where
\[
K(m, t) := t^{1-\xi}\left( \int_0^t (t - \tau)^{\eta - 1} e^{-m(t - \tau)} d\tau \right)^{1/\beta'} = t^{\eta - 1/\beta'} \left( \int_0^1 (1 - \sigma)^{\eta - 1} e^{-m(1 - \sigma)} d\sigma \right)^{1/\beta'}.
\]
Fix any \( \delta \in (0, T) \). Then \( K(t, m) \to 0 \) as \( m \to \infty \) by Lebesgue’s theorem, uniformly with respect to \( t \in [\delta, T) \). If \( 0 < \delta \leq \delta \) then
\[
K(t, m) \leq c\delta^{\eta - 1/\beta'}.
\]
Thus, given \( R > 0 \), it follows that we can fix \( m > 0 \) such that
\[
\|T_a\|_{C_{1-\xi,m}([0,T], L_{q_2}(\Omega))} \leq 1/2
\]
for all \( a \in L_\beta(Q) \) satisfying \( \|a\|_{L_\beta(Q)} < R \). Now we infer from (viii) and (ix) that
\[
y = (1 - T_a)^{-1}(U \ast u + Uy^0) \in C_{1-\xi,m}([0,T], L_{q_2}(\Omega))
\]
for \( y^0 \in L_{q_1}(\Omega), \|a\|_{L_\beta(Q)} < R \) and that the map
\[
L_\beta(Q) \times L_2(Q) \times L_{q_1}(\Omega) \to C_{1-\xi,m}([0,T], L_{q_2}(\Omega)), \quad (a, u, y^0) \mapsto y
\]
is analytic and bounded on bounded sets. Using this property for the couple 
\((q_1, q_2) := (q’, 2)\) and, subsequently, for \((q_1, q_2) := (2, q)\), we see that the map

\[ L_\beta(Q) \times L_2(Q) \times L_{q^*}(\Omega) \rightarrow L_q(\Omega), \quad (a, u, y^0) \mapsto y(T) \]

is analytic and bounded on bounded sets. This concludes the proof. □

**Remark 4.2.** Lemma 4.1 guarantees the solvability of (2.13): notice that

\[ q < 2 \]

Due to

\[ q \]

Proof of Theorem 2.6. Choose \( v \in U_{ad} \), \( \mu \in [0, 1] \) and let \( y_\mu \) be the solution of

\[ (2.5) \]

with \( u \) replaced by \( u + \mu(v - u) \). If \( \mu \) is small enough, say \( \mu \leq \mu_0 \), then due to

the stability estimates in Theorem A.1 and the regularity results in Theorem 2.3,

the solution \( y_\mu \) is global and satisfies

\[
\|y_\mu - y\|_{L_2(Q)} + \|y_\mu(\cdot, T) - y(\cdot, T)\|_{L_q(\Omega)} \leq C\mu\|v - u\|_{L_2(Q)}.
\] (4.7)

Assume \( \mu \leq \mu_0 \) and set \( z_\mu := (y_\mu - y)/\mu \). Then \( z_\mu \) solves the problem

\[
\begin{aligned}
\partial_t z_\mu - \Delta z_\mu &= a_\mu z_\mu + (v - u), & x \in \Omega, \ t \in J, \\
Bz_\mu &= 0, & x \in \Gamma, \ t \in J,
\end{aligned}
\] (4.8)

where \( a_\mu := \lambda \int_0^1 |y + \theta(y_\mu - y)|^{\lambda - 1} \, d\theta \). Let \( z \) be the solution of

\[
\begin{aligned}
\partial_t z - \Delta z &= a z + (v - u), & x \in \Omega, \ t \in J, \\
Bz &= 0, & x \in \Gamma, \ t \in J,
\end{aligned}
\]

where \( a := \lambda |y|^{\lambda - 1} \). Set \( \beta := 2\lambda/(\lambda - 1) \). Since \( a_\mu \to a \) in \( L_\beta(Q) \) as \( \mu \to 0 \), Lemma 4.1 implies

\[ z_\mu(\cdot, T) \to z(\cdot, T) \quad \text{in} \quad L_q(\Omega). \] (4.9)

Set

\[
\mathcal{I}_1(\mu) := \int_\Omega |y_\mu(\cdot, T) - y^*|^q \, dx, \quad \mathcal{I}_2(\mu) := N \int_Q (u + \mu(v - u))^2 \, dx \, dt.
\]

The mapping \( L_q(\Omega) \to \mathbb{R} : \varphi \mapsto \int_\Omega |\varphi - y^*|^q \, dx \) is convex. Hence

\[
q \int_\Omega |y(\cdot, T) - y^*|^q \, dx \leq q \int_\Omega |y_\mu(\cdot, T) - y^*|^q \, dx \leq \frac{\mathcal{I}_1(\mu) - \mathcal{I}_1(0)}{\mu} \leq q \int_\Omega |y_\mu(\cdot, T) - y^*|^q \, dx.
\]

Since (4.7) implies

\[ |y_\mu(\cdot, T) - y^*|^q \rightarrow |y(\cdot, T) - y^*|^q \quad \text{in} \quad L_q(\Omega) \]

and (4.9) is true, we see that \( \mathcal{I}_1 \) is right differentiable at 0 and

\[ \mathcal{I}_1(0^+) = \int_\Omega p(\cdot, T)z(\cdot, T) \, dx. \]

We have also \( \mathcal{I}_2(0) = 2N \int_Q u(v - u) \, dx \, dt \) and

\[ (\mathcal{I}_1 + \mathcal{I}_2)(\mu) = J(y_\mu, u + \mu(v - u)) \geq J(y, u) = (\mathcal{I}_1 + \mathcal{I}_2)(0), \]

hence

\[ \int_\Omega p(\cdot, T)z(\cdot, T) \, dx + 2N \int_Q u(v - u) \, dx \, dt \geq 0. \]
Consequently, it is sufficient to show that
\[
\int_{\Omega} p(\cdot,T)z(\cdot,T) \, dx = \int_{Q} p(v-u) \, dx \, dt.
\]
Let \( \varphi_k \in \mathcal{D}(\Omega) \) be such that \( \varphi_k \to p(\cdot,T) \) in \( L_p(\Omega) \) and \( a_k \in \mathcal{D}(Q) \) be such that \( a_k \to a \) in \( L_p(Q) \). Let \( p_k \) be the solution of (2.13) with \( a = \lambda|y|^{\lambda-1} \) replaced by \( a_k \) and the final condition replaced by \( p_k(\cdot,T) = \varphi_k \). Then \( p_k \) is smooth and \( \|p_k\| \to \|p\| \) in \( L_2(Q) \) due to Lemma 4.1. Notice that \( z \in L_{2\lambda}(Q) \) due to Theorem A.1 (cf. the beginning of the proof of Theorem 2.3), hence \( az \in L_2(Q) \) and the maximal Sobolev regularity implies \( \partial_t z, \Delta z \in L_2(Q) \). We have
\[
\int_{Q} p_k(v-u) \, dx \, dt = \int_{Q} p_k(\partial_t z - \Delta z - az) \, dx \, dt
\]
\[
= \int_{Q} (-\partial_t p_k - \Delta p_k - ap_k)z \, dx \, dt + \int_{\Omega} \varphi_k z(\cdot,T) \, dx
\]
\[
= \int_{Q} (a_k - a) p_k z \, dx \, dt + \int_{\Omega} \varphi_k z(\cdot,T) \, dx \to \int_{\Omega} p(\cdot,T)z(\cdot,T) \, dx,
\]
since \( p_k \) stay bounded in \( L_2(Q) \) due to Lemma 4.1. Now
\[
\int_{Q} p_k(v-u) \, dx \, dt \to \int_{Q} p(v-u) \, dx \, dt
\]
concludes the proof.

\[\square\]

5. The Case of a Multiplicative Control

**Proof of Theorem 2.8.** The proof is almost the same as in Theorem 2.3 (but the solutions \( y \) are more regular now). The only nontrivial modification is required in the estimate of the function \( V \) and the \( L_2(Q) \)-norm of \( \partial_t y \) in the proof of Lemma 3.2.

Hence, assume \( y \in C([0,T + t_1],W^1_2(\Omega)) \) is a solution of (2.14), where \( t_1 > 0 \) is fixed. Since \( \mathbb{U}_{\text{ad}} \) is bounded in \( L_{\infty}(Q) \), there exists a constant \( M \) such that \( \|u\|_{L_{\infty}(Q)} \leq M \) for all \( u \in \mathbb{U}_{\text{ad}} \). Let \( V \) be defined as in the proof of Lemma 3.2. Then
\[
V'(t) = -\int_{\Omega} (\partial_t y)^2(t) \, dx + \int_{\Omega} uy \partial_y(t) \, dx
\]
\[
\leq \frac{M^2}{2} \int_{\Omega} y^2(t) \, dx - \frac{1}{2} \int_{\Omega} (\partial_t y)^2(t) \, dx.
\]
Let \( \tau < 1, \ T + t_1, \ \text{and} \ t \in [0,T] \). Denoting \( C_0 := \int_{\Omega} y^2(x,0) \, dx \), we have
\[
\int_{\Omega} y^2(t) \, dx \leq C_0 + 2 \int_{0}^{t} \int_{\Omega} y \partial_t y \, dx \, dt \leq C_0 + \int_{0}^{T} \int_{\Omega} y^2 \, dx \, dt + \int_{0}^{T} \int_{\Omega} (\partial_t y)^2 \, dx \, dt.
\]
Integrating this estimate over \( t \in [0,T] \), we get
\[
\int_{0}^{T} \int_{\Omega} y^2 \, dx \, dt \leq C_0 T + T \int_{0}^{T} \int_{\Omega} y^2 \, dx \, dt + \tau \int_{0}^{T} \int_{\Omega} (\partial_t y)^2 \, dx \, dt,
\]
hence
\[
\int_{0}^{T} \int_{\Omega} y^2 \, dx \, dt \leq \frac{C_0 T}{1 - \tau} + \tau \int_{0}^{T} \int_{\Omega} (\partial_t y)^2 \, dx \, dt.
\]
(5.2)
Let $\tau_1 \in (0,1)$ be defined by $\tau_1 M^2 = \frac{1}{2}$ and $\tau \in [0,\tau_1]$ (enlarging $M$ we may assume $\tau_1 \leq T + t_1$). Then integrating (5.1) and using (5.2) we arrive at

$$V(\tau) - V(0) \leq \frac{C_0}{4} - \frac{1}{4} \int_0^\tau (\partial_t y)^2 \, dx \, dt, \quad \tau \in [0,\tau_1].$$

(5.3)

This estimate guarantees $V(t) \leq V(0) + C_0/4$ on $[0,\tau_1]$.

Fix $\delta \in (0, t_1 \wedge \tau_1)$ and assume $V(t_0) \ll -1$ for some $t_0 \in [0,\tau_1 - \delta]$. Then (5.3) implies $V(t) \leq -K \ll -1$ for all $t \in [\tau_1 - \delta, \tau_1]$. As in (3.15) we have

$$\frac{1}{2} \frac{d}{dt} \int_\Omega y^2 \, dx = -2V(t) + c_1 \int_\Omega |y|^{\lambda + 1} \, dx + \int_\Omega uy^2 \, dx$$

$$\geq K + c_2 \left( \int_\Omega y^2 \, dx \right)^{(\lambda+1)/2}$$

for any $t \in [\tau_1 - \delta, \tau_1]$. In the same way as in the proof of Lemma 3.2, this inequality yields a contradiction if $K = K(\lambda, c_2, \delta)$ is large enough. Consequently,

$$V(t) \geq -C \quad \text{for all } t \in [0,\tau_1 - \delta].$$

(5.4)

Now (5.3) implies $\int_0^{\tau_1 - \delta} \int_\Omega (\partial_t y)^2 \, dx \, dt \leq C$, hence $\int_0^{\tau_1} y^2(t) \, dx \leq C$ for $t$ belonging to $[0,\tau_1 - \delta]$. In particular, $\int_\Omega y^2(\tau_1 - \delta) \, dx \leq C_1$, where $C_1$ does not depend on $u$.

Repeating the estimates above on the interval $[\tau_1 - \delta, 2\tau_1 - \delta]$ instead of $[0,\tau_1]$ and then on $[2\tau_1 - 2\delta, 3\tau_1 - 2\delta]$ etc, we obtain the desired bounds for $V(t), \|y(t)\|_{L^2(\Omega)}$, $t \in J$, and $\|\partial_t y\|_{L^2(J)}$.

\[\square\]

6. Parabolic Systems

Proof of Theorem 2.10. Let $\varphi_1 > 0$ be an eigenfunction corresponding to the first eigenvalue $\mu_1$ of the problem $-\Delta \varphi_1 = \mu_\varphi$ in $\Omega$, $\mathcal{B}_1 \varphi = 0$ on $\Gamma$. Notice that $\varphi_1$ is a positive constant if $\mathcal{B}_1 = \partial_r$, hence the weighted Lebesgue space $L^p(\Omega, \varphi_1(x) \, dx)$ equals $L^1(\Omega)$ in this case.

We shall prove that

(i) any bound of $y_1(t)$ in $L^p(\Omega, \varphi_1(x) \, dx)$ or $L^p(\Omega)$, $p \geq 1$, implies a bound of $y_2(t)$ in the same space;

(ii) the space $X := L_1(\Omega, \varphi_1(x) \, dx) \times L_1(\Omega, \varphi_1(x) \, dx)$ is a continuation space for problem (2.15), that is, if the solution $y$ is defined on $[0, T^*)$, $T^* > 0$, and $\|y(T^*)\|_X \leq M$ then this solution can be continued for $t \in [T^*, T^* + \tau]$, where $\tau = \tau(M) > 0$. In addition, $\|u(t)\|_{L^\infty(\Omega) \times L^\infty(\Omega)} \leq C(\delta, M)$ for any $t \in [T^* + \delta, T^* + \tau]$ and $\delta > 0$;

(iii) all global solutions of problem (2.15) with $u$ bounded in $L_r(J, L^+_\infty(\Omega))$ and $y_1(T)$ bounded in $L_q(\Omega)$ are uniformly bounded in $L^\infty(Q)$.

Then the conclusion follows similarly as in the proof of Theorem 2.3.

(i) Let $u \in U_{ad}$. Set $w := y_2^2/2 - by_2 - ay_1$. One can easily verify

$$\partial_t w - \Delta w \leq -au \leq 0,$$

hence the comparison principle guarantees $w \leq C$ in $Q$, where $C$ does not depend on $u$. This estimate implies

$$y_2^2 \leq C(1 + y_1),$$

(6.1)

and the conclusion follows.

(ii) Set $z := ay_1 + by_2$. Then

$$\partial_t z - \Delta z = ay_1 y_2 + au \leq C(1 + z^{3/2}) + au.$$

(6.2)
Since \( n < 4 \) if \( B = \partial_\nu \) and \( n < 3 \) if \( B = \gamma \), the problem \( \partial_t \tilde{z} - \Delta \tilde{z} = C(1 + |z|^{3/2}) + au \), \( B \tilde{z} = 0 \), is well posed in \( X_1 := L_1(\Omega, \varphi_1(x) \, dx) \) due to [30] and [13], respectively. More precisely, if \( \|\tilde{z}(0)\|_{X_1} \leq M \) then there exists \( \tau = \tau(M) > 0 \) such that the solution \( \tilde{z} \) exists on \( [0, \tau] \) and satisfies \( \|\tilde{z}(t)\|_{L_\infty(\Omega)} \leq C(\delta, M) \) for any \( t \in [\delta, \tau] \) and \( \delta > 0 \). A comparison argument shows that the same estimate is true for the function \( z \). In particular, the space \( X \) is a continuation space for (2.15) in the sense described above.

(iii) Now assume that \( u \) belongs to a bounded set in \( \mathbb{W}^G \subset L_r(J, L^+_{s}(\Omega)) \) and \( y_1(T) \) is bounded in \( L_1(\Omega) \). The above arguments show that the solution \( y \) can be continued for \( t \in [0, T + \tau] \), where \( \tau > 0 \) does not depend on \( u \) and \( u(x, t) := 0 \) if \( t > T \). Multiplying the second equation in (2.15) with \( \varphi_1 \) and using (6.1) we obtain

\[
\partial_t \int_\Omega y_2 \varphi_1 \, dx + \mu_1 \int_\Omega y_2 \varphi_1 \, dx = a \int_\Omega y_1 \varphi_1 \, dx \geq c \int_\Omega y_2^2 \varphi_1 \, dx - C \geq c \left( \int_\Omega y_2 \varphi_1 \, dx \right)^2 - C
\]

for any \( t \in [0, T + \tau] \). Using standard blow-up arguments (cf. the arguments following (3.15) in the proof of Lemma 3.2), this estimate guarantees a uniform bound for \( y_2(t) \), \( t \in [0, T + \tau/2] \), in the weighted space \( L_1(\Omega, \varphi_1(x) \, dx) \). Integrating the second equation in (2.15) we obtain now

\[
\int_0^{T + \tau/2} \int_\Omega y_1 \varphi_1 \, dx \, dt \leq C. \tag{6.3}
\]

The first equation in (2.15) implies

\[
\int_\Omega y_1 \varphi_1 \, dx \bigg|_{t_1}^{t_2} + (\mu_1 + b) \int_{t_1}^{t_2} \int_\Omega y_1 \varphi_1 \, dx \, dt \geq 0,
\]

hence using (6.3) we deduce

\[
\int_\Omega y_1(t_2) \varphi_1 \, dx \geq \int_\Omega y_1(t_1) \varphi_1 \, dx - C \tag{6.4}
\]

for any \( t_1, t_2 \in [0, T + \tau/2], \ t_2 > t_1 \).

Obviously, (6.3) and (6.4) imply a uniform estimate for \( y_1(t), \ t \in J \), in the space \( L_1(\Omega, \varphi_1(x) \, dx) \). Now (i) and (ii) imply uniform bounds for \( y_1, y_2 \) in \( L_\infty([\delta, T] \times \Omega) \) for any \( \delta > 0 \). Since the bounds for \( y_1, y_2 \) in \( L_\infty([0, \delta] \times \Omega) \) for \( \delta > 0 \) small enough are guaranteed by the well posedness of (2.15) in \( L_\infty(\Omega) \times L_\infty(\Omega) \) and the boundedness of \( u \) in \( L_r(J, L_\infty(\Omega)) \), the conclusion follows.

**Appendix: The Basic Existence, Uniqueness, and Stability Theorem for Semilinear Problems**

For the reader’s convenience we collect here the main existence, uniqueness and stability results for strong solutions of the semilinear problem

\[
\dot{y} + A_0 y = F(y) \text{ in } [0, T], \quad y(0) = y_0^0, \tag{A.1}
\]

where \( A = A_0 \) be the isomorphism between \( W^{s}_{q,B} \) and \( W^{s-2}_{q,B} \) mentioned in Section 2. They follow from [6, Theorems 3.3, 3.4] and [5, Theorems 5, 7(ii)]. Analogous results are true in the case of systems.
We write $C^1_b^{-}(Y, X)$ for the space of all maps from $Y$ into $X$ which are uniformly Lipschitz continuous on bounded sets. If $X$ and $Y$ are spaces of functions defined on $[0, T]$, then $F : X \to Y$ is said to possess the Volterra property if, given any $u \in X$ and $t \in (0, T)$, the restriction of $F(u)$ to $[0, t]$ depends on the values of $u|_{[0, t]}$ only.

Theorem A.1. Assume
\[ s, \sigma \notin S_q, \quad 0 \leq s < \sigma < 2, \]  
and suppose that $r > 1, \ r \neq 2/(\sigma - s), \ \sigma - 2/r \notin S_q, \ y^0 \in Y^0 := W^{\sigma - 2/r}_q$. Denote $X_t := L_r([0, t], W^{\sigma - 2}_q)$. If $r < 2/\sigma - s$ fix $p \in [1, 2/(\sigma - s + 2/r))$ and set $Y_t := L_p([0, t], W^p_q)$. Let $F \in C^1_b^{-}(Y_t, X_t)$ have the Volterra property. If $r < 2/\sigma - s$ or $r > 2/(\sigma - s)$ then (A.1) has a unique strong $L_p(W^p_q)$ or $C^p(W^p_q)$-solution $y(y^0, F)$, respectively, defined on the maximal existence interval $[0, t(y^0, F))$. If $y(y^0, F)$ is global. Let $\omega_1 > 0$, and let $\omega_2 : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function,
\[
\begin{align*}
\|y^0\|_{Y^n} + \|F(0)\|_{X_T} &\leq \omega_1, \\
\|F(y_1) - F(y_2)\|_{X_T} &\leq \omega_2(R)\|y_1 - y_2\|_{Y_T}.
\end{align*}
\]  
for any $R > 0$ and $y_1, y_2 \in Y_T$ whose norms are bounded by $R$. Fix $R > \|y(y^0, F)\|_{Y_1}$. Then there exist positive constants $\varepsilon, c$ (depending only on $R, t, \omega_1, \omega_2$) with the following property: If $\tilde{y}^0 \in Y^0$, $\tilde{F} \in C^1_b^{-}(Y_T, X_T)$ has the Volterra property, $\tilde{y}^0$ and $\tilde{F}$ satisfy (A.3) and
\[
\|y^0 - \tilde{y}^0\|_{Y^n} + \sup_{\|y\|_{Y_T} \leq R} \|(F - \tilde{F})(y)\|_{X_T} \leq \varepsilon,
\]  
than $t \leq t(\tilde{y}^0, \tilde{F})$, $y(\tilde{y}^0, \tilde{F}) \in Y_t$ and
\[
\|y(y^0, F) - y(\tilde{y}^0, \tilde{F})\|_{Y_T} \leq c\left(\|y^0 - \tilde{y}^0\|_{Y^n} + \sup_{\|y\|_{Y_T} \leq R} \|(F - \tilde{F})(y)\|_{X_T}\right).
\]
If $y = y(y^0, F)$ is global then
\[
y \in L_r(J, W^p_{q, B}) \cap W^1_r(J, W^{\sigma - 2}_{q, B}) \quad (A.4)
\]  
for any $\tilde{\sigma} < \sigma$, and the norm of $y$ in this space can be estimated by a constant $C = C(\|F(y)\|_{X_T}, y^0, y^n).$

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